Reparametrization invariance of B decay amplitudes and implications for new physics searches in B decays

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When studying B decays within the Standard Model, it is customary to use the unitarity of the CKM matrix in order to write the decay amplitudes in terms of only two of the three weak phases which appear in the various diagrams. Occasionally, it is mentioned that those two weak phases can be used in order to describe any decay amplitude, even beyond the Standard Model. Here we point out that, when describing a generic decay amplitude, the two weak phases can be chosen completely at will, and we study the behavior of the decay amplitudes under changes in the two weak phases chosen as a basis. Of course, physical observables cannot depend on such reparametrizations. This has an impact in discussions of the SM and in attempts to parametrize new physics effects in the decay amplitudes. We illustrate these issues by looking at $B \to \psi K_S$ and the isospin analysis in $B \to \pi\pi$.

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I. INTRODUCTION

In the Standard Model (SM) of electroweak interactions, CP violation appears through one single irremovable phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1], making it a rather predictive theory. The goal of B physics experiments is to exploit this feature and uncover new physics effects. Reviews may be found, for example, in [2, 3, 4].

This program is complicated by the presence of uncertain hadronic matrix elements relating the quark field operators utilized in writing down the theory with the hadrons detected by experiment. In some cases, such hadronic matrix elements can be removed and one is able to relate experimental observables with parameters in the original Lagrangian of the electroweak theory. In the context of CP violating asymmetries, the situation is sometimes described by the following statements: i) "If the decay amplitude depends on only one weak phase, then we can relate experiment with a parameter appearing in the original weak Lagrangian"; or ii) "If the decay amplitude is written in terms of two weak phases, then we cannot relate experiment with a parameter in the original Lagrangian". These statements are *imprecise*! One of the side-benefits of our analysis is the correction of these sentences.

In this article, we clarify what can (and cannot) be said about the weak phases entering a given decay amplitude:

- We point out that a given decay amplitude can be described by *any two weak phases*, $\{\phi_{A1}, \phi_{A2}\}$, chosen completely at random (as long as they do not differ by a multiple of 180°);
- We distinguish "experimental weak phases" from "theoretical weak phases", explaining that the former can be measured if and only if there is no direct CP violation;
- We see what happens when we change the basis utilized to describe the weak phases from $\{\phi_{A1}, \phi_{A2}\}$ into $\{\phi'_{A1}, \phi'_{A2}\}$. Since physical results cannot change under such a reparametrization, we refer to this property as "reparametrization invariance" of the decay amplitudes;
- We discuss the impact that reparametrization invariance has on searches for new physics in B decays;
- And, we relate our observations with statements scattered in the literature and applicable in very particular special cases, explaining their generalizations or limitations.

In section II we show that any two weak phases can be used to describe a generic B decay amplitude. In section III we show that one can ascertain experimentally whether the decay amplitude can be parametrized exclusively with a single weak phase and we illustrate with a few examples that such information may not be enough to determine a weak phase in the original electroweak Lagrangian. In section IV we see how reparametrization invariance affects the analysis of $B \to \pi\pi$ decays, in the SM and in the presence of new physics, turning to the decays $B \to \psi K$ in section V. In section VI we discuss some general features of observables determined experimentally to depend on a single weak phase, which, moreover, coincides with that predicted in the SM; we identify the types of new physics which may (or may not) be consistent with such results. In section VII we present our conclusions.

II. PARAMETRIZING THE WEAK PHASE CONTENT OF THE DECAY AMPLITUDES

A. Model independent analysis

Let us consider the decay of a B meson into some specific final state f. For the moment, B stands for B^+, B_d^0 or B_s^0. When discussing generic features of the decay amplitudes without reference to any particular model, it has become commonplace to parametrize the decay amplitudes as

$$A_f = M_1 e^{i\phi_{A1}} e^{i\delta_1} + M_2 e^{i\phi_{A2}} e^{i\delta_2}, \tag{1}$$

$$\bar{A}_{\bar{f}} = M_1 e^{-i\phi_{A1}} e^{i\delta_1} + M_2 e^{-i\phi_{A2}} e^{i\delta_2}, \tag{2}$$

where ϕ_{A1} and ϕ_{A2} are two CP-odd weak phases; M_1 and M_2 are the magnitudes of the corresponding terms; and δ_1 and δ_2 are the corresponding CP-even strong phases. These expressions apply to the decays of a (neutral or charged) B meson into the final state f and the charge-conjugated decay, respectively. For the decay of a neutral B meson into a CP eigenstate with CP eigenvalue $\eta_f = \pm 1$, the RHS of Eq. (2) appears multiplied by η_f.

In this context, it is sometimes mentioned that any third weak phase may be written in terms of the first two [5]. Indeed, it is easy to show that the impact of a third weak phase ϕ_{A3} in A_f and $\bar{A}_{\bar{f}}$ can be described in terms of ϕ_{A1} and ϕ_{A2}, as long as there are parameters a and b such that

$$\begin{aligned} e^{i\phi_{A3}} &= a e^{i\phi_{A1}} + b e^{i\phi_{A2}}, \\ e^{-i\phi_{A3}} &= a e^{-i\phi_{A1}} + b e^{-i\phi_{A2}}, \end{aligned} \tag{3}$$

are both satisfied. The solutions are

$$\begin{aligned} a &= \frac{\sin(\phi_{A3} - \phi_{A2})}{\sin(\phi_{A1} - \phi_{A2})}, \\ b &= \frac{\sin(\phi_{A3} - \phi_{A1})}{\sin(\phi_{A2} - \phi_{A1})}, \end{aligned} \tag{4}$$

which are valid if $\phi_{A1} - \phi_{A2} \neq n\pi$, with n integer, meaning that (obviously) the same cannot be done with only one weak phase.

This result can be used to write any amplitude, with an arbitrary number N of distinct weak phases, in terms of only two. Indeed,

$$\begin{aligned} A_f &= \tilde{M}_1 e^{i\phi_{A1}} e^{i\tilde{\delta}_1} + \tilde{M}_2 e^{i\phi_{A2}} e^{i\tilde{\delta}_2} + \sum_{k=3}^{N} \tilde{M}_k e^{i\phi_{Ak}} e^{i\tilde{\delta}_k} \\ &= M_1 e^{i\phi_{A1}} e^{i\delta_1} + M_2 e^{i\phi_{A2}} e^{i\delta_2}, \end{aligned} \tag{5}$$

if

$$\begin{aligned} M_1 e^{i\delta_1} &= \tilde{M}_1 e^{i\tilde{\delta}_1} + \sum_{k=3}^{N} a_k \tilde{M}_k e^{i\tilde{\delta}_k}, \\ M_2 e^{i\delta_2} &= \tilde{M}_2 e^{i\tilde{\delta}_2} + \sum_{k=3}^{N} b_k \tilde{M}_k e^{i\tilde{\delta}_k}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} a_k &= \frac{\sin(\phi_{Ak} - \phi_{A2})}{\sin(\phi_{A1} - \phi_{A2})}, \\ b_k &= \frac{\sin(\phi_{Ak} - \phi_{A1})}{\sin(\phi_{A2} - \phi_{A1})}. \end{aligned} \tag{7}$$

Two questions now arises. First question: which two weak phases do we take as our basis $\{\phi_{A1}, \phi_{A2}\}$? As we shall recall below, for each decay amplitude there are three choices which appear natural within the SM. But the derivation leading to Eq. (5) made no explicit reference to the weak phases $\{\phi_{A1}, \phi_{A2}\}$ chosen; it didn't even refer to any particular model for the weak interactions. It is true that, within some particular model, we may look at its

Lagrangian for inspiration. But we *need not* do that. We may choose for our basis any pair of weak phases (as long as they do not differ by a multiple of 180°); say, $\{0°, 90°\}$, or even $\{5°, 10°\}$.

Second question: what happens when we describe the decay amplitudes with different sets of weak phases $\{\phi_{A1}, \phi_{A2}\}$ as our basis? Consider a second set of weak phases $\{\phi'_{A1}, \phi'_{A2}\}$. Using Eqs. (3) and (4), it is easy to show that

$$\begin{aligned} A_f &= M_1 e^{i\phi_{A1}} e^{i\delta_1} + M_2 e^{i\phi_{A2}} e^{i\delta_2}, \\ &= M'_1 e^{i\phi'_{A1}} e^{i\delta'_1} + M'_2 e^{i\phi'_{A2}} e^{i\delta'_2}, \end{aligned} \tag{8}$$

as long as

$$\begin{aligned} M'_1 e^{i\delta'_1} &= M_1 e^{i\delta_1} \frac{\sin(\phi_{A1} - \phi'_{A2})}{\sin(\phi'_{A1} - \phi'_{A2})} + M_2 e^{i\delta_2} \frac{\sin(\phi_{A2} - \phi'_{A2})}{\sin(\phi'_{A1} - \phi'_{A2})}, \\ M'_2 e^{i\delta'_2} &= M_1 e^{i\delta_1} \frac{\sin(\phi_{A1} - \phi'_{A1})}{\sin(\phi'_{A2} - \phi'_{A1})} + M_2 e^{i\delta_2} \frac{\sin(\phi_{A2} - \phi'_{A1})}{\sin(\phi'_{A2} - \phi'_{A1})}. \end{aligned} \tag{9}$$

Eqs. (9) tell us how to relate the parameters needed to describe the decay amplitudes with two different choices for the pair of weak phases used as a basis [6]. We stress that these weak phases may be chosen completely at will. Any set will do.

Of course, physical results cannot change under such a reparametrization; we refer to this property as "reparametrization invariance". But, this property implies that we must be careful with our wording and interpretations, especially when discussing new physics effects. This is strikingly clear in those situations usually described as depending on a single weak phase, to be studied in section VI.

B. Remarks on the Standard Model

We can use the results of the previous section in order to place in a more general context some statements commonly made about the SM. In the SM, the $\bar{b} \to \bar{q}$ transitions ($q = d, s$) involve the three CKM structures $V_{ub}^* V_{uq}$, $V_{cb}^* V_{cq}$, and $V_{tb}^* V_{tq}$. A generic decay amplitude may be written as

$$A(\bar{b} \to \bar{q}) = V_{ub}^* V_{uq} A_u + V_{cb}^* V_{cq} A_c + V_{tb}^* V_{tq} A_t, \tag{10}$$

where the A_i ($i = u, c, t$) involve the relevant hadronic matrix elements with the corresponding CP-even strong phases. But, the unitarity of the CKM matrix,

$$V_{ub}^* V_{uq} + V_{cb}^* V_{cq} + V_{tb}^* V_{tq} = 0, \tag{11}$$

can be used to express everything in terms of only two weak phases. Of course, there are three such possibilities:

$$\begin{aligned} A(\bar{b} \to \bar{q}) &= V_{ub}^* V_{uq} (A_u - A_t) + V_{cb}^* V_{cq} (A_c - A_t), && (12) \\ &= V_{ub}^* V_{uq} (A_u - A_c) + V_{tb}^* V_{tq} (A_t - A_c), && (13) \\ &= V_{cb}^* V_{cq} (A_c - A_u) + V_{tb}^* V_{tq} (A_t - A_u). && (14) \end{aligned}$$

This strategy is followed universally. In the context of $B_d \to \pi^+\pi^-$ decays, the parametrization in Eq. (12) is known as the "c-convention" and the parametrization in Eq. (13) is known as the "t-convention"; the relation among them has been discussed in detail by Gronau and Rosner in [7]. We will name the parametrization in Eq. (14) the "p-convention" (for penguin), as it does not contain the CKM structure usually associated with the tree level diagram. In the SM, the amplitudes A_i could be calculated exactly if we knew how to calculate the corresponding hadronic matrix elements [8].

Therefore, in the context of the SM, Eqs. (12) through (14) provide us with three natural choices for the pair of weak phases $\{\phi_{A1}, \phi_{A2}\}$ chosen as the basis for Eq. (5). For example, it the t-convention of Eq. (13), we would take

$$\begin{aligned} M_1 e^{i\phi_{A1}} e^{i\delta_1} &= V_{ub}^* V_{uq} (A_u - A_c), \\ M_2 e^{i\phi_{A2}} e^{i\delta_2} &= V_{tb}^* V_{tq} (A_t - A_c). \end{aligned} \tag{15}$$

But, although the unitarity of the CKM matrix was utilized in reaching Eqs. (12)–(14), unitarity is not needed in order to justify any of these basis choices. For example, we can use the weak phases in $V_{ub}^* V_{uq}$ and $V_{tb}^* V_{tq}$ as a basis, regardless of whether the CKM is unitary or not. Moreover, although these (three) choices are natural and useful within the SM, they are not mandatory; not even within the SM. We stress our main point: one may choose for the basis any pair of weak phases (as long as they do not differ by a multiple of 180°); say, $\{0°, 90°\}$, or even $\{5°, 10°\}$.

III. "EXPERIMENTAL" WEAK PHASES VERSUS "THEORY" WEAK PHASES

A. "Experimental" determination of the presence of a single weak phase

It turns out that one can determine experimentally, at least in principle, if the decay amplitude of a neutral B meson can (or not) be written in terms of a single weak phase. In order to clarify this statement, we recall that the full description of neutral meson decays involves Eqs. (1), (2), and also the mixing parameter

$$\frac{q_B}{p_B} = e^{2i\phi_M}, \tag{16}$$

in the combination

$$\begin{aligned} \lambda_f &= \frac{q_B}{p_B}\frac{\bar{A}_f}{A_f} \qquad (17)\\ &= \eta_f e^{-2i\phi_1}\frac{1+re^{i(\phi_1-\phi_2)}e^{i\delta}}{1+re^{-i(\phi_1-\phi_2)}e^{i\delta}}\;, \qquad (18)\end{aligned}$$

where $\phi_1 \equiv \phi_{A1}-\phi_M$, $\phi_2 \equiv \phi_{A2}-\phi_M$, $\delta = \delta_2-\delta_1$, and $r = M_2/M_1$. We have assumed that $|q_B/p_B| = 1$, meaning that the CP violation in $B-\overline{B}$ mixing is negligible. To set the notation, we recall in the appendix that λ_f is measurable from the decay rates through

$$\begin{aligned} S_f &\equiv \frac{2\mathrm{Im}(\lambda_f)}{1+|\lambda_f|^2} = -\eta_f\frac{\sin(2\phi_1)+2r\sin(\phi_1+\phi_2)\cos\delta+r^2\sin(2\phi_2)}{1+2r\cos(\phi_1-\phi_2)\cos\delta+r^2}, \qquad (19)\\ C_f &\equiv \frac{1-|\lambda_f|^2}{1+|\lambda_f|^2} = \frac{2r\sin(\phi_1-\phi_2)\sin\delta}{1+2r\cos(\phi_1-\phi_2)\cos\delta+r^2}, \qquad (20)\end{aligned}$$

since

$$\lambda_f = \frac{1}{1+C_f}\left(\pm\sqrt{1-C_f^2-S_f^2}+iS_f\right) \tag{21}$$

For simplicity, we will assume in the following that S_f and C_f can be measured with absolute precision.

We now claim that $C_f = 0$ *if and only if the decay amplitude is dominated by a single weak phase*. Moreover, *in such cases* S_f *determines that weak phase*, up to discrete ambiguities. It is clear that a decay dominated by a single weak phase leads to $C_f = 0$, so we only have to show the converse. Let us assume that $C_f = 0$. From Eq. (20), this implies that: i) $r = 0$ (and there is only one amplitude/weak phase); or that ii) $\phi_1 = \phi_2$ (and there is only one weak phase); or that iii) $\delta_1 = \delta_2$ [9]. In the last case, we can always find a magnitude M_3 and a weak phase ϕ_{A3} such that

$$\begin{aligned} A_f &= \left(M_1e^{i\phi_{A1}}+M_2e^{i\phi_{A2}}\right)e^{i\delta_1} = M_3e^{i\phi_{A3}}e^{i\delta_1}, \qquad (22)\\ \bar{A}_{\bar{f}} &= \eta_f\left(M_1e^{-i\phi_{A1}}+M_2e^{-i\phi_{A2}}\right)e^{i\delta_1} = \eta_f M_3e^{-i\phi_{A3}}e^{i\delta_1}. \qquad (23)\end{aligned}$$

These equalities are satisfied by

$$\begin{aligned} M_3^2 &= M_1^2+M_2^2+2M_1M_2\cos(\phi_1-\phi_2), \qquad (24)\\ e^{-2i\phi_{A3}} &= \frac{M_1e^{-i\phi_{A1}}+M_2e^{-i\phi_{A2}}}{M_1e^{i\phi_{A1}}+M_2e^{i\phi_{A2}}} = \frac{1+re^{i(\phi_{A1}-\phi_{A2})}}{1+re^{-i(\phi_{A1}-\phi_{A2})}}e^{-2i\phi_{A1}}, \qquad (25)\end{aligned}$$

from which we can always determine ϕ_{A3}, because the numerator and the denominator on the RHS of Eq. (25) are complex conjugate. (Of course, the same will not hold if $\delta_1 \neq \delta_2$.) In cases i) and ii) $S_f = -\eta_f\sin(2\phi_1)$; in case iii) $S_f = -\eta_f\sin(2\phi_3)$, where $\phi_3 = \phi_{A3}-\phi_M$. This completes our proof.

We stress the significance of our result: if $C_f = 0$ then we are sure that the amplitudes may be written in terms of only one weak phase, which, moreover, is measured through S_f. This occurs even for case iii) which was originally written as containing two distinct weak phases. Our result seems to contradict the usual simplified statement that "if the decay amplitude is determined by only one weak phase, then we can relate experiment with theory; if more than one weak phase is involved, then we cannot". The subtle, yet crucial, point is not whether we may write the decay amplitudes in terms of only one weak phase (a fact we have just shown can be ascertained experimentally) but, rather, whether we may write the decay amplitudes in terms of only one weak phase *which we can identify from the theoretical Lagrangian.*

B. Theory faces experiment

1. *Two "theory" weak phases can look like one "experimental" weak phase*

Let us consider some decay, such as $B_d \to \pi\pi$ in the SM, which receives contributions from a tree and a penguin diagram with different weak phases, ϕ_{A1} and ϕ_{A2}, respectively, in some chosen phase convention. Now imagine that, by some accident, their relative strong phase vanishes. (Of course, this is a very simplified picture, but it will illustrate our point.) In that case, Eqs. (22)–(25) guarantee that we may rewrite the corresponding decay amplitude as depending on only one weak phase, ϕ_{A3}. The problem is that we cannot turn that "experimental" information into knowledge about the weak phases ϕ_1 and ϕ_2 which appeared in our "theoretical" Lagrangian. This can be seen clearly in Eq. (25): even if we knew ϕ_1 from elsewhere, we would still require knowledge of r in order to extract ϕ_2 from the "experimental" determination of ϕ_3. And, unfortunately, $r = M_2/M_1$ depends on the ratio of uncertain hadronic matrix elements.

In this case, although we know from experiment that the decay amplitude may be rewritten in terms of a single weak phase, (without full knowledge of r) we have no way of turning that information into a determination on the weak phases present in the theoretical Lagrangian.

2. *One "theory" weak phase can be written as two "theory" weak phases*

Let us now consider some decay amplitude usually described by a single weak phase ϕ_{A3}, such as the decay amplitude for $B_d \to \psi K_S$ within the SM. Clearly, Eq. (3) implies that we could have chosen to describe the same amplitude with two weak phases which, as we stress in this article, could be chosen completely at will (we will stop mentioning that the two weak phases chosen cannot differ by a multiple of 180°). For example, we could use Eq. (3) in order to rewrite the single weak phase in terms of, say, $\phi_{A1} = 5°$ and $\phi_{A2} = 10°$. Using Eqs. (3) and (4) we find

$$\begin{aligned} A_f = M_3 e^{i\phi_{A3}} e^{i\delta_3} &= \frac{M_3 e^{i\delta_3}}{\sin 5°}\left[-\sin\left(\phi_{A3} - 10°\right) e^{i\,5°} + \sin\left(\phi_{A3} - 5°\right) e^{i\,10°}\right], \\ \bar{A}_f = M_3 e^{-i\phi_{A3}} e^{i\delta_3} &= \frac{M_3 e^{i\delta_3}}{\sin 5°}\left[-\sin\left(\phi_{A3} - 10°\right) e^{-i\,5°} + \sin\left(\phi_{A3} - 5°\right) e^{-i\,10°}\right]. \end{aligned} \tag{26}$$

In this case, the amplitudes corresponding to those weak phases must obey

$$r e^{i\delta} = \frac{M_2}{M_1} e^{i(\delta_2 - \delta_1)} = -\frac{\sin\left(\phi_{A3} - 5°\right)}{\sin\left(\phi_{A3} - 10°\right)}, \tag{27}$$

from which δ is either 0 or π and we get back the case iii) discussed above. Again, the key point is whether we can relate the weak phase measured by experiment with a parameter in the theoretical Lagrangian. In this case we assumed from the start that we could; we assumed that we could relate ϕ_{A3} with a weak phase in the theoretical Lagrangian. Rewriting things in terms of two weak phases, as in Eq. (26), was just a nasty and unneeded complication.

One might worry whether, when written in terms of the two weak phases $\{\phi_{A1}, \phi_{A2}\} = \{5°, 10°\}$, the experimental observables S_f and C_f remain consistent with the presence of a single weak phase. Specially given how complicated Eqs. (19) and (20) become. However, going back to Eqs. (19) and (20), and substituting $\phi_1 = 5° - \phi_M$, $\phi_2 = 10° - \phi_M$, and $re^{i\delta}$ by Eq. (27), we do indeed recover $S_f = -\eta_f \sin(2\phi_3)$ and $C_f = 0$.

C. Which weak phases can we measure?

The most striking result of the previous arguments is the following: *a priori, the weak phases appearing in the parametrization of the decay amplitudes have no physical meaning.* Said otherwise, the fact that some weak phase is written in the decomposition of A_f does not, by itself, guarantee that that weak phase will be observable. This is the only possible conclusion from the fact that we can *choose at will* the two weak phases $\{\phi_{A1}, \phi_{A2}\}$ for our basis, *c.f.* Eqs. (8), and that there are infinite such choices.

Although we have already proved this result, it is interesting to revisit it in the following way. Let us compare the two basis $\{\phi_{A1}, \phi_{A2}\}$ and $\{\phi'_{A1}, \phi_{A2}\}$, with $\phi_{A1} \neq \phi'_{A1}$. Imagine that there were an algorithm allowing us to write ϕ_{A1} as a function of physical observables. Then, given the similarity of the functional forms in Eqs. (8), we would be able to extract ϕ'_{A1} with exactly the same function of the physical observables. But that would lead to $\phi_{A1} = \phi'_{A1}$, which contradicts our assumption. A very particular version of this argument has been used by London, Sinha and Sinha in [10], when comparing the c- and t-conventions in $B \to \pi\pi$ decays. Here we have shown that this result is true in complete generality; it affects whatever weak phase we use in the decay amplitude, because we can use any phase ϕ_{A1} in the parametrization of any decay amplitude (as long as we use also a second weak phase $\phi_{A2} \neq \phi_{A1} + n\pi$). What we cannot do is parametrize a *generic* decay amplitude using *exclusively* the weak phase ϕ_{A1}.

Thus far, we have shown the following:

- Experimentally, if $C_f = 0$ (corresponding to $|\lambda_f| = 1$), then one can write the decay amplitude in terms of only one weak phase, and that phase is measurable through S_f.
- Theoretically, one usually writes that decay amplitude in terms of the weak phases appearing in the electroweak Lagrangian.
- If the theoretical description of the decay amplitude involves only one weak phase from the electroweak Lagrangian, then we can identify that phase with the phase measured experimentally.
- Otherwise, we have an "experimental" weak phase which we cannot turn into a determination of the "theory" weak phases used in the decomposition of the decay amplitude.

These issues are easier to understand by considering a specific decay within the SM.

IV. REPARAMETRIZATION INVARIANCE IN $B \to \pi\pi$ DECAYS

A. Decay amplitudes in $B \to \pi\pi$

In the SM, the $\bar{b} \to \bar{d}$ decays leading to $B \to \pi\pi$ can get contributions proportional to any of the CKM structures in Eq. (10):

$$\begin{aligned} V_{ub}^* V_{ud} &\sim A\lambda^3 R_b e^{i\gamma}, && (28)\\ V_{cb}^* V_{cd} &\sim -A\lambda^3, && (29)\\ V_{tb}^* V_{td} &\sim A\lambda^3 R_t e^{-i\beta}. && (30) \end{aligned}$$

The quantities appearing on the RHS are defined in the appendix in a rephasing invariant way, but a convenient phase convention has been used in equating the RHS to the LHS. Substitution of Eqs. (28)-(30) in Eq. (11) leads to the usual form of the unitarity triangle

$$R_b e^{i\gamma} + R_t e^{-i\beta} = 1. \tag{31}$$

Thus, if inspired by the weak phases which appear in the SM description of $B \to \pi\pi$ decays, we are lead to choosing two among the weak phases γ, π, and $-\beta$. In the c-convention of Eq. (12), the phases are γ and π; in the t-convention of Eq. (13), the phases are γ and $-\beta$; and, in the p-convention of Eq. (14), the phases are π and $-\beta$.

Of course, due to reparametrization invariance, we may equally well choose any other set of two phases as our basis; we need not look for the SM for inspiration. For example, we could again choose to write the decay amplitudes in terms of $\{\phi_{A1}, \phi_{A2}\} = \{5^\circ, 10^\circ\}$. Any two different sets are related to each other as in Eqs. (9).

Let us apply this relation to the two sets $\{\phi_{A1}, \phi_{A2}\} = \{\gamma, \pi\}$, and $\{\phi'_{A1}, \phi'_{A2}\} = \{\gamma, -\beta\}$. Direct application of Eqs. (9) leads to

$$\begin{aligned} T_t e^{i\delta_t^T} &= T_c e^{i\delta_c^T} - \frac{\sin\beta}{\sin(\beta+\gamma)} P_c e^{i\delta_c^P}, \\ P_t e^{i\delta_t^P} &= -\frac{\sin\gamma}{\sin(\beta+\gamma)} P_c e^{i\delta_c^P}, && (32) \end{aligned}$$

where we have used the conventional notation $M_1 = T_c$, $M_2 = P_c$, $M'_1 = T_t$, and $M'_2 = P_t$ [11]. Moreover, $\delta_1 = \delta_c^T$, $\delta'_1 = \delta_t^T$, $\delta'_2 = \delta_t^P$, and our $\delta_2 = \delta_c^P$ has the opposite sign of that used by Gronau and Rosner in [7], because we define the weak phase of the penguin contribution in the c-convention as π, rather than 0 as done there. Eqs. (32) reproduce Eqs. (6) through (10) of reference [7]. However, here it becomes clear that those relations have absolutely nothing to do with the unitarity of the CKM matrix, which was clearly not used in deriving Eqs. (9). Eqs. (32) result merely from our freedom to reparametrize the decay amplitudes.

In some sense, the relations in Eqs. (32) do not even bear any relation to the SM. Because this might be difficult to accept, let us explain it in detail. Imagine that there is no new physics contribution affecting the determinations of $|V_{ub}|$, Δm_d, and the CP violating asymmetry in $B_d \to \psi K_S$. These measurements determine the Wolfenstein parameters η and ρ, and, thus, the phases β and γ, albeit with errors. Let us now assume that there is a substantial new physics contribution to $B_d \to \pi^+\pi^-$. Then, we may still parametrize the decay amplitude in terms of the known phases $\{\phi_{A1}, \phi_{A2}\} = \{\gamma, \pi\}$, or, alternatively, in terms of the known phases $\{\phi'_{A1}, \phi'_{A2}\} = \{\gamma, -\beta\}$. And Eqs. (32) still tell us how to go from one parametrization to the next. Nevertheless, there is a difference between the unreal scenario we are considering here and the SM. In the unreal scenario considered here we have absolutely no access to the magnitudes T_t, P_t, T_c, and P_c. In contrast, in the SM we might, in principle, calculate these magnitudes within some procedure, such as QCD factorization [12] or perturbative QCD [13].

An exercise similar to the one leading to the relations in Eqs. (32) allows us to relate the quantities in the p-convention with the others. We find

$$\begin{aligned} M_{1p} e^{i\delta_{1p}} &= -\frac{\sin(\beta+\gamma)}{\sin\beta} T_c e^{i\delta_c^T} + P_c e^{i\delta_c^P}, \\ M_{2p} e^{i\delta_{2p}} &= -\frac{\sin\gamma}{\sin\beta} T_c e^{i\delta_c^T}, \end{aligned} \tag{33}$$

and

$$\begin{aligned} M_{1p} e^{i\delta_{1p}} &= -\frac{\sin(\beta+\gamma)}{\sin\beta} T_t e^{i\delta_t^T}, \\ M_{2p} e^{i\delta_{2p}} &= -\frac{\sin\gamma}{\sin\beta} T_t e^{i\delta_t^T} + P_t e^{i\delta_t^P}, \end{aligned} \tag{34}$$

where we have denoted the magnitudes and strong phases in the p-convention by M_{1p}, M_{2p}, δ_{1p}, and δ_{2p} [11].

B. Parameter counting for the CP asymmetry in $B_d \to \pi^+\pi^-$

Let us go back to the c-convention,

$$A_{\pi^+\pi^-} = T_c\, e^{i\delta_c^T} e^{i\gamma} + P_c\, e^{i\delta_c^P} e^{i\pi}, \tag{35}$$

and use the new notation $\phi_M = -\tilde{\beta}$ for the weak phase in mixing. From Eq. (18), we find

$$\lambda_{\pi^+\pi^-} = e^{-2i\tilde{\beta}}\, \frac{e^{-i\gamma} + z_c}{e^{i\gamma} + z_c}, \tag{36}$$

where

$$z_c = e^{i\pi}\, \frac{P_c\, e^{i\delta_c^P}}{T_c\, e^{i\delta_c^T}}. \tag{37}$$

Notice that this parametrization is completely general. Any new physics model (with $|q_B/p_B| = 1$) can be brought to this form. If the new physics affects the phase of the mixing but does not affect substantially the $\bar{b} \to \bar{c}c\bar{s}$ decay amplitudes, then $\tilde{\beta}$ is the phase measured in the decays $B_d \to \psi K$. In that case, the two measurements contained in $\lambda_{\pi^+\pi^-}$ (*i.e.*, its magnitude and phase, or, alternatively, $S_{\pi^+\pi^-}$ and $C_{\pi^+\pi^-}$) depend on three parameters: γ, the phase and the magnitude of z_c.

Similarly, in the t-convention

$$A_{\pi^+\pi^-} = T_t\, e^{i\delta_t^T} e^{i\gamma} + P_t\, e^{i\delta_t^P} e^{-i\beta}, \tag{38}$$

and

$$\lambda_{\pi^+\pi^-} = e^{-2i\tilde{\beta}}\, \frac{e^{-i\gamma} + z_t\, e^{i\beta}}{e^{i\gamma} + z_t\, e^{-i\beta}} = e^{-2i(\tilde{\beta}-\beta)}\, \frac{e^{-i(\beta+\gamma)} + z_t}{e^{i(\beta+\gamma)} + z_t}, \tag{39}$$

where

$$z_t = \frac{P_t\, e^{i\delta_t^P}}{T_t\, e^{i\delta_t^T}}. \tag{40}$$

In this notation, assuming that $\tilde{\beta}$ has been measured, the two observables depend on four parameters: γ, β, the phase and magnitude of z_t.

In the SM, $\tilde{\beta} \equiv \beta$ and the parameter counting in the two notations coincides. But, in the presence of new physics in the mixing, the parameter counting does not coincide; the t-convention seems to have one more unknown. Why does this occur? We have assumed that $\tilde{\beta}$ is measured in $B_d \to \psi K$ decays. So, we only have as many unknowns as those present in $\bar{A}_f/A_f$. When the decay amplitudes are written in terms of only two weak phases, there are four unknowns: the two weak phases $\{\phi_{A1}, \phi_{A2}\}$; the ratio of magnitudes; and the difference of strong phases. In the c-convention, we choose for the first phase the theory variable γ, and for the second phase the constant phase π. In contrast, in the t-convention, we choose for the first phase the theory variable γ, and for the second phase another theory variable $-\beta$. So, natural as they may be, the different choices lead to different parameter counting.

Using Eqs. (32) we find the relation between the two hadronic parameters to be

$$z_c = \frac{z_t \sin(\beta+\gamma)}{\sin\gamma - z_t \sin\beta}. \tag{41}$$

So, in a way, the three parameters which appeared in the t-convention as z_t and β are, in the c-convention, reorganized into the two parameters in z_c. It is also interesting to understand why the extra weak phase β goes completely unnoticed when one stays within the SM and calculates (with some prescription) the hadronic matrix elements in the two conventions. The reason lies in the unitarity of the CKM matrix, due to which

$$\begin{aligned} R_b &= \frac{\sin\beta}{\sin(\beta+\gamma)}, \\ R_t &= \frac{\sin\gamma}{\sin(\beta+\gamma)}. \end{aligned} \tag{42}$$

As a result,

$$z_c = \frac{z_t}{R_t - z_t\, R_b}. \tag{43}$$

Eq. (41) seemed to involve the weak phases. However, due to CKM unitarity, this dependence on the weak phases is hidden as CP-conserving quantities R_t and R_b in the relation between the hadronic parameters in the two conventions (z_c and z_t). This reflects the fact that the model calculations of z_c depend on the CP-conserving constraints on the CKM matrix. Thus, any knowledge about γ obtained by combining the c-convention in Eq. (36) with some model calculation of z_c provides a consistency check within the SM, but it may not yield an independent determination of the phase γ of the generalized CKM matrix, valid in models with non-unitary CKM matrices.

As a further example, we write

$$\begin{aligned} A_{\pi^+\pi^-} &= A_5\, e^{i\delta_5} e^{i\,5^\circ} + A_{10}\, e^{i\delta_{10}} e^{i\,10^\circ}, \\ \bar{A}_{\pi^+\pi^-} &= A_5\, e^{i\delta_5} e^{-i\,5^\circ} + A_{10}\, e^{i\delta_{10}} e^{-i\,10^\circ}, \end{aligned} \tag{44}$$

and we use Eqs. (9) in order to relate the basis set $\{\phi_{A1}, \phi_{A2}\} = \{\gamma, \pi\}$, with $\{\phi'_{A1}, \phi'_{A2}\} = \{5^\circ, 10^\circ\}$. We find

$$\begin{aligned} -A_5\, e^{i\delta_5} &= \frac{\sin(\gamma - 10^\circ)}{\sin(5^\circ)} T_c\, e^{i\delta_c^T} + 2\cos(5^\circ) P_c\, e^{i\delta_c^P}, \\ A_{10}\, e^{i\delta_{10}} &= \frac{\sin(\gamma - 5^\circ)}{\sin(5^\circ)} T_c\, e^{i\delta_c^T} + P_c\, e^{i\delta_c^P}, \end{aligned} \tag{45}$$

Therefore, in the $\{5^\circ, 10^\circ\}$ basis

$$\lambda_{\pi^+\pi^-} = e^{-2i\tilde{\beta}}\, \frac{e^{-i5^\circ} + z\, e^{-i10^\circ}}{e^{i5^\circ} + z\, e^{i10^\circ}} = e^{-2i(\tilde{\beta}-5^\circ)}\, \frac{1 + z\, e^{-i5^\circ}}{1 + z\, e^{i5^\circ}}, \tag{46}$$

where

$$z = \frac{A_{10}\, e^{i\delta_{10}}}{A_5\, e^{i\delta_5}} = \frac{-\sin(\gamma - 5^\circ) + z_c \sin(5^\circ)}{\sin(\gamma - 10^\circ) - z_c \sin(10^\circ)}. \tag{47}$$

Assuming that $\tilde{\beta}$ has been measured, the two observables in $\lambda_{\pi^+\pi^-}$ depend on only two parameters: the phase and magnitude of z.

For completeness, we include the p-convention case, where $\{\phi_{A1}, \phi_{A2}\} = \{\pi, -\beta\}$. We find

$$\lambda_{\pi^+\pi^-} = e^{-2i\tilde{\beta}}\, \frac{1 + z_p\, e^{i\beta}}{1 + z_p\, e^{-i\beta}}, \tag{48}$$

where

$$\begin{aligned} z_p &= e^{i\pi}\, \frac{M_{2p}\, e^{i\delta_{2p}}}{M_{1p}\, e^{i\delta_{1p}}} \\ &= -\frac{\sin\gamma}{\sin(\beta+\gamma) + z_c \sin\beta} = \frac{-\sin\gamma + z_t \sin\beta}{\sin(\beta+\gamma)}, \end{aligned} \tag{49}$$

which, if the CKM is unitary, turns into

$$z_p = -\frac{R_t}{1 + z_c R_b} = -R_t + z_t R_b\ . \tag{50}$$

To summarize, let us assume that $\tilde{\beta}$ has been measured, and that we wish to interpret the two measurements contained in $\lambda_{\pi^+\pi^-}$ (*i.e.*, its magnitude and phase, or, alternatively, $S_{\pi^+\pi^-}$ and $C_{\pi^+\pi^-}$). We can fit these two observables in a variety of ways: with two quantities (z), in the $\{5^\circ, 10^\circ\}$ basis; with three quantities (z_c and γ), in the c-convention; or with four quantities (z_t, γ, and β), in the t-convention. As we have just explained, there is no inconsistency, despite the different parameter counting in each case.

C. The isospin analysis in $B \to \pi\pi$

In this section, we extend the Gronau-London isospin analysis [14], allowing for the presence of new physics effects. We face it not as a way to "trap the penguin" but, rather, as a way to determine

$$\lambda_{\pi^+\pi^0} = \frac{q}{p}\frac{A(B^- \to \pi^-\pi^0)}{A(B^+ \to \pi^+\pi^0)}. \tag{51}$$

In principle, the expression on the RHS of Eq. (51) does not make sense; it is no even rephasing invariant under independent phase transformations of B_d^0, $\overline{B_d^0}$, B^+, and B^-. However, if one assumes isospin symmetry, then B^+ must transform with B_d^0 and B^- with $\overline{B_d^0}$, and Eq. (51) becomes rephasing invariant. Although $\lambda_{\pi^+\pi^0}$ is not measurable directly by any single experiment, since it involves both neutral and charged B states, we will now show that it can be determined experimentally through the standard isospin analysis, even in the presence new physics in both mixing and some type of new physics in decay (see below).

In the isospin analysis of Gronau and London one requires the observables C_{+-}, S_{+-}, B_{+-}, C_{+0}, B_{+0}, C_{00}, and B_{00}. Here and henceforth, the sub-indexes refer to the charges of the final state, and we have used

$$B_f = \frac{|A_f|^2 + |\bar{A}_f|^2}{2}, \tag{52}$$

which is proportional to the untagged (charge averaged) branching ratio into the final state f. Eqs. (20) and (52) may be inverted,

$$\begin{aligned} |A_f|^2 &= B_f(1 + C_f), \\ |\bar{A}_f|^2 &= B_f(1 - C_f), \end{aligned} \tag{53}$$

meaning that the magnitudes of the decay amplitudes are determined from the observables C_f and B_f obtained in time-integrated decay rates.

Although we allow for new physics in the decay amplitudes, we will assume that it obeys two properties: i) that it does not produce large $\Delta I = 5/2$ contributions to the decay amplitudes; and ii) that it does not produce large isospin-violating contributions. Under these conditions, the decay amplitudes obey two triangle relations,

$$\begin{aligned} A_{+0} &= \frac{1}{\sqrt{2}} A_{+-} + A_{00}, \\ \bar{A}_{+0} &= \frac{1}{\sqrt{2}} \bar{A}_{+-} + \bar{A}_{00}. \end{aligned} \tag{54}$$

Since we know the magnitudes of all the decay amplitudes, we may calculate

$$\begin{aligned}
R &= \mathrm{Re}\left(A_{+0}A^*_{+-}\right) = \frac{|A_{+0}|^2 + 1/2|A_{+-}|^2 - |A_{00}|^2}{\sqrt{2}},\\
\bar{R} &= \mathrm{Re}\left(\bar{A}_{+0}\bar{A}^*_{+-}\right) = \frac{|\bar{A}_{+0}|^2 + 1/2|\bar{A}_{+-}|^2 - |\bar{A}_{00}|^2}{\sqrt{2}},\\
\rho &= |A_{+0}A^*_{+-}|,\\
\bar{\rho} &= |\bar{A}_{+0}\bar{A}^*_{+-}|,
\end{aligned} \tag{55}$$

from which we can determine the complex vectors

$$\begin{aligned}
A_{+0}A^*_{+-} &= R \pm i\sqrt{\rho^2 - R^2},\\
\bar{A}_{+0}\bar{A}^*_{+-} &= \bar{R} \pm i\sqrt{\bar{\rho}^2 - \bar{R}^2}.
\end{aligned} \tag{56}$$

Recall that the measurements of C_{+-} and B_{+-} yield $|A_{+-}|$ and $|\bar{A}_{+-}|$. We continue to assume that $|q/p| = 1$, thus determining $|\lambda_{+-}|$. Combining this with S_{+-} determines also the phase of λ_{+-}, up to a two-fold discrete ambiguity, *c.f.* Eq. (21). Therefore, λ_{+-} is known experimentally. Now we notice that

$$\lambda_{+0}\lambda^*_{+-} = \left|\frac{q}{p}\right|^2 \frac{\bar{A}_{+0}\bar{A}^*_{+-}}{A_{+0}A^*_{+-}}, \tag{57}$$

leading to

$$\lambda_{+0} = \frac{1}{\lambda^*_{+-}} \frac{\bar{R} \pm i\sqrt{\bar{\rho}^2 - \bar{R}^2}}{R \pm i\sqrt{\rho^2 - R^2}}. \tag{58}$$

Thus, under the assumptions about the new physics mentioned above, the isospin analysis allows us to determine the complex quantity λ_{+0} (up to an eight-fold discrete ambiguity), which is the result we wanted to prove.

As a particular case, we consider first how this analysis applies to the SM. There, the $\Delta I = 5/2$ contributions [16], the electroweak penguins [17] and other isospin violating contributions to this channel are small [18]. Neglecting them, the isospin triangle relations in Eqs. (54) remain. Also, the penguin diagrams only contribute to the $\Delta I = 1/2$ term which, moreover, does not contribute to A_{+0}. As a result, A_{+0} has the phase γ of the tree level diagram and the "reconstructed observable" is

$$\lambda_{\pi^+\pi^0} = e^{2i\alpha}, \tag{59}$$

where $\alpha = \pi - \beta - \gamma$. Thus, in the SM, one predicts that $|\lambda_{+0}| = 1$ (*i.e.*, $C_{+0} = 0$), and that the phase reconstructed from the measurable quantities through Eq. (58) is the CKM phase α. (We may allow for models in which the new physics contributes exclusively to a new phase in the mixing by substituting α by $\tilde{\alpha} = \pi - \tilde{\beta} - \gamma$.) In the SM, this does not coincide with the phase measured in $B_d \to \pi^+\pi^-$ decays,

$$\lambda_{\pi^+\pi^-} = |\lambda_{\pi^+\pi^-}| e^{2i\alpha_{\rm eff}}, \tag{60}$$

and Eq. (58) reads

$$e^{2i\alpha} = e^{2i\alpha_{\rm eff}} e^{-2i\delta_\alpha}. \tag{61}$$

The factor of two multiplying α in the exponent leads to a further two-fold ambiguity, meaning that this isospin analysis determines α with a sixteen-fold discrete ambiguity. It is well known that the SM isospin construction may be used in order to place bounds on δ_α (and, thus, on how much the $\alpha_{\rm eff}$ measured in $B_d \to \pi^+\pi^-$ decays differs from the CKM phase α) even if the observable C_{00} is not known to the required precision [15].

We should also mention that there is a relation between λ_{+0} and λ_{00} similar to the one between λ_{+0} and λ_{+-} in Eq. (58). This means that the observables B_{+-}, C_{+-}, S_{+-}, B_{00}, C_{00}, B_{+0}, and C_{+0} determine not only λ_{+0} but also S_{00}. If the measurement of S_{00} were feasible, we would have a cross-check on the isospin construction, as well as a reduction in the ambiguity in the determination of α to four-fold.

In the SM, Eq. (58) is just a new way of interpreting the Gronau-London isospin analysis. However, Eq. (58) tells us what portions of the isospin analysis remain in more general models; namely, the determination of λ_{+0}.

Let us now turn to the interpretation of this general isospin analysis, in the light of reparametrization invariance. If $C_{+0} = 0$, then we know from subsection III A that λ_{+0} determines one "experimental" weak phase. In theories in which there is really only one weak phase mediating the decay, then λ_{+0} measures that weak phase. But, as mentioned in subsection III B 1, $C_{+0} = 0$ might be due to a negligible strong phase difference between two amplitudes with different weak phases. In either case, the SM is excluded if the weak phase of λ_{+0} differs from the SM 2α (up to electroweak penguins). Of course, the SM is trivially excluded if C_{+0} is sufficiently different from zero.

Let us now imagine that λ_{+0} has been determined, has unit magnitude and that its phase agrees (within errors) with the SM 2α. Does this guarantee that the SM is the correct description of the decay? We will return to this question at the end of section VI.

V. THE DECAY $B \to \psi K$

In the SM, the $\bar{b} \to \bar{s}$ decays leading to $B \to \psi K$ can get contributions proportional to any of the CKM structures in Eq. (10):

$$V_{ub}^* V_{us} \sim A\lambda^4 R_b e^{i(\gamma+\chi')}, \tag{62}$$

$$V_{cb}^* V_{cs} \sim A\lambda^2, \tag{63}$$

$$V_{tb}^* V_{ts} \sim -A\lambda^2 R_t e^{i\chi}. \tag{64}$$

The quantities appearing on the RHS are defined in the appendix in a rephasing invariant way, but a convenient phase convention has been used in equating the RHS to the LHS. The phases already reflect the structure of a possible non-unitary CKM matrix, but the magnitudes reflect the SM (unitarity) constraints. Substitution of Eqs. (62)-(64) in Eq. (11) leads to

$$-\lambda^2 R_b e^{i(\gamma+\chi')} + e^{i\chi} = 1, \tag{65}$$

which represents a very "squashed" unitarity triangle. Thus, if inspired by the weak phases which appear in the SM description of $B \to \psi K$ decays, we are lead to choosing two among the weak phases $\gamma + \chi'$, 0, and $\chi + \pi$. The equivalent to the c-convention in Eq. (35) is

$$A_{\psi K} = M_{1c}\, e^{i\delta_{1c}} e^{i(\gamma+\chi')} + M_{2c}\, e^{i\delta_{2c}}; \tag{66}$$

the equivalent to the t-convention in Eq. (38) is

$$A_{\psi K} = M_{1t}\, e^{i\delta_{1t}} e^{i(\gamma+\chi')} + M_{2t}\, e^{i\delta_{2t}} e^{i(\chi+\pi)}; \tag{67}$$

and the equivalent to the p-convention is

$$A_{\psi K} = M_{1p}\, e^{i\delta_{1p}} + M_{2p}\, e^{i\delta_{2p}} e^{i(\chi+\pi)}. \tag{68}$$

But we may equally well use any other two weak phases as our basis. The relation between the several options follows the analysis in sections IV A and IV B.

In the SM, the $\bar{b} \to \bar{s}$ decays are predicted to depend (almost) on a single weak phase. A rough argument is usually presented based on the CKM structures in Eqs. (62)–(64), and can be seen clearly in Eq. (68), where the second phase, $\chi \sim \lambda^2$, is very close to the first phase, 0. To revisit this argument in the other two conventions we must notice that, although the phase difference is large, the SM predicts a hierarchy in the magnitudes of those cases. Indeed, what matters for the difference of λ_f from the ideal case in which it measures a pure weak phase ($\phi_2 = \phi_{A2} - \phi_M$) is the product [21]

$$\frac{M_1}{M_2} \sin(\phi_{A1} - \phi_{A2}). \tag{69}$$

Table I shows the hierarchy of the two magnitudes imposed by the magnitudes of the CKM matrix elements and the phase difference in the three conventions. The product is always small, leading to the conclusion that, barring a large compensating hierarchy arising from the hadronic matrix elements, the SM predicts that $\lambda_{\psi K_S}$ does measure a single weak phase. Several calculations of the hadronic matrix elements involved show that these do not offset the hierarchy seen in Table I. As a result, using the c–type convention within the SM, one finds

$$\lambda_{\psi K_S} = -e^{-2i(\tilde{\beta}+\chi')} \frac{1 + |M_{1c}/M_{2c}| e^{-i(\gamma+\chi')} e^{i(\delta_{1c}-\delta_{2c})}}{1 + |M_{1c}/M_{2c}| e^{i(\gamma+\chi')} e^{i(\delta_{1c}-\delta_{2c})}} \tag{70}$$

$$\approx -e^{-2i(\tilde{\beta}+\chi')}, \tag{71}$$

TABLE I: CKM predictions for the (CKM part of the) ratio of magnitudes and for the difference of weak phases between the two terms, in the various conventions.

convention	$(M_1/M_2)_{\text{CKM part}}$	$\sin(\phi_{A1} - \phi_{A2})$	product
c-type	$\sim R_b \lambda^2$	$\sin(\gamma + \chi') \sim 1$	$\sim \lambda^2$
t-type	$\sim R_b \lambda^2$	$\sin(\gamma + \chi' - \chi - \pi) \sim 1$	$\sim \lambda^2$
p-type	~ 1	$\sin(-\chi - \pi) \sim \lambda^2$	$\sim \lambda^2$

where $|M_{1c}/M_{2c}| \ll 1$ was used in obtaining the second line. (We mention in passing that the decay $B_d \to \phi K_S$ may be written in a similar fashion. However, there is some experimental evidence that, in that case, $|M_{1c}/M_{2c}|$ may not be much smaller than unity, meaning that the presence of a second weak phase must be taken into account.)

The fact that $\lambda_{\psi K_S}$ is given by a single weak phase and that that phase agrees with the SM expectation (2β) is confirmed experimentally to very high accuracy [22]:

$$\begin{aligned} |\lambda_{B_d \to (c\bar{c})K}| &= 0.969 \pm 0.028 \ , \\ \sin\left(\arg \lambda_{B_d \to (c\bar{c})K}\right) &= 0.725 \pm 0.037 \ . \end{aligned} \tag{72}$$

Since λ_{+0} and $\lambda_{\psi K_S}$ depend on a single weak phase in the SM, we now turn to a detailed analysis of those situations, in the light of reparametrization invariance.

VI. DECAYS WHICH DEPEND ON A SINGLE WEAK PHASE

A. Reparametrization invariance versus rephasing invariance

Before we proceed, we should clarify a few questions. First: so far, we have been using the weak phases in decay, ϕ_{A1} and ϕ_{A2}, and the weak phase in mixing, ϕ_M. However, these quantities are not invariant under a rephasing of the meson states or of the quark field operators [19]. The quantities which are rephasing invariant are $\phi_1 = \phi_{A1} - \phi_M$ and $\phi_2 = \phi_{A2} - \phi_M$, which show up in Eq. (18). A similar argument applies to $\delta = \delta_2 - \delta_1$. For this reason, it is sometimes better to combine Eqs. (1), (2), and (16) into

$$\sqrt{\frac{q_B}{p_B}} \bar{A}_f \ = \ \eta_f \, e^{i\delta_1} \left(M_1 e^{-i\phi_1} + M_2 e^{-i\phi_2} e^{i\delta} \right) , \tag{73}$$

$$\sqrt{\frac{p_B}{q_B}} A_f \ = \ e^{i\delta_1} \left(M_1 e^{i\phi_1} + M_2 e^{i\phi_2} e^{i\delta} \right) , \tag{74}$$

which (δ_1 aside) refer only to rephasing invariant quantities.

Second: this rephasing invariance is *not* the same as the reparametrization invariance we discuss in this article. To see this, we consider some decay amplitude with phases ϕ_{A1} and ϕ_{A2}. It is true that, through rephasings, we may change these phases into $\phi_{A1} - \phi'$ and $\phi_{A2} - \phi'$, respectively. However, the difference between the two phases remains fixed. This is *not* the case when we consider a general reparametrization from the phases $\{\phi_{A1}, \phi_{A2}\}$ into some other basis $\{\phi'_{A1}, \phi'_{A2}\}$, such that $\phi'_{A2} - \phi'_{A1} \neq \phi_{A2} - \phi_{A1}$. Section IV A provides ample illustrations of this point.

Third: under reparametrization of the decay amplitudes, the rephasing invariant phases ϕ_1 and ϕ_2 are changed into $\phi'_1 = \phi'_{A1} - \phi_M$ and $\phi'_2 = \phi'_{A2} - \phi_M$, respectively. As we stressed when discussing the phases in the decay amplitudes, this means that, in general, these phases are not measurable in a single decay. What we can measure is the phase of λ_f, which, as we showed in section III A, measures a single weak phase if and only if $C_f = 0$.

B. Interpreting results which depend on a single weak phase

Imagine that $C_f = 0$ and that S_f has been measured. Then, as we showed in section III A, λ_f depends on a single "experimental" weak phase $\phi_3 = \phi_{A3} - \phi_M$,

$$\lambda_f = \eta_f e^{-2i\phi_3}. \tag{75}$$

We may now ask what type of theories reproduce this result. Clearly, a theory in which the decay is given by a single weak phase and has

$$e^{-i\delta_3}\sqrt{\frac{q_B}{p_B}}\bar{A}_f = \eta_f\, M_3\, e^{-i\phi_3} \tag{76}$$

reproduces this result. (In the usual parlance of phases in mixing, ϕ_M, and phases in decay, ϕ_{A3}, this may be obtained with different choices for these phases, ϕ'_{A3} and ϕ'_M, as long as $\phi_3 = \phi_{A3} - \phi_M = \phi'_{A3} - \phi'_M$.)

Can this result be obtained in a different way with the parametrization in Eqs. (73) and (74)? Eq. (25) answers this question. It shows that a theory with two distinct weak phases can reproduce the "experimental" result of Eq. (75) as long as the two diagrams do not have a strong phase difference and their weak phases obey Eq. (25).

We will now address a more subtle question. Let us assume that one knows that λ_f is given by a single weak phase within the SM, as in Eq. (75). This occurs, for instances, in λ_{+0} discussed at the end of section IV C, or in $\lambda_{\psi K_S}$ discussed in section V. In each case, we know what the SM diagrams are and that, when there are several, they share (at least very approximately) their weak phase. We also assume that the experiments confirm that this is indeed given by a single weak phase ($C_f = 0$) and that this phase coincides with the SM prediction (S_f has been measured and it agrees with the SM expectation). Now we entertain the possibility that there is only one new physics diagram contributing to this decay, with a new weak phase ϕ_{A4}, and that there is no new contribution to the mixing. Then ϕ_3 has the SM value and there is now a new weak phase $\phi_4 = \phi_{A4} - \phi_M \neq \phi_3$. Clearly, because we wish to reproduce λ_f, the strong phase difference between the two diagrams must vanish. But then, the new weak phase must be such that

$$e^{-2i\phi_3} = \frac{M_3 e^{-i\phi_3} + M_4 e^{-i\phi_4}}{M_3 e^{i\phi_3} + M_4 e^{i\phi_4}} = \frac{1 + r e^{i(\phi_3-\phi_4)}}{1 + r e^{-i(\phi_3-\phi_4)}}\, e^{-2i\phi_3}. \tag{77}$$

This equation is only possible if $\phi_4 = \phi_3 + n\pi$, contradicting our hypothesis. This means that in experiments which:

- are predicted in the SM to depend on a single weak phase;
- the "experimental" weak phase has been measured (*i.e.*, $C_f = 0$ and S_f has been measured);
- and the weak phase measured coincides with that predicted by the SM;

there is absolutely no possibility that the new physics brings a new weak phase exclusively to the decay which is different from the SM one (even if the strong phase difference between the two diagrams vanishes). It is easy to see that having more than one new physics diagram does not help [20].

The only way out of this conclusion occurs if the new physics contributes with a new weak phase to the decay and also with a new weak phase to the mixing. In that case, the full theory has a new mixing phase ϕ'_M, leading to two weak phases $\phi'_3 \neq \phi_3$ and ϕ'_4 and Eq. (77) becomes

$$e^{-2i\phi_3} = \frac{M_3 e^{-i\phi'_3} + M_4 e^{-i\phi'_4}}{M_3 e^{i\phi'_3} + M_4 e^{i\phi'_4}} = \frac{1 + r e^{i(\phi'_3-\phi'_4)}}{1 + r e^{-i(\phi'_3-\phi'_4)}}\, e^{-2i\phi'_3}, \tag{78}$$

which might have nontrivial solutions.

VII. CONCLUSIONS

In this article, we point out that a generic $B \to f$ decay amplitude may be written as a sum of two terms, corresponding to a pair of weak phases $\{\phi_{A1}, \phi_{A2}\}$ chosen completely at will (as long as they do not differ by a multiple of 180°). Clearly, physical observables may not depend on this choice; we designate this property by "reparametrization invariance".

We explore some of the unusual features of reparametrization invariance. For example, we show that the relations between the c-convention and t-convention in $B \to \pi\pi$ decays have nothing to do with CKM unitarity and that Eqs. (32) would hold even if "β" and "γ" were merely names for some weak phases with no connection to the SM whatsoever. This allows us to explain the apparent discrepancy in the parameter counting of $\lambda_{\pi^+\pi^-}$ for some specific choices for the weak phases utilized as a basis, *c.f.* Eqs. (36), (39), and (46). We have extended the isospin analysis of Gronau and London to cases with new physics in the mixing and (some types of) new physics in the decay amplitudes, viewing it as a way to measure $\lambda_{\pi^+\pi^0}$.

As a result of reparametrization invariance, the weak phases used as a basis cannot (in general) be measured in any single decay. To study the one exception, we showed that $C_f = 0$ if and only if the decay amplitude is dominated by a single weak phase. Thus, decays in which $C_f = 0$ do determine one "experimental" weak phase. Nevertheless, this can only be turned into knowledge about a SM weak phase in those cases in which the SM description of the decay depends on only one weak phase. Finally, in experiments with $C_f = 0$, which are predicted in the SM to depend on a single weak phase and do indeed reproduce the phase expected (known from other constraints on the unitary CKM matrix) there is absolutely no possibility that the new physics brings a new weak phase exclusively to the decay which is different from the SM one (even if the strong phase difference between the two diagrams vanishes). This "theorem" might only be evaded if the new physics contributes with a new weak phase to the decay and also with a new weak phase to the mixing.

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[6] Notice that, in equating the first line of Eq. (8) to its second line, a specific phase choice has been made for the relative phase between the two conventions. Thus, the strong phases in the two conventions become tied together. For example, one may no longer choose the phases δ_1 and δ_1' independently. This can be seen clearly in Eqs. (9), where setting $\delta_1 = \delta_1' = 0$ would lead to the crazy conclusion that $\delta_2 = \delta_2' = 0$, thus precluding direct CP violation.
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$$\begin{aligned}
T_c e^{i\delta_c^T} e^{i\gamma} &= V_{ub}^* V_{ud} \left(A_u - A_t\right), \\
P_c e^{i\delta_c^P} e^{i\pi} &= V_{cb}^* V_{cd} \left(A_c - A_t\right), \\
T_t e^{i\delta_t^T} e^{i\gamma} &= V_{ub}^* V_{ud} \left(A_u - A_c\right), \\
P_t e^{i\delta_t^P} e^{-i\beta} &= V_{tb}^* V_{td} \left(A_t - A_c\right), \\
M_{1p} e^{i\delta_{1p}} e^{i\pi} &= V_{cb}^* V_{cd} \left(A_c - A_u\right), \\
M_{2p} e^{i\delta_{2p}} e^{-i\beta} &= V_{tb}^* V_{td} \left(A_t - A_u\right).
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[20] Imagine some decay having a SM contribution with magnitude $\tilde{M}_3$, weak phase ϕ_3, and strong phase $\tilde{\delta}_3$. Now imagine that there are several new physics contributions, with different magnitudes $\tilde{M}_k$, weak phases ϕ_k, and strong phases $\tilde{\delta}_k$ $(k = 4, 5, \ldots)$,

$$\begin{aligned} \sqrt{\frac{q_B}{p_B}}\bar{A}_f &= \eta_f \sum_k \tilde{M}_k\, e^{-i\phi_k} e^{i\tilde{\delta}_k}, \\ \sqrt{\frac{p_B}{q_B}}A_f &= \sum_k \tilde{M}_k\, e^{i\phi_k} e^{i\tilde{\delta}_k}. \end{aligned} \tag{80}$$

Using a reasoning similar to the one in Eqs. (5)–(7), we may write everything in terms of two weak phases, which we choose as the SM phase ϕ_3 and the phase ϕ_4 of the first new physics diagram. This reduces the general problem to the simple case in Eq. (77).

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APPENDIX: NOTATION

1. Observables in decay rates

The time-dependent decay rate of a neutral meson into the final state f may be written as

$$\begin{aligned} \Gamma[B^0(t) \to f] &= \frac{|A_f|^2 + |\bar{A}_f|^2}{2}\, e^{-\Gamma t} \left\{ \cosh\left(\frac{\Delta\Gamma\, t}{2}\right) + D_f \sinh\left(\frac{\Delta\Gamma\, t}{2}\right) \right. \\ &\qquad \left. + C_f \cos(\Delta m\, t) - S_f \sin(\Delta m\, t) \right\}, \\ \Gamma[\overline{B^0}(t) \to f] &= \frac{|A_f|^2 + |\bar{A}_f|^2}{2}\, e^{-\Gamma t} \left\{ \cosh\left(\frac{\Delta\Gamma\, t}{2}\right) + D_f \sinh\left(\frac{\Delta\Gamma\, t}{2}\right) \right. \\ &\qquad \left. - C_f \cos(\Delta m\, t) + S_f \sin(\Delta m\, t) \right\}, \end{aligned} \tag{A.1}$$

where

$$D_f \equiv \frac{2\mathrm{Re}(\lambda_f)}{1 + |\lambda_f|^2} \tag{A.2}$$

$$C_f \equiv \frac{1 - |\lambda_f|^2}{1 + |\lambda_f|^2} \tag{A.3}$$

$$S_f \equiv \frac{2\mathrm{Im}(\lambda_f)}{1 + |\lambda_f|^2}. \tag{A.4}$$

Therefore,

$$\lambda_f = \frac{1}{1 + C_f}\,(D_f + iS_f) \tag{A.5}$$

is a physical observable, and

$$D_f^2 + C_f^2 + S_f^2 = 1. \tag{A.6}$$

If the width difference is too small (as it happens in the B_d system), then we can set $\Delta\Gamma = 0$ and D_f is not measured. It can be inferred from Eq. (A.6) with a twofold ambiguity, meaning that λ_f is determined from Eq. (A.5) with that twofold ambiguity.

2. Phase structure of a generalized CKM matrix

It is interesting to note that, even if one goes beyond the SM, there are only four irremovable phases in the generalized CKM matrix [2], which we may chose to be

$$\beta \equiv \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \tag{A.7}$$

$$\gamma \equiv \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right), \tag{A.8}$$

$$\chi \equiv \arg\left(-\frac{V_{cb}V_{cs}^*}{V_{tb}V_{ts}^*}\right), \tag{A.9}$$

$$\chi' \equiv \arg\left(-\frac{V_{us}V_{ud}^*}{V_{cs}V_{cd}^*}\right). \tag{A.10}$$

Such a generalized CKM matrix will cease to be unitary, but we may parametrize its phase structure as [2]

$$\arg V = \begin{pmatrix} 0 & \chi' & -\gamma \\ \pi & 0 & 0 \\ -\beta & \pi+\chi & 0 \end{pmatrix}, \tag{A.11}$$

where a convenient phase convention has been chosen. The phase α is *defined* by $\alpha = \pi - \beta - \gamma$. In the SM, these phases are related to the Wolfenstein parameters through

$$R_t e^{-i\beta} \approx 1 - \rho - i\eta, \tag{A.12}$$

$$R_b e^{-i\gamma} \approx \rho - i\eta, \tag{A.13}$$

$$\chi \approx \lambda^2\eta, \tag{A.14}$$

$$\chi' \approx A^2\lambda^4\eta, \tag{A.15}$$

where

$$R_b = \left|\frac{V_{ud}V_{ub}}{V_{cd}V_{cb}}\right|, \tag{A.16}$$

$$R_t = \left|\frac{V_{td}V_{tb}}{V_{cd}V_{cb}}\right|. \tag{A.17}$$

This allows us to use a schematic form for the SM CKM matrix which keeps this phase structure but only expands each magnitude to leading order in λ. This is not a consistent expansion, but allows us to see where the phases would come in if we expanded everything up to a sufficient high power of λ [4]:

$$V \approx \begin{pmatrix} 1 & \lambda e^{i\chi'} & A\lambda^3 R_b e^{-i\gamma} \\ -\lambda & 1 & A\lambda^2 \\ A\lambda^3 R_t e^{-i\beta} & -A\lambda^2 e^{i\chi} & 1 \end{pmatrix}. \tag{A.18}$$

The ranges for χ and χ' are discussed in reference [23] for some models of new physics.