A COMMENT ON THE RELATIONSHIP BETWEEN
DIFFERENTIAL AND DIMENSIONAL RENORMALIZATION *

Gerald Dunne

Mathematics Department
and
Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

and

Nuria Rius

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

arXiv:hep-th/9206038v1 9 Jun 1992

ABSTRACT

We show that there is a very simple relationship between differential and dimensional renormalization of low-order Feynman graphs in renormalizable massless quantum field theories. The beauty of the differential approach is that it achieves the same finite results as dimensional renormalization without the need to modify the space time dimension.

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The method of "differential regularization and renormalization" was introduced in Ref. [1] by Freedman, Johnson and Latorre (FJL) as a novel technique for regulating the ultraviolet divergences of quantum field theories and computing finite renormalized correlation functions. The method is applied to real space (rather than momentum space) Feynman diagrams, and requires no explicit cut-off and no explicit counterterms. Renormalized amplitudes are calculated directly, and these satisfy Callan–Symanzik-type equations which determine the renormalization group functions $\beta(g)$ and $\gamma(g)$ [1,2]. More recent investigations have explored the compatibility with supersymmetry [3] and gauge invariance [4], the extension to *massive* quantum field theories [5], and the issue of lower-dimensional gauge theories [6].

In this Letter we explain how differential regularization and renormalization is related to dimensional regularization and renormalization. Such an understanding is clearly important since dimensional regularization and renormalization is the best understood and most widely-used procedure for analyzing renormalizable quantum field theories (particularly those involving *gauge* symmetry). Furthermore, a major motivation [1,2] for the development of differential renormalization is the observation that (at least formally) the method is compatible with both gauge and chiral symmetry, and may therefore prove more convenient than dimensional renormalization for chiral gauge theories as well as dimension-specific theories such as Chern–Simons, Wess–Zumino–Witten, *etc*.

We shall illustrate the relationship between differential and dimensional regularization and renormalization in three types of renormalizable *massless* quantum field theories:

(i) massless scalar theory (ϕ^4 in four dimensions, ϕ^3 in six dimensions, and ϕ^6 in three dimensions);

(ii) pure Yang–Mills theory in four-dimensions, using the background field method; and

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(iii) massless four-dimensional quantum electrodynamics.

We shall restrict our attention to low-order graphs in order to pinpoint most clearly the *essential* correspondence between differential and dimensional renormalization (furthermore, much of the initial "processing" of higher-order graphs is common to both approaches, and has been discussed in detail in Refs. [1,2]). We shall also clarify the issue of gauge invariance in massless QED, where differential renormalization introduces (in principle separate) mass scales for the vertex and self-energy graphs which must however be correlated in a specific way to satisfy the one-loop Ward identity. This is to be contrasted with the dimensional renormalization approach in which there is a *single* mass scale μ (arising from the fact that the coupling constant acquires a non-zero engineering dimension) and the Ward identity is satisfied automatically (in, for example, the minimal subtraction scheme).

The essential idea [1] of differential regularization and renormalization stems from the observation that most primitively divergent low-order Feynman graphs are well-defined in real space except for short-distance singularities at coincident points. If these singularities are too severe, the graph will not have a well-defined Fourier transform and higher-order graphs of which the original graph is a sub-graph will contain divergent integrals — hence the need for some form of regularization. The strategy of differential regularization and renormalization is to *isolate the coincident point singularities and to reduce the degree of divergence of these singularities by expressing the singular terms as derivatives of less singular terms*. In practice, this strategy is implemented in two distinct stages.

In the first stage *no dimensionful regulator is needed*. In this initial stage one may use straightforward algebraic manipulations and simple differential identities in order to isolate the coincident point singularities and to extract derivatives so as to reduce the degree of

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divergence. In massless quantum field theories (in $d \neq 2$) we typically encounter power-law-type singularities in bare graphs, and these may have their degree of divergence reduced by using the *identity*

$$|x|^{-p} = \frac{\Box\, |x|^{-p+2}}{(-p+2)\,(d-p)} \quad , \tag{1}$$

In general (and, as we shall see, in particular for renormalizable theories) one confront the singularity $|x|^{-d}$, whose degree of divergence may only be reduced further by the introduction of an arbitrary (but essential) logarithmic mass scale.*† It is clear that one cannot use Eq. (1) as it stands because of the $\frac{1}{d-p}$ pole.

In the second stage of differential regularization and renormalization one proceeds by *defining* the regulated form:

$$\left.|x|^{-d}\right|_{\text{reg}} := \frac{1}{2(2-d)}\Box\left(\frac{\ln M^2|x|^2}{|x|^{d-2}}\right) \quad , \qquad (d \neq 2) \quad . \tag{2}$$

This relation is certainly true away from the origin, and, furthermore, the dependence on the arbitrary (but necessary) mass scale M yields the appropriate δ-function singularity at the origin:

$$M\frac{d}{dM}\left(\left.|x|^{-d}\right|_{\text{reg}}\right) = \frac{2\pi^{d/2}}{\Gamma\left(\dfrac{d}{2}\right)}\delta^{(d)}(x) \quad . \tag{3}$$

To obtain Eq. (3) we have used the fact that the massless scalar Green's function in d $(\neq 2)$ dimensions is

$$G(x) = -\frac{\Gamma\left(\dfrac{d}{2}-1\right)}{4\pi^{d/2}}|x|^{2-d} \quad , \tag{4}$$

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* Note also that $|x|^{-d}$ is the borderline between powers of $|x|$ which have or do not have a well-defined Fourier transform (see below, Eq. (7)).

† In a *super-renormalizable* theory this may not be necessary — it may be possible to apply differential renormalization without introducing a mass scale. See Ref. [6] for an example.

and it satisfies

$$\Box G(x) = \delta^{(d)}(x) \ . \tag{5}$$

It is worth stressing that in differential regularization and renormalization there are *no infinite (or finite) counterterms* — one may simply replace $|x|^{-d}$ by its regulated form as in (2). Then the scaling relation (3) is the key to deriving the Callan–Symanzik equations satisfied by the renormalization amplitudes [1]. In the differential approach, one also integrates by parts freely within graphs and in taking Fourier transforms [1]. The consistency of this procedure and its compatibility with unitarity has been explicitly checked (in massless ϕ^4 theory through three-loop order) in a cutoff version of differential renormalization [7].

This two-stage implementation of the differential renormalization strategy has been applied with impressive success and efficiency to many examples of renormalizable quantum field theories [1–7]. We now show that this same strategy may be implemented using real-space *dimensional* regularization and renormalization. Recall that real-space dimensional renormalization of quantum field theories, known as the "method of uniqueness," [8,9,10] is an enormously powerful calculational tool with which multi-loop computations may be performed with far greater ease than with conventional (principally momentum space) techniques. Here, however, our emphasis and motivation are quite different — we are interested in the comparison with differential renormalization, and consequently we are interested in not only the infinite parts of graphs, but also their regulated finite parts.

The first important point to realize is that the *initial* stage of "processing" a bare Feynman graph (as outlined above) does not require a regulating mass scale, and may be performed in *exactly* the same manner in an arbitrary dimension d of space-time. It is only in the second stage, when one encounters the singularity $|x|^{-d}$, that any difference between dimensional and

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differential regularization arises. The dimensional regularization approach described here is simply another way of regulating this singularity. In fact, if the theory is formulated in the non-integer space-time dimension $d = D - 2\epsilon$ (where D is the "real" integer dimension and ϵ is infinitesimal) then the initial processing of the bare graph does not lead to a singularity $|x|^{-d}$ but to a singularity $|x|^{-d+r\epsilon}$, where $r\epsilon$ is some integer multiple of ϵ. Indeed, since in a massless $d = (D - 2\epsilon)$-dimensional theory the coupling constant acquires a mass dimension proportional to ϵ, one in fact encounters a singularity $\mu^{r\epsilon}|x|^{-d+r\epsilon}$ where μ is a universal dimensionful parameter associated with the coupling constant.* Now one is free to use the identity (1) directly, isolating the $\frac{1}{d-p}$ pole as a $1/\epsilon$ pole:

$$
\begin{aligned}
\mu^{r\epsilon}|x|^{-d+r\epsilon} &= \frac{1}{\epsilon}\mu^{r\epsilon}\frac{1}{r(2 - d + r\epsilon)}\Box|x|^{-d+r\epsilon+2} \\
&= -\frac{1}{\epsilon}\frac{4\pi^{d/2}}{r\left(2 - d + r\epsilon\right)\Gamma\left(\dfrac{d}{2} - 1\right)}\delta^{(d)}(x) + \frac{1}{2(2-d)}\Box\left(\frac{\ln\mu^2|x|^2}{|x|^{d-2}}\right) + \mathcal{O}(\epsilon) \quad .
\end{aligned}
\tag{6}
$$

Thus, in the dimensional approach, the $x = 0$ singularity of the left-hand side of Eq. (6) is regulated as a "counterterm" $\delta^{(d)}(x)$, with both a $1/\epsilon$ pole and a finite ϵ^0 coefficient, and a finite regulated ϵ^0 term of exactly the same form as the differential regularization expression (2) (with the dimensional regularization mass scale μ identified with the differential regularization mass scale M). As a matter of terminology, we shall refer to the $1/\epsilon$ pole counterterm as the "infinite counterterm," the ϵ^0 counterterm as the "finite counterterm," and the ϵ^0 regulated term involving μ as the "non-counterterm finite part." This is precisely analogous to the situation in conventional momentum space dimensional regularization, where the divergences of divergent momentum integrals are regulated as Laurent expansions in ϵ.

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* The fact that the "correct" power of μ appears with the bare graph is a special feature of renormalizable theories.

Indeed, we may equivalently present Eq. (6) in momentum space form by Fourier transforming the left-hand side:

$$
\int d^d x \, e^{ik \cdot x} \mu^{r\epsilon} |x|^{-d+r\epsilon} = \frac{2^{r\epsilon} \pi^{d/2} \Gamma\left(\dfrac{r\epsilon}{2}\right)}{\Gamma\left(\dfrac{d - r\epsilon}{2}\right)} \left(\frac{\mu}{|k|}\right)^{r\epsilon}
$$

$$
= -\frac{1}{\epsilon} \frac{4\pi^{d/2}}{r(2 - d + r\epsilon) \Gamma\left(\dfrac{d}{2} - 1\right)} \tag{7}
$$

$$
+ \frac{\pi^{d/2}}{\Gamma\left(\dfrac{d}{2}\right)} \left(-\gamma_E + \psi\left(\frac{d}{2} - 1\right) + \ln\left(\frac{2\mu^2}{|k|^2}\right)\right) + \mathcal{O}(\epsilon) \quad .
$$

Here $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ is the digamma function and $\gamma_E = -\psi(1) = 0.5772\ldots$ is Euler's constant. This agrees with the Fourier transform of the right-hand side of Eq. (6) computed by freely integrating by parts (as is usual in dimensional regularization). In fact, using Eq. (7) it is straightforward to convert *any* of the real space expressions we discuss in this paper into the corresponding momentum space formulas.

Renormalization now proceeds exactly as usual in the dimensional regularization approach (albeit in real space rather than in momentum space) — *i.e.* by defining an appropriate subtraction scheme (see *e.g.* Ref. [11]). Note from Eq. (3) that the scaling dependence of the "non-counterterm finite part" produces contributions of the "finite counterterm" form. In more conventional language this simply corresponds to adjusting the renormalization scale.

There are, of course, minor technical complications (see below) in gauge theories involving some tensor algebra and Lie algebra manipulations and in fermionic theories involving some Dirac matrix algebra, but the *essence* of the correspondence between differential and dimensional regularization (at low order) lies in Eqs. (2), (3) and (6).

1. Massless Scalar Theories

Consider the Euclidean space massless scalar theory with action

$$S = \int d^d x \left(\frac{1}{2} \phi \Box \phi - \frac{\lambda}{n!} \phi^n \right) \tag{8}$$

where $d \neq 2$ and $n \neq 2$. The scalar Green's function is as in (4) and (5) and so, apart from tadpole contributions, the lowest order two-point function is (see Fig. 1)

$$\begin{aligned}
\Gamma(x) &= \frac{\lambda^2}{(n-1)!} \left(G(x) \right)^{n-1} \\
&= \frac{\lambda^2}{(n-1)!} \left(-\frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{d/2}} \right)^{n-1} |x|^{(2-d)(n-1)} \quad .
\end{aligned} \tag{9}$$

For relevant values of d and n, this graph is singular at the origin and it may have its degree of divergence reduced by applying Eq. (1). For the theory to be renormalizable (at this order) this must lead to a $\Box \delta(x)$ counterterm — this requirement relates the space-time dimension d and the interaction power n as:

$$(2-d)(2-n) = 4 \tag{10}$$

(where we have used that $\Box |x|^{2-d} \sim \delta^{(d)}(x)$). Thus, for example, we have ϕ^4 theory in four dimensions, ϕ^3 in six dimensions or ϕ^6 in three dimensions. In these cases, the two-point function is proportional to $|x|^{-2-d}$. Applying formula (1) once, we arrive at a $|x|^{-d}$ singularity. Then using (2) we may simply write down the finite renormalized amplitude in the differential approach:

$$\left. \Gamma(x) \right|_{\text{diff. ren.}} = -\frac{\lambda^2}{(n-1)!} \left(-\frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{d/2}} \right)^{n-1} \frac{1}{4d(d-2)} \Box \Box \left(\frac{\ln M^2 |x|^2}{|x|^{d-2}} \right) \quad . \tag{11}$$

In the dimensional approach, with $d = D - 2\epsilon$ (and the integer part, D, of the space-time dimension satisfies (10)) the coupling constant λ is replaced by $\lambda \to \lambda_0 \mu^{(n-2)\epsilon}$, where λ_0 is dimensionless and μ is a universal mass scale.* Then

$$\Gamma(x) = \frac{\lambda_0^2 \mu^{2(n-2)\epsilon}}{(n-1)!} \left(-\frac{\Gamma\left(\frac{D}{2} - 1 - \epsilon\right)}{4\pi^{D/2-\epsilon}} \right)^{n-1} |x|^{-2-d+2(n-2)\epsilon} \quad . \tag{12}$$

Using the relation (6) and expanding in powers of ϵ we obtain

$$\begin{aligned}
\Gamma(x)\Big|_{\text{dim ren.}} &= \frac{\lambda_0^2}{(n-1)!} \left(-\frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2}} \right)^{n-2} \\
&\quad \times \frac{1}{16D} \left(-\frac{1}{\epsilon} + (n-2)\psi\left(\frac{D}{2} - 1\right) - (n-2)\ln\pi - \frac{4(2D^2 - D - 2)}{D(D-2)^2} \right) \Box\delta(x) \\
&\quad - \frac{\lambda_0^2}{(n-1)!} \left(-\frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2}} \right)^{n-1} \frac{1}{4D(D-2)} \Box\,\Box \left(\frac{\ln\mu^2|x|^2}{|x|^{D-2}} \right) + \mathcal{O}(\epsilon) \quad .
\end{aligned} \tag{13}$$

Thus we see that the dimensional renormalization two-point function agrees with the differential renormalization result if we choose our dimensional renormalization scheme to be one of subtracting both the infinite and finite counterterms, leaving just the non-counterterm finite part, with the scales M and μ identified. And because of the scaling relation (3), any other scheme which retains any part of the finite counterterm simply corresponds to rescaling the arbitrary mass scale M.

2.   Four-dimensional Yang–Mills Theory in Background Field Method

In Ref. [1], Freedman, Johnson and Latorre (FJL) used differential regularization and renormalization to compute the one-loop two-point function in the background field method.

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* Another way to regulate this massless $\lambda\phi^n$ theory is to keep d at its integer value but take n to be non-integer, $n = N + \delta$ with N an integer: this is the "δ-expansion" of Bender and collaborators [12].

Recall [13], that due to the explicit background field gauge invariance of the background field method, the two-point function is sufficient to determine the renormalization group β-function. The one-loop contribution to the effective action from the gluon and ghost loops in Fig. 2 is (after performing the Lie algebra contractions)

$$
\begin{aligned}
\Omega_2(B) &= \frac{1}{2}\left[(2.\text{a}) + (2.\text{b})\right] \\
&= -\frac{g_0^2 \mu^{4-d}}{2} C_A \int d^d x\, d^d y\, B_\mu^a(x) B_\nu^a(y) \\
&\quad \times \left[-\frac{d}{4}\partial_\mu \partial_\nu \left(G(x-y)^2\right) + d\left(\partial_\mu G(x-y)\right)\left(\partial_\nu G(x-y)\right)\right. \\
&\quad \left. + 4\partial_\mu \partial_\nu \left(G(x-y)^2\right) - 4\delta_{\mu\nu}\Box\left(G(x-y)^2\right)\right] \ .
\end{aligned}
\tag{14}
$$

Here B_μ^a are the background gauge field potentials,* C_A is the quadratic Casimir in the adjoint representation, and all derivatives are with respect to x. $G(x-y)$ is the massless scalar Green's function as in Eq. (4). After performing some simple tensor algebra in (14) we find the manifestly transverse expression

$$
\begin{aligned}
\Omega_2(B) &= -\frac{g_0^2 \mu^{4-d}}{2} C_A \left(-\frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{d/2}}\right)^2 \frac{\left(8 - \frac{15}{2}d\right)}{(2-2d)} \\
&\quad \times \int d^d x\, d^d y\, B_\mu^a(x) B_\nu^a(y) \left(\partial_\mu \partial_\nu - \delta_{\mu\nu}\Box\right)|x-y|^{4-2d} \ .
\end{aligned}
\tag{15}
$$

Now, setting $d = 4 - 2\epsilon$ and then using the relation (6) once we obtain

$$
\begin{aligned}
\Omega_2(B) &= -\frac{g_0^2}{2}\mu^{2\epsilon} C_A \left(\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\right)^2 \frac{(-22+15\epsilon)}{(4\epsilon-6)}\int d^d x\, d^d y\, B_\mu^a(x) B_\nu^a(y)\left(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box\right)|x-y|^{-4+4\epsilon} \\
&= -\frac{11 C_A}{96}\frac{g_0^2}{\pi^2}\left(\frac{1}{\epsilon} + \frac{131}{66} + \gamma_E + \ln\pi\right)\int d^d x\, d^d y\, B_\mu^a(x) B_\nu^a(y)\left(\partial_\mu\partial_\nu - \Box\delta_{\mu\nu}\right)\delta(x-y) \\
&\quad + \frac{11 C_A}{24}\left(\frac{g_0}{4\pi^2}\right)^2 \int d^d x\, d^d y\, B_\mu^a(x) B_\nu^a(y)\left(\partial_\mu\partial_\nu - \Box\delta_{\mu\nu}\right)\Box\left(\frac{\ln\mu^2|x-y|^2}{|x-y|^2}\right) + \mathcal{O}(\epsilon) \ .
\end{aligned}
\tag{16}
$$

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* Note that B_μ^a coincides with A_μ^a of Ref. [13], while in Ref. [1] $B_\mu^a = gA_\mu^a$.

The $1/\epsilon$ pole part of (16) (from which one may deduce the one-loop β-function) agrees with the momentum space dimensional regularization computation [13], and the non-counterterm finite part (also from which one may deduce the one-loop β-function) agrees with the differential regularization result of FJL (see Eq. (II.G.22) of Ref. [1]), with $M = \mu$. Once again, in terms of *dimensional* renormalization, *differential* renormalization consists of a subtraction scheme in which only the non-counterterm finite part is retained as the renormalized amplitude.

3. Massless Four-Dimensional QED

In d dimensions, the massless fermion Green's function is

$$S(x) = -\left(\gamma_\mu \frac{\partial}{\partial x^\mu}\right) G(x) \tag{17}$$

where $G(x)$ is the massless scalar Green's function in (4), and so $S(x)$ satisfies

$$\gamma \cdot \frac{\partial}{\partial x} S(x) = \Box G(x) = \delta^{(d)}(x) \quad . \tag{18}$$

Then the one-loop self-energy graph in $d = 4 - 2\epsilon$ dimensions is (see Fig. 3)

$$\Sigma(x) = -\left(e_o \mu^\epsilon\right)^2 \gamma_\mu S(x) \gamma_\nu \delta_{\mu\nu} G(x) \quad . \tag{19}$$

Using the Dirac matrix relation $\gamma_\mu \gamma_\nu \gamma_\mu = (d-2)\gamma_\nu$, we find

$$
\begin{aligned}
\Sigma(x) &= e_0^2 \mu^{2\epsilon}(1-\epsilon)\left(\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\right)^2 \gamma \cdot \frac{\partial}{\partial x}|x|^{-4+4\epsilon} \\
&= \frac{e_0^2}{16\pi^2}\left(\frac{1}{\epsilon} + [\gamma_E + 1 + \ln\pi]\right)\gamma \cdot \frac{\partial}{\partial x}\delta(x) - \frac{e_0^2}{64\pi^2}\gamma \cdot \frac{\partial}{\partial x}\Box\left(\frac{\ln\mu^2|x|^2}{|x|^2}\right) + \mathcal{O}(\epsilon) \quad ,
\end{aligned}
\tag{20}
$$

where in the last step we have used the relation (6). This agrees with the standard (*e.g.* Ref. [11]) one-loop momentum space dimensional regularization result, and the non-counterterm finite piece agrees with the differential renormalization result of FJL [1], with the differential renormalization mass scale M_Σ taken equal to μ.

The analysis of the one-loop QED vertex function (see Fig. 4) in real space requires some preliminary algebraic and tensor manipulations in order to isolate the short-distance singularities. (The reason for this is simply that the vertex effectively depends on two space-time coordinates, rather than just one as in all the previous examples.) However, these "stage one" manipulations are exactly the same as in the differential regularization approach (see [1]), except that we work in d rather than four dimensions. The vertex is (see Fig. 4)

$$
\begin{aligned}
V_j(x, y, z) &= \left(e_0 \mu^{2-d/2} \right)^3 \gamma_\mu S(x-z) \gamma_j S(z-y) \gamma_\nu \delta_{\mu\nu} G(x-y) \\
&= e_0^3 \mu^{3\epsilon} \left(2\gamma_b \gamma_j \gamma_a - 2\epsilon \gamma_a \gamma_j \gamma_b \right) \left(\frac{\partial}{\partial x^a} G(x-z) \right) \left(\frac{\partial}{\partial z^b} G(z-y) \right) G(x-y) \;,
\end{aligned}
\tag{21}
$$

where in the last line we have specialized to $d = 4 - 2\epsilon$ dimensions and we have used the Dirac matrix identity: $\gamma_\mu \gamma_a \gamma_j \gamma_b \gamma_\mu = 2\gamma_b \gamma_j \gamma_a - (4-d)\gamma_a \gamma_j \gamma_b$.

Using the notation of FJL, we write the vertex as

$$
V_j(x, y, z) \equiv -e_0^3 \mu^{3\epsilon} \left(2\gamma_b \gamma_j \gamma_a - 2\epsilon \gamma_a \gamma_j \gamma_b \right) V_{ab}(x-z, y-z) \;,
\tag{22}
$$

thereby defining

$$
V_{ab}(u, v) \equiv \left(\frac{\partial}{\partial u^a} G(u) \right) \left(\frac{\partial}{\partial v^b} G(v) \right) G(u-v) \;.
$$

Exactly as in FJL, one then isolates the singularity in V_{ab} within the trace part by writing (for example)

$$
\begin{aligned}
V_{ab}(u, v) &= \frac{\partial}{\partial u^a} \left(G(u) \left(\frac{\partial}{\partial v^b} G(v) \right) G(u-v) \right) - \frac{\partial}{\partial v^b} \left(G(u)G(v) \left(\frac{\partial}{\partial u^a} G(u-v) \right) \right) \\
&\quad + G(u)G(v) \left(\frac{\partial}{\partial u^a} \frac{\partial}{\partial v^b} - \frac{\delta^{ab}}{d} \frac{\partial}{\partial u} \cdot \frac{\partial}{\partial v} \right) G(u-v) \\
&\quad + \frac{\delta^{ab}}{d} G(u)G(v) \frac{\partial}{\partial u} \cdot \frac{\partial}{\partial v} G(u-v) \\
&\equiv \widetilde{V}_{ab}(u, v) + \frac{\delta^{ab}}{d} G(u)G(v) \frac{\partial}{\partial u} \cdot \frac{\partial}{\partial v} G(u-v) \;.
\end{aligned}
\tag{23}
$$

Note [1] that $\widetilde{V}_{ab}(u,v)$ has a finite Fourier transform in four dimensions, so we only need to regulate the remainder, $V_{ab} - \widetilde{V}_{ab}$. Using the defining relation (5) for the scalar Green's function we find

$$V_{ab} - \widetilde{V}_{ab} = -\frac{\delta^{ab}}{(4-2\epsilon)} \left(\frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \right)^2 \delta(u-v)|u|^{-4+4\epsilon} \ . \tag{24}$$

In the *differential* approach one would have $\epsilon \equiv 0$ and so one would proceed (see [1]) by using the regulated form of $|u|^{-4}$ given by Eq. (2). In the *dimensional* regularization approach we expand $|u|^{-4+4\epsilon}$ as in Eq. (6) (with a factor $\mu^{2\epsilon}$ "borrowed" from the full vertex expression in (22); remembering that the vertex has an *overall* mass factor μ^ϵ due to the mass dimension of the coupling $e = e_0 \mu^\epsilon$) to obtain

$$
\begin{aligned}
V_{ab}(u,v) = {}& \widetilde{V}_{ab}(u,v) - \frac{\delta^{ab}\mu^{-2\epsilon}}{64\pi^2} \left(\frac{1}{\epsilon} + \left[\gamma_E + \frac{5}{2} + \ln\pi \right] \right) \delta(u)\delta(v) \\
& + \frac{\delta^{ab}\mu^{-2\epsilon}}{256\pi^4} \delta(u-v) \Box \left(\frac{\ln \mu^2|u|^2}{|u|^2} \right) \\
= {}& -\frac{1}{64\pi^6} V_{ab}^{\mathrm{FJL}}(u,v) \Big|_{\mathrm{reg}} - \frac{\delta^{ab}\mu^{-2\epsilon}}{64\pi^2} \left(\frac{1}{\epsilon} + \left[\gamma_E + \frac{5}{2} + \ln\pi \right] \right) \delta(u)\delta(v) \ .
\end{aligned}
\tag{25}
$$

If we set $d = 4$, then $V_{ab}^{\mathrm{FJL}}(u,v)\big|_{\mathrm{reg}}$ is the differential regulated form of V_{ab} found by FJL (see Eq. (II.C.9) of Ref. [1]). Inserting (25) into the full vertex expression (22) we find

$$V_j(x,y,z) = V_j^{\mathrm{FJL}}(x,y,z) \Big|_{\mathrm{reg}} + e_0\mu^\epsilon \frac{e_0^2}{16\pi^2} \left(\frac{1}{\epsilon} + \left[\gamma_E + \frac{1}{2} + \ln\pi \right] \right) \gamma_j \delta(x-z)\delta(z-y) \ , \tag{26}$$

where $V_j^{\mathrm{FJL}}\big|_{\mathrm{reg}}$ is the final differential regulated form in [1], with the differential renormalization mass scale for the vertex, M_V, taken equal to μ. (Notice that the $\left[\gamma_E + \frac{5}{2} + \ln\pi \right] \epsilon^0$ coefficient in (25) becomes $\left[\gamma_E + \frac{1}{2} + \ln\pi \right]$ in (26) because of the contraction of the $\frac{1}{\epsilon}$ pole in V_{ab} with the $\epsilon\gamma_a\gamma_j\gamma_b$ prefactor in the full vertex expression in (22).) Once again we see that the differential regularization and renormalization approach yields the non-counterterm finite part of the dimensional regularization approach.

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It is straightforward to check that the dimensional renormalization vertex in (26) and self-energy in (20) satisfy the one-loop Ward identity

$$\frac{\partial}{\partial z^j} V_j(x, y, z) = \big(\delta(z - x) - \delta(z - y)\big)\Sigma(x - y) \quad . \tag{27}$$

This is immediately clear for the $\frac{1}{\epsilon}$ pole part of the counterterms, but takes a little work for the other terms. Indeed, at first sight, comparing the $\left[\gamma_E + \frac{1}{2} + \ln \pi\right]$ factor in (26) with the "corresponding" $\left[\gamma_E + 1 + \ln \pi\right]$ factor in (20), it looks as though there is a discrepancy in the finite part. However, differentiating $V_j^{\text{FJL}}\big|_{\text{reg}}$ produces an extra finite counterterm contribution which "corrects" matters. In contrast, in the differential renormalization approach there are *no counterterms at all* in Σ or V_j, so this extra finite "counterterm" contribution to $\frac{\partial}{\partial z_j} V_j$ must be cancelled. This may be achieved simply by rescaling the masses — if $\ln(M_\Sigma / M_V) = \frac{1}{4}$ the one-loop Ward identity is satisfied [1].

The implication of this is simply that while dimensional regularization and renormalization (for example, with a pole subtraction renormalization prescription) automatically respects gauge invariance (*i.e.* the Ward identity is satisfied), the differential regularization and renormalization procedure (which has *no counterterms* at all) will, in general, require correlations between the mass scales appearing in various regulated graphs in order to preserve gauge invariance. This is not in the least surprising, as it is clear from Eqs. (2) and (3) that adjusting the differential renormalization mass scale in a renormalizable theory produces finite counterterms.

To conclude, we have shown that there is a very simple relationship between differential and dimensional renormalization in low order graphs. It is not unreasonable to expect that a systematic application of these ideas to higher order graphs may yield a proof that differential

renormalization works just as well as dimensional renormalization to all orders. However, the more interesting task is to apply differential renormalization to theories in which dimensional regularization is problematical.

ACKNOWLEDGEMENTS

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FIGURE CAPTIONS

Fig. 1: Lowest order two-point function in a $\lambda\phi^n$ massless scalar field theory. There are $(n-1)$ internal lines.

Fig. 2: One-loop graphs involving two external background fields. Wavy lines indicate quantum gluons and dashed lines indicate ghosts.

Fig. 3: One-loop self-energy graph in QED. Straight lines indicate fermion propagators and wavy lines indicate photon propagators.

Fig. 4: One-loop vertex graph in QED. The index j is the space-time index of the vertex.