

July 2004

B and B_s decay constants from moments of Finite Energy Sum Rules in QCD¹

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Abstract

We use an appropriate combination of moments of finite energy sum rules in QCD in order to compute the B_q -meson decay constants f_B and f_{B_s} . We perform the calculation using a two-loop computation of the imaginary part of the pseudoscalar two point function in terms of the running bottom quark mass. The results are stable with the so called QCD duality threshold and they are in agreement with the estimates obtained from Borel transform QCD sum rules and lattice computations.

PACS: 12.38.Bx, 12.38.Lg.

¹Supported by MCYT-FEDER under contract FPA2002-00612.

1 Introduction

Since the pioneering work of Shifman, Vainshtein and Zacharov [1], Laplace sum rules have been successfully applied to calculate all sort of parameters of the hadronic spectrum. The main advantage of this type of sum rules is that a Borel transform applied to the correlation function enhances the contribution of the low lying resonances of the hadronic spectrum, which properties are to be determined. On the other hand, it reinforces the convergence of the QCD asymptotic calculation in the high energy domain. The price to pay with this method is the appearance of the so-called Borel parameter which has to be fixed by stability arguments. Other particular sum rules based on Hilbert transform, inverse moments..., have also been used to suitably deal with other particular problems.

In this note we introduce a method based on positive moments of QCD finite energy sum rules. Traditionally, this type of sum rules have the disadvantage of reducing the contribution of the low energy part of the hadronic spectrum, whereas they enhance the QCD high energy region. Although they are easy to handle, one needs to fix a QCD duality threshold, where theoretical calculations are accurate enough and at the same time the low energy region admits a suitable hadronic parametrization. Nevertheless, it is not always easy to fix the value of this duality threshold in the sense that the results have to be independent of this value. The method we propose is to combine different moments of finite energy sum rules in order to get a polynomial weight for the correlation function such that its contribution becomes negligible when integrated from the continuous physical threshold until the QCD duality threshold. In this way, the low lying resonance region is enhanced in the hadronic spectrum and, on the other hand, the role of the QCD duality threshold becomes less relevant on the final results. We will be more explicit on this in the next section.

Here we use our method to evaluate the decay constants of the lightest pseudoscalar bottom mesons (f_B and f_{B_s}), which parametrize the B_q -meson matrix elements of the pseudoscalar current:

$$\langle \Omega | (M_b + m_q) \bar{q} i \gamma_5 b(0) | B_q \rangle = f_{B_q} M_{B_q}^2.$$

These decay constants have received recently a lot of attention since they enter in hadronic matrix elements of $B - \bar{B}$ mixing, and its accurate evaluation would facilitate a better determination of the B_B mixing factor from recent experiments carried out in the B-factories. They also appear in the leptonic B decay widths and its knowledge could provide a good determination of the $|V_{qb}|$ matrix element in future experiments. Calculations of these decay constants have been performed since the eighties, with results in the range of $f_B = 160 - 210$ MeV and $f_{B_s}/f_B = 1.09 - 1.22$ from Borel transform techniques [2, 4, 5, 6]. Computations in lattice QCD give also results in a wide range: $f_B = 161 - 218$ MeV and $f_{B_s}/f_B = 1.11 - 1.16$ [8, 9] (for a review and a collection of the results, see [7]). As we see, in these range of values there is still room for improvement.

The plan of this note is the following: in the next section we briefly review the theoretical method proposed, in the third section we discuss the theoretical and experimental inputs used in the calculation and in the fourth one we present our results for the decay constants with a discussion of the errors. We finish the paper giving the conclusions.

2 The method

The two point function relevant to our problem is:

$$\Pi(s = q^2) = i \int dx e^{iqx} \langle \Omega | T(j_5(x)j_5(0)) | \Omega \rangle,$$

where $\langle \Omega |$ is the physical vacuum and the current $j_5(x)$ is the divergence of the axial-vector current:

$$j_5(x) = (M_Q + m_q) : \bar{q}(x) i\gamma_5 Q(x) :$$

M_Q is the mass of the heavy quark $Q(x)$ which will be the bottom quark in our case, whereas m_q stands for the light quark mass, up, down or strange. In order to write down the sum rules relevant to our calculation, we apply Cauchy's theorem to the two point correlation function $\Pi(s)$, weighted with a polynomial $P(s)$ as indicated in the following:

$$\frac{1}{2\pi i} \oint_{\Gamma} s^i P(s) \Pi(s) ds = 0, \quad (1)$$

(the power s^i , with $i \geq 0$, is introduced here for convenience, as it will become apparent in the following).

The integration path Γ is extended along a circle of radius $|s| = s_0$, and along both sides of the physical cut starting at the physical threshold $s_{\text{ph.}}$, i. e. running in the interval $s \in [s_{\text{ph.}}, s_0]$. Neither the polynomial $P(s)$ nor the power of the integration variable change the analytical properties of $\Pi(s)$, so that we obtain the following sum rule:

$$\frac{1}{\pi} \int_{s_{\text{ph.}}}^{s_0} s^i P(s) \text{Im} \Pi(s) ds = - \frac{1}{2\pi i} \oint_{|s|=s_0} s^i P(s) \Pi(s) ds \quad (2)$$

On the left hand side of this equation, we enter the experimental information of $\text{Im} \Pi(s)$, starting from the physical threshold $s_{\text{ph.}}$ till the integration radius s_0 , whereas, on the right hand side, we consider the asymptotic QCD theoretical calculations to be plugged into the integration contour of radius s_0 . Therefore, this radius makes the compromise where the contribution of the hadronic spectrum can be approximated by the QCD calculation. This is our QCD duality threshold that we refer to in the previous section.

The asymptotic expansion of QCD ($\Pi^{\text{QCD}}(s)$) can be split in two parts, including the perturbative and non-perturbative terms, as follows:

$$\Pi^{\text{QCD}}(s) = \Pi^{\text{pert.}}(s) + \Pi^{\text{nonpert.}}(s), \quad (3)$$

At this stage, we consider that $\Pi^{\text{pert.}}(s)$ is an analytic function of s , with a real cut starting at $s_{\text{QCD}} = (M_b + m_q)^2$, therefore, we can use again the Cauchy's theorem to convert the integration of $\Pi^{\text{pert.}}(s)$ along the circle $|s| = s_0$ into an integration of the corresponding absorptive part along the QCD cut,

$$\begin{aligned} & \frac{1}{\pi} \int_{s_{\text{ph.}}}^{s_0} s^i P(s) \text{Im} \Pi(s) ds = \\ &= \frac{1}{\pi} \int_{(M_b + m_q)^2}^{s_0} s^i P(s) \text{Im} \Pi^{\text{pert.}}(s) ds - \frac{1}{2\pi i} \oint_{|s|=s_0} s^i P(s) \Pi^{\text{nonpert.}}(s) ds \end{aligned} \quad (4)$$

On the right hand side of equation (4) we take for the perturbative spectral function the exact two-loop QCD calculation [2]

$$\begin{aligned} & \frac{1}{\pi} \text{Im} \Pi^{\text{pert.}}(s) = \\ &= \frac{3}{8\pi^2} (M_b + m_q)^2 s \left(1 - \frac{M_b^2}{s}\right)^2 \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \frac{2}{3} \left[4 \text{dilog} \left(\frac{M_b^2}{s} \right) \right. \right. \\ &+ 2 \ln \left(\frac{M_b^2}{s} \right) \ln \left(1 - \frac{M_b^2}{s} \right) - \left(5 - \frac{2M_b^2}{s} \right) \ln \left(1 - \frac{M_b^2}{s} \right) \\ &+ \left[\left(1 - 2 \frac{M_b^2}{s} \right) \left(3 - \frac{M_b^2}{s} \right) \ln \left(\frac{M_b^2}{s} \right) + \frac{17}{2} - \frac{33M_b^2}{2s} \right. \\ &\left. \left. - 3 \left(1 - \frac{3M_b^2}{s} \right) \ln \left(\frac{M_b^2}{\mu^2} \right) \right] \left(1 - \frac{M_b^2}{s} \right)^{-1} \right\} \end{aligned} \quad (5)$$

where $M_b = M_b(\mu)$ is the running mass of the b quark in the \overline{MS} scheme. It is known that the expansion in terms of the running mass converges much faster, in a wide range of the renormalization scale, than the one in terms of the bottom pole mass, as notice, for instance, in [4].

For the non perturbative part $\Pi^{\text{nonpert.}}(s)$ we take the contribution coming from the vacuum expectation values of non perturbative operators up to dimension six [3, 2]:

$$\begin{aligned} \Pi^{\text{nonpert.}}(s) &= \frac{M_{\text{pole}}^2}{s - M_{\text{pole}}^2} \left[M_{\text{pole}} \langle \bar{q}q \rangle - \frac{1}{12} \langle \frac{\alpha_s}{\pi} G^2 \rangle \right] - \\ &- \frac{1}{2} M_{\text{pole}}^3 \left[\frac{1}{(s - M_{\text{pole}}^2)^2} + \frac{M_{\text{pole}}^2}{(s - M_{\text{pole}}^2)^3} \right] \langle \bar{q}\sigma Gq \rangle - \end{aligned}$$

$$- \frac{8}{27}\pi M_{\text{pole}}^2 \left[\frac{2}{(s - M_{\text{pole}}^2)^2} + \frac{M_{\text{pole}}^2}{(s - M_{\text{pole}}^2)^3} - \frac{M_{\text{pole}}^4}{(s - M_{\text{pole}}^2)^4} \right] \alpha_s < \bar{q}q >^2 . \quad (6)$$

The contour integration of the non perturbative part is easily done by means of the residues theorem.

Again, for calculational purposes, we consider in this non perturbative expansion the relation between the pole mass M_{pole} and the running mass M_b in the appropriate order of the coupling constant [4, 11, 12].

Finally, on the left hand side of equation (2), we parametrize the absorptive part of the two point correlation function by means of a narrow width approximation of the lightest B_q resonance plus the hadronic continuum of the $b\bar{q}$ channel starting at s_{cont} , above the resonance region:

$$\frac{1}{\pi} \text{Im}\Pi(s) = M_{B_q}^4 f_{B_q}^2 \delta(s - M_{B_q}^2) + \frac{1}{\pi} \text{Im}\Pi^{\text{cont.}} \theta(s - s_{\text{cont.}}) \quad (7)$$

where M_{B_q} and f_{B_q} are respectively the mass and the decay constants of the lowest lying pseudoscalar meson B_q .

Looking back to equation (2) and taking into account all the theoretical parameters as well as the mass of the B_q -meson as our inputs in the calculation, we see that the decay constant can be computed as far as we had a good control of the hadronic continuum contribution of the experimental side.

Since this is not the case, to cope with this problem we make an appropriate choice of the polynomial ($P(s)$) in that equation. We take:

$$P(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \dots + a_n s^n, \quad (8)$$

such that its coefficients are fixed by imposing a normalization condition at threshold $P(s_{\text{ph.}} = M_{B_q}^2) = 1$, and requiring that should vanish in the range $[s_{\text{cont.}}, s_0]$ in a least square sense, i.e.,

$$\int_{s_{\text{cont.}}}^{s_0} s^k P(s) ds = 0 \text{ for } k = 0, \dots, n \quad (9)$$

These conditions exactly cancel the continuum contribution as far as $\text{Im } \Pi^{\text{cont.}}$ can be well approximated by an n degree polynomial. On the other hand, by virtue of the normalization condition, it will enhance the role of the B_q resonance. Notice however that by increasing the degree of the polynomial, $P(s)$, we will require the knowledge of further terms in the non perturbative series, which are unknown. Therefore, a compromise criterion on the choice of the polynomial degree has to be taken.

To check the consistency of the method in this work, we have considered a second and third degree polynomials and the results are fully compatible within the range of the errors introduced by the inputs of the calculation. We also have checked explicitly that a smooth continuum contribution has no influence in the result. This procedure was previously used in the calculation of the

charm mass from the $c\bar{c}$ experimental data. The continuum data from the BES II collaboration [13] had no influence when an appropriate polynomial was included [14]. Employing the same technique, very accurate prediction of the bottom quark mass was also obtained using the experimental information of the upsilon system [15].

After these considerations we proceed with the analytical calculation of the decay constant f_{B_q} , neglecting the contribution that comes from $\text{Im}\Pi^{\text{cont.}}$. For different powers of s^i we obtain different sum rules which, in principle, should give the same result for the decay constant f_{B_q} . Notice however that, by increasing the value of i we get larger contribution from both, the large s region of the spectral function and higher orders in the non perturbative QCD expansion². Since these pieces have the main experimental and theoretical uncertainties, we will consider the calculation coming from the sum rules corresponding to the powers with $i = 0$ and $i = 1$ which, in any case it is enough for our purposes.

$$M_{B_q}^4 f_{B_q}^2 = \frac{1}{\pi} \int_{(M_b+m_q)^2}^{s_0} P(s) \text{Im}\Pi^{\text{pert.}}(s) ds - \text{Res} \{ P(s) \Pi^{\text{nonpert.}}(s), s = M_b^2 \} \quad (10)$$

for $i = 0$ ("first sum rule") and

$$M_{B_q}^6 f_{B_q}^2 = \frac{1}{\pi} \int_{(M_b+m_q)^2}^{s_0} s P(s) \text{Im}\Pi^{\text{pert.}}(s) ds - \text{Res} \{ s P(s) \Pi^{\text{nonpert.}}(s), s = M_b^2 \} \quad (11)$$

for $i = 1$ ("second sum rule"). Let us emphasize that in these two sum rules there are two unknowns, the decay constant f_{B_q} and the QCD duality threshold s_0 , the last one appears in the upper integration limit and also in the coefficients of the polynomial $P(s)$. Therefore, we can use both sum rules to determine f_{B_q} as well as s_0 . To employ a couple of sum rules in order to fix the QCD duality threshold (s_0) is a usual procedure [2, 4], but, as we will see, with our method we have the additional advantage that the value obtained for s_0 is very stable, in the sense that any change of this value would not affect appreciably the result of the decay constant f_{B_q} .

3 Results

In the calculation of f_B we take $m_q = 0$ in the factor $(M_b + m_q)^2$ of the correlation function, and in the low integration limit of equation (4), whereas for f_{B_s} we keep the strange quark mass different from zero although its contribution turns out to be negligible.

The experimental inputs are as follows. For q being the light quarks $q = u, d$ we have the physical threshold $s_{\text{ph.}}$, at the squared mass of the lowest lying resonance in the $b\bar{q}$ channel :

$$s_{\text{ph.}} = M_B^2 = 5.279^2 \text{ GeV}^2. \quad (12)$$

²The results obtained with higher moments are compatible with the ones found here. However they are less stable with the duality parameter s_0 and, hence, less accurate.

The continuum threshold $s_{\text{cont.}}$ is taken at the $B\pi\pi$ intermediate state, in an $I = \frac{1}{2}$ s-wave, i. e.

$$s_{\text{cont.}} = (M_B + 2m_\pi)^2 = 30.90\text{GeV}^2. \quad (13)$$

For the strange s quark we take:

$$\begin{aligned} s_{\text{ph.}} &= M_{B_s}^2 = 5.369^2 \text{ GeV}^2 \\ s_{\text{cont.}} &= (M_{B_s} + 2m_\pi)^2 = 31.92 \text{ GeV}^2. \end{aligned} \quad (14)$$

In the theoretical side of the sum rule the inputs we take are as follows. The strong coupling constant is taken at the scale of the electroweak Z boson mass [17]

$$\alpha_s(M_Z) = 0.118 \pm 0.003 \quad (15)$$

and run down to the computation scale using the four loop formulas of reference [16]. The values, with the corresponding error bars, for the quark and gluon condensates (see for example [4]) and the mass of the strange quark [18] are:

$$\begin{aligned} \langle \bar{q}q \rangle (2 \text{ GeV}) &= (-267 \pm 17 \text{ MeV})^3, \\ \langle \frac{\alpha_s}{\pi} G G \rangle &= 0.024 \pm 0.012 \text{ GeV}, \\ \langle \bar{q}\sigma G q \rangle &= m_0^2 \langle \bar{q}q \rangle, \quad (\text{with } m_0^2 = 0.8 \pm 0.2 \text{ GeV}) \\ \langle \bar{s}s \rangle &= (0.8 \pm 0.3) \langle \bar{q}q \rangle \\ m_s (2 \text{ GeV}) &= 120 \pm 50 \text{ MeV} \end{aligned} \quad (16)$$

We also need to fix the renormalization scale μ in the equations (5) and (6). For definiteness we take $\mu = M_b(M_b)$, although in order to find the stability of the results under the renormalization scale we will study its variation for values of this scale in a range appropriate for our discussion, namely, $(\Lambda_{QCD}^2 \ll \mu^2 < s_0)$.

Finally for the bottom quark, the value $M_b \approx 4.20 \text{ GeV}$ is nowadays generally accepted. We take the result of [15] which has also been obtained with the sum rule method described here: $M_b(M_b) = 4.19 \pm 0.05 \text{ GeV}$.

Now, we proceed in the way described before. Firstly, we compute f_B as a function of s_0 with the two different sum rules (2), for $i = 0, 1$. Then, we fix s_0 at the value where both sum rules give the same result for f_B . With this usual procedure, and for a second degree polynomial, we find $s_0 = 48.5 \text{ MeV}^2$ and $f_B = 185 \text{ MeV}$.

From the theoretical inputs quoted above the main source of errors comes from the bottom mass which, in the range given above, produces a variation in the decay constant of $\mp 11 \text{ MeV}$. Other source of errors are the quark condensates³ which affect the result by $\pm 4 \text{ MeV}$. Since the decay constants are physical observables, the results should be independent of the renormalization

³As for the contribution of the condensates, the only relevant comes from the lowest dimension one which gives an 8% of the total result. The higher dimension terms considered here doesn't give a sizeable contribution giving a hint on the convergence of the condensate series in this approach.

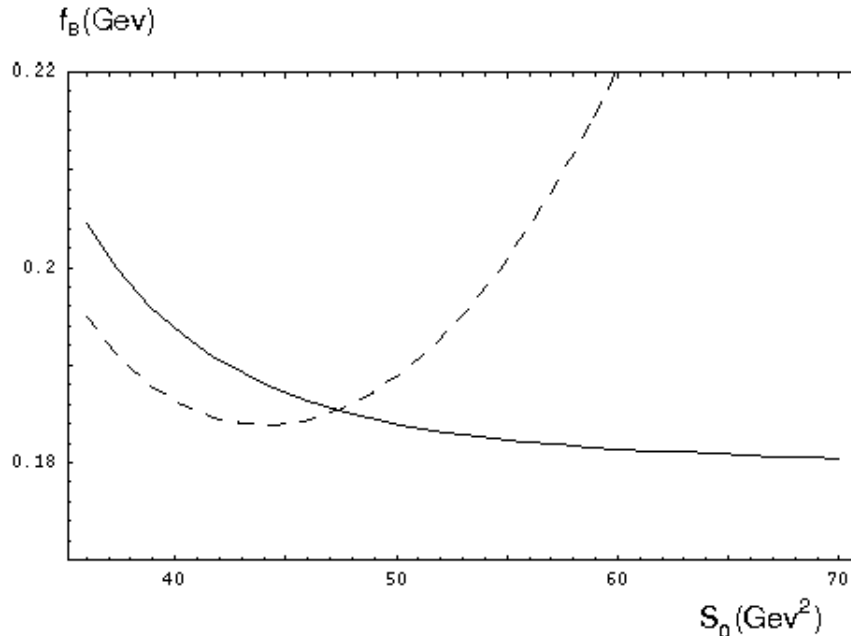


Figure 1: Decay constant f_B as a function of the integration radius s_0 for $M_b(M_b) = 4.19$ GeV. With a second degree polynomial in the sum rule (2), the dashed line represents the case $i = 1$ and the solid line the case $i = 0$.

scale μ . However, here we have used an approximation in the asymptotic spectral function of QCD, taken only the two-loop order (5,6). Therefore a residual dependence on the renormalization scale is expected. In order to quantify the uncertainty of fixing the scale at the bottom mass, we vary the scale in the range $\mu \in [3, 6]$ GeV, introducing an uncertainty of ± 18 MeV in the result. This dependence in the renormalization scale is expected to lower down if higher orders in the coupling constant in (5,6) are taken into account.

Adding quadratically all these errors, we finally quote the following result for the decay constant of the light meson B :

$$f_B = 185 \pm 22 \text{ MeV}. \quad (17)$$

Notice in figure 1 that with $i = 1$ (dashed line) and a second degree polynomial there is a stable value (minimum), $f_B = 183$ MeV, at $s_0 = 44$ GeV². But for $i = 0$ (solid line) the result $f_B = 180$ MeV, is an stable value (inflection point) which is practically constant around $s_0 = 60$ GeV². However, to compare with other results from QCD sum rules, we take the crossing point of figure 1 as our final result. The stable results of both curves are completely compatible

within error bars in such a way that we could have considered the value for f_B in the wide range of the stability region of s_0 .

Proceeding in the same fashion for the B_s meson, with the only change given by the nonzero mass of the light s quark, we find the decay constant f_{B_s} . In this case, as can be appreciated in figure 2, the s_0 value where the two sum rules give the same result for f_{B_s} is $s_0 = 49.6 \text{ GeV}^2$.

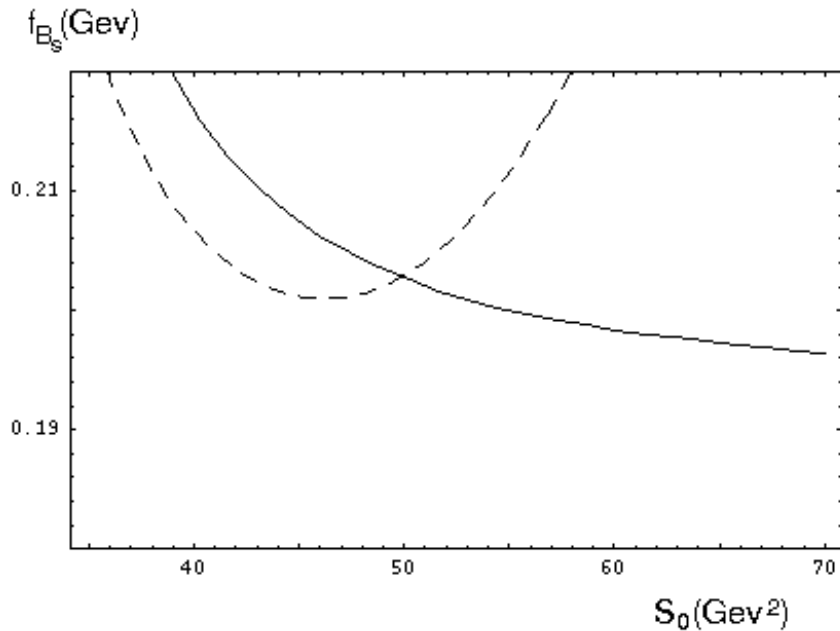


Figure 2: Decay constant f_{B_s} as a function of the integration radius s_0 for $M_b(M_b) = 4.19 \text{ GeV}$. With a second degree polynomial in the sum rule (2), the dashed line represents the case $i = 1$ and the solid line the case $i = 0$.

The result in the intersection point is:

$$f_{B_s} = 202 \pm 24 \text{ MeV}. \quad (18)$$

where a similar analysis of errors has been considered. The only new ingredient is the uncertainty coming from the strange quark mass, which contribution turns out to be negligible.

It is of special interest the ratio of the decay constants f_{B_s} and f_B , which

should be 1 in the chiral limit. We find

$$\frac{f_{B_s}}{f_B} = 1.09 \pm 0.01 \quad (19)$$

We are free to remark that in the calculation of this ratio, the uncertainties of the theoretical parameters are correlated, this is why the final error becomes very small.

4 Conclusions

In this note we have computed the decay constant of B_q -mesons for q either the strange s or the u or d massless quarks. We have used a suitable combination of moments of QCD finite energy sum rules in order to minimize the shortcomings of the available experimental data. On the theoretical side of the sum rule, we have used the pseudoscalar two point function calculated up to two-loop in perturbative QCD and up to dimension six condensates in the non perturbative QCD expansion. Instead of the commonly adopted pole mass of the bottom quark, we use the running mass to get a good convergence of the perturbative series. We have a good control of the results against the duality threshold s_0 , which turn to be very stable.

The results found, taking the running mass of the bottom quark $M_b(M_b) = 4.19 \pm 0.05$ GeV are given in equations (17,18) and we collect here for convenience:

$$f_B = 185 \pm 22 \text{ MeV}, \quad f_{B_s} = 202 \pm 24 \text{ MeV}.$$

where the error bars come from the uncertainty in the theoretical parameters (16) as well as the residual dependence on the renormalization scale. We notice that the results are very sensitive to the value of the running mass, giving most of the theoretical uncertainty. On the other hand they turn out to be quite stable against the variations of the other input parameters, in particular the integration radius s_0 .

In this treatment we could not include the three-loop corrections in the two point correlation function, since the complete analytical QCD expression along the cut is not known. Despite of this lack of information, one can interpolate the three-loop low energy QCD calculations with the high energy ones [4, 10]. The results do not differ much from the exact two-loop calculation. Another possibility, is to include the three-loop high energy expansion along the integration circle $|s| = s_0$ which is known to certain accuracy. We have also performed this calculation somewhere else, having results very close to the ones found here.

Within the error bars, our results agree with other results in the literature, obtained with different sum rule methods and with lattice computations [2, 4, 5, 6, 8, 9]. However, we advocate for values of the decay constants in the lower band.

References

- [1] M. A. Shifman, A.I. Vainshtein and V. I. Zakharov, Nucl.Phys. **B147** (1979) 519, **B147** (1979) 385, **B147** (1979) 448.
- [2] L. J. Reinders, Phys. Rev. **D38** (1988) 947.
S. Narison, *QCD Spectral Sum Rules*, World Scientific lecture notes in Physics; vol. 26. Singapore, 1989.
- [3] T.M. Aliev, V.L. Eletsky, Sov.J.Nucl.Phys.**38** (1983) 936, Yad.Fiz.**38** (1983) 1537.
- [4] M. Jamin and O. Lange, Phys. Rev. **D65** (2002) 056003.
- [5] S. Narison, Phys. Lett. **B520** (2001) 115.
- [6] C. A. Dominguez and N. Paver, Phys. Lett. **B197** (1987) 423.
- [7] P. Colangelo (INFN, Bari), A. Khodjamirian, Boris Ioffe Festschrift At the Frontier of Particle Physics. Handbook of QCD vol. 3, ed. by M. Shifman (World Scientific, Singapore, 2001).
- [8] A. Abada et al. (APE coll.), Nucl. Phys. (Proc. Suppl.) **B83** (2000) 268.
- [9] K. C. Bowler (UKQCD coll.), Nucl. Phys. **B619** (2001) 507.
- [10] K.G. Chetyrkin, M. Steinhauser, Phys.Lett.**B502** (2001), Eur. Phys. J. **C21** (2001) 319.
- [11] K. G. Chetyrkin and M. Steinhauser, Nucl. Phys. **B573** (2000) 617.
- [12] J. A. M. Vermaseren, S. A. Larin and T. van Ritbergen, Phys. Lett. **B405** (1997) 327.
- [13] J. Z. Bai et al. (BES collaboration), Phys. Rev. Lett. **88** (2002) 101802.
- [14] J. Peñarrocha and K. Schilcher. Phys. Lett. **B515** (2001) 291.
- [15] J. Bordes, J. Peñarrocha and K. Schilcher. Phys. Lett. **B562** (2003) 81.
- [16] G. Rodrigo, A. Pich and A. Santamaria, Phys.Lett. **B424** (1998) 367.
- [17] S. Bethke, J. Phys. **G26** (2000) R27.
- [18] K. Hagiwara et al. (PDG) Phys. Rev. **D66** (2002) 010001.