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# Bethe-Salpeter Approach for Meson-Meson Scattering in Chiral Perturbation Theory.

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**Abstract**

The Bethe-Salpeter equation restores exact elastic unitarity in the $s-$ channel by summing up an infinite set of chiral loops. We use this equation to show how a chiral expansion can be undertaken by successive approximations to the potential which should be iterated. Renormalizability of the amplitudes in a broad sense can be achieved by allowing for an infinite set of counter-terms as it is the case in ordinary Chiral Perturbation Theory. Within this framework we calculate the $\pi\pi$ scattering amplitudes both for $s-$ and $p-$waves at lowest order in the proposed expansion where a successful description of the low-lying resonances ($\sigma$ and $\rho$) and threshold parameters is obtained. We also extract the $SU(2)$ low energy parameters $\bar{l}_{1,2,3,4}$ from our amplitudes.

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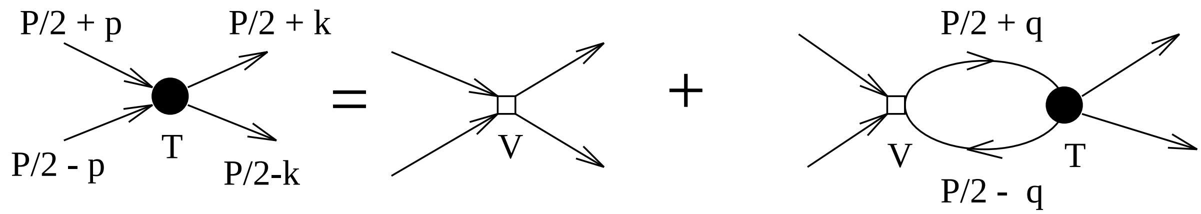


Figure 1: Diagrammatic representation of the BSE equation. It is also sketched the used kinematics.

# 1 Introduction

Chiral Perturbation Theory (ChPT) to finite order is unable to describe resonances. Actually, it is rather the opposite, resonances determine the bulk of the $\mathcal{O}(p^4)$ parameters [1, 2]. Such a description requires the use of a non-perturbative scheme. Several approaches have been suggested, Pade Re-summation (PR) [3], Large $N_f-$Expansion (LNE) [4], Inverse Amplitude Method (IAM) [5], Current Algebra Unitarization (CAU) [6], Dispersion Relations (DP) [7], Roy Equations [8], Coupled Channel Lippmann-Schwinger Approach (CCLS) [9] and hybrid approaches [10]. Besides their advantages and success to describe the data in the low-lying resonance region, any of them has specific drawbacks. In all above approaches except by LNE and CCLS it is not clear which is the ChPT series of diagrams which has been summed up. This is not the case for the CCLS approach, but there a three momentum cut-off is introduced, hence breaking translational Lorentz invariance and therefore the scattering amplitude can be only evaluated in the the Center of Mass (CM) frame. On the other hand, though the LNE and CAU approaches preserve crossing symmetry, both of them violate unitarity. Likewise, those approaches which preserve exact unitarity violate crossing symmetry.

In this paper we propose the use of the Bethe-Salpeter equation (BSE) to sum up an infinite set of diagrams without use of cut-offs in the physical amplitudes. We will show that this re-summation restores exact elastic unitarity in the $s-$ channel and it naturally leads to the appearance of the experimentally observed resonances. Besides, crossing symmetry can be restored perturbatively. Our approach, at lowest order and in the chiral limit reproduces the bubble re-summation undertaken in Ref. [11].

# 2 The Bethe-Salpeter Equation.

For the sake of simplicity, we neglect coupled channels contributions and thus the BSE for the scattering of two identical pseudo-scalar mesons of mass $m$ and kinematics described in Fig. 1, reads

$$T_P(p,k) \quad = \quad V_P(p,k) + \mathrm{i}\int \frac{d^4q}{(2\pi)^4} T_P(q,k)\Delta(q+P/2,m)\Delta(-q+P/2,m)V_P(p,q) \quad (1)$$

where $T_P(p,k)$ and $V_P(p,k)$ are the total scattering amplitude[1] and the two particle irreducible amplitude respectively. Besides, $\Delta$ is the exact pseudoscalar meson propagator. Note that, to solve the above equation both the off-shell potential and amplitude are required. Clearly, for the exact *potential* $V$ the BSE provides an exact solution of the scattering amplitude $T$ [12]. Obviously an exact solution for $T$ is not accessible, since $V$ and $\Delta$ are not exactly known. We propose an expansion along the lines of ChPT both for the exact potential ($V$) and the exact propagator ($\Delta$). Thus at lowest order in this expansion, $V$ should be replaced by the $\mathcal{O}(p^2)$ chiral amplitude (${}^{(2)}T$) and $\Delta$ by the free meson propagator, $\Delta^0(r,m) = (r^2 - m^2 + \mathrm{i}\epsilon)^{-1}$. Even at lowest order, by solving Eq. (1) we sum up an infinite set of diagrams[2]. Our expansion is related to the approach recently pursued for low energy $NN-$scattering where higher order $t-$ and $u-$channel contributions to the potential are suppressed in the heavy nucleon mass limit [13].

To illustrate the procedure, let us consider elastic $\pi\pi$ scattering in the $s-$ and $p-$waves. There, for comparison with the experimental phase shifts, $\delta_{IJ}(s)$, we define the projection over each partial wave $J$ for each isospin channel $I$

$$T_{IJ}(s) = \frac{1}{2}\int_{-1}^{+1} P_J(\cos\theta)\, T_P^I(p,k)\; d(\cos\theta) \;=\; \frac{\mathrm{i}8\pi s}{\lambda^{\frac{1}{2}}(s,m^2,m^2)}\left[e^{2\mathrm{i}\delta_{IJ}(s)} - 1\right] \qquad (2)$$

where $\theta$ is the angle between $\vec{p}$ and $\vec{k}$, $P_J$ are Legendre polynomials and $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$.

## 2.1 Isoscalar $s-$wave $\pi\pi$ Scattering.

At lowest order, the off-shell potential $V$ in this channel is approximated by

$$V_P^0(p,k) \approx^{(2)} T_P^0(p,k) \;=\; \frac{5m^2 - 3s - 2(p^2+k^2)}{2f^2} \qquad (3)$$

where $m$ is the pion mass, for which we take 134.98 MeV, and $f$ the pion decay constant, for which we take 92.4 MeV. To solve Eq. (1) with the above potential we propose a solution of the form

$$T_P^0(p,k) \;=\; A(s) + B(s)(p^2+k^2) + C(s)p^2k^2 \qquad (4)$$

where $A, B$ and $C$ are functions to be determined. Note that, as a simple one loop calculation shows, there appears a new off-shell dependence ($p^2k^2$) not present in the $\mathcal{O}(p^2)$ potential ${}^{(2)}T$. That is similar to what happens in standard ChPT [1].

[1]The normalization of the amplitude $T$ is determined by its relation with the differential cross section in the CM system of the two identical mesons and it is given by $d\sigma/d\Omega = |T_P(p,k)|^2/64\pi^2 s$, where $s = P^2$. The phase of the amplitude $T$ is such that the optical theorem reads $\mathrm{Im}T_P(p,p) = -\sigma_{\rm tot}(s^2 - 4s\,m^2)^{1/2}$, with $\sigma_{\rm tot}$ the total cross section.

[2]Note that our approach is different to the LNE one [4]. There the bubbles are summed up by means of a collective field with no well defined isospin. Hence, crossing symmetry is maintained but elastic unitarity in the $s-$channel is violated.

The above ansatz reduces the BSE to a linear algebraic system of equations which provides the full off-shell scattering amplitude. The resulting inverse amplitude on the mass shell and in the CM frame ($\vec{P} = 0, p^0 = k^0 = 0, P^0 = \sqrt{s}$) reads

$$T_{00}^{-1}(s) \quad = \quad -I_0(s) + \frac{2\left(f^2 + I_2(4m^2)\right)^2}{2I_4(4m^2) + (m^2 - 2s)f^2 + (s - 4m^2)I_2(4m^2)} \tag{5}$$

where

$$I_{2n}(s) = \mathrm{i} \int \frac{d^4q}{(2\pi)^4} \frac{(q^2)^n}{\left[(q - \frac{P}{2})^2 - m^2 + \mathrm{i}\epsilon\right]\left[(q + \frac{P}{2})^2 - m^2 + \mathrm{i}\epsilon\right]} \tag{6}$$

$I_0, I_2$ and $I_4$ are ultraviolet divergent integrals and thus Eq. (5) requires renormalization. We have made use of translational and Lorentz invariance which relate the integrals $I_2(s)$ and $I_4(s)$ with $I_0(s)$ and the divergent constants $I_2(4m^2)$ and $I_4(4m^2)$. Note also that $I_0(s)$ is only logarithmically divergent and it only requires one subtraction, ie., $\bar{I}_0(s) = I_0(s) - I_0(4m^2)$ is finite and it is given by

$$\bar{I}_0(s) \quad = \quad \frac{1}{(4\pi)^2}\sqrt{1 - \frac{4m^2}{s}} \log \frac{\sqrt{1 - \frac{4m^2}{s}} + 1}{\sqrt{1 - \frac{4m^2}{s}} - 1} \tag{7}$$

where the complex phase of the argument of the log is taken in the interval $[-\pi, \pi[$

To renormalize the amplitude given in Eq. (5), we note that in the spirit of an Effective Field Theory (EFT) all possible counter-terms should be considered. This can be achieved in our case in a perturbative manner, making use of the formal expansion of the bare amplitude $T = V + VG_0V + VG_0VG_0V + \cdots$, where $G_0$ is the two particle propagator. Thus, a counter-term series should be added to the bare amplitude such that the sum of both becomes finite. At each order in the perturbative expansion, the divergent part of the counter-term series is completely determined. However, the finite piece remains arbitrary as long as the used *potential* $V$ and the pion propagator are approximated rather than being the exact ones. Our renormalization scheme is such that the renormalized amplitude can be cast, again, as in Eq. (5). This amounts in practice, to interpret the previously divergent quantities $I_{0,2,4}(4m^2)$ as renormalized free parameters. After having renormalized, we add a superscript $R$ to differentiate between the previously divergent, $I_{0,2,4}(4m^2)$, and now finite quantities, $I^R_{0,2,4}(4m^2)$. These parameters and therefore the renormalized amplitude can be expressed in terms of physical (measurable) magnitudes. In principle, these quantities should be understood in terms of the underlying QCD dynamics, but in practice it seems more convenient so far to fit $I^R_{0,2,4}(4m^2)$ to the available data. The threshold properties of the amplitude (scattering length, effective range, etc..) can then be determined from them. Besides the pion properties $m$ and $f$, at this order in the expansion we have three parameters. The appearance of three new parameters is not surprising because the highest divergence we find is quartic ($I_4(s)$) and therefore to make it convergent we need to perform three subtractions. This situation is similar to what happens in standard ChPT where one needs to include at the one loop level some new low-energy parameters ($\bar{l}_i$) [1]. In fact, if $t-$ and $u-$ channel unitarity corrections are

Taylor expanded around threshold, a comparison of our (now) finite amplitude, Eq. (5), to the $\mathcal{O}(p^4)$ $\pi\pi$ amplitude in terms of these parameters, $\bar{l}_{1,2,3,4}$, becomes possible. We do this explicitly in the Appendix.

At the lowest order in the expansion proposed in this work, we approximate, in the scattering region $s > 4m^2$, the $\mathcal{O}(1/f^4)$ $t-$ and $u-$ channel unitarity corrections (function $h_{IJ}$ in Eq. (15)) by a Taylor expansion around threshold to order $(s-4m^2)^2$. At next order in our expansion (when the full $\mathcal{O}(p^4)$-corrections are included both in the *potential* and in the pion propagator) we will recover the full $t-$ and $u-$ channel unitarity logs at $\mathcal{O}(1/f^4)$, and at the next order ($\mathcal{O}(1/f^6)$) we will be approximating these logs by a Taylor expansion to order $(s-4m^2)^3$. Thus the analytical structure of the amplitude derived from the left hand cut is only recovered perturbatively. This is in common to other approaches (PR, DP, IAM $\cdots$) fullfiling exact unitarity in the s-channel, as discussed in [5], [7],[14].

## 2.2 Isovector $p-$wave $\pi\pi$ Scattering

At lowest order, the off-shell potential $V$ in this channel is approximated by

$$V_P^1(p,k) \approx^{(2)} T_P^1(p,k) \;=\; \frac{2p\cdot k}{f^2} \tag{8}$$

As before, to solve Eq. (1) with the above potential we propose a solution of the form

$$T_P^1(p,k) \;=\; M(s)p\cdot k + N(s)(p\cdot P)(k\cdot P) \tag{9}$$

where $M$ and $N$ are functions to be determined. Note that, as expected from our previous discussion for the $s-$wave, there appears a new off-shell dependence $((p\cdot P)(k\cdot P))$ not present in the $\mathcal{O}(p^2)$ potential. Again, this ansatz reduces the BSE to a linear algebraic system of equations which provides the full off-shell scattering amplitude. The resulting inverse amplitude on the mass shell, after angular momentum projection, and in CM frame reads

$$T_{11}^{-1}(s) = \;=\; -I_0(s) + \frac{2I_2(4m^2) - 6f^2}{s-4m^2} \tag{10}$$

Similarly to the $s-$wave case, the above equation presents divergences which need to be consistently removed in terms, for instance, of the scattering volume and effective range in the $p$-wave. Because the highest divergence present is quadratic only two subtractions are needed and hence only two undetermined parameters appear. As we will discuss below, crossing symmetry at $\mathcal{O}(1/f^4)$ provides a relationship between the parameters entering in different isospin channels. We emphasize once more, that the proliferation of undetermined parameters is not a drawback as compared to standard ChPT.

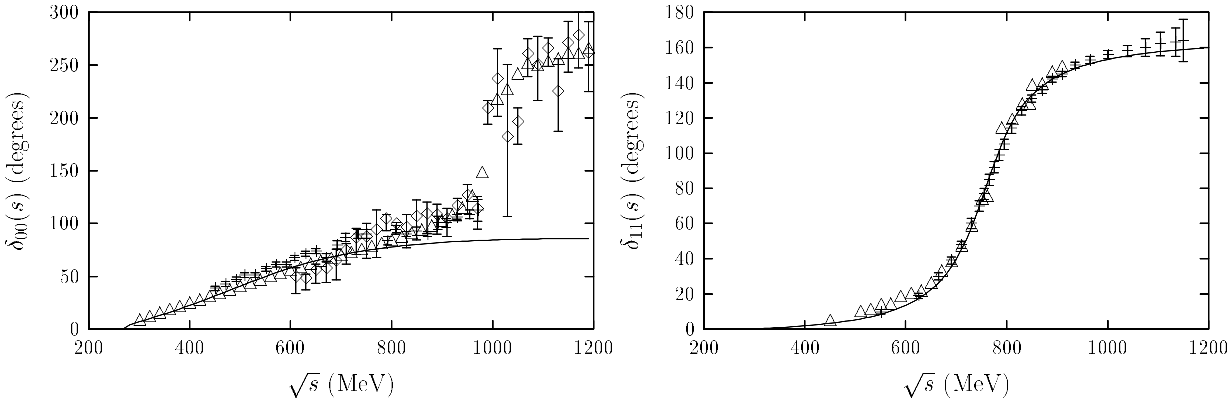


Figure 2: $s-$ (left) and $p-$wave (right) $\pi\pi$ phase shifts as a function of the total CM energy $\sqrt{s}$. Left panel: Triangles, squares and crosses stand for the experimental analysis of Refs. [15], [16] and [17], respectively. Right panel: Triangles and crosses stand for the experimental analysis of Refs. [18] and [19], respectively. Solid lines are the fit (see Table 1) of our $s$-channel unitarized $\mathcal{O}(p^2)-$model to the data of Refs. [15] and [19].

# 3 Results

The solutions of Eqs. (5) and (10) satisfy elastic unitarity. Indeed, the imaginary part of $T_{IJ}^{-1}(s)$ is determined by the imaginary part of $\bar{I}_0(s)$ and in the physical region is given by

$$\mathrm{Im}T_{IJ}^{-1}(s+\mathrm{i}\epsilon) = -\mathrm{Im}\bar{I}_0(s+\mathrm{i}\epsilon) \quad = \quad \frac{\lambda^{\frac{1}{2}}(s,m^2,m^2)}{16\pi s} \tag{11}$$

in agreement with the unitarity requirement provided $m$ is the physical pion mass. That makes possible to extract the phase shifts unambiguously within our framework.

As we have already mentioned, our approach violates crossing symmetry. At order $\mathcal{O}(1/f^4)$ our isoscalar $s-$ and isovector $p-$wave amplitudes are polynomials of degree two in the variable $(s-4m^2)$, with a total of five (3+2) arbitrary coefficients (see Eq. (17)), and there are no logarithmical corrections to account for $t-$ and $u-$channel unitarity corrections ($h_{IJ}$ term in Eq. (15)). Far from the left hand cut, these latter corrections can be expanded in a Taylor series to order $(s-4m^2)^2$, but in that case the one loop $SU(2)$ ChPT $s-$ and isovector $p-$wave amplitudes can be cast as second order polynomials in the variable $(s-4m^2)$, with a total of four ($\bar{l}_{1,2,3,4}$) arbitrary coefficients [1]. To restore, in this approximation, crossing symmetry in our amplitudes requires the existence of a relationship between our five undetermined parameters. This relation reads (see the Appendix)

$$75I_2^{R,s}/2m^2 + 8I_0^{R,p} + 33I_0^{R,s} + 5I_2^{R,p}/m^2 + \frac{10157}{1920\pi^2} = 0 \tag{12}$$

Once this constraint is implemented in our model, there exists a linear relation between our remaining four undetermined parameters and the most commonly used $\bar{l}_{1,2,3,4}$ parameters (Eq. (18)).

In Fig. 2 we show the agreement of our model with the experimental phase shifts, both in the $s-$ (left) and $p-$(right)waves. We fit the four undetermined parameters[3] $I^{R,s}_{0,4}(4m^2)$ and $I^{R,p}_{0,2}(4m^2)$ to the scalar and vector data. Results of the combined fit can be found in Table 1. Values of $\chi^2/\mathrm{dof}$ and the threshold parameters deduced from our formulae are also shown in Table 1. As we see, the vector channel is satisfactorily described up to 1 GeV, whereas the scalar channel is well reproduced up to 0.8 GeV. In the latter case, and for these high energies, one should also include the mixing with the $K\bar{K}$ channel as pointed out recently in Refs. [9]–[10]. Regarding the deduced threshold parameters we find agreement within experimental uncertainties. For completeness we quote the deduced values for the low energy coefficients (see Eq. (18) in the Appendix)

$$\bar{l}_1 = -0.90 \pm 0.09,\ \bar{l}_2 = 6.03 \pm 0.08,\ \bar{l}_3 = 2 \pm 7,\ \bar{l}_4 = 3.9 \pm 0.4 \tag{13}$$

if a fit to the data of Refs. [15], [18] is performed[4]. As we see they are in reasonable good agreement with those obtained in standard ChPT [1], which also use the same set of data to analyze $\pi\pi$ scattering. When the data of Ref.[19] for the $p-$wave are fitted we find compatible values for $\bar{l}_{1,2}$ and incompatible values for $\bar{l}_{3,4}$. We prefer to use the data of Ref. [18] to estimate the low energy parameters $\bar{l}_{1,2,3,4}$, since they come much closer to threshold than those of Ref. [19] ($\sqrt{s} = 350$ MeV versus 450 MeV). Furthermore both sets of data are incompatible up to about $\sqrt{s} = 650$ MeV.

Once we have determined the parameters $\bar{l}_{1,2,3,4}$, the $I = 2, J = 0$ lowest order BSE amplitude is completely fixed, as demanded from crossing symmetry. It is to say, the corresponding integrals which appear in this channel are completely determined by the $\bar{l}\,'s$ or equivalently by the $\sigma-$ and $\rho-$channel integrals. We have evaluated the corresponding phase-shift in the $I = 2, J = 0$ channel, using the values of Eq. (13), and compared to the data of Ref. [20] finding a remarkable agreement up to about 1450 MeV ($\chi^2/(\mathrm{num.\ data}) = 0.9$) and threshold parameters ($m\, a_{20} = -0.0390 \pm 0.0018$, and $m^3\, b_{20} = -0.0701 \pm 0.0010$) in agreement within uncertainties with the experimental values.

Comparison to recent two-loop calculations [21] would also be possible. To make such a comparison really meaningful, we should include in our analysis higher waves than the $s-$ and $p-$waves considered here, which would necessarily require to include $\mathcal{O}(p^4)$ corrections to both the *potential* and the pion propagator. This point is out of the scope of this work and will be subject of future research [22]. Nevertheless, we would like to point out that the lowest order explored in this letter, though very simple, it already leads to reasonable sizes for the two loop contributions to the amplitudes. Thus for instance, for the $s-$wave scattering length we obtain $a_{00}/a_{00}^{\mathrm{tree}} = 1 + 0.27\,(\mathrm{one\ loop}) + 0.07\,(\mathrm{two\ loops}) + \cdots$, in a reasonable agreement with the findings of Bijnens *et al.,* in Ref. [21].

Recent work, Refs. [9]–[10], suggests that off-shellness can be ignored when solving the BSE, since it only amounts to a pion mass and pion decay constant renormalization. This on-shell procedure corresponds to setting our $I^R_2(4m^2)$ and $I^R_4(4m^2)$ parameters to

[3] We use Eq. (12) to express $I^{R,s}_2$ in terms of $I^{R,p}_{0,2}$ and $I^{R,s}_0$.

[4] We remind to the reader that we have assumed a 5% error in the data of these two references. If we had assumed a 1(10)% errors, the statistical fluctuations quoted in Eq. (13) would have decreased (increased) roughly by a factor of 5 (2).

| | Data Refs. [15], [19] | | Data Refs. [15], [18] | |
|---|---|---|---|---|
| | $J=I=0$ | $J=I=1$ | $J=I=0$ | $J=I=1$ |
| $I_0^R(4m^2)$ | $-0.0323 \pm 0.0005$ | $-0.1157 \pm 0.0017$ | $-0.0289 \pm 0.0005$ | $-0.0980 \pm 0.0017$ |
| $I_2^R(4m^2)$ | – | $0.011 \pm 0.006$ | – | $0.072 \pm 0.005$ |
| $I_4^R(4m^2)$ | $-0.031 \pm 0.004$ | – | $-0.003 \pm 0.003$ | – |
| $m^{2J+1}a_{IJ}$ | $0.233 \pm 0.011$ | $0.0292 \pm 0.0005$ | $0.204 \pm 0.008$ | $0.0356 \pm 0.0006$ |
| $m^{2J+3}b_{IJ}$ | $0.237 \pm 0.007$ | $0.0050 \pm 0.0002$ | $0.267 \pm 0.006$ | $0.0062 \pm 0.0002$ |
| $\chi^2/\mathrm{dof}$ | 1.6 | | 3.9 | |
| $I_2^{R,s}(4m^2)$ | $0.0082 \pm 0.0008$ | | $-0.0016 \pm 0.0007$ | |

Table 1: Fitted ($I_n^R(4m^2)$) parameters to the experimental data of Refs. [15] ($J=I=0$), [18] and [19] ($J=I=1$). $I_n(4m^2)$ are given in units of $(2m)^n$ and $I_2^{R,s}(4m^2)$ is calculated from Eq. (12). Errors in the fitted parameters are statistical and have been obtained by increasing the value of $\chi^2$ by one unit. We also give the threshold parameters $a_{IJ}$ and $b_{IJ}$ obtained from an expansion of the scattering amplitude, $\mathrm{Re}T_{IJ} = -16\pi m(s/4-m^2)^J[a_{IJ} + b_{IJ}(s/4-m^2) + \cdots]$ close to threshold. In the the $s-$wave channel and due to the lack of error estimates in Ref. [15], we have assumed a rule of thumb error of 5% in the data and carry out the fit up to 900 MeV. We have chosen this set of data because the data of Refs. [16] and [17] seem to be inconsistent between each other at low energies and cover near threshold a narrower region than that covered by the analysis of Ref. [15]. For the $p-$wave, the data of Refs. [18] (here again, we assume an error of 5% in the data, because the lack of error estimates of the original analysis ) and [19] disagree again, specially close to threshold.

zero, and taking $f$ and $m$ as the physical parameters. Direct inspection of our expressions shows that one can not simultaneously absorb the $\sigma-$ and $\rho-$channel divergences by a redefinition of the $m$ and $f$ parameters. Since the model of Refs. [9]–[10] has been constrained not to have these new parameters ($I_{2,4}^R(4m^2)$) generated by the off-shellness it is impossible for them to simultaneously reproduce the $\sigma-$ and $\rho-$channels; a polynomial, with some new parameters, has to be added to the on shell-constrained solution of the BSE to properly describe the resonance region in the $J=I=1$ channel. The inclusion of this polynomial with free parameters is justified [9]–[10] within the IAM approach [5]. We would like to stress here that considering explicitly the off-shellness automatically embodies the IAM, as it is invoked in Refs. [9] and [10], and thus incorporates such a polynomial.

# 4 Conclusions and Outlook.

We have solved the BSE for $\pi\pi$ scattering in the ladder approximation. This is to say using the *potential* and pion propagator at lowest order in the chiral expansion. This calculation produces unitary amplitudes in the $s-$channel and describes satisfactorily $s-$ and $p-$ wave phase-shifts from threshold up to the region of low-lying resonances. The present approach can be extended in principle to higher orders in the chiral expansion, i.e., including in the *potential* and in the pion propagator terms of order $\mathcal{O}(p^4)$ and higher. The new divergences will become more severe and more subtraction constants will be needed. That will be translated in an increasing number of free parameters, as it is the case also of standard ChPT. To be more specific, for instance, the $\mathcal{O}(p^4)$ *potential* contains unitarity corrections in the $t-$ and $u-$ channels, (the corresponding $s-$channel correction to the amplitude is not two particle irreducible). Hence, the solution of the BSE with this *potential* will automatically implement exact unitarity in the $s-$channel and perturbative unitarity in the $t-$ and $u-$channels.

The inclusion of unitarity corrections in the $t-$ and $u-$ channels in the *potential* makes the practical solution of the BSE a cumbersome task and it is currently underway [22]. However, if one neglects these corrections in the $t-$ and $u-$ channels the BSE can be solved using a much simpler algebraic procedure as shown here for the lowest order case. In that case only unitarity in the $s-$channel will be restored and crossing symmetry will be violated. Thus, one should expect a general solution, below the four pion production threshold, of the form

$$T_{IJ}^{-1}(s) = -\bar{I}_0(s) + \frac{P_n^{IJ}(s-4m^2)}{Q_n^{IJ}(s-4m^2)} \tag{14}$$

where $P_n^{IJ}(x)$ and $Q_n^{IJ}(x)$ are polynomials[5] of order $n$ (corresponding to the order $\mathcal{O}(p^{2n})$ in the proposed expansion) with real coefficients. Most of these coefficients have to be fitted to the data to accomplish with the renormalization program. The resemblance to a sort of *pade* approximant for the inverse amplitude, although not exactly in the form proposed in Ref. [3], is striking. On the other hand, Eq. (14) provides a model of the type $N/D$ [23] for the amplitude, where $D$ has a right cut and the function $N$ is approximated by a polynomial with no cuts. These ideas have been recently examined in Ref. [24].

Finally, we should mention that to describe the $s-$wave at higher energies than 0.8 GeV, a coupled channel formalism needs to be used. Such an improvement of our model can be easily implemented, at least at the lowest order $\mathcal{O}(p^2)$.

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# Appendix

The $\mathcal{O}(p^2)+\mathcal{O}(p^4)$ on-shell $SU(2)-$ ChPT isoscalar $s-$ and isovector $p-$ wave $\pi\pi$ amplitudes, expressed in terms of the renormalization invariant parameters $\bar{l}_{1,2,3,4}$, are given

[5]Note that the first non-vanishing coefficient in $Q_n$ corresponds to the power $(s-4m^2)^J$.

by [1]

$$T_{IJ}(s) \;=\; V_1^{IJ}(s)/f^2 + \left(V_2^{IJ}(s) + \bar{I}_0(s)\times[\,V_1^{IJ}(s)\,]^2\right)/f^4 \tag{15}$$

$$\begin{aligned}
V_1^{IJ}(s) &= \begin{cases} \frac{m^2-2s}{2} & I=J=0 \\ \frac{4m^2-s}{6} & I=J=1 \end{cases}\\
V_2^{IJ}(s) &= -\frac{1}{192\pi^2}g_{IJ} + \frac{1}{12}h_{IJ}\\
h_{IJ} &= \int_{-1}^{1}\frac{d(\cos\theta)}{2}\left(f_{IJ}(t)\bar{I}_0(t) + f_{IJ}(u)\bar{I}_0(u)\right)P_J(\cos\theta)\\
&= \begin{cases} \frac{5m^4}{4\pi^2} + \frac{101m^2(s-4m^2)}{96\pi^2} + \frac{191m^2(s-4m^2)^2}{288\pi^2} + \cdots & I=J=0 \\ \frac{89m^2(s-4m^2)}{288\pi^2} - \frac{37m^2(s-4m^2)^2}{2880\pi^2} + \cdots & I=J=1 \end{cases}
\end{aligned}$$

where $t=-2(s/4-m^2)(1-\cos\theta)$, $u=-2(s/4-m^2)(1+\cos\theta)$ and

$$f_{IJ}(x) \;=\; \begin{cases} 10x^2 + x(2s-32m^2) + 37m^4 - 8sm^2 & I=J=0 \\ 2x^2 + x(s+2m^2) - m^4 - 4sm^2 & I=J=1 \end{cases} \tag{16}$$

$$g_{IJ} \;=\; \begin{cases} m^4\left(40\bar{l}_1 + 80\bar{l}_2 - 15\bar{l}_3 + 84\bar{l}_4 + 125\right) + m^2(s-4m^2)\times \\ \left(32\bar{l}_1 + 48\bar{l}_2 + 24\bar{l}_4 + \frac{232}{3}\right) + (s-4m^2)^2\left(\frac{22}{3}\bar{l}_1 + \frac{28}{3}\bar{l}_2 + \frac{142}{9}\right) & I=J=0 \\ \frac{s-4m^2}{3}\left[4m^2\left(-2\bar{l}_1 + 2\bar{l}_2 + 3\bar{l}_4 + 1\right) + (s-4m^2)\left(-2\bar{l}_1 + 2\bar{l}_2 + 1\right)\right] & I=J=1 \end{cases}$$

On the other hand, expanding up to order $\mathcal{O}(p^4)$ the amplitudes of Eqs. (5) and (10) we find,

$$T_{IJ}(s) \;=\; V_1^{IJ}(s)/f^2 + \left(W^{IJ}(s) + \bar{I}_0(s)\times[\,V_1^{IJ}(s)\,]^2\right)/f^4 \tag{17}$$

$$W^{IJ}(s) \;=\; \begin{cases} I_4^{R,s} + 7m^2 I_2^{R,s} + \frac{5}{2}(s-4m^2)I_2^{R,s} + I_0^{R,s}\times[\,V_1^{00}(s)\,]^2 & I=J=0 \\ V_1^{11}(s)\left(\frac{1}{3}I_2^{R,p} + I_0^{R,p}\,V_1^{11}(s)\right) & I=J=1 \end{cases}$$

Taylor expanding around threshold the function $h_{IJ}$ and identifying the functions $V_2^{IJ}$ of Eq. (15) and $W_2^{IJ}$ of Eq. (17) one obtains the constraint given in Eq. (12) and the following relations:

$$\begin{aligned}
\bar{l}_1 &= \frac{107}{750} + \pi^2\left(\frac{112}{25}I_0^{R,p} - \frac{288}{25}I_0^{R,s}\right) && (18)\\
\bar{l}_2 &= -\frac{1997}{3000} - \pi^2\left(\frac{352}{100}I_0^{R,p} + \frac{1152}{100}I_0^{R,s}\right)
\end{aligned}$$

$$\bar{l}_3 = -\frac{2761}{3750} + \pi^2 \left( \frac{1472}{375} I_0^{R,p} - \frac{1776}{125} I_0^{R,s} + \frac{224}{75m^2} I_2^{R,p} + \frac{64}{5m^4} I_4^{R,s} \right)$$

$$\bar{l}_4 = \frac{173}{120} + \pi^2 \left( \frac{16}{3} I_0^{R,p} + \frac{8}{3m^2} I_2^{R,p} \right)$$

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