

Precision Tau Physics

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Abstract

Precise measurements of the lepton properties provide stringent tests of the Standard Model and accurate determinations of its parameters. We overview the present status of τ physics, highlighting the most recent developments, and discuss the prospects for future improvements. The leptonic decays of the τ lepton probe the structure of the weak currents and the universality of their couplings to the W boson. The universality of the leptonic Z couplings has also been tested through $Z \rightarrow \ell^+ \ell^-$ decays. The hadronic τ decay modes constitute an ideal tool for studying low-energy effects of the strong interaction in very clean conditions. Accurate determinations of the QCD coupling and the Cabibbo mixing V_{us} have been obtained with τ data. The large mass of the τ opens the possibility to study many kinematically-allowed exclusive decay modes and extract relevant dynamical information. Violations of flavour and CP conservation laws can also be searched for with τ decays. Related subjects such as μ decays, the electron and muon anomalous magnetic moments, neutrino mixing and B -meson decays into τ leptons are briefly covered. Being one of the fermions most strongly coupled to the scalar sector, the τ lepton is playing now a very important role at the LHC as a tool to test the Higgs properties and search for new physics at higher scales.

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1 Introduction

Since its discovery [1] in 1975 at the SPEAR e^+e^- storage ring, the τ lepton has been a subject of extensive experimental study [2–20]. The very clean sample of boosted $\tau^+\tau^-$ events accumulated at the Z peak, together with the large statistics collected in the Υ region, have considerably improved the statistical accuracy of the τ measurements and, more importantly, have brought a new level of systematic understanding, allowing us to make sensible tests of the τ properties. On the theoretical side, a lot of effort has been invested to improve our understanding of the τ dynamics. The basic τ properties were already known, before its actual discovery [1], thanks to the pioneering paper of Tsai [21]. The detailed study of higher-order electroweak corrections and QCD contributions has promoted the physics of the τ lepton to the level of precision tests.

The τ lepton is a member of the third fermion generation which decays into particles belonging to the first and second ones. Thus, τ physics could provide some clues to the puzzle of the recurring families of leptons and quarks. In fact, one naively expects the heavier fermions to be more sensitive to whatever dynamics is responsible for the fermion mass generation. The pure leptonic or semileptonic character of τ decays provides a clean laboratory to test the structure of the weak currents and the universality of their couplings to the gauge bosons. Moreover, the τ is the only known lepton massive enough to decay into hadrons; its semileptonic decays are then an ideal tool for studying strong interaction effects in very clean conditions.

All experimental results obtained so far confirm the Standard Model (SM) scenario, in which the τ is a sequential lepton with its own quantum number and associated neutrino. The increased sensitivities of the most recent experiments result in interesting limits on possible new physics contributions to the τ decay amplitudes. In the following, the present knowledge on the τ lepton and the prospects for further improvements are analysed. Rather than giving a detailed review of experimental results, the emphasis is put on the physics which can be investigated with the τ data. Exhaustive information on more experimental aspects can be found in Refs. [22] and [23].

The leptonic τ decays can be accurately predicted in the SM. The relevant expressions are analysed in section 2, where they are compared with the most recent measurements of the μ and τ leptonic decay widths, and used to test the universality of the leptonic W couplings in section 3, which also includes the universality tests performed with π , K and W decays. The Lorentz structure of the leptonic charged-current interactions is further discussed in section 4. While the high-precision muon data shows nicely that the bulk of the μ decay amplitude is indeed of the predicted $V - A$ type, the Lorentz structure of the τ decay is not yet determined by data; nevertheless, useful constraints on hypothetical new-physics contributions have been established. Section 5 describes the leptonic electroweak precision tests performed at the Z peak, confirming the family-universality of the leptonic Z couplings and the existence of (only) three SM neutrino flavours.

The hadronic decays of the τ lepton allow us to investigate the hadronic weak currents and test low-energy aspects of the strong interaction. The exclusive decay modes are discussed in section 6, which shows that at very low energies the chiral symmetry of QCD determines the coupling of any number of pseudoscalar mesons to the left-handed quark current. The measured hadronic distributions in τ decay provide crucial information on the resonance dynamics, which dominates at higher momentum transfer. Section 7 discusses the short-distance QCD analysis of the inclusive hadronic width of the τ lepton. The total hadronic width is currently known with four-loop accuracy, providing a very precise determination of the QCD coupling at the τ mass scale and, therefore, a very significant test of asymptotic freedom from its comparison with determinations performed at much higher energies. The inclusive hadronic distribution gives, in addition, important information on non-perturbative QCD parameters. The semi-inclusive hadronic decay width into Cabibbo-suppressed modes is analysed in section 8, where a quite competitive determination of $|V_{us}|$ is obtained; the accuracy of this result could be considerably improved in the future with much higher statistics.

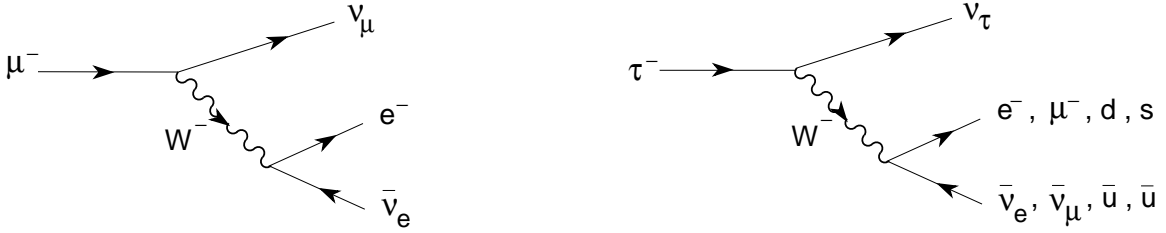


Figure 1: Feynman diagrams for $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$ and $\tau^- \rightarrow \nu_\tau X^-$ ($X^- = e^- \bar{\nu}_e, \mu^- \bar{\nu}_\mu, d\bar{u}, s\bar{u}$).

Together with hadronic e^+e^- data, the hadronic τ -decay distributions are needed to determine the SM prediction for the μ anomalous magnetic moment. Section 9 presents an overview of the e , μ and τ magnetic, electric and weak dipole moments, which are expected to have a high sensitivity to physics beyond the SM. The τ lepton constitutes a superb probe to search for new-physics signals. The current status of CP-violating asymmetries in τ decays is described in section 10, while section 11 discusses the production of τ leptons in B decays, which is sensitive to new-physics contributions with couplings proportional to fermion masses. The large τ mass allows one to investigate lepton-flavour and lepton-number violation, through a broad range of kinematically-allowed decay modes, complementing the high-precision searches performed in μ decay. The current experimental limits are given in section 12; they provide stringent constraints on flavour models beyond the SM.

Processes with τ leptons in the final state are playing now an important role at the LHC, either to characterize the Higgs properties or to search for new particles at higher scales. The current status is briefly described in section 13, before concluding with a few summarizing comments in section 14.

2 Lepton Decays

The decays of the charged leptons, μ^- and τ^- , proceed through the W -exchange diagrams shown in Fig. 1, with the universal SM strength associated with the charged-current interactions:

$$\mathcal{L}_{\text{CC}} = -\frac{g}{2\sqrt{2}} W_\mu^\dagger \left\{ \sum_\ell \bar{\nu}_\ell \gamma^\mu (1 - \gamma_5) \ell + \bar{u} \gamma^\mu (1 - \gamma_5) (V_{ud} d + V_{us} s) \right\} + \text{h.c.} \quad (1)$$

The momentum transfer carried by the intermediate W^- is very small compared to M_W . Therefore, the vector-boson propagator shrinks to a point and can be well approximated through a local four-fermion interaction governed by the Fermi coupling constant $G_F/\sqrt{2} = g^2/(8M_W^2)$. The leptonic decay widths are given by

$$\Gamma_{\ell \rightarrow \ell'} \equiv \Gamma[\ell^- \rightarrow \ell'^- \bar{\nu}_{\ell'} \nu_\ell (\gamma)] = \frac{G_{\ell'\ell}^2 m_\ell^5}{192\pi^3} f(m_{\ell'}^2/m_\ell^2) \left(1 + \delta_{\text{RC}}^{\ell'\ell}\right) \quad (2)$$

where $f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \log x$, and [24–32]

$$\delta_{\text{RC}}^{\ell'\ell} = \frac{\alpha}{2\pi} \left[\frac{25}{4} - \pi^2 + \mathcal{O}\left(\frac{m_{\ell'}^2}{m_\ell^2}\right) \right] + \dots \quad (3)$$

takes into account radiative QED corrections, which are known to $\mathcal{O}(\alpha^2)$. The tiny neutrino masses have been neglected and (γ) represents additional photons or lepton pairs which have been included inclusively in $\delta_{\text{RC}}^{\ell'\ell}$. Higher-order electroweak corrections and the non-local structure of the W propagator, are usually incorporated into the effective coupling [33, 34]

$$G_{\ell'\ell}^2 = \left[\frac{g^2}{4\sqrt{2}M_W^2} (1 + \Delta r) \right]^2 \left[1 + \frac{3}{5} \frac{m_\ell^2}{M_W^2} + \frac{9}{5} \frac{m_{\ell'}^2}{M_W^2} + \mathcal{O}\left(\frac{m_{\ell'}^4}{M_W^2 m_\ell^2}\right) \right], \quad (4)$$

Table 1: Average values of some basic τ parameters [22, 23, 36].

m_τ	(1776.82 ± 0.16) MeV
τ_τ	$(290.29 \pm 0.53) \times 10^{-15}$ s
$\text{Br}(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)$	$(17.818 \pm 0.041)\%$
$\text{Br}(\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu)$	$(17.392 \pm 0.040)\%$
B_μ/B_e	0.9761 ± 0.0028
$\text{Br}(\tau^- \rightarrow \nu_\tau \pi^-)$	$(10.811 \pm 0.053)\%$
$\text{Br}(\tau^- \rightarrow \nu_\tau K^-)$	$(0.6955 \pm 0.0096)\%$

so that $G_{e\mu}$ coincides with the Fermi coupling defined in the $V - A$ theory.

Since $\tau_\mu^{-1} = \Gamma[\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu(\gamma)]$, the Fermi coupling is determined by the muon lifetime. The MuLan collaboration has recently achieved a very accurate measurement of τ_μ with a precision of 1.0 parts per million [35], more than 15 times as precise as any previous experiment. It is the most accurate particle lifetime ever measured and, consequently, dominates the world average $\tau_\mu = 2.196\,981\,1(22) \times 10^{-6}$ s [23]. Combined with the electron and muon masses, $m_e = 0.510\,998\,928(11)$ MeV and $m_\mu = 105.658\,371\,5(35)$ MeV [23], it implies

$$G_F \equiv G_{e\mu} = (1.166\,378\,7 \pm 0.000\,000\,6) \times 10^{-5} \text{ GeV}^{-2}. \quad (5)$$

Owing to its much heavier mass, the τ lepton has several final states which are kinematically allowed: $\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e$, $\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu$, $\tau^- \rightarrow \nu_\tau d \bar{u}$ and $\tau^- \rightarrow \nu_\tau s \bar{u}$. The universality of the W couplings implies that all these decay modes have equal amplitudes (if final fermion masses and QCD interactions are neglected), except for an additional $N_C |V_{ui}|^2$ factor ($i = d, s$) in the semileptonic channels, where $N_C = 3$ is the number of quark colours. Taking into account the unitarity of the quark mixing matrix, $|V_{ud}|^2 + |V_{us}|^2 = 1 - |V_{ub}|^2 \approx 1$, one easily gets a lowest-order estimate for the τ lifetime,

$$\tau_\tau \equiv \frac{1}{\Gamma(\tau)} \approx \left\{ \Gamma(\mu) \left(\frac{m_\tau}{m_\mu} \right)^5 [2 + N_C (|V_{ud}|^2 + |V_{us}|^2)] \right\}^{-1} \approx \frac{1}{5} \tau_\mu \left(\frac{m_\mu}{m_\tau} \right)^5 = 3.3 \times 10^{-13} \text{ s}, \quad (6)$$

while the branching ratios for the different channels are expected to be approximately ($\ell = e, \mu$)

$$B_\ell \equiv \text{Br}(\tau^- \rightarrow \nu_\tau \ell^- \bar{\nu}_\ell) \simeq \frac{1}{5} = 20\%, \quad \frac{\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)} \simeq N_C = 3. \quad (7)$$

The agreement with the experimental values given in table 1 provides strong evidence for the quark colour degree of freedom. The numerical differences are mainly due to the missing QCD corrections which enhance the hadronic τ decay width by about 20% (see section 7).

Using the value of G_F measured in μ decay and taking into account the final fermion masses and higher-order corrections, Eq. (2) provides a precise relation between the τ lifetime and the leptonic branching ratios:

$$B_e = \frac{B_\mu}{0.972559 \pm 0.000005} = \frac{\tau_\tau}{(1632.9 \pm 0.6) \times 10^{-15} \text{ s}}. \quad (8)$$

The quoted errors reflect the present uncertainty of 0.16 MeV in the value of m_τ .

The predicted value of B_μ/B_e is in fair agreement with the measured ratio given in table 1. The small deviation (1.3σ) originates in the recent and precise BaBar measurement $B_\mu/B_e = 0.9796 \pm 0.0039$ [37], which is 1.8σ away from the theoretical value. The previous world-average without B-factory measurements, $B_\mu/B_e = 0.9725 \pm 0.0039$, was in perfect agreement with the SM prediction. As shown

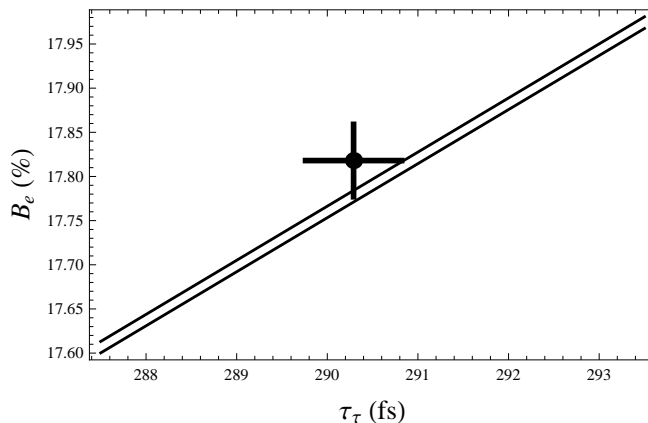


Figure 2: Relation between B_e and τ_τ . The diagonal band corresponds to Eq. (8).

in Fig. 2, the relation between B_e and τ_τ is also well satisfied by the present data. Notice, that this relation is very sensitive to the value of the τ mass.

Using the relations in Eq. (8), the measured values of B_e , B_μ and τ_τ can be combined together to obtain an improved determination of the electronic branching fraction within the SM, *i.e.*, assuming the universality of the W^\pm couplings:

$$B_e^{\text{uni}} = (17.818 \pm 0.024)\%. \quad (9)$$

The τ leptonic branching fractions and the τ lifetime are known with a precision of 0.2%, far away from the impressive 10^{-6} accuracy achieved for the muon lifetime. Belle has just released the first B-factory measurement of the τ lifetime, $\tau_\tau = (290.17 \pm 0.53 \pm 0.33) \times 10^{-15}$ s [36], with a total uncertainty smaller than the PDG 2012 average $\tau_\tau = (290.6 \pm 1.0) \times 10^{-15}$ s [23], leading to the new world average in table 1. The Belle result is based on $e^+e^- \rightarrow \tau^+\tau^-$ events collected on the $\Upsilon(4S)$ resonance, where both τ leptons decay to $3\pi\nu_\tau$.

The τ mass is only known at the 10^{-4} level. Making an energy scan of $\sigma(e^+e^- \rightarrow \tau^+\tau^-)$ around the $\tau^+\tau^-$ production threshold [38–40], the BES-III experiment aims to reach an accuracy better than 0.1 MeV [41]. Uncertainties comparable to the m_τ world average have been already reached in a preliminary analysis, using around 20 pb^{-1} of τ data [42].

3 Lepton Universality

In the SM all lepton doublets have identical couplings to the W boson. Comparing the measured decay widths of leptonic or semileptonic decays which only differ in the lepton flavour, one can test experimentally that the W interaction is indeed the same, *i.e.*, that $g_e = g_\mu = g_\tau \equiv g$. The B_μ/B_e ratio constrains $|g_\mu/g_e|$, while the B_e/τ_τ relation provides information on $|g_\tau/g_\mu|$. The present results are shown in table 2, together with the constraints obtained from leptonic π , K and W decays.

The τ determination of $|g_\mu/g_e|$ is already as precise ($\sim 0.15\%$) as the one obtained from $\pi_{\ell 2}$ decays and comparable accuracies have been reached recently with $K_{\ell 2}$ and $K_{\ell 3}$ data. The ratios ($P = \pi, K$)

$$R_{P \rightarrow e/\mu} \equiv \frac{\Gamma[P^- \rightarrow e^- \bar{\nu}_e(\gamma)]}{\Gamma[P^- \rightarrow \mu^- \bar{\nu}_\mu(\gamma)]} = \left| \frac{g_e}{g_\mu} \right|^2 \frac{m_e^2}{m_\mu^2} \left(\frac{1 - m_e^2/m_P^2}{1 - m_\mu^2/m_P^2} \right)^2 (1 + \delta R_{P \rightarrow e/\mu}) \quad (10)$$

have been calculated and measured with high accuracy. Within the SM, the leptonic decay rate of a pseudoscalar meson is helicity suppressed as a consequence of the $V-A$ structure of the charged currents,

Table 2: Experimental determinations of the ratios $g_\ell/g_{\ell'}$.

	$\Gamma_{\tau \rightarrow \mu}/\Gamma_{\tau \rightarrow e}$	$\Gamma_{\pi \rightarrow \mu}/\Gamma_{\pi \rightarrow e}$	$\Gamma_{K \rightarrow \mu}/\Gamma_{K \rightarrow e}$	$\Gamma_{K \rightarrow \pi\mu}/\Gamma_{K \rightarrow \pi e}$	$\Gamma_{W \rightarrow \mu}/\Gamma_{W \rightarrow e}$
$ g_\mu/g_e $	1.0018 (14)	1.0021 (16)	0.9978 (20)	1.0010 (25)	0.996 (10)
	$\Gamma_{\tau \rightarrow e}/\Gamma_{\mu \rightarrow e}$	$\Gamma_{\tau \rightarrow \pi}/\Gamma_{\pi \rightarrow \mu}$	$\Gamma_{\tau \rightarrow K}/\Gamma_{K \rightarrow \mu}$	$\Gamma_{W \rightarrow \tau}/\Gamma_{W \rightarrow \mu}$	
$ g_\tau/g_\mu $	1.0011 (15)	0.9962 (27)	0.9858 (70)	1.034 (13)	
	$\Gamma_{\tau \rightarrow \mu}/\Gamma_{\mu \rightarrow e}$	$\Gamma_{W \rightarrow \tau}/\Gamma_{W \rightarrow e}$			
$ g_\tau/g_e $	1.0030 (15)	1.031 (13)			

making these ratios sensitive probes of new-physics interactions. The known radiative corrections $\delta R_{P \rightarrow e/\mu}$ include a summation of leading QED logarithms $\alpha^n \log^n(m_\mu/m_e)$ [43, 44] and a systematic two-loop calculation of $\mathcal{O}(e^2 p^4)$ effects within Chiral Perturbation Theory [45]. The comparison between the SM predictions [45]

$$R_{\pi \rightarrow e/\mu}^{\text{SM}} = (1.2352 \pm 0.0001) \times 10^{-4}, \quad R_{K \rightarrow e/\mu}^{\text{SM}} = (2.477 \pm 0.001) \times 10^{-5}, \quad (11)$$

and the experimental $R_{\pi \rightarrow e/\mu}$ [46–48] and $R_{K \rightarrow e/\mu}$ ratios [49, 50],

$$R_{\pi \rightarrow e/\mu} = (1.230 \pm 0.004) \times 10^{-4}, \quad R_{K \rightarrow e/\mu} = (2.488 \pm 0.010) \times 10^{-5}, \quad (12)$$

gives the results quoted in table 2. Ongoing experiments at PSI [51] and TRIUMF [52] are expected to improve the experimental precision of $R_{\pi \rightarrow e/\mu}$ from 0.3% to 0.05%.

The $K \rightarrow \pi \ell \bar{\nu}_\ell$ decays proceed without any helicity suppression; the differences between the muon and electron modes stem mainly from isospin and phase space [53]. Both sets of $K_{\ell 3}$ data are then used to determine the Cabibbo mixing between the quarks of the first and second generations. Comparing the V_{us} values obtained from $K \rightarrow \pi e \bar{\nu}_e$ and $K \rightarrow \pi \mu \bar{\nu}_\mu$ decays, one obtains the corresponding $|g_\mu/g_e|$ value in table 2 [54].

The decay modes $\tau^- \rightarrow \nu_\tau \pi^-$ and $\tau^- \rightarrow \nu_\tau K^-$ can also be used to test universality through the ratios

$$R_{\tau/P} \equiv \frac{\Gamma(\tau^- \rightarrow \nu_\tau P^-)}{\Gamma(P^- \rightarrow \mu^- \bar{\nu}_\mu)} = \left| \frac{g_\tau}{g_\mu} \right|^2 \frac{m_\tau^3}{2m_P m_\mu^2} \frac{(1 - m_P^2/m_\tau^2)^2}{(1 - m_\mu^2/m_P^2)^2} (1 + \delta R_{\tau/P}), \quad (13)$$

where the dependence on the hadronic matrix elements (the so-called meson decay constants f_P) factors out. Owing to the different energy scales involved, the radiative corrections to the $\tau^- \rightarrow \nu_\tau P^-$ amplitudes are however not the same than the corresponding effects in $P^- \rightarrow \mu^- \bar{\nu}_\mu$. The size of the relative correction has been roughly estimated to be [43, 55]:

$$\delta R_{\tau/\pi} = (0.16 \pm 0.14)\%, \quad \delta R_{\tau/K} = (0.90 \pm 0.22)\%. \quad (14)$$

Using these numbers, the measured $\tau^- \rightarrow \pi^- \nu_\tau$ and $\tau^- \rightarrow K^- \nu_\tau$ decay rates imply the $|g_\tau/g_\mu|$ ratios given in table 2.

Due to the limited statistics available, the direct leptonic decays of the W boson only test universality at the 1% level. The LEP data contains a slight excess of events in $W \rightarrow \nu_\tau \tau$, implying a 2.6σ (2.4σ) deviation from universality in $|g_\tau/g_\mu|$ ($|g_\tau/g_e|$) [23, 56]. This discrepancy cannot be easily understood, given the stringent limits on $|g_\tau/g_{e,\mu}|$ from W -mediated decays [57]. Future high-energy e^+e^- colliders should clarify whether this is a real physical effect or just a statistical fluctuation.

The present data verify the universality of the leptonic charged-current couplings to the 0.15% level. It is important to realize the complementarity of the different universality tests. The pure leptonic decay modes probe the couplings of a transverse W . In contrast, the semileptonic decays $P^- \rightarrow \ell^- \bar{\nu}_\ell$ and $\tau^- \rightarrow \nu_\tau P^-$ are only sensitive to the spin-0 piece of the charged current; thus, they could unveil the presence of possible scalar-exchange contributions with Yukawa-like couplings proportional to some power of the charged-lepton mass. One can easily imagine new-physics scenarios which would modify differently the two types of leptonic couplings. For instance, in the usual two-Higgs doublet models charged-scalar exchange generates a correction to the ratio B_μ/B_e , but $R_{P \rightarrow e/\mu}$ remains unaffected [58].

4 Lorentz Structure of the Charged Current

Let us consider the leptonic decays $\ell^- \rightarrow \ell'^- \bar{\nu}_{\ell'} \nu_\ell$, where the lepton pair (ℓ, ℓ') may be (μ, e) , (τ, e) or (τ, μ) . With high statistics, these leptonic decay modes allow us to investigate the Lorentz structure of the decay amplitudes through the analysis of the energy and angular distribution of the final charged lepton, complemented with polarization information whenever available.

The most general, local, derivative-free, lepton-number conserving, four-lepton interaction Hamiltonian, consistent with locality and Lorentz invariance [59–65],

$$\mathcal{H} = 4 \frac{G_{\ell\ell'}}{\sqrt{2}} \sum_{n,\epsilon,\omega} g_{\epsilon\omega}^n [\bar{\ell}'_\epsilon \Gamma^n (\nu_{\ell'})_\sigma] \left[\overline{(\nu_\ell)_\lambda} \Gamma_n \ell_\omega \right], \quad (15)$$

contains ten complex coupling constants or, since a common phase is arbitrary, nineteen independent real parameters which could be different for each leptonic decay. The subindices $\epsilon, \omega, \sigma, \lambda$ label the chiralities (left-handed, right-handed) of the corresponding fermions, and $n = S, V, T$ the type of interaction: scalar ($\Gamma^S = I$), vector ($\Gamma^V = \gamma^\mu$), tensor ($\Gamma^T = \sigma^{\mu\nu}/\sqrt{2}$). For given n, ϵ, ω , the neutrino chiralities σ and λ are uniquely determined. Taking out a common factor $G_{\ell\ell'}$, which is determined by the total decay rate, the coupling constants $g_{\epsilon\omega}^n$ are normalized to [63]

$$1 = \frac{1}{4} (|g_{RR}^S|^2 + |g_{RL}^S|^2 + |g_{LR}^S|^2 + |g_{LL}^S|^2) + 3 (|g_{RL}^T|^2 + |g_{LR}^T|^2) + (|g_{RR}^V|^2 + |g_{RL}^V|^2 + |g_{LR}^V|^2 + |g_{LL}^V|^2). \quad (16)$$

Thus, $|g_{\epsilon\omega}^S| \leq 2$, $|g_{\epsilon\omega}^V| \leq 1$ and $|g_{\epsilon\omega}^T| \leq 1/\sqrt{3}$. It is convenient to introduce the probabilities [63]

$$Q_{\epsilon\omega} = \frac{1}{4} |g_{\epsilon\omega}^S|^2 + |g_{\epsilon\omega}^V|^2 + 3 (1 - \delta_{\epsilon\omega}) |g_{\epsilon\omega}^T|^2 \quad (17)$$

for the decay of an ω -handed ℓ^- into an ϵ -handed daughter lepton. In the Standard Model, $g_{LL}^V = 1$ and all other $g_{\epsilon\omega}^n = 0$.

For an initial lepton polarization \mathcal{P}_ℓ , the final charged-lepton distribution in the decaying-lepton rest frame is usually parametrized in the form [60, 61]

$$\frac{d^2\Gamma_{\ell \rightarrow \ell'}}{dx d\cos\theta} = \frac{m_\ell \omega^4}{2\pi^3} G_{\ell\ell'}^2 \sqrt{x^2 - x_0^2} \left\{ F(x) - \frac{\xi}{3} \mathcal{P}_\ell \sqrt{x^2 - x_0^2} \cos\theta A(x) \right\}, \quad (18)$$

where θ is the angle between the ℓ^- spin and the final charged-lepton momentum, $\omega \equiv (m_\ell^2 + m_{\ell'}^2)/2m_\ell$ is the maximum ℓ'^- energy for massless neutrinos, $x \equiv E_{\ell'}/\omega$ is the reduced energy, $x_0 \equiv m_{\ell'}/\omega$ and

$$F(x) = x(1-x) + \frac{2}{9} \rho (4x^2 - 3x - x_0^2) + \eta x_0(1-x),$$

$$A(x) = 1 - x + \frac{2}{3} \delta \left(4x - 4 + \sqrt{1 - x_0^2} \right). \quad (19)$$

For an unpolarized lepton ℓ , the distribution is characterized by the so-called Michel [59] parameter ρ and the low-energy parameter η . Two more parameters, ξ and δ , can be determined when the initial lepton polarization is known. If the polarization of the final charged lepton is also measured, 5 additional independent parameters (ξ' , ξ'' , η'' , α' , β') appear [23]. In the SM, $\rho = \delta = 3/4$, $\eta = \eta'' = \alpha' = \beta' = 0$ and $\xi = \xi' = \xi'' = 1$.

For massless neutrinos, the total decay rate is given by [65]

$$\Gamma_{\ell \rightarrow \ell'} = \frac{\widehat{G}_{\ell'\ell}^2 m_\ell^5}{192\pi^3} f(m_{\ell'}^2/m_\ell^2) \left(1 + \delta_{\text{RC}}^{\ell'\ell}\right), \quad (20)$$

where

$$\widehat{G}_{\ell'\ell} \equiv G_{\ell'\ell} \sqrt{1 + 4\eta \frac{m_{\ell'} g(m_{\ell'}^2/m_\ell^2)}{m_\ell f(m_{\ell'}^2/m_\ell^2)}}, \quad (21)$$

$g(z) = 1 + 9z - 9z^2 - z^3 + 6z(1+z)\log z$, and the SM radiative correction $\delta_{\text{RC}}^{\ell'\ell}$ has been included. Since we assume that the SM provides the dominant contribution to the decay rate, any additional higher-order correction beyond the effective Hamiltonian (15) would be a subleading effect.

The normalization $G_{e\mu}$ corresponds to the Fermi coupling G_F , measured in μ decay. The B_μ/B_e and $B_e\tau_\mu/\tau_\tau$ universality tests, discussed in the previous section, actually probe the ratios $|\widehat{G}_{\mu\tau}/\widehat{G}_{e\tau}|$ and $|\widehat{G}_{e\tau}/\widehat{G}_{e\mu}|$, respectively. The experimental determination of $G_{e\mu}$ is then sensitive to the uncertainty in $\eta_{\mu \rightarrow e}$.

In terms of the $g_{\epsilon w}^n$ couplings, the shape parameters in Eqs. (18) and (19) are:

$$\begin{aligned} \rho &= \frac{3}{4}(\beta^+ + \beta^-) + (\gamma^+ + \gamma^-), \\ \xi &= 3(\alpha^- - \alpha^+) + (\beta^- - \beta^+) + \frac{7}{3}(\gamma^+ - \gamma^-), \\ \xi\delta &= \frac{3}{4}(\beta^- - \beta^+) + (\gamma^+ - \gamma^-), \\ \eta &= \frac{1}{2} \text{Re}[g_{LL}^V g_{RR}^{S*} + g_{RR}^V g_{LL}^{S*} + g_{LR}^V (g_{RL}^{S*} + 6g_{RL}^{T*}) + g_{RL}^V (g_{LR}^{S*} + 6g_{LR}^{T*})], \end{aligned} \quad (22)$$

where [66]

$$\begin{aligned} \alpha^+ &\equiv |g_{RL}^V|^2 + \frac{1}{16}|g_{RL}^S + 6g_{RL}^T|^2, & \alpha^- &\equiv |g_{LR}^V|^2 + \frac{1}{16}|g_{LR}^S + 6g_{LR}^T|^2, \\ \beta^+ &\equiv |g_{RR}^V|^2 + \frac{1}{4}|g_{RR}^S|^2, & \beta^- &\equiv |g_{LL}^V|^2 + \frac{1}{4}|g_{LL}^S|^2, \\ \gamma^+ &\equiv \frac{3}{16}|g_{RL}^S - 2g_{RL}^T|^2, & \gamma^- &\equiv \frac{3}{16}|g_{LR}^S - 2g_{LR}^T|^2, \end{aligned} \quad (23)$$

are positive-definite combinations of decay constants, corresponding to a final right-handed ($\alpha^+, \beta^+, \gamma^+$) or left-handed ($\alpha^-, \beta^-, \gamma^-$) lepton. The normalization constraint (16) is equivalent to $\alpha^+ + \alpha^- + \beta^+ + \beta^- + \gamma^+ + \gamma^- = 1$. The probabilities $Q_{\epsilon w}$ can be extracted from the measurable shape parameters:

$$\begin{aligned} Q_{LL} &= \beta^- = \frac{1}{4} \left(-3 + \frac{16}{3}\rho - \frac{1}{3}\xi + \frac{16}{9}\xi\delta + \xi' + \xi'' \right), \\ Q_{RR} &= \beta^+ = \frac{1}{4} \left(-3 + \frac{16}{3}\rho + \frac{1}{3}\xi - \frac{16}{9}\xi\delta - \xi' + \xi'' \right), \\ Q_{LR} &= \alpha^- + \gamma^- = \frac{1}{4} \left(5 - \frac{16}{3}\rho + \frac{1}{3}\xi - \frac{16}{9}\xi\delta + \xi' - \xi'' \right), \\ Q_{RL} &= \alpha^+ + \gamma^+ = \frac{1}{4} \left(5 - \frac{16}{3}\rho - \frac{1}{3}\xi + \frac{16}{9}\xi\delta - \xi' - \xi'' \right). \end{aligned} \quad (24)$$

Table 3: 90% CL experimental bounds for the $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$ couplings [76]

$ g_{RR}^S < 0.035$	$ g_{LR}^S < 0.050$	$ g_{RL}^S < 0.412$	$ g_{LL}^S < 0.550$
$ g_{RR}^V < 0.017$	$ g_{LR}^V < 0.023$	$ g_{RL}^V < 0.104$	$ g_{LL}^V > 0.960$
$ g_{RR}^T \equiv 0$	$ g_{LR}^T < 0.015$	$ g_{RL}^T < 0.103$	$ g_{LL}^T \equiv 0$
$ g_{LR}^S + 6 g_{LR}^T < 0.143$	$ g_{LR}^S + 2 g_{LR}^T < 0.108$	$ g_{LR}^S - 2 g_{LR}^T < 0.070$	
$ g_{RL}^S + 6 g_{RL}^T < 0.418$	$ g_{RL}^S + 2 g_{RL}^T < 0.417$	$ g_{RL}^S - 2 g_{RL}^T < 0.418$	
$Q_{RR} + Q_{LR} < 8.2 \times 10^{-4}$			

Upper bounds on any of these probabilities translate into corresponding limits for all couplings with the given chiralities. Particularly powerful are the relations:

$$\begin{aligned}
 Q_{\ell_R} &\equiv Q_{RR} + Q_{LR} = \frac{1}{2} \left[1 + \frac{\xi}{3} - \frac{16}{9} (\xi\delta) \right], \\
 Q_{\ell'_R} &\equiv Q_{RR} + Q_{RL} = \frac{1}{2} (1 - \xi').
 \end{aligned}
 \tag{25}$$

The total probability for the decay of an initial right-handed lepton is determined by ξ and $\xi\delta$, while a single polarization parameter fixes the probability to decay into a final right-handed lepton. Other useful positive-definite quantities are:

$$\begin{aligned}
 \frac{3}{2} \beta^+ + 2\gamma^- &= \rho - \xi\delta, \\
 Q_{LR} + Q_{RL} &= \frac{1}{2} \left[5 - \frac{16}{3} \rho - \xi'' \right].
 \end{aligned}
 \tag{26}$$

The first one provides direct bounds on $|g_{RR}^V|$, $|g_{RR}^S|$ and $|g_{LR}^S - 2 g_{LR}^T|$, and shows that $\rho \geq \xi\delta$. A precise measurement of the polarization parameter ξ'' would imply, through the second identity, upper limits on all couplings $g_{\epsilon\omega}^n$ with $\epsilon \neq \omega$.

For μ decay, where precise measurements of the polarizations of both μ and e have been performed, there exist [63] upper bounds on Q_{RR} , Q_{LR} and Q_{RL} , and a lower limit on Q_{LL} . They imply corresponding upper bounds on the 8 couplings $|g_{RR}^n|$, $|g_{LR}^n|$ and $|g_{RL}^n|$. The measurements of the μ^- and the e^- do not allow to determine $|g_{LL}^S|$ and $|g_{LL}^V|$ separately [63, 67]. Nevertheless, a lower limit on $|g_{LL}^V|$ is obtained from the inverse muon decay because $\sigma(\nu_\mu e^- \rightarrow \mu^- \nu_e) \propto |g_{LL}^V|^2$ [63]. Since the helicity of the ν_μ in pion decay is experimentally known [68, 69] to be $-1/2$ with high precision, $|2 h_{\nu_\mu} + 1| < 0.0041$ (90% CL) [70], the $|g_{LL}^S|$ contribution to $\sigma(\nu_\mu e^- \rightarrow \mu^- \nu_e)$ is negligible; contributions from other $g_{\epsilon\omega}^n$ couplings [71] are severely suppressed by the μ -decay constraints. Once a lower bound on $|g_{LL}^V|$ has been set, the relation $Q_{LL} < 1$ provides the upper limit $|g_{LL}^S|^2 < 4(1 - |g_{LL}^V|^2)$. The present 90% CL bounds on the μ -decay couplings [72–75] are given in table 3. These limits show nicely that the bulk of the μ -decay transition amplitude is indeed of the predicted V–A type.

The experimental analysis of the τ decay parameters is necessarily different from the one applied to the muon, because of the much shorter τ lifetime. The measurement of the τ polarization and the parameters ξ and δ is still possible due to the fact that the spins of the $\tau^+ \tau^-$ pair produced in $e^+ e^-$ annihilation are strongly correlated [21, 77–85]. Another possibility is to use the beam polarization, as done by SLD [86]. However, the polarization of the charged lepton emitted in the τ decay has never been measured. In principle, this could be done for the decay $\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau$ by stopping the muons and detecting their decay products [82]. An alternative method would be [87] to use the radiative decays

Table 4: Michel parameters [23]. The last column assumes identical couplings for $\ell = e, \mu$. $\xi_{\mu \rightarrow e}$ refers to the product $\xi_{\mu \rightarrow e} \mathcal{P}_\mu$, where $\mathcal{P}_\mu \approx 1$ is the longitudinal polarization of the μ from π decay.

	$\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$	$\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau$	$\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau$	$\tau^- \rightarrow \ell^- \bar{\nu}_\ell \nu_\tau$
ρ	0.74979 ± 0.00026	0.763 ± 0.020	0.747 ± 0.010	0.745 ± 0.008
η	0.057 ± 0.034	0.094 ± 0.073	—	0.013 ± 0.020
ξ	$1.0009 \begin{smallmatrix} + 0.0016 \\ - 0.0007 \end{smallmatrix}$	1.030 ± 0.059	0.994 ± 0.040	0.985 ± 0.030
$\xi\delta$	$0.7511 \begin{smallmatrix} + 0.0012 \\ - 0.0006 \end{smallmatrix}$	0.778 ± 0.037	0.734 ± 0.028	0.746 ± 0.021
ξ'	1.00 ± 0.04	—	—	—
ξ''	0.65 ± 0.36	—	—	—

Table 5: 95% CL experimental bounds for the leptonic τ -decay couplings [101]

$\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau$			
$ g_{RR}^S < 0.70$	$ g_{LR}^S < 0.99$	$ g_{RL}^S \leq 2$	$ g_{LL}^S \leq 2$
$ g_{RR}^V < 0.17$	$ g_{LR}^V < 0.13$	$ g_{RL}^V < 0.52$	$ g_{LL}^V \leq 1$
$ g_{RR}^T \equiv 0$	$ g_{LR}^T < 0.082$	$ g_{RL}^T < 0.51$	$ g_{LL}^T \equiv 0$
$\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau$			
$ g_{RR}^S < 0.72$	$ g_{LR}^S < 0.95$	$ g_{RL}^S \leq 2$	$ g_{LL}^S \leq 2$
$ g_{RR}^V < 0.18$	$ g_{LR}^V < 0.12$	$ g_{RL}^V < 0.52$	$ g_{LL}^V \leq 1$
$ g_{RR}^T \equiv 0$	$ g_{LR}^T < 0.079$	$ g_{RL}^T < 0.51$	$ g_{LL}^T \equiv 0$

$\tau \rightarrow \ell^- \bar{\nu}_\ell \nu_\tau \gamma$ ($\ell = e, \mu$), since the distribution of the photons emitted by the daughter lepton is sensitive to the lepton polarization. The measurement of the inverse decay $\nu_\tau \ell^- \rightarrow \tau^- \nu_\ell$ looks far out of reach.

The experimental status on the τ -decay Michel parameters [86, 88–95] is shown in table 4. For comparison, the more accurate values measured in μ decay [72, 76, 96–100] are also given. Table 5 gives the resulting 95% CL bounds on the τ -decay couplings.

If lepton universality is assumed, the leptonic decay ratios B_μ/B_e and $B_e \tau_\mu/\tau_\tau$ provide limits on the low-energy parameter η . The best sensitivity [102] comes from $\hat{G}_{\mu\tau}$, where the term proportional to η is not suppressed by the small m_e/m_ℓ factor. The world-averaged value of the B_μ/B_e ratio implies then:

$$\eta_{\tau \rightarrow \ell} = 0.016 \pm 0.013, \quad (27)$$

which only assumes e/μ universality. This determination is more accurate than the $\mu \rightarrow e$ and $\tau \rightarrow \mu$ ones in table 4, obtained from the shape of the energy distribution. It is also slightly more accurate than the value quoted in the table for $\tau \rightarrow \ell$. A non-zero value of η would show that there are at least two different couplings with opposite chiralities for the charged leptons. Assuming the V–A coupling g_{LL}^V to be dominant, the second one would be a scalar coupling g_{RR}^S . To first order in new physics contributions, $\eta \approx \text{Re}(g_{RR}^S)/2$; Eq. (27) puts then the 95% CL bound: $-0.019 < \text{Re}(g_{RR}^S) < 0.083$.

Table 6: Limits on the couplings $g_{\epsilon\omega}^n$, assuming that there are no tensor couplings. The τ -decay (μ -decay) values are at 95% CL (90% CL).

	$ g_{RR}^S $	$ g_{LR}^S $	$ g_{RL}^S $	$ g_{LL}^S $	$ g_{RR}^V $	$ g_{LR}^V $	$ g_{RL}^V $	$ g_{LL}^V $
$\mu \rightarrow e$	< 0.035	< 0.048	< 0.412	< 0.550	< 0.017	< 0.023	< 0.017	> 0.960
$\tau \rightarrow \mu$	< 0.39	< 0.39	≤ 2	≤ 2	< 0.18	< 0.12	< 0.14	≤ 1
$\tau \rightarrow e$	< 0.42	< 0.42	≤ 2	≤ 2	< 0.17	< 0.13	< 0.13	≤ 1
$\tau \rightarrow l$	< 0.34	< 0.34	≤ 2	≤ 2	< 0.17	< 0.13	< 0.12	≤ 1

Table 7: Bounds on the $g_{\epsilon\omega}^V$ couplings, assuming that (non-standard) W -exchange is the only relevant interaction. The τ -decay (μ -decay) limits are at 95% CL (90% CL). Numbers within parentheses use μ -decay data through cross-channel identities.

	$ g_{RR}^V $	$ g_{LR}^V $	$ g_{RL}^V $	$ g_{LL}^V $
$\mu \rightarrow e$	< 0.0004	< 0.023	< 0.017	> 0.999
$\tau \rightarrow \mu$	< 0.017 (0.003)	< 0.12	< 0.14 (0.023)	> 0.983
$\tau \rightarrow e$	< 0.017 (0.002)	< 0.13	< 0.13 (0.017)	> 0.983

4.1 Model-Dependent Constraints

The sensitivity of the present τ data is not good enough to get strong constraints from a completely general analysis of the four-fermion Hamiltonian. Nevertheless, better limits can be obtained within particular models. For instance, let us assume that there are no tensor couplings, *i.e.*, $g_{\epsilon\omega}^T = 0$. This condition is satisfied in any model where the interactions are mediated by vector bosons and/or charged scalars. In this case, the quantities $(1 - \frac{4}{3}\rho)$, $(1 - \frac{4}{3}\xi\delta)$ and $(1 - \frac{4}{3}\rho) + \frac{1}{2}(1 - \xi)$ reduce to sums of $|g_{\epsilon\omega}^n|^2$, which are positive semidefinite; *i.e.*, in the absence of tensor couplings, $\rho \leq \frac{3}{4}$, $\xi\delta \leq \frac{3}{4}$ and $(1 - \xi) > 2(\frac{4}{3}\rho - 1)$ [65]. The corresponding limits on the couplings $g_{\epsilon\omega}^n$ are shown in table 6. For μ decay one obtains a sizeable improvement of $|g_{RL}^V|$ by a factor of 6, and a 4% reduction of the $|g_{LR}^S|$ upper bound. In the τ decay modes, stronger bounds are obtained for $|g_{RR}^S|$, $|g_{LR}^S|$ and $|g_{RL}^V|$.

If one only considers W -mediated interactions, but admitting the possibility that the W couples non-universally to leptons of any chirality, the effective couplings factorize into the product of two leptonic W couplings, *i.e.*, $g_{\ell_e\ell_\omega}^V = \kappa_{\ell_e}^* \kappa_{\ell_\omega}$. This implies additional relations among the couplings of the effective Hamiltonian, such as $g_{LR}^V g_{RL}^V = g_{LL}^V g_{RR}^V$ [71, 103], which hold within any of the three channels, $\mu \rightarrow e$, $\tau \rightarrow e$ and $\tau \rightarrow \mu$. Moreover, the effective couplings of the different decay modes get related through identities like $g_{\mu L\tau L}^V g_{e L\tau R}^V = g_{\mu L\tau R}^V g_{e L\tau L}^V$ [65]. The resulting limits are given in table 7. Notice the strong lower bounds on the g_{LL}^V couplings, which are a consequence of the normalization condition (16). If the lepton decays are mediated by a single vector-particle (the W), the effective interaction is constrained in all cases to be mostly left-handed with high accuracy.

For W -mediated interactions, the hadronic decay modes $\tau^- \rightarrow \nu_\tau h^-$ can also be used to test the structure of the $\tau\nu_\tau W$ vertex, if one assumes that the W coupling to the light quarks is the SM one. The effective Hamiltonian contains only two vector couplings g_λ , with λ being the τ (and ν_τ) chirality, with the normalization $|g_L|^2 + |g_R|^2 = 1$. The \mathcal{P}_τ dependent part of the decay amplitude is then proportional to $\xi_h = |g_L|^2 - |g_R|^2$, which plays a role analogous to the leptonic decay constant ξ . The parameter ξ_h determines the mean ν_τ helicity times a factor of -2 ; $\xi_h = 1$ in the SM. The analysis of $\tau^+\tau^-$ decay

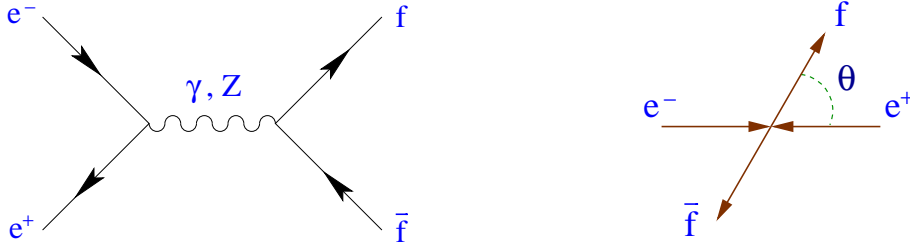


Figure 3: Tree-level contributions to $e^+e^- \rightarrow \bar{f}f$ and kinematical configuration in the centre-of-mass system.

correlations in leptonic-hadronic and hadronic-hadronic decay modes, using the π , ρ and a_1 hadronic final states [86, 88, 89, 91–93, 104–107], gives $\xi_h = 0.995 \pm 0.007$ [101]. This implies the 95% CL bounds $|g_L| > 0.995$ and $|g_R| < 0.10$.

5 Neutral-Current Couplings

In the SM, tau pair production in e^+e^- annihilation proceeds through the electromagnetic and weak neutral-current interactions, shown in Fig. 3: $e^+e^- \rightarrow \gamma^*, Z^* \rightarrow \tau^+\tau^-$. At high energies, where the Z contribution is important, the study of the production cross section allows to extract information on the lepton electroweak parameters. The Z coupling to the fermionic neutral current is given by [108]

$$\mathcal{L}_{\text{NC}}^Z = -\frac{g}{2 \cos \theta_W} Z_\mu \sum_f \bar{f} \gamma^\mu (v_f - a_f \gamma_5) f, \quad (28)$$

where $v_f = T_3^f (1 - 4|Q_f| \sin^2 \theta_W)$ and $a_f = T_3^f$, with $T_3^f = \pm \frac{1}{2}$ the corresponding weak isospin. Thus, the weak neutral couplings are predicted to be the same for all fermions with equal electric charge Q_f .

For unpolarized e^+ and e^- beams, the differential $e^+e^- \rightarrow \ell^+\ell^-$ cross section can be written as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8s} \{A(1 + \cos^2 \theta) + B \cos \theta - h_\ell [C(1 + \cos^2 \theta) + D \cos \theta]\}, \quad (29)$$

where $h_\ell = \pm 1$ denotes the sign of the ℓ^- helicity and θ is the scattering angle between e^- and ℓ^- in the centre-of-mass reference system. At lowest order,

$$A = 1 + 2 v_e v_\ell \text{Re}(\chi) + (v_e^2 + a_e^2)(v_\ell^2 + a_\ell^2) |\chi|^2, \quad (30)$$

$$B = 4 a_e a_\ell \text{Re}(\chi) + 8 v_e a_e v_\ell a_\ell |\chi|^2, \quad (31)$$

$$C = 2 v_e a_\ell \text{Re}(\chi) + 2 (v_e^2 + a_e^2) v_\ell a_\ell |\chi|^2, \quad (32)$$

$$D = 4 a_e v_\ell \text{Re}(\chi) + 4 v_e a_e (v_\ell^2 + a_\ell^2) |\chi|^2, \quad (33)$$

and χ contains the Z propagator

$$\chi = \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha} \frac{s}{s - M_Z^2 + is\Gamma_Z/M_Z}. \quad (34)$$

The coefficients A , B , C and D can be experimentally determined, by measuring the total cross section, the forward-backward asymmetry, the polarization asymmetry and the forward-backward polarization asymmetry, respectively:

$$\sigma(s) = \frac{4\pi\alpha^2}{3s} A, \quad \mathcal{A}_{\text{FB}}(s) \equiv \frac{N_F - N_B}{N_F + N_B} = \frac{3}{8} \frac{B}{A},$$

$$\begin{aligned}\mathcal{A}_{\text{Pol}}(s) &\equiv \frac{\sigma^{(h_\ell=+1)} - \sigma^{(h_\ell=-1)}}{\sigma^{(h_\ell=+1)} + \sigma^{(h_\ell=-1)}} = -\frac{C}{A}, \\ \mathcal{A}_{\text{FB,Pol}}(s) &\equiv \frac{N_F^{(h_\ell=+1)} - N_F^{(h_\ell=-1)} - N_B^{(h_\ell=+1)} + N_B^{(h_\ell=-1)}}{N_F^{(h_\ell=+1)} + N_F^{(h_\ell=-1)} + N_B^{(h_\ell=+1)} + N_B^{(h_\ell=-1)}} = -\frac{3}{8} \frac{D}{A}.\end{aligned}\quad (35)$$

Here, N_F and N_B denote the number of negative leptons emerging in the forward and backward hemispheres, respectively, with respect to the electron direction.

For $s = M_Z^2$, the real part of the Z propagator vanishes and the photon-exchange terms can be neglected in comparison with the Z -exchange contributions ($\Gamma_Z^2/M_Z^2 \ll 1$). Eqs. (35) become then,

$$\begin{aligned}\sigma^{0,\ell} &\equiv \sigma(M_Z^2) = \frac{12\pi}{M_Z^2} \frac{\Gamma_e \Gamma_\ell}{\Gamma_Z^2}, & \mathcal{A}_{\text{FB}}^{0,\ell} &\equiv \mathcal{A}_{\text{FB}}(M_Z^2) = \frac{3}{4} \mathcal{P}_e \mathcal{P}_\ell, \\ \mathcal{A}_{\text{Pol}}^{0,\ell} &\equiv \mathcal{A}_{\text{Pol}}(M_Z^2) = \mathcal{P}_\ell, & \mathcal{A}_{\text{FB,Pol}}^{0,\ell} &\equiv \mathcal{A}_{\text{FB,Pol}}(M_Z^2) = \frac{3}{4} \mathcal{P}_e,\end{aligned}\quad (36)$$

where Γ_ℓ is the Z partial decay width to the $\ell^+\ell^-$ final state, and

$$\mathcal{P}_\ell \equiv \frac{-2v_\ell a_\ell}{v_\ell^2 + a_\ell^2}\quad (37)$$

is the average longitudinal polarization of the lepton ℓ^- , which only depends on the ratio of the vector and axial-vector couplings.

Small higher-order corrections can produce large variations on the predicted lepton polarization because $|v_\ell| = \frac{1}{2}|1 - 4 \sin^2 \theta_W| \ll 1$. Therefore, \mathcal{P}_ℓ is a very sensitive function of $\sin^2 \theta_W$, providing an interesting window to probe electroweak quantum effects. Subtracting from the data initial-state QED corrections, γ -exchange and γ - Z interference contributions, the tiny electroweak boxes and corrections for $s \neq M_Z^2$, one can use the equations given above as effective expressions in terms of loop-corrected couplings or, equivalently, an effective electroweak mixing angle: $\sin^2 \theta_{\text{eff}}^{\text{lept}} \equiv \frac{1}{4}(1 - v_\ell/a_\ell)$.

The Z partial decay width to the $\ell^+\ell^-$ final state,

$$\Gamma_\ell \equiv \Gamma(Z \rightarrow \ell^+\ell^-) = \frac{G_F M_Z^3}{6\pi\sqrt{2}} (v_\ell^2 + a_\ell^2) (1 + \delta_{\text{RC}}^Z),\quad (38)$$

determines the sum $(v_\ell^2 + a_\ell^2)$, while the ratio v_ℓ/a_ℓ is derived from the asymmetries.¹ The absolute signs of v_ℓ and a_ℓ are established by the convention $a_e < 0$.

The measurement of the final polarization asymmetries can (only) be done for $\ell = \tau$, because the spin polarizations of the τ^+ and τ^- are reflected in the distorted distribution of their decay products. Therefore, \mathcal{P}_τ and \mathcal{P}_e can be determined from a measurement of the spectrum of the final charged particles in the decay of one τ , or by studying the correlated distributions between the final products of both taus [79, 84, 85].

With polarized e^+e^- beams, which were available at SLC, one can also study the left-right asymmetry between the cross sections for initial left- and right-handed electrons, and the corresponding forward-backward left-right asymmetry:

$$\mathcal{A}_{\text{LR}}^0 \equiv \mathcal{A}_{\text{LR}}(M_Z^2) = \frac{\sigma_L(M_Z^2) - \sigma_R(M_Z^2)}{\sigma_L(M_Z^2) + \sigma_R(M_Z^2)} = -\mathcal{P}_e, \quad \mathcal{A}_{\text{FB,LR}}^{0,\ell} \equiv \mathcal{A}_{\text{FB,LR}}(M_Z^2) = -\frac{3}{4} \mathcal{P}_\ell.\quad (39)$$

At the Z peak, $\mathcal{A}_{\text{LR}}^0$ measures the average initial lepton polarization, \mathcal{P}_e , without any need for final particle identification, while $\mathcal{A}_{\text{FB,LR}}^{0,\ell}$ provides a direct determination of the final fermion polarization.

¹ The asymmetries determine two possible solutions: $v_\ell/a_\ell = -(1 \pm \sqrt{1 - \mathcal{P}_\ell^2})/\mathcal{P}_\ell$. This ambiguity can be solved with the energy dependence of $\mathcal{A}_{\text{FB}}(s)$ [109] or through the measurement of the transverse spin-spin correlation of the two taus in $Z \rightarrow \tau^+\tau^-$ which determines $C_{TT} = (a_\ell^2 - v_\ell^2)/(a_\ell^2 + v_\ell^2)$ [83]. The experimental value $C_{TT} = 1.01 \pm 0.12$ [110, 111] requires $|v_\tau/a_\tau| \ll 1$.

Table 8: Measured values of the leptonic Z decay widths, forward-backward asymmetries and longitudinal polarization of the final ℓ^- from polarization asymmetries [23, 109, 112]. The last column shows the combined result (for a massless lepton) assuming lepton universality. The resulting effective vector and axial-vector Z couplings are also given.

	e	μ	τ	ℓ
Γ_ℓ (MeV)	83.91 (12)	83.99 (18)	84.08 (22)	83.984 (86)
$\mathcal{A}_{\text{FB}}^{0,\ell}$ (%)	1.45 (25)	1.69 (13)	1.88 (17)	1.71 (10)
\mathcal{P}_ℓ	-0.1515 (19)	-0.142 (15)	-0.143 (4)	-0.1499 (18)
v_ℓ	-0.03817 (47)	-0.0367 (23)	-0.0366 (10)	-0.03783 (41)
a_ℓ	-0.50111 (35)	-0.50120 (54)	-0.50204 (64)	-0.50123 (26)
$v_\mu/v_e = 0.961$ (61)		$a_\mu/a_e = 1.0002$ (13)		
$v_\tau/v_e = 0.959$ (29)		$a_\tau/a_e = 1.0019$ (15)		

Table 8 shows the present experimental results for the leptonic Z decay widths and asymmetries, and the corresponding effective vector and axial-vector couplings of the Z boson. Lepton universality appears to be well satisfied, although there is a 1.9σ difference between the values of \mathcal{P}_e (dominated by the SLD measurement of \mathcal{A}_{LR}^0) and \mathcal{P}_τ (dominated by the measured $\mathcal{A}_{\text{Pol}}^{0,\tau}$ at LEP). Assuming universality, the independent determinations of \mathcal{P}_ℓ from the forward-backward asymmetries, $|\mathcal{P}_\ell^{\text{FB}}| = (4\mathcal{A}_{\text{FB}}^{0,\ell}/3)^{1/2} = 0.151 \pm 0.004$, the LEP τ polarization asymmetries, $\mathcal{P}_\ell^{\text{LEP}} = -0.1465 \pm 0.0033$, and the SLD left-right asymmetries, $\mathcal{P}_\ell^{\text{SLD}} = -0.1513 \pm 0.0021$, are in good agreement, giving the combined average $\mathcal{P}_\ell = -0.1501 \pm 0.0016$ [109, 112].

The resulting 68% probability contours are shown in Figs. 4. The left figure compares Γ_ℓ and the effective electroweak mixing angle determined from all LEP-I and SLD asymmetries, $\sin^2 \theta_{\text{eff}}^{\text{lept}} = 0.23153 \pm 0.00016$ [109, 112], with the SM prediction as a function of m_t and M_H . The right figure gives the corresponding effective vector and axial-vector couplings v_ℓ and a_ℓ . The precision electroweak measurements require a low value of the Higgs mass, in nice agreement with the recent LHC discovery of a Higgs-like boson with $M_H = 126$ GeV [113, 114]. These figures provide also strong evidence of the electroweak radiative corrections. The good agreement with the SM predictions is lost if only the QED vacuum polarization contribution is taken into account, as indicated by the point with an arrow. Notice that the uncertainty induced by the input value of $\alpha(M_Z^2)^{-1} = 128.95 \pm 0.05$ is sizeable. The measured couplings of the three charged leptons confirm lepton universality in the neutral-current sector. The solid contour combines the three determinations, assuming universality.

The measurement of $\mathcal{A}_{\text{Pol}}^{0,\tau}$ and $\mathcal{A}_{\text{FB,Pol}}^{0,\tau}$ assumes that the τ decay proceeds through the SM charged-current interaction. A more general analysis should take into account the fact that the τ decay width depends on the product $\xi\mathcal{P}_\tau$, where ξ is the corresponding Michel parameter in leptonic decays or the equivalent quantity ξ_h in the semileptonic modes. A separate measurement of ξ and \mathcal{P}_τ can be performed, analysing the correlated distribution of the $\tau^+\tau^-$ decays [84, 85]. The LEP collaborations found in this way $\mathcal{P}_\tau = -0.132 \pm 0.019$ (ALEPH) [115], $\mathcal{P}_\tau = -0.131 \pm 0.014$ (DELPHI) [89] and $\mathcal{P}_\tau = -0.164 \pm 0.016$ (L3) [91], in agreement with the more precise value quoted in table 8.

The neutrino couplings can also be determined from the invisible Z decay width, by assuming three identical neutrino generations with left-handed couplings, and fixing the sign from neutrino scattering data: $v_{\nu_\ell} = a_{\nu_\ell} = 0.50076 \pm 0.00076$. Alternatively, one can use the SM prediction for Γ_{inv} to get a

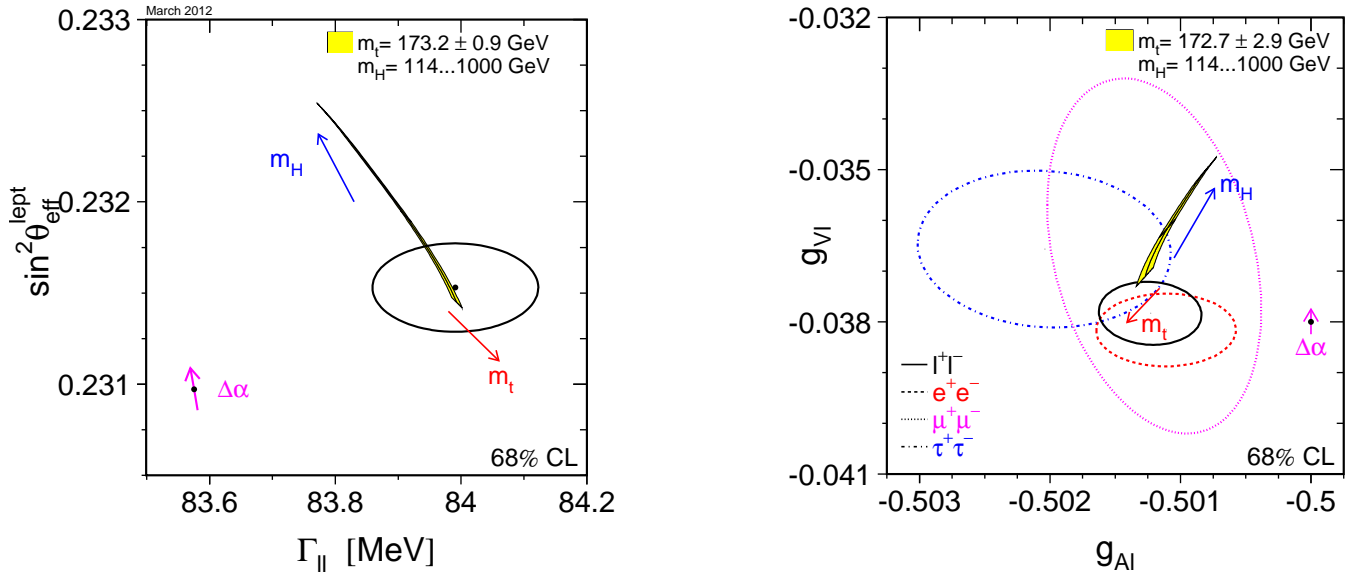


Figure 4: Combined LEP and SLD measurements of $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ and Γ_ℓ (left) and the corresponding effective vector and axial-vector couplings v_ℓ and a_ℓ (right). The shaded region shows the SM prediction. The arrows point in the direction of increasing values of m_t and M_H . The point shows the predicted values if, among the electroweak radiative corrections, only the photon vacuum polarization is included. Its arrow indicates the variation induced by the uncertainty in $\alpha(M_Z^2)$ [109, 112].

determination of the number of light neutrino flavours [109, 112]:

$$N_\nu = 2.9840 \pm 0.0082. \quad (40)$$

The universality of the neutrino couplings has been tested with $\nu_\mu e$ and $\nu_e e$ scattering data, which fix the ν_μ and ν_e couplings to the Z : $v_{\nu_\mu} = a_{\nu_\mu} = 0.502 \pm 0.017$ and $v_{\nu_e} = a_{\nu_e} = 0.528 \pm 0.085$ [116].

The first direct observation of the τ neutrino was accomplished at Fermilab, twenty-five years after the τ discovery, using the ν_τ production chain $p+N \rightarrow D_s+X$, followed by $D_s \rightarrow \tau^- \bar{\nu}_\tau$ and $\tau^- \rightarrow \nu_\tau+Y$. The DONUT experiment observed a total of nine $\nu_\tau + N \rightarrow \tau + Z$ events [117], making manifest the existence of a third neutrino associated with the tau flavour.

6 Hadronic Decays

The τ is the only known lepton massive enough to decay into hadrons. Its semileptonic decays are ideally suited to investigate the hadronic weak currents and perform low-energy tests of the strong interaction. The decay $\tau^- \rightarrow \nu_\tau H^-$ probes the matrix element of the left-handed charged current between the vacuum and the final hadronic state H^- ,

$$\langle H^- | (V_{ud}^* \bar{d} + V_{us}^* \bar{s}) \gamma^\mu (1 - \gamma_5) u | 0 \rangle. \quad (41)$$

Contrary to the well-known process $e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons}$, which only tests the electromagnetic vector current, the semileptonic τ decays offer the possibility to study the properties of both vector and axial-vector currents, with Cabibbo-allowed and Cabibbo-suppressed final states [118].

For the Cabibbo-allowed modes with $J^P = 1^-$, the matrix element of the vector charged current can also be obtained, through an isospin rotation, from the isovector part of the e^+e^- annihilation cross section into hadrons, which measures the hadronic matrix element of the $I = 1$ component of the electromagnetic current,

$$\langle V^0 | (\bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d) | 0 \rangle. \quad (42)$$

The $\tau \rightarrow \nu_\tau V^-$ decay width is then expressed as an integral over the cross section of the corresponding $e^+e^- \rightarrow V^0$ process [21, 119]:

$$R_{\tau \rightarrow V} \equiv \frac{\Gamma(\tau^- \rightarrow \nu_\tau V^-)}{\Gamma_{\tau \rightarrow e}} = \frac{3 \cos^2 \theta_C}{2\pi\alpha^2 m_\tau^8} S_{\text{EW}} \int_0^{m_\tau^2} ds (m_\tau^2 - s)^2 (m_\tau^2 + 2s) s \sigma_{e^+e^- \rightarrow V^0}^{I=1}(s), \quad (43)$$

where the factor $S_{\text{EW}} = 1.0201 \pm 0.0003$ contains the renormalization-group improved electroweak correction, including a next-to-leading order resummation of large logarithms [28, 120, 121]. The available data on $e^+e^- \rightarrow V^0$, can then be compared with the corresponding $\tau \rightarrow \nu_\tau V^-$ decay width and invariant-mass distribution [122–127]. The e^+e^- data contain in addition an isoscalar ($I = 0$) component, which cannot be accessed through τ decays, and needs to be subtracted in (43).

Hadronic τ decays and the e^+e^- annihilation into hadrons offer very good laboratories to improve our present understanding of the non-perturbative QCD dynamics. The general form factors characterizing the hadronic amplitudes can be experimentally extracted from the Dalitz-plot distributions of the final hadrons [128]. The exhaustive analysis of these processes provides a very valuable data basis to confront with theoretical models.

6.1 Chiral Dynamics

In the absence of quark masses the QCD Lagrangian splits into two independent chirality (left/right) sectors, with their own quark flavour symmetries. With three light quarks (u, d, s), the QCD Lagrangian is then approximately invariant under chiral $SU(3)_L \otimes SU(3)_R$ rotations in flavour space. However, the vacuum is not symmetric under the chiral group. Thus, chiral symmetry breaks down to the usual eightfold-way $SU(3)_V$, generating the appearance of eight Goldstone bosons in the hadronic spectrum, which can be identified with the lightest pseudoscalar octet; their small masses being generated by the quark mass matrix, which explicitly breaks chiral symmetry. The Goldstone nature of the pseudoscalar octet implies strong constraints on their interactions, which can be worked out through an expansion in powers of momenta over the chiral symmetry-breaking scale [129]. At low momentum transfer, the coupling of any number of Goldstones (π, K, η) to the V–A current can be rigorously calculated with the effective field theory techniques of Chiral Perturbation Theory (χ PT) [130–133].

In the low-energy effective chiral realization of 3-flavour QCD, the vector $V_\mu^{ij} = \bar{q}^j \gamma_\mu q^i$ and axial-vector $A_\mu^{ij} = \bar{q}^j \gamma_\mu \gamma_5 q^i$ ($i, j = u, d, s$) quark currents take the form [118, 132]:

$$\begin{aligned} V_\mu &= -i \left(\Phi \overset{\leftrightarrow}{\partial}_\mu \Phi \right) + \mathcal{O}(\Phi^4) + \mathcal{O}(p^3) \\ &\quad - \frac{iN_C}{6\sqrt{2}\pi^2 f^3} \varepsilon_{\mu\nu\alpha\beta} \left\{ \partial^\nu \Phi \partial^\alpha \Phi \partial^\beta \Phi + \mathcal{O}(\Phi^5) + \mathcal{O}(p^5) \right\}, \\ A_\mu &= -\sqrt{2}f \partial_\mu \Phi + \frac{\sqrt{2}}{3f} \left[\Phi, \left(\Phi \overset{\leftrightarrow}{\partial}_\mu \Phi \right) \right] + \mathcal{O}(\Phi^5) + \mathcal{O}(p^3) \\ &\quad + \frac{N_C}{12\pi^2 f^4} \varepsilon_{\mu\nu\alpha\beta} \left\{ \partial^\nu \Phi \partial^\alpha \Phi \left(\Phi \overset{\leftrightarrow}{\partial}^\beta \Phi \right) + \mathcal{O}(\Phi^6) + \mathcal{O}(p^5) \right\}, \end{aligned} \quad (44)$$

where the odd-parity pieces proportional to the Levi-Civita pseudotensor are generated by the Wess–Zumino–Witten term of the chiral Lagrangian [134, 135], which incorporates the non-abelian chiral anomaly of QCD. The 3×3 matrix

$$\Phi(x) \equiv \frac{\vec{\lambda}}{\sqrt{2}} \vec{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \quad (45)$$

parametrizes the pseudoscalar octet fields. Thus, at lowest order in momenta, the couplings of the Goldstones to the electroweak currents can be calculated in a straightforward way in terms of the single parameter f . In particular, f determines the pion decay constant in the massless quark limit: $f_\pi = f + \mathcal{O}(m_q) = (92.2 \pm 0.1) \text{ MeV}$.

At very low momentum transfer (low invariant mass of the final hadrons), the $\tau^- \rightarrow \nu_\tau(P_1 \dots P_N)^-$ amplitudes ($P_i = \pi, K, \eta$) can be easily obtained from Eqs. (44), which provide the lowest-order contribution in the momentum expansion. The one-loop χ PT corrections are known [130–132, 136–138] for the lowest-multiplicity states ($\pi, K, 2\pi, K\bar{K}, K\pi, K\eta, 3\pi, 4\pi$). Moreover, a two-loop calculation for the 2π decay mode is also available [136]. Therefore, exclusive hadronic τ decay data at low values of q^2 can be compared with rigorous QCD predictions. There exist also well-grounded theoretical results (based on a $1/M_\rho$ expansion) for decays such as $\tau^- \rightarrow \nu_\tau(\rho\pi)^-, \tau^- \rightarrow \nu_\tau(K^*\pi)^-$ and $\tau^- \rightarrow \nu_\tau(\omega\pi)^-$, but only in the kinematical configuration where the pion is soft in the vector-meson rest frame [139]. Tau decays involve, however, high values of momentum transfer where the χ PT predictions no longer apply. The relevant hadronic dynamics is governed by the non-perturbative regime of QCD, in the resonance region, which makes very difficult to perform first-principle calculations for exclusive decays. Nevertheless, one can still construct reasonable approximations, taking into account the low-energy chiral theorems.

In the first theoretical analyses of hadronic τ decays, the chiral predictions were often extrapolated to higher values of q^2 by suitable final-state-interaction enhancements, through resonance form factors normalized to one at zero invariant mass [118, 125, 140–144]. This simple prescription takes into account phenomenologically the resonance structures present in each channel and provides a useful description of the τ amplitudes in terms of a few hadronic parameters which can be adjusted to the data. Therefore it has been incorporated into the TAUOLA Monte Carlo library [145] and has been extensively used to study the main τ decay modes [146–150]. However, the resonance modelization is too naive to be considered as an actual implementation of the QCD dynamics. The addition of resonance form factors to the chiral low-energy amplitudes does not guarantee that the chiral symmetry constraints on the resonance couplings are correctly implemented [151]. Quite often, the numerical predictions could be drastically changed by varying some free parameter or modifying the form-factor ansatz.

The proper way of including higher-mass states into the effective chiral theory was elaborated in Refs. [152, 153] and has been further developed in more recent works [154–160]. Using the techniques of the resulting Resonance Chiral Theory (R χ T), it is possible to perform a systematic analysis of the τ decay amplitudes within an effective field theory realization of the underlying QCD Lagrangian.

6.2 Two-body Semileptonic Decays

For the decay modes with lowest multiplicity, $\tau^- \rightarrow \nu_\tau\pi^-$ and $\tau^- \rightarrow \nu_\tau K^-$, the relevant matrix elements

$$\langle \pi^-(p) | \bar{d}\gamma^\mu \gamma_5 u | 0 \rangle = -i\sqrt{2} f_\pi p^\mu, \quad \langle K^-(p) | \bar{s}\gamma^\mu \gamma_5 u | 0 \rangle = -i\sqrt{2} f_K p^\mu, \quad (46)$$

are already known from the measured decays $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ and $K^- \rightarrow \mu^- \bar{\nu}_\mu$. The corresponding τ decay widths can then be predicted rather accurately through Eq. (13). As shown in table 2, these predictions are in good agreement with the measured values, and provide a quite precise test of charged-current universality.

Alternatively, the ratio between the measured $\tau^- \rightarrow \nu_\tau K^-$ and $\tau^- \rightarrow \nu_\tau \pi^-$ decay widths can be used to determine the ratio of the corresponding hadronic matrix elements:

$$\frac{|V_{us}| f_K}{|V_{ud}| f_\pi} = \frac{m_\tau^2 - m_\pi^2}{m_\tau^2 - m_K^2} \left\{ \frac{\text{Br}(\tau^- \rightarrow \nu_\tau K^-)}{\text{Br}(\tau^- \rightarrow \nu_\tau \pi^-)} \frac{1 + \delta R_{\tau/\pi}}{1 + \delta R_{\tau/K}} \frac{1}{1 + \delta R_{K/\pi}} \right\}^{1/2} = 0.2737 \pm 0.0021. \quad (47)$$

We have used the radiative corrections in Eq. (14), together with the corresponding correction to the meson decay ratio $R_{K/\pi} \equiv \Gamma(K^- \rightarrow \mu^- \bar{\nu}_\mu)/\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu)$:

$$\delta R_{K/\pi} = -(0.0069 \pm 0.0017) - (0.0044 \pm 0.0015) = -(0.0113 \pm 0.0023). \quad (48)$$

The first number is of electromagnetic origin [161,162], while the second accounts for the strong isospin-breaking correction [161], so that f_K and f_π denote the meson decay constants in the isospin limit. The result (47) is consistent with the value $(|V_{us}| f_K)/(|V_{ud}| f_\pi) = 0.2763 \pm 0.0005$, obtained from $R_{K/\pi}$ [161], but it has a much larger uncertainty. Taking the lattice average $f_K/f_\pi = 1.193 \pm 0.005$ [163] and $|V_{ud}| = 0.97425 \pm 0.00022$ from super-allowed nuclear beta decays [164], one gets then a determination of the Cabibbo mixing:

$$\frac{|V_{us}|}{|V_{ud}|} = \begin{cases} 0.2294 \pm 0.0020 \\ 0.2316 \pm 0.0011 \end{cases}, \quad |V_{us}| = \begin{cases} 0.2235 \pm 0.0019 & (\tau \rightarrow K/\pi) \\ 0.2256 \pm 0.0010 & (K/\pi \rightarrow \mu) \end{cases}. \quad (49)$$

6.3 Decays into Two Hadrons

The decay into two pseudoscalar mesons, $\tau^- \rightarrow \nu_\tau P^- P'^0$, is mediated by the vector current. The relevant hadronic matrix element can be parametrized in terms of two form factors:

$$\langle P^- P'^0 | \bar{d}_i \gamma^\mu u | 0 \rangle = C_{PP'} \left\{ \left(p_- - p_0 - \frac{\Delta_{PP'}}{s} q \right)^\mu F_V^{PP'}(s) + \frac{\Delta_{PP'}}{s} q^\mu F_S^{PP'}(s) \right\}, \quad (50)$$

where p_-^μ and p_0^μ are the momenta of the charged and neutral pseudoscalars, respectively, $q^\mu = (p_- + p_0)^\mu$ is the momentum transfer and $s = q^2$. The two Lorentz structures correspond to $J^P = 1^-$ and 0^+ transitions. The scalar contribution is suppressed by the mass-squared difference $\Delta_{PP'} = m_{P^-}^2 - m_{P'^0}^2$ because the vector current is conserved in the limit of equal quark masses. The global normalization coefficient $C_{PP'}$ has been chosen so that the vector form factor $F_V^{PP'}(s)$ is one at lowest order in χ PT. Thus,

$$C_{\pi\pi} = \sqrt{2}, \quad C_{K\bar{K}} = -1, \quad C_{K\pi} = \frac{1}{\sqrt{2}}, \quad C_{\pi\bar{K}} = -1, \quad C_{K\eta_8} = \sqrt{\frac{3}{2}}, \quad (51)$$

while d_i refers to the corresponding down-type quark d or s .

The hadronic invariant-mass distribution is given by

$$\frac{d\Gamma}{ds} = \frac{G_F^2 |V_{ui}|^2 m_\tau^3}{768\pi^3} S_{EW}^{\text{had}} C_{PP'}^2 \left(1 - \frac{s}{m_\tau^2}\right)^2 \left\{ \left(1 + 2 \frac{s}{m_\tau^2}\right) \lambda_{PP'}^{3/2} |F_V^{PP'}(s)|^2 + 3 \frac{\Delta_{PP'}^2}{s^2} \lambda_{PP'}^{1/2} |F_S^{PP'}(s)|^2 \right\}, \quad (52)$$

where $\lambda_{PP'} \equiv \lambda(s, m_{P^-}^2, m_{P'^0}^2)/s^2$ and $S_{EW}^{\text{had}} = 1.0157 \pm 0.0003$ accounts for the short-distance electroweak corrections [28, 120, 121]. Long-distance electromagnetic corrections and isospin-breaking contributions are channel dependent and have been only studied in a model-dependent way for the $\pi\pi$ [165, 166] and $K\pi$ [167, 168] final states.

6.3.1 $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$

In the limit of isospin symmetry the two-pion final state does not receive any scalar contribution. The decay $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$ is then governed by the so-called pion form factor $F_\pi(s) \equiv F_V^{\pi\pi}(s)$. A dynamical understanding of $F_\pi(s)$ can be achieved [157, 169–172], using analyticity, unitarity and some general properties of QCD, such as chiral symmetry [131–133] and the short-distance asymptotic behaviour [152–154, 156]. Putting all these fundamental ingredients together, one can express the pion form factor in the simple form [169]

$$F_\pi(s) = \frac{M_\rho^2}{M_\rho^2 - s - iM_\rho\Gamma_\rho(s)} \exp \left\{ -\frac{s}{96\pi^2 f_\pi^2} \text{Re}[A(s)] \right\}, \quad (53)$$

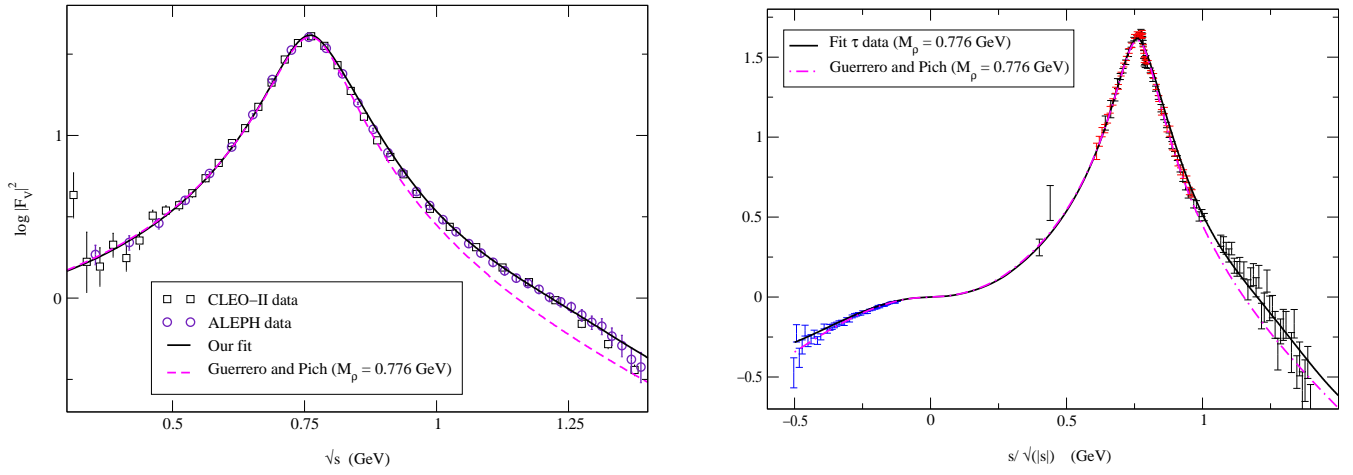


Figure 5: Pion form factor from ALEPH [178] and CLEO [179] τ data (left) and $e^+e^- \rightarrow \pi^+\pi^-$ [180] and $e^-\pi^- \rightarrow e^-\pi^-$ [181] data (right), compared with theoretical predictions [169,171]. The dashed lines correspond to the result in Eq. (53).

where

$$A(s) \equiv \log \left(\frac{m_\pi^2}{M_\rho^2} \right) + 8 \frac{m_\pi^2}{s} - \frac{5}{3} + \sigma_\pi^3 \log \left(\frac{\sigma_\pi + 1}{\sigma_\pi - 1} \right) \quad (54)$$

contains the one-loop chiral logarithms [130], which account for the final-state interactions of the two pions, $\sigma_\pi \equiv \sqrt{1 - 4m_\pi^2/s}$ and the off-shell ρ width is given by $\Gamma_\rho(s) = \theta(s - 4m_\pi^2) \sigma_\pi^3 M_\rho s / (96\pi f_\pi^2)$ [169,170].

In the limit of an infinite number of QCD colours [173,174], $F_\pi(s)$ is described by an infinite sum of narrow-width vector resonance contributions [154,175,176]. Eq. (53) only contains the contribution from the lightest ρ state, which is the dominant one below 1 GeV. The normalization $F_\pi(0) = 1$ is a consequence of the conservation of the electromagnetic current, while the short-distance properties of QCD require the form factor to vanish at infinite momentum. The large- N_C propagator has been dressed with pion loop corrections, which are subleading in $1/N_C$, in such a way that a Taylor expansion in powers of s/M_ρ^2 correctly reproduces the rigorous one-loop χ PT prediction at very low energies. The constraints from analyticity and unitarity allow us to perform a resummation of the one-loop chiral logarithms through an Omnès exponential [177]; using a Dyson summation, the absorptive part of these corrections has been reabsorbed into the ρ width in order to regulate the resonance pole. Thus, Eq. (53) extends the validity domain of the χ PT prediction by combining together a series of basic theoretical requirements on $F_\pi(s)$. This analytical prediction, which only depends on M_ρ , m_π and the pion decay constant f_π , is compared with the data in Fig. 5. The agreement is rather impressive and extends to negative s values, where the $e^-\pi^-$ elastic data sits.

The modifications induced by kaon loops, heavier ρ resonance contributions and additional next-to-leading in $1/N_C$ corrections can be easily included; one gets then a more accurate approximation at the price of having more free parameters which decrease the predictive power [157,171,172]. This gives a better description of the ρ' shoulder around 1.2 GeV (continuous lines in Fig. 5). A clear signal for the $\rho''(1700)$ resonance in $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$ events has been reported by Belle, with a data sample 20 times larger than in previous experiments [182]. Fig. 6 shows a recent fit to the Belle data, including the effects from the ρ' and ρ'' states [183].

The $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$ decay amplitude can be related through an isospin rotation with the isovector piece of $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$. Thus, for $2m_\pi^2 < s < m_\tau^2$, the pion form factor can be obtained from the two sets of data. For many years there have been sizeable discrepancies between the different experimental measurements, suggesting that systematic uncertainties have been probably underestimated. The amplitudes measured in e^+e^- annihilation experiments [180,184–188] are slightly lower than the

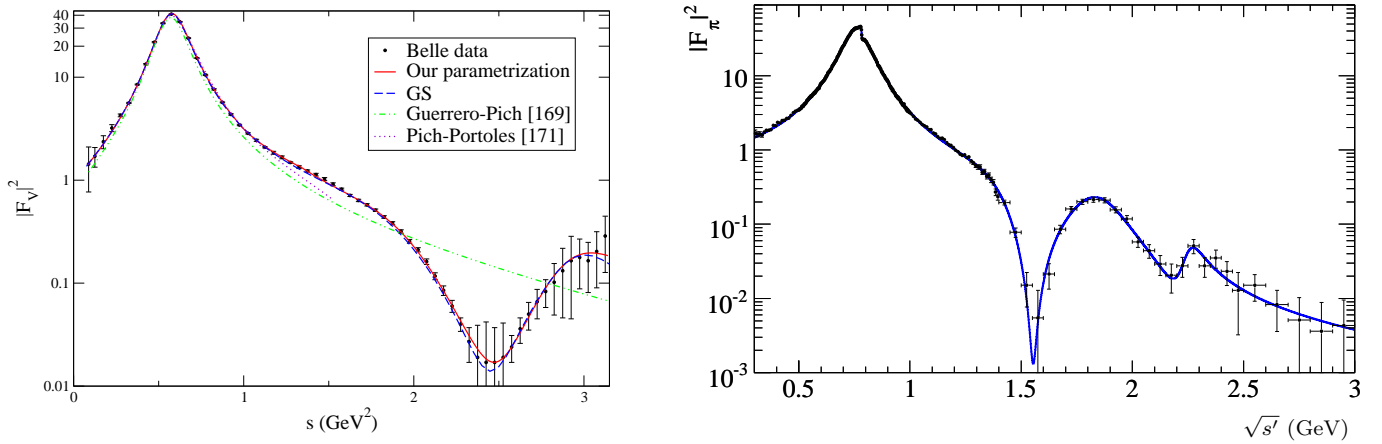


Figure 6: Pion form factor extracted from Belle $\tau^- \rightarrow \nu_\tau \pi^- \pi^0$ data [182] (left). The curves correspond to different phenomenological approximations, including ρ , ρ' and ρ'' contributions [183]. The BaBar measurement from $e^+e^- \rightarrow \pi^+\pi^-\gamma$ events [184] (right) shows also clear ρ''' and ω ($I = 0$) signals.

corresponding τ decay measurements [178, 179, 182, 189], and the difference does not seem to be fully accounted for through the calculated isospin-violating effects [165, 166]. In addition to the direct energy scan adopted in most e^+e^- measurements, some recent experiments use the so-called radiative return method to extract $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$ from the $e^+e^- \rightarrow \pi^+\pi^-\gamma$ data sample, probing different ranges of $\pi^+\pi^-$ invariant masses through the radiated photon (initial state radiation) [190–193]. Using this method, the most recent BaBar data [184] has reduced considerably the tension with τ -decay measurements, approaching the tau results, but discrepancies persist with the KLOE $e^+e^- \rightarrow \pi^+\pi^-\gamma$ data [187].

Taking into account isospin-breaking corrections [165, 166], one predicts from e^+e^- data $\text{Br}(\tau^- \rightarrow \nu_\tau \pi^- \pi^0) = (24.94 \pm 0.25)\%$ [194], which is 2.1σ smaller than the average of direct τ decay measurements $(25.504 \pm 0.092)\%$ [22]. The largest (smallest) discrepancy of 2.7σ (1.2σ) between prediction and direct measurement is exhibited by KLOE (BaBar) [194]. This discrepancy translates into different estimates of the hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon that will be discussed in section 9.1. Accurate measurements of $F_\pi(s)$ are a critical ingredient of the SM prediction for the muon $g - 2$, which reinforces the need for new precise e^+e^- and τ data sets to clarify the present experimental situation.

6.3.2 $\tau^- \rightarrow \nu_\tau(K\pi)^-$

Owing to the different final pseudoscalar masses, the decays $\tau^- \rightarrow \nu_\tau K^- \pi^0$ and $\tau^- \rightarrow \nu_\tau \pi^- \bar{K}^0$ receive contributions from two different form factors with $J^P = 1^-$ and 0^+ . The same vector and scalar form factors are probed in $K_{\ell 3}$ processes, but tau decays are sensitive to a much broader (and different) range of invariant hadronic masses.

A detailed study of these decays has been made in Refs. [150, 167, 195–197]. The vector form factor $F_V^{K\pi}(s)$ can be described in an analogous way to $F_\pi(s)$, while the scalar component $F_S^{K\pi}(s)$ takes also into account additional information from $K\pi$ scattering data through dispersion relations [198–200]. Fig. 7 compares the distribution of $\tau^- \rightarrow \nu_\tau \pi^- K_S$ events measured by Belle [201], with a theoretical fit using the R χ T description of $F_V^{K\pi}(s)$ with two resonances [195]. The scalar component is predicted to give only a small contribution to the total decay width, $\text{Br}[\tau \rightarrow \nu_\tau (K\pi)_{S\text{-wave}}] = (3.88 \pm 0.19) \times 10^{-4}$, but it is sizeable at very low invariant mass. As expected, the $K^*(892)$ resonance generates the dominant contribution with a small correction from the $K^*(1410)$ state at higher invariant mass. The Belle data shows a bump at 0.682–0.705 GeV (points 5, 6 and 7) which is not supported by the theoretical description and, therefore, it is not included in the fit; this bump does not seem to be present in the

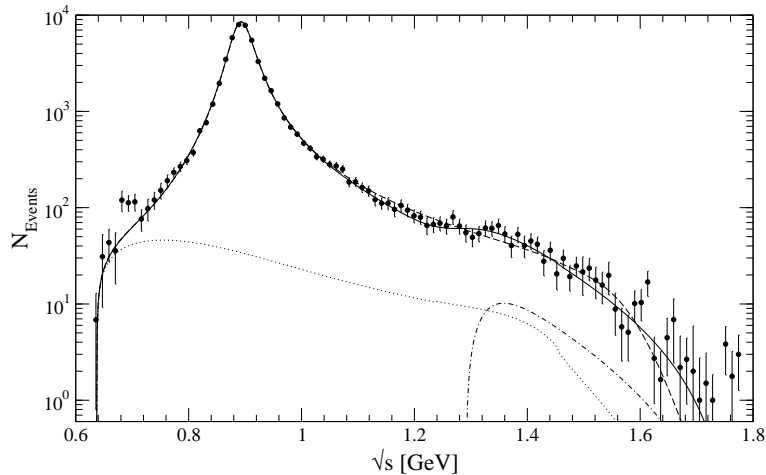


Figure 7: Distribution of $\tau^- \rightarrow \nu_\tau \pi^- K_S$ events measured by Belle [201]. The solid line shows the theoretical fit of Ref. [195], including a $R\chi T$ description of $F_V^{\pi K}(s)$ with two resonances and $F_S^{\pi K}(s)$ from [199]. The scalar and $K^*(1410)$ contributions are indicated by the dotted and dash-dotted lines.

preliminary BaBar data [202].

The fit to the $\tau^- \rightarrow \nu_\tau \pi^- K_S$ spectrum allows one to predict the slope and curvature of the vector form factor, with a precision comparable (slightly better) to the one attained in $K_{\ell 3}$ decays, and the absolute normalization, *i.e.*, the $\tau^- \rightarrow \nu_\tau \pi^- \bar{K}^0$ and $\tau^- \rightarrow \nu_\tau K^- \pi^0$ branching ratios [167, 195–197]. The most recent analysis makes a combined fit to $\tau^- \rightarrow \nu_\tau \pi^- K_S$ and $K_{\ell 3}$ data, using dispersive parametrizations of the two form factors and including electromagnetic and isospin-violating corrections which are needed for $K_{\ell 3}$, and finds the predictions [167]:

$$\begin{aligned} \text{Br}(\tau^- \rightarrow \nu_\tau \pi^- \bar{K}^0)_{\text{th}} &= (0.857 \pm 0.030)\%, \\ \text{Br}(\tau^- \rightarrow \nu_\tau K^- \pi^0)_{\text{th}} &= (0.471 \pm 0.018)\%. \end{aligned} \quad (55)$$

These numbers are slightly higher than the present experimental world averages, $\text{Br}(\tau^- \rightarrow \nu_\tau \pi^- \bar{K}^0) = (0.8206 \pm 0.0182)\%$ and $\text{Br}(\tau^- \rightarrow \nu_\tau K^- \pi^0) = (0.4322 \pm 0.0149)\%$ [22], confirming the earlier observation made in Ref. [195]. The implications of these results for the determination of $|V_{us}|$ will be discussed later in section 8.

The decays $\tau \rightarrow K^- \eta \nu_\tau$ and $\tau \rightarrow K^- \eta' \nu_\tau$ have been also studied recently in Ref. [203]. These decay modes are quite suppressed. The first one has been measured at the B factories [204, 205] with an average branching fraction $\text{Br}(\tau \rightarrow K^- \eta \nu_\tau) = (1.53 \pm 0.08) \times 10^{-4}$ [22], while only an upper limit exists in the $K^- \eta'$ case, $\text{Br}(\tau \rightarrow K^- \eta' \nu_\tau) < 2.4 \times 10^{-6}$ (90% CL) [206].

6.3.3 $\tau^- \rightarrow \nu_\tau \pi^- \eta$

The $\pi^- \eta$ final state has $I^G = 1^-$ and $J^P = 0^+$ or 1^- (for S or P wave, respectively). Its observation in τ decays would therefore indicate either a violation of G-parity (the Cabibbo-allowed vector current has even G-parity) or new physics incorporating second-class currents. In the SM, this decay can proceed through the violation of isospin provided by the light quark mass difference $m_d - m_u$ or through electromagnetic corrections, and it is expected to be very suppressed [142, 207, 208]. At lowest order in χ PT the coupling to the vector current is governed by the well-known $\pi^0 - \eta$ mixing, giving rise to constant (and equal) vector and scalar form factors:

$$F_V^{\pi\eta}(s)_{\text{LO}} = F_S^{\pi\eta}(s)_{\text{LO}} = \frac{\sqrt{3}(m_d - m_u)}{4(m_s - \hat{m})} \approx 0.99 \times 10^{-2}, \quad (56)$$

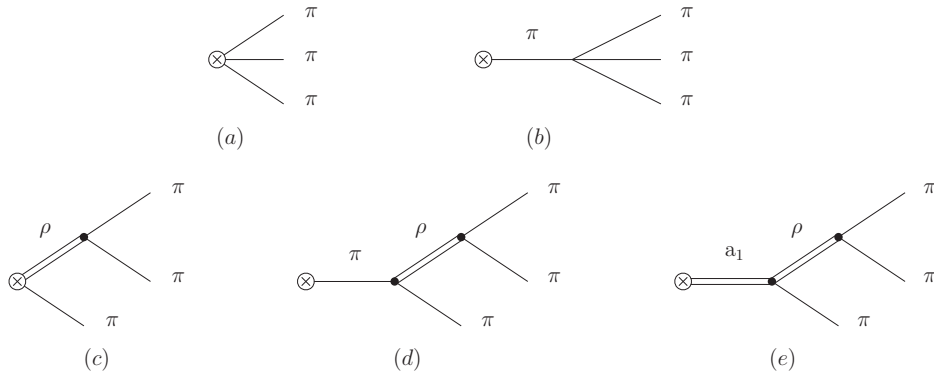


Figure 8: Tree-level Feynman diagrams contributing to $\tau^- \rightarrow \nu_\tau(3\pi)^-$.

where $\hat{m} = (m_d + m_u)/2$. This result gets significantly enhanced by next-to-leading-order chiral corrections and electromagnetic contributions. The value of the two form factors at $s = 0$ can be related in a simple and very elegant way with the ratio $F_V^{K^-\pi^0}(0)/F_V^{\pi^-K^0}(0)$ [209]. Exploiting the $K_{\ell 3}$ information [54], one can then obtain a very precise prediction of the $\pi\eta$ form factors at $s = 0$ [210]:

$$F_V^{\pi\eta}(0) = F_S^{\pi\eta}(0) = (1.49 \pm 0.23) \times 10^{-2}. \quad (57)$$

Inserting this information into subtracted dispersion relations, together with the known χ PT constraints, the two form factors have been recently estimated in the relevant kinematical domain, implying the following τ decay branching fraction [210]:

$$\text{Br}(\tau^- \rightarrow \nu_\tau \pi^- \eta)_{\text{th}} = (0.48^{+0.30}_{-0.20}) \times 10^{-5}. \quad (58)$$

This theoretical prediction, which is dominated by the scalar contribution, is in the lower range of previous evaluations [142, 207–209, 211, 212] and a factor 20 smaller than the present experimental upper bound: $\text{Br}(\tau^- \rightarrow \nu_\tau \pi^- \eta) < 9.9 \times 10^{-5}$ (95% CL) [204]. A slightly better experimental limit exists for the $\pi^- \eta'$ final state: $\text{Br}(\tau^- \rightarrow \nu_\tau \pi^- \eta') < 4.0 \times 10^{-6}$ (90% CL) [206].

6.4 Higher-Multiplicity Decays

Higher-multiplicity modes involve a richer dynamical structure, providing a very valuable experimental window into the non-perturbative hadronization of the QCD currents. However, accounting for the strong final-state interaction is not an easy game when three or more hadrons are present. There exist tree-level R χ T calculations for most τ decays into three mesons, and even some final states with four pseudoscalars have been estimated, but the corrections induced by chiral loops are not yet implemented (except for $\tau \rightarrow \nu_\tau 3\pi$ [136] and $\tau \rightarrow \nu_\tau 4\pi$ [137], at very low q^2). The present predictions correspond to the limit of an infinity number of QCD colours; the only subleading correction in the $1/N_C$ expansion which is taken into account is the finite width of the hadronic resonances. Although the accuracy of these R χ T approximations is still moderate, they provide a direct connection with the fundamental QCD theory and constitute a very valuable starting point to analyse the measured observables. Including suitable modifications to account phenomenologically for the contributions not yet included, one gets useful parametrizations to fit the experimental data in a sensible way, which nowadays are being incorporated into the TAUOLA library [213]. From the high-statistics τ decay (and e^+e^-) data samples, it is then possible to extract important information on the hadronic structure, allowing one to improve the theoretical tools and get a better control of the strong interaction in the resonance region.

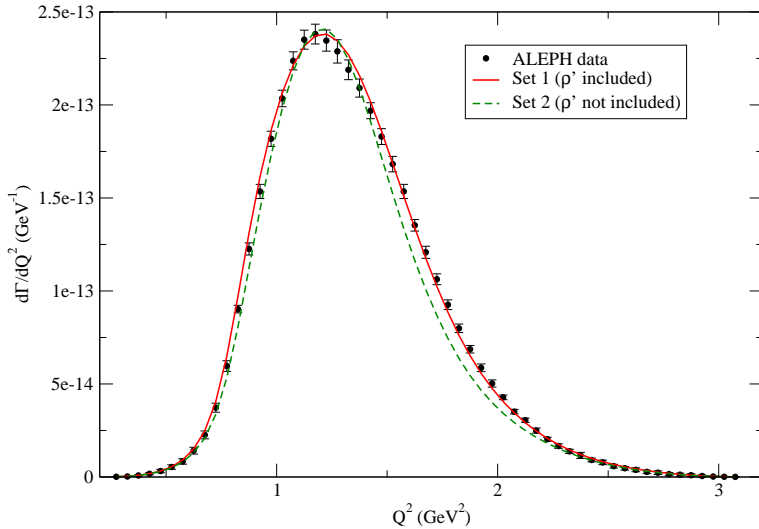


Figure 9: ALEPH $\tau^- \rightarrow \nu_\tau \pi^- \pi^+ \pi^-$ data [216] compared with the fitted theoretical distribution [214].

The first detailed studies of $\tau^- \rightarrow \nu_\tau \pi^+ \pi^- \pi^-$ and $\tau^- \rightarrow \nu_\tau \pi^0 \pi^0 \pi^-$ were made in Refs. [125, 140]. A $R\chi T$ analysis was later performed in Refs. [214, 215], including a theoretical description of the measured [106, 189, 216–218] $J^P = 1^+$ structure functions [128]. Fig. 8 shows the relevant tree-level Feynman diagrams with resonance exchanges. The theoretical analysis includes dynamical contributions from the $\rho(770)$, $\rho(1450)$ and $a_1(1260)$ resonances, taking into account their energy-dependent widths. This turns out to be an important physical effect for broad resonant structures, modifying in a sizeable way the values of their fitted mass and width parameters. The fit to the ALEPH data [216], shown in Fig. 9, gives as central values $M_{a_1} = 1.120$ GeV and $\Gamma_{a_1}(M_{a_1}^2) = 0.483$ GeV [214], quite different from the numbers quoted by the PDG [23] which do not take into account the energy dependence of the resonance width.

The $\tau^- \rightarrow \nu_\tau (3\pi)^-$ decay amplitudes contain two interfering contributions, corresponding to the exchange of the two identical pions in the final state, which generate a parity-violating angular asymmetry [125, 219, 220], making possible to determine the sign of the ν_τ helicity to be -1 [105–107].

Detailed phenomenological analyses exist for other decays such as $\tau \rightarrow \nu_\tau K \bar{K} \pi$ [221] and $\tau \rightarrow \nu_\tau K 2\pi$ [222], which involve both vector and axial-vector currents, and $\tau \rightarrow \eta^{(\prime)} 2\pi \nu_\tau$ [223], which is driven by the vector current (up to tiny isospin-breaking effects). The $R\chi T$ analysis of $\tau \rightarrow \nu_\tau K \bar{K} \pi$ [221] suggests that the vector contribution dominates in this mode, in agreement with the early study in Ref. [144], but at variance with Refs. [224, 225] which advocate a larger axial component. A precise experimental determination of the hadronic invariant-mass distribution would help to disentangle the two contributions. The vector-current amplitudes contributing to these decays can be also studied through the corresponding e^+e^- annihilation processes [226].

The $R\chi T$ techniques have been also applied to more involved transitions such as $e^+e^- \rightarrow 3\pi$ [226], $\tau \rightarrow \nu_\tau 4\pi$ and $e^+e^- \rightarrow 4\pi$ [137, 138] and radiative τ decays with one-meson in the final state [227, 228].

6.4.1 Experimental status

A big effort is underway to fully understand the rich pattern of hadronic decay modes of the τ [22, 23]. Tables 9 and 10 show the present world averages for the Cabibbo allowed and Cabibbo suppressed branching ratios, using the 40 “base modes” defined by the *Heavy Flavor Averaging Group* (HFAG) [22]. The huge data samples accumulated at the B factories have allowed for a sizeable reduction of the statistical errors, so systematic uncertainties dominate in most cases. The global HFAG fit to the world τ data has a $\chi^2/\text{d.o.f.} = 143.5/118$, corresponding to a confidence level $\text{CL} = 5.5\%$ [22]. Adding the two leptonic

Table 9: Cabibbo-allowed hadronic τ branching ratios [22]. h^\pm stands for π^\pm or K^\pm .

X^-	$\text{Br}(\tau^- \rightarrow \nu_\tau X^-)$ (%)	X^-	$\text{Br}(\tau^- \rightarrow \nu_\tau X^-)$ (%)
π^-	(10.811 ± 0.053)	$\pi^- \pi^+ \pi^-$ (ex. K^0, ω)	(8.9719 ± 0.0511)
$\pi^- \pi^0$	(25.504 ± 0.092)	$\pi^- \pi^+ \pi^- \pi^0$ (ex. K^0, ω)	(2.7659 ± 0.0710)
$\pi^- 2\pi^0$ (ex. K^0)	(9.2414 ± 0.0997)	$h^- h^+ h^- 2\pi^0$ (ex. K^0, ω, η)	(0.0973 ± 0.0354)
$\pi^- 3\pi^0$ (ex. K^0)	(1.0322 ± 0.0749)	$h^- h^+ h^- 3\pi^0$	(0.0320 ± 0.0031)
$h^- 4\pi^0$ (ex. K^0, η)	(0.1091 ± 0.0391)	$\pi^- K^- K^+$	(0.1435 ± 0.0027)
$K^- K^0$	(0.1591 ± 0.0157)	$\pi^- K^- K^+ \pi^0$	(0.0061 ± 0.0018)
$K^- K^0 \pi^0$	(0.1450 ± 0.0071)	$3h^- 2h^+$ (ex. K^0)	(0.0823 ± 0.0031)
$\pi^- K_S^0 K_S^0$	(0.0240 ± 0.0050)	$3h^- 2h^+ \pi^0$ (ex. K^0)	(0.0198 ± 0.0024)
$\pi^- K_S^0 K_L^0$	(0.1082 ± 0.0203)	$\pi^- \pi^0 \eta$	(0.1386 ± 0.0072)
$\pi^- K_L^0 K_L^0$	(0.0240 ± 0.0050)	$\pi^- \omega$	(1.9535 ± 0.0647)
$\pi^- K^0 \bar{K}^0 \pi^0$	(0.0310 ± 0.0230)	$h^- \pi^0 \omega$	(0.4049 ± 0.0418)
$a_1^- (\rightarrow \pi^- \gamma)$	(0.0400 ± 0.0200)		
$\text{Br}(\tau^- \rightarrow \nu_\tau X^-) = (61.85 \pm 0.11)\%$			

branching ratios, given in table 1, the sum of all fitted branching ratios is consistent with one [22],

$$1 - \sum_i \text{Br}(\tau^- \rightarrow \nu_\tau^- X_i) = (0.0704 \pm 0.1060)\%, \quad (59)$$

but the unitarity constraint is not used in the fit.

Several inconsistencies are known to exist in the τ branching fraction measurements, but the sources of these inconsistencies are unknown. In particular, the recent decrease of many experimental branching ratios is worrisome. There are 20 branching fractions measured at the B factories for which older measurements exist. As pointed out by the PDG [23], 18 of these 20 B-factory branching fractions are smaller than the previous non-B-factory values. The average normalized difference between the two sets of measurements is -1.30σ . Moreover, the BaBar and Belle results differ significantly for the 6 decay modes measured by both experiments. New measurements and refined analyses are clearly needed.

Recent progress includes the measurement of many high-multiplicity 3- and 5-prong decays [206], modes with K_S [$\pi^- K_S(\pi^0), \pi^- K_S K_S(\pi^0), K^- K_S(\pi^0)$] [229, 230] and analyses of hadronic distributions in decays into two and three mesons [230, 231].

7 The Inclusive τ Hadronic Width

The inclusive character of the total τ hadronic width renders possible [232–234] an accurate calculation of the ratio [(γ) represents additional photons or lepton pairs]

$$R_\tau \equiv \frac{\Gamma[\tau^- \rightarrow \nu_\tau \text{ hadrons}(\gamma)]}{\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma)]}, \quad (60)$$

using standard field theoretic methods. The theoretical analysis involves the two-point correlation functions for the vector $V_{ij}^\mu = \bar{\psi}_j \gamma^\mu \psi_i$ and axial-vector $A_{ij}^\mu = \bar{\psi}_j \gamma^\mu \gamma_5 \psi_i$ colour-singlet quark currents ($i, j = u, d, s$; $\mathcal{J} = V, A$):

$$\Pi_{ij, \mathcal{J}}^{\mu\nu}(q) \equiv i \int d^4 x e^{iqx} \langle 0 | T(\mathcal{J}_{ij}^\mu(x) \mathcal{J}_{ij}^\nu(0)^\dagger) | 0 \rangle, \quad (61)$$

Table 10: Cabibbo-suppressed hadronic τ branching ratios [22]. h^\pm stands for π^\pm or K^\pm .

X_s^-	$\text{Br}(\tau^- \rightarrow \nu_\tau X_s^-)$ (%)	X_s^-	$\text{Br}(\tau^- \rightarrow \nu_\tau X_s^-)$ (%)
K^-	(0.6955 ± 0.0096)	$K^- \eta$	(0.0153 ± 0.0008)
$K^- \pi^0$	(0.4322 ± 0.0149)	$K^- \pi^0 \eta$	(0.0048 ± 0.0012)
$K^- 2\pi^0$ (ex. K^0)	(0.0630 ± 0.0222)	$\pi^- \bar{K}^0 \eta$	(0.0094 ± 0.0015)
$K^- 3\pi^0$ (ex. K^0, η)	(0.0419 ± 0.0218)	$K^- \omega$	(0.0410 ± 0.0092)
$\pi^- \bar{K}^0$	(0.8206 ± 0.0182)	$K^- \phi$ ($\phi \rightarrow K\bar{K}$)	(0.0037 ± 0.0014)
$\pi^- \bar{K}^0 \pi^0$	(0.3649 ± 0.0108)	$K^- \pi^- \pi^+$ (ex. K^0, ω)	(0.2923 ± 0.0068)
$\pi^- \bar{K}^0 2\pi^0$	(0.0269 ± 0.0230)	$K^- \pi^- \pi^+ \pi^0$ (ex. K^0, ω, η)	(0.0411 ± 0.0143)
$\bar{K}^0 h^- h^+ h^-$	(0.0222 ± 0.0202)		
$\text{Br}(\tau^- \rightarrow \nu_\tau X_s^-) = (2.875 \pm 0.050)\%$			

which have the Lorentz decompositions

$$\Pi_{ij,\mathcal{J}}^{\mu\nu}(q) = (-g^{\mu\nu}q^2 + q^\mu q^\nu) \Pi_{ij,\mathcal{J}}^{(1)}(q^2) + q^\mu q^\nu \Pi_{ij,\mathcal{J}}^{(0)}(q^2), \quad (62)$$

where the superscript ($J = 0, 1$) denotes the angular momentum in the hadronic rest frame.

The imaginary parts of the correlators $\Pi_{ij,\mathcal{J}}^{(J)}(q^2)$ are proportional to the spectral functions for hadrons with the corresponding quantum numbers. The hadronic decay rate of the τ can be written as an integral of these spectral functions over the invariant mass s of the final-state hadrons:

$$R_\tau = 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im}\Pi^{(1)}(s) + \text{Im}\Pi^{(0)}(s) \right]. \quad (63)$$

The appropriate combinations of correlators are

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2 \left(\Pi_{ud,V}^{(J)}(s) + \Pi_{ud,A}^{(J)}(s) \right) + |V_{us}|^2 \left(\Pi_{us,V}^{(J)}(s) + \Pi_{us,A}^{(J)}(s) \right). \quad (64)$$

We can separate the inclusive contributions associated with specific quark currents:

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}. \quad (65)$$

$R_{\tau,V}$ and $R_{\tau,A}$ correspond to the first two terms in (64), while $R_{\tau,S}$ contains the remaining Cabibbo-suppressed contributions. Non-strange hadronic decays of the τ are resolved experimentally into vector ($R_{\tau,V}$) and axial-vector ($R_{\tau,A}$) contributions according to whether the hadronic final state includes an even or odd number of pions. Strange decays ($R_{\tau,S}$) are of course identified by the presence of an odd number of kaons in the final state.

Since the hadronic spectral functions are sensitive to the non-perturbative effects of QCD that bind quarks into hadrons, the integrand in Eq. (63) cannot be calculated at present from QCD. Nevertheless the integral itself can be calculated systematically by exploiting the analytic properties of the correlators $\Pi^{(J)}(s)$. They are analytic functions of s except along the positive real s axis, where their imaginary parts have discontinuities. Using the closed contour in Fig. 10, R_τ can then be expressed as a contour integral in the complex s plane running counter-clockwise around the circle $|s| = m_\tau^2$ [232–234]:

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(0+1)}(s) - 2\frac{s}{m_\tau^2} \Pi^{(0)}(s) \right]. \quad (66)$$

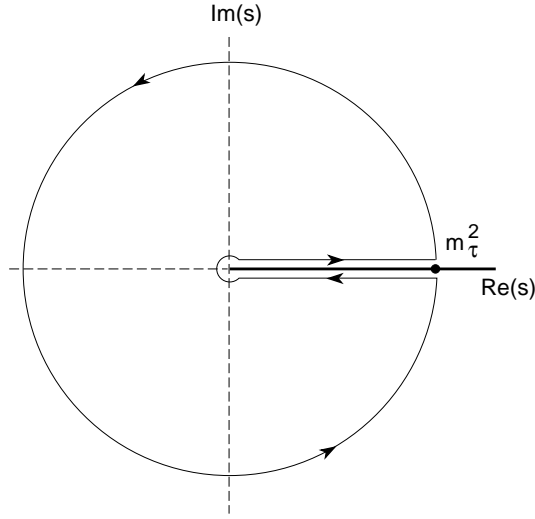


Figure 10: Integration contour in the complex s plane, used to obtain Eq. (66).

The advantage of expression (66) over (63) for R_τ is that it requires the correlators only for complex s of order m_τ^2 , which is significantly larger than the scale associated with non-perturbative effects in QCD. The short-distance Operator Product Expansion (OPE) can therefore be used to organize the perturbative and non-perturbative contributions to the correlators into a systematic expansion in powers of $1/s$ [235],

$$\Pi^{(J)}(s) = \sum_{D=2n} \sum_{\dim \mathcal{O}=D} \frac{\mathcal{C}^{(J)}(s, \mu) \langle \mathcal{O}(\mu) \rangle}{(-s)^{D/2}}, \quad (67)$$

where the inner sum is over local gauge-invariant scalar operators of dimension $D = 0, 2, 4, \dots$. The parameter μ is an arbitrary factorization scale, which separates long-distance non-perturbative effects, which are absorbed into the vacuum matrix elements $\langle \mathcal{O}(\mu) \rangle$, from short-distance effects, which are included in the Wilson coefficients $\mathcal{C}^{(J)}(s, \mu)$. The $D = 0$ term (unit operator) corresponds to the pure perturbative contributions, neglecting quark masses. The leading quark-mass corrections generate the $D = 2$ term. The first dynamical operators involving non-perturbative physics appear at $D = 4$.

Inserting the functions (67) into (66) and evaluating the contour integral, R_τ can be expressed as an expansion in powers of $1/m_\tau^2$, with coefficients that depend only logarithmically on m_τ . The uncertainties associated with the use of the OPE near the time-like axis are heavily suppressed because the integrand in Eq. (66) includes a factor $(1 - s/m_\tau^2)^2$, which provides a double zero at $s = m_\tau^2$, making negligible the contribution from the region near the branch cut.

It is convenient to express the corrections to R_τ from dimension- D operators in terms of the fractional corrections $\delta_{ij, \mathcal{J}}^{(D)}$ to the naive contribution from the current with quantum numbers ij, \mathcal{J} [234]:

$$R_{\tau, V/A} = \frac{3}{2} |V_{ud}|^2 S_{\text{EW}} \left(1 + \delta_P + \sum_{D=2,4,\dots} \delta_{ud, V/A}^{(D)} \right), \quad (68)$$

$$R_{\tau, S} = 3 |V_{us}|^2 S_{\text{EW}} \left(1 + \delta_P + \sum_{D=2,4,\dots} \delta_{us}^{(D)} \right), \quad (69)$$

where $\delta_{ij}^{(D)} = (\delta_{ij, V}^{(D)} + \delta_{ij, A}^{(D)})/2$ is the average of the vector and axial-vector corrections, the factor $S_{\text{EW}} = 1.0201 \pm 0.0003$ contains the renormalization-group improved electroweak correction [28, 120, 121], and $\delta_P = \delta_{ij, \mathcal{J}}^{(0)}$ is the purely perturbative QCD correction, neglecting quark masses, which is the same for all the components of R_τ .

Table 11: Perturbative coefficients of the Adler function expansion and FOPT approximation to δ_P .

n	1	2	3	4	5
K_n	1	1.6398	6.3710	49.0757	
g_n	0	3.5625	19.9949	78.0029	307.78
r_n	1	5.2023	26.3659	127.079	

In the chiral limit ($m_u = m_d = m_s = 0$), the vector and axial-vector currents are conserved. This implies $s\Pi^{(0)}(s) = 0$. Therefore, only the correlator $\Pi^{(0+1)}(s)$ contributes to Eq. (66). Since $(1-x)^2(1+2x) = 1 - 3x^2 + 2x^3$ [$x \equiv s/m_\tau^2$], Cauchy's theorem guarantees that, up to tiny logarithmic running corrections, the only non-perturbative contributions to the circle integration in (66) originate from operators of dimensions $D = 6$ and 8 . The usually leading $D = 4$ operators can only contribute to R_τ with an additional suppression factor of $\mathcal{O}(\alpha_s^2)$, which makes their effect negligible [234].

The Cabibbo-allowed component of R_τ ,

$$R_{\tau,V+A} = N_C |V_{ud}|^2 S_{\text{EW}} \{1 + \delta_P + \delta_{\text{NP}}\}, \quad (70)$$

is then a very clean observable to test perturbative QCD. Quark mass effects are tiny and the non-perturbative correction $\delta_{\text{NP}} = \sum_{D \geq 2} \delta_{ud}^{(D)}$ is heavily suppressed by six powers of the τ mass. The dominant contribution is the perturbative correction $\delta_P \sim 20\%$, which turns out to be very sensitive to α_s , allowing for an accurate determination of the fundamental QCD coupling [232, 234]. The recent calculation of the $\mathcal{O}(\alpha_s^4)$ contribution to $\Pi^{(0+1)}(s)$ [236] has triggered a renewed theoretical interest on the $\alpha_s(m_\tau^2)$ determination [17, 225, 237–249], since it allows to push the accuracy to the four-loop level.

7.1 Perturbative Contribution to R_τ

In the chiral limit, the result is more conveniently expressed in terms of the logarithmic derivative of the two-point correlation function of the vector (axial) current, $\Pi(s) = \Pi_{i \neq j, V}^{(0+1)}(s) = \Pi_{i \neq j, A}^{(0+1)}(s)$, the so-called Adler function [250], which satisfies an homogeneous renormalization-group equation:

$$D(s) \equiv -s \frac{d}{ds} \Pi(s) = \frac{N_C}{12\pi^2} \left\{ 1 + \sum_{n=1} K_n \left(\frac{\alpha_s(-s)}{\pi} \right)^n \right\}. \quad (71)$$

With the choice of renormalization scale $\mu^2 = -s$, all logarithmic corrections proportional to powers of $\log(-s/\mu^2)$ have been summed into the running coupling. For three active quark flavours and in the $\overline{\text{MS}}$ scheme, the known ($n \leq 4$) K_n coefficients take the values indicated in table 11 [236, 251, 252].

The perturbative component of R_τ is given by

$$\delta_P = \sum_{n=1} K_n A^{(n)}(\alpha_s) = \sum_{n=1} (K_n + g_n) a_\tau^n \equiv \sum_{n=1} r_n a_\tau^n, \quad (72)$$

where the functions [253]

$$A^{(n)}(\alpha_s) = \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{s} \left(\frac{\alpha_s(-s)}{\pi} \right)^n \left(1 - 2 \frac{s}{m_\tau^2} + 2 \frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8} \right) \quad (73)$$

are contour integrals in the complex plane, which only depend on $a_\tau \equiv \alpha_s(m_\tau^2)/\pi$. Using the exact solution (up to unknown $\beta_{n>4}$ contributions) for $\alpha_s(-s)$ given by the renormalization-group β -function equation, they can be numerically computed with a very high accuracy [253]. Table 12 gives the

Table 12: Exact results for $A^{(n)}(\alpha_s)$ ($n \leq 4$) at different β -function approximations, and corresponding values of $\delta_P = \sum_{n=1}^4 K_n A^{(n)}(\alpha_s)$, for $a_\tau \equiv \alpha_s(m_\tau^2)/\pi = 0.11$. The last row shows the FOPT estimates at $\mathcal{O}(a_\tau^4)$.

	$A^{(1)}(\alpha_s)$	$A^{(2)}(\alpha_s)$	$A^{(3)}(\alpha_s)$	$A^{(4)}(\alpha_s)$	δ_P
$\beta_{n>1} = 0$	0.14828	0.01925	0.00225	0.00024	0.20578
$\beta_{n>2} = 0$	0.15103	0.01905	0.00209	0.00020	0.20537
$\beta_{n>3} = 0$	0.15093	0.01882	0.00202	0.00019	0.20389
$\beta_{n>4} = 0$	0.15058	0.01865	0.00198	0.00018	0.20273
$\mathcal{O}(a_\tau^4)$	0.16115	0.02431	0.00290	0.00015	0.22665

numerical values for $A^{(n)}(\alpha_s)$ ($n \leq 4$) obtained at the one-, two-, three- and four-loop approximations (i.e. $\beta_{n>1} = 0$, $\beta_{n>2} = 0$, $\beta_{n>3} = 0$ and $\beta_{n>4} = 0$, respectively), together with the corresponding results for $\delta_P = \sum_{n=1}^4 K_n A^{(n)}(\alpha_s)$, taking $a_\tau = 0.11$. The perturbative convergence is very good and the results are stable under changes of the renormalization scale.

The integrals $A^{(n)}(\alpha_s)$ can be expanded in powers of a_τ , $A^{(n)}(\alpha_s) = a_\tau^n + \mathcal{O}(a_\tau^{n+1})$. One obtains in this way the naive perturbative expansion given in the r.h.s of Eq. (72). This approximation is known as *fixed-order perturbation theory* (FOPT), while the improved expression, keeping the non-expanded values of $A^{(n)}(\alpha_s)$, is usually called *contour-improved perturbation theory* (CIPT) [253, 254].

As shown in the last row of table 12, even at $\mathcal{O}(a_\tau^4)$, FOPT gives a rather bad approximation to the integrals $A^{(n)}(\alpha_s)$, overestimating δ_P by 11% at $a_\tau = 0.11$. The long running of $\alpha_s(-s)$ along the circle $|s| = m_\tau^2$ generates the very large g_n coefficients, given in table 11, which depend on $K_{m<n}$ and $\beta_{m<n}$ [253]. These corrections are much larger than the original K_n contributions. The origin of this bad behaviour can be understood analytically at one loop [253]. In FOPT one makes within the contour integral the series expansion $(\log(-s/m_\tau^2) = i\phi, \phi \in [-\pi, \pi])$

$$\frac{\alpha_s(-s)}{\pi} \approx \frac{a_\tau}{1 - i\beta_1 a_\tau \phi/2} \approx a_\tau \sum_n \left(\frac{i}{2} \beta_1 a_\tau \phi \right)^n, \quad (74)$$

which is only convergent for $a_\tau < 0.14$. At the four-loop level the radius of convergence is slightly smaller than the physical value of a_τ . Thus, FOPT gives rise to a non-convergent series, which suffers from a large renormalization-scale dependence. The long running along the circle makes compulsory to resum the large logarithms, $\log^n(-s/m_\tau^2)$, using the renormalization group. This is precisely what CIPT does, and results in a well-behaved perturbative series with a mild dependence on the renormalization scale.

7.1.1 Higher-order perturbative contributions and renormalon hypothesis

A lot of effort has been devoted to estimate the size of the unknown higher-order corrections to δ_P [17, 225, 237–242, 247, 248, 255–265]. The perturbative expansion of the Adler function is expected to be an asymptotic series. If its Borel transform, $B(t) \equiv \sum_{n=0} K_{n+1} t^n/n!$, were well-behaved, one could define $D(s)$ through the Borel integral

$$D(s) = \frac{1}{4\pi^2} \left\{ 1 + \int_0^\infty dt e^{-\pi t/\alpha_s(s)} B(t) \right\}. \quad (75)$$

However, $B(t)$ has pole singularities at positive (infrared renormalons) and negative (ultraviolet renormalons) integer values of the variable $u \equiv -\beta_1 t/2$, with the exception of $u = 1$ [266]. The infrared renormalons at $u = +n$ are related to OPE corrections of dimension $D = 2n$; an infrared singularity

at $u = +1$ does not exist because there are no gauge-invariant operators with $D = 2$. The renormalon poles closer to the origin dominate the large-order behaviour of $D(s)$.

It has been argued that, once in the asymptotic regime (large n), the renormalonic behaviour of the K_n coefficients could induce cancellations with the running g_n corrections, which would be missed by CIPT. In that case, FOPT could approach faster the ‘true’ result provided by the Borel summation of the full renormalon series [237]. This happens actually in the large- β_1 limit [257–259], which however does not approximate well the known K_n coefficients. A large $u = 2$ renormalon effect could also induce such a cancellation, because the contour integral eliminates contributions from $D = 4$ operators. Models of higher-order corrections with this behaviour have been advocated, mixing different types of renormalons ($n = -1, 2$ and 3) plus a linear polynomial, and fitting the free parameters to the known $K_{n \leq 4}$ coefficients with the assumption $K_5 = 283$ [237]. For a given value of $\alpha_s(m_\tau^2)$, one gets in this way a larger δ_P than in CIPT. The result looks however model dependent [260, 261].

The implications of a renormalonic behaviour have also been studied in a more model-independent way, using an optimal conformal mapping in the Borel plane, which achieves the best asymptotic rate of convergence, and properly implementing the CIPT procedure within the Borel transform [239–241]. Assuming that the known fourth-order series is already governed by the $u = -1$ and $u = 2$ renormalons, the conformal mapping generates a full series expansion ($K_5 = 256, K_6 = 2929 \dots$) which results, after Borel summation, in a larger value of δ_P ; i.e., the $K_{n > 4}$ terms give a positive contribution to δ_P . One obtains then numerical results close to the FOPT value.

Renormalons provide an interesting guide to possible higher-order corrections, making apparent that the associated uncertainties have to be carefully estimated. However, one should keep in mind the adopted assumptions. In fact, there are no visible signs of renormalonic behaviour in the presently known series: the $n = -1$ ultraviolet renormalon is expected to dominate the asymptotic regime, implying an alternating series, while all known K_n coefficients have the same sign. One could either assume that renormalons only become relevant at higher orders, for instance at $n = 7$, and apply the conformal mapping with arbitrary input values for K_5 and K_6 . Different assumptions about these two unknown coefficients would result in different predicted values for δ_P .

A different reshuffling of the perturbative series, not related to renormalons, can be obtained [247] through an expansion in terms of the β function and its derivatives, instead of the usual expansion in powers of the strong coupling. This procedure results in a different estimate of higher-order corrections, leading to a weaker dependence on the renormalization scale and a value of δ_P similar to the standard CIPT result.

7.2 Non-Perturbative Corrections

Compared with the perturbative uncertainties, the power-suppressed contributions to R_τ are quite small. Quark mass effects [234, 267–269] are tiny for the Cabibbo-allowed current and amount to a negligible correction smaller than 10^{-4} . Non-perturbative effects are suppressed by six powers of the τ mass [234]. Moreover, the $D = 6$ contributions to the vector and axial-vector correlators are expected to have opposite signs (the vacuum-saturation approximation gives $\delta_{ud,V}^{(6)}/\delta_{ud,A}^{(6)} \sim -7/11$), leading to a partial cancellation in $\delta_{ud}^{(6)}$. Thus, the non-perturbative contribution to the total $(V + A)$ Cabibbo-allowed decay width is smaller than the separate vector and axial-vector corrections. The estimated theoretical value is $\delta_{NP} = -0.007 \pm 0.004$ [234].

The numerical size of non-perturbative effects can be determined from the measured invariant-mass distribution of the final hadrons in τ decay [270]. Although the distributions themselves cannot be predicted at present, certain weighted integrals of the hadronic spectral functions can be calculated in

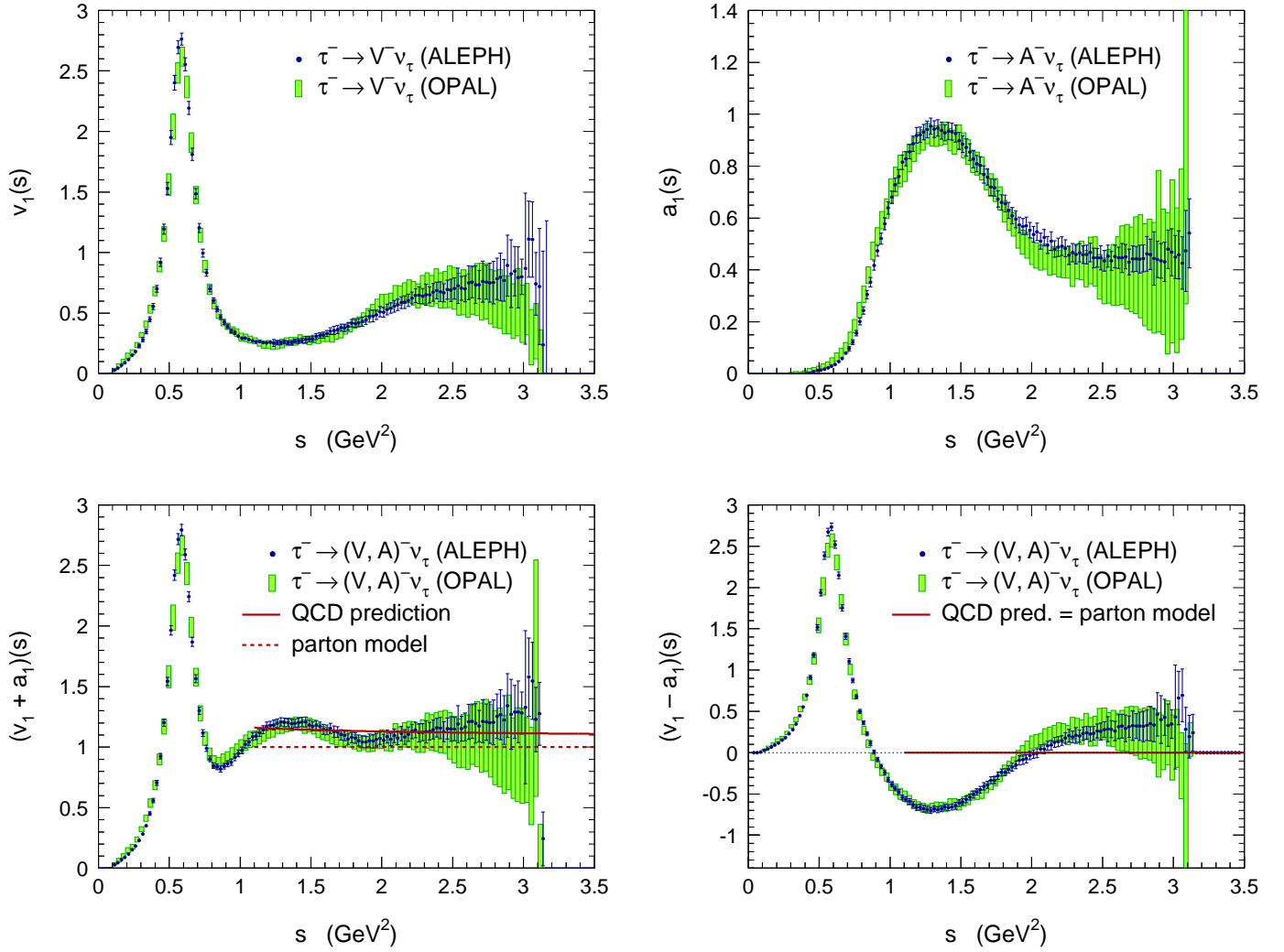


Figure 11: Vector (upper-left) and axial (upper-right) spectral functions measured by ALEPH [189] (blue data points), compared with the corresponding OPAL distributions [271] (green rectangular blocks). The lower plots show the sum (left) and difference (right) of the vector and axial spectral functions, together with the perturbative QCD predictions (continuous red lines). Figures taken from Ref. [17].

the same way as R_τ . The analyticity properties of $\Pi_{ij,\mathcal{J}}^{(J)}$ imply [118,270]:

$$\int_0^{s_0} ds w(s) \text{Im}\Pi_{ij,\mathcal{J}}^{(J)} = \frac{i}{2} \oint_{|s|=s_0} ds w(s) \Pi_{ij,\mathcal{J}}^{(J)}, \quad (76)$$

with $w(s)$ an arbitrary weight function without singularities in the region $|s| \leq s_0$. Generally speaking, the accuracy of the theoretical predictions can be much worse than the one of R_τ , because non-perturbative effects are not necessarily suppressed; in fact, they can become very sizeable with appropriately chosen weight functions. But this is precisely what makes these integrals interesting: they can be used to measure the parameters characterizing the non-perturbative dynamics and perform relevant tests of QCD in the strong-coupling regime. Notice that weights of the form $(s/m_\tau^2)^n$ project the OPE contribution of dimension $D = 2n + 2$.

To perform an experimental analysis, it is convenient to use moments of the directly measured invariant-mass distribution, for instance [270] ($k, l \geq 0$)

$$R_{\tau,V+A}^{kl}(s_0) \equiv \int_0^{s_0} ds \left(1 - \frac{s}{s_0}\right)^k \left(\frac{s}{m_\tau^2}\right)^l \frac{dR_{\tau,V+A}}{ds}. \quad (77)$$

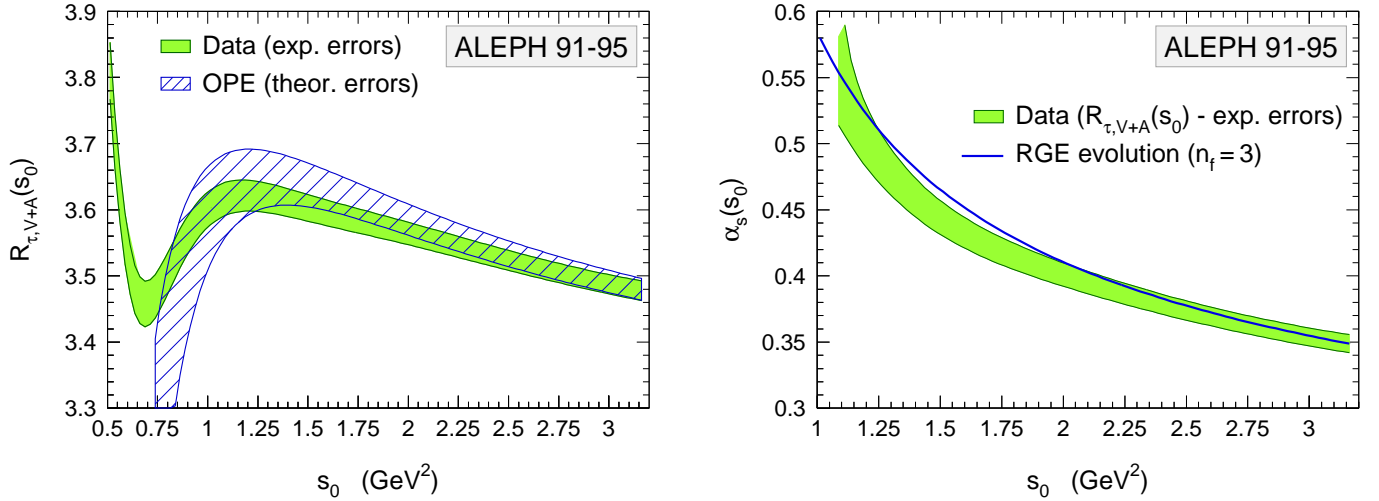


Figure 12: Dependence on s_0 of $R_{\tau, V+A}(s_0)$ (left) and $\alpha_s(s_0)$ (right) [17]. The ALEPH fit (green bands) is compared with the theoretical CIPT prediction.

The factor $(1 - s/s_0)^k$ supplements $(1 - s/m_\tau^2)^2$ for $s_0 \neq m_\tau^2$, in order to squeeze the integrand at the crossing of the positive real axis and, therefore, improves the reliability of the OPE analysis; moreover, for $s_0 = m_\tau^2$ it reduces the contribution from the tail of the distribution, which is badly defined experimentally. A combined fit of different $R_{\tau, V+A}^{kl}(m_\tau^2)$ moments results in experimental values for $\alpha_s(m_\tau^2)$ and for the coefficients of the inverse power corrections in the OPE. $R_{\tau, V+A}^{00}(m_\tau^2) = R_{\tau, V+A}$ uses the overall normalization of the hadronic distribution, while the ratios $D_{\tau, V+A}^{kl}(m_\tau^2) = R_{\tau, V+A}^{kl}(m_\tau^2)/R_{\tau, V+A}$ are based on the shape of the s distribution and are more dependent on non-perturbative effects [270].

The spectral functions measured by ALEPH [189] and OPAL [271] are shown in Figs. 11, with the normalization $v_1(s) = 2\pi \text{Im}\Pi_{ud, V}^{(1)}(s)$ and $a_1(s) = 2\pi \text{Im}\Pi_{ud, A}^{(1)}(s)$. There is good agreement between both experiments, although the ALEPH results are more precise. The errors increase in the high s region because of the kinematical phase-space suppression close to the end-point of the τ decay distribution. The ρ and a_1 resonances are clearly visible. One also observes the expected smearing of resonance structures above 1 GeV in the $V + A$ distribution, where the data approaches the perturbative QCD prediction. The opposite behaviour is seen in the $V - A$ spectral function, which is a purely non-perturbative quantity. The total $V + A$ distribution was previously studied by CLEO [272], but with much larger uncertainties.

The predicted suppression [234] of the non-perturbative corrections to R_τ has been confirmed by ALEPH [189, 216, 217], CLEO [272] and OPAL [271]. The presently most complete and precise experimental analysis, performed with the ALEPH data, obtains [17, 225]

$$\delta_{\text{NP}} = -0.0059 \pm 0.0014, \quad (78)$$

in good agreement with the theoretical expectations and previous experimental determinations. Larger non-perturbative effects have been found for the separate currents, $\delta_{\text{NP}}^V = 0.0189 \pm 0.0025$ and $\delta_{\text{NP}}^A = -0.0311 \pm 0.0016$ [225], with the expected size and signs. Using the CIPT prescription, the fit to the whole ALEPH distribution gives $\alpha_s(m_\tau^2) = 0.344 \pm 0.009$, while the separate vector and axial data result in $\alpha_s(m_\tau^2) = 0.347 \pm 0.010$ and $\alpha_s(m_\tau^2) = 0.335 \pm 0.011$, respectively [225].

Evaluating the integral $R_{\tau, V+A}^{00}(s_0)$ with an arbitrary upper limit of integration $s_0 \leq m_\tau^2$, one can test that the experimental s_0 dependence agrees well with the theoretical predictions up to rather low values of s_0 [17, 273]. Equivalently, from the measured $R_\tau^{00}(s_0)$ distribution one obtains $\alpha_s(s_0)$ as a function of the scale s_0 . As shown in Fig. 12, the result exhibits an impressive agreement with the running predicted at four-loop order by QCD. A similar test was performed longtime ago for $R_{\tau, V}$, using the vector spectral function measured in $e^+e^- \rightarrow \text{hadrons}$, and varying the value of the τ mass [127].

All previous analyses were based on weight functions proportional to powers of $(1 - s/s_0)^k$ (pinched weights), in order to suppress the OPE uncertainties near the real axis [17, 225, 261, 274–278]. Using instead non-pinched weights, one can also analyse the role of duality violations in generic moments of the spectral distributions. These effects are negligible in $R_{\tau, V+A}$, since they are smaller than the errors induced by δ_{NP} which are in turn subdominant with respect to the leading perturbative uncertainties; however, they could be more relevant for other moments. Although the present data are not good enough to perform accurate tests, interesting results start to emerge. A recent fit to rescaled OPAL data (adjusted to reflect current values of exclusive hadronic τ -decay branching ratios), with moments chosen to maximize duality violations, finds $\delta_{\text{NP}} = -0.003 \pm 0.012$ and $\alpha_s(m_\tau^2) = 0.333 \pm 0.018$ (CIPT+FOPT) [244], in agreement but less precise than the result obtained from the ALEPH invariant-mass distribution.

7.3 Updated Determination of the QCD Coupling

The measured τ leptonic branching fractions imply a quite precise value for the normalized hadronic decay width: $R_\tau = (1 - B_\mu - B_e)/B_e = 3.636 \pm 0.011$. A slightly more accurate number can be obtained assuming lepton universality, *i.e.*, the relations in Eq. (8). The information from the τ lifetime and leptonic branching fractions can then be combined to derive an improved value for the electron branching ratio, given in Eq. (9), and R_τ :

$$R_\tau^{\text{uni}} = \frac{1}{B_e^{\text{uni}}} - 1.972559 = 3.6397 \pm 0.0076. \quad (79)$$

The ratio R_τ can be also extracted from the sum of all hadronic decay modes. Using B_e^{uni} and accounting for all statistical correlations [22], one finds

$$R_\tau = \frac{\sum_{\text{had}} \text{Br}(\tau^- \rightarrow \nu_\tau X_{\text{had}}^-)}{B_e^{\text{uni}}} = 3.6326 \pm 0.0084, \quad (80)$$

in good agreement with the previous value and with a similar error. The advantage of this last procedure is that, using the information given in tables 9 and 10, one can separate the Cabibbo-allowed and Cabibbo-suppressed components:

$$R_{\tau, V+A} = 3.4712 \pm 0.0079, \quad R_{\tau, S} = 0.1614 \pm 0.0028. \quad (81)$$

Subtracting $R_{\tau, S}$ from R_τ^{uni} in (79), results in $R_{\tau, V+A} = 3.4783 \pm 0.0081$, consistent with (81) and of similar accuracy; however, this number would absorb the effect of any unobserved hadronic states (or unaccounted systematics) entirely in $R_{\tau, V+A}$, while they could also be strange final states. Therefore, we follow the procedure advocated by the HFAG [22] and use the direct determination of $R_{\tau, V+A}$ in Eq. (81). Taking $V_{ud} = 0.97425 \pm 0.00022$ [164], it implies

$$\delta_{\text{P}} + \delta_{\text{NP}} = 0.1950 \pm 0.0028. \quad (82)$$

Subtracting now the small non-perturbative contribution in Eq. (78), one obtains

$$\delta_{\text{P}} = 0.2009 \pm 0.0031. \quad (83)$$

The main uncertainty in the τ determination of the strong coupling originates in the treatment of higher-order perturbative corrections. We estimate them [248] adding the fifth-order term $K_5 A^{(5)}(\alpha_s)$ with $K_5 = 275 \pm 400$; the central value is in the range favoured by renormalon models, but our conservative uncertainty is large enough to allow for a correction of opposite sign. We also include the 5-loop variation with changes of the renormalization scale in the range $\mu^2/(-s) \in [0.4, 2.0]$. The

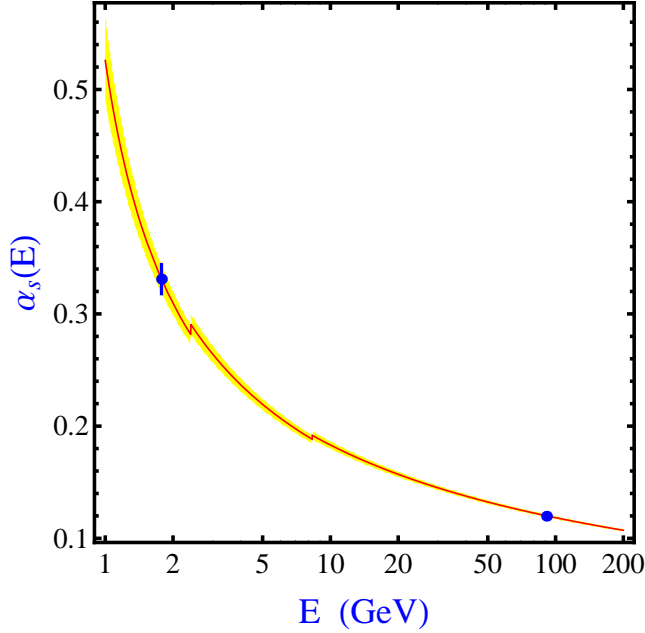


Figure 13: Evolution of the strong coupling $\alpha_s(m_\tau^2)$ to higher energies and comparison with the direct measurement at $s = M_Z^2$.

error induced by the truncation of the β function at fourth-order is estimated through the variation of the results at five loops, assuming $\beta_5 = \pm\beta_4^2/\beta_3 = \mp 443$; in CIPT this slightly changes the values of $A^{(n)}(\alpha_s)$, while in FOPT it increases the scale dependence. The FOPT procedure shows as expected a much more sizeable μ dependence, but it is less sensitive to the uncertainties on δ_P and K_5 . We add linearly the three theoretical uncertainties (K_5 , μ , β_5) and combine their sum in quadrature with the ‘experimental’ error on δ_P .

Using CIPT one gets $\alpha_s(m_\tau^2) = 0.341 \pm 0.013$, while FOPT would give $\alpha_s(m_\tau^2) = 0.319 \pm 0.014$ [249]. Combining the two results, but keeping conservatively the smallest error, we get

$$\alpha_s^{(n_f=3)}(m_\tau^2) = 0.331 \pm 0.013, \quad (84)$$

significantly larger (16σ) than the result obtained from the Z hadronic width, $\alpha_s^{(n_f=5)}(M_Z^2) = 0.1197 \pm 0.0028$ [23] (n_f denotes the relevant number of quark flavours at the given energy scale). After evolution up to the scale M_Z [279–283], the strong coupling in (84) decreases to $\alpha_s^{(n_f=5)}(M_Z^2) = 0.1200 \pm 0.0015$, in excellent agreement with the direct measurement at the Z peak. The comparison of these two determinations, shown in Fig. 13, provides a beautiful test of the predicted QCD running; *i.e.*, a very significant experimental verification of asymptotic freedom:

$$\alpha_s^{(n_f=5)}(M_Z^2) \Big|_\tau - \alpha_s^{(n_f=5)}(M_Z^2) \Big|_Z = 0.0003 \pm 0.0015_\tau \pm 0.0028_Z. \quad (85)$$

7.4 Chiral Sum Rules

For massless quarks, the chiral invariance of QCD guarantees that the two-point correlation function of a left-handed and a right-handed quark currents, $\Pi_{LR}(s) = \Pi_{ud,V}^{(0+1)}(s) - \Pi_{ud,A}^{(0+1)}(s)$, vanishes identically to all orders in perturbation theory. The non-zero value of $\Pi_{LR}(s)$ originates in the spontaneous breaking of chiral symmetry by the QCD vacuum. At large momenta, the corresponding OPE only receives contributions from operators with dimension $D \geq 6$, which implies $\lim_{s \rightarrow \infty} s^2 \Pi_{LR}(s) = 0$. The non-zero

up and down quark masses induce tiny corrections with dimensions two and four, which are negligible at high energies. At very low momenta, χ PT dictates the low-energy expansion of $\Pi_{LR}(s)$ in terms of the pion decay constant and the χ PT couplings L_{10} [$\mathcal{O}(p^4)$] and C_{87} [$\mathcal{O}(p^6)$]; the needed expressions are known to two loops in the chiral expansion [284].

Analyticity relates the short- and long-distance regimes through the dispersion relation

$$\frac{1}{2\pi i} \oint_{|s|=s_0} ds w(s) \Pi_{LR}(s) = - \int_{s_{th}}^{s_0} ds w(s) \rho(s) + 2f_\pi^2 w(m_\pi^2) + \text{Res}[w(s) \Pi_{LR}(s), s=0], \quad (86)$$

where $\rho(s) \equiv \frac{1}{\pi} \text{Im}\Pi_{LR}(s)$ and $w(s)$ is an arbitrary weight function that is analytic in the whole complex plane except at the origin, where it can have poles. The last term in (86) accounts for the possible residue at the origin.

For $s_0 \leq m_\tau^2$, the integral along the real axis can be evaluated with the measured tau spectral functions. Taking $w(s) = s^n$ with $n \geq 0$, there is no residue at the origin and, with s_0 large enough so that the OPE can be applied in the entire circle $|s| = s_0$, the OPE coefficients are directly related to the spectral function integration. With $n = 0$ and 1, there is no OPE contribution in the chiral limit and one gets the celebrated first and second Weinberg sum rules [285]:

$$\lim_{s_0 \rightarrow \infty} I_1(s_0) = f_\pi^2, \quad \lim_{s_0 \rightarrow \infty} I_2(s_0) = f_\pi^2 m_\pi^2 \approx 0. \quad (87)$$

where

$$I_{n+1}(s_0) = \frac{1}{2} \int_0^{s_0} ds s^n \rho(s). \quad (88)$$

For negative values of the integer n , the OPE does not contribute either while the residues at zero are determined by the χ PT low-energy couplings, which can be then experimentally determined [286].

The integrals $I_1(s_0)$ and $I_2(s_0)$, calculated with ALEPH [17, 189] and OPAL [271] data, are shown in Figs. 14. Although there is good agreement between the two sets of spectral functions, the ALEPH data are more precise, specially in the higher s_0 region close to the kinematical τ decay end-point. One clearly observes the ρ and a_1 resonance structures, which generate oscillations around the expected asymptotic value. Within their large uncertainties, the OPAL data are already consistent with (87) at $s_0 = 3 \text{ GeV}^2$; however, the more precise ALEPH data indicate that asymptotics is not yet reached. Therefore, additional information on the spectral function above m_τ^2 is required.

The absence of perturbative contributions makes the relation (86) and ideal tool to investigate possible quark-hadron duality effects, formally defined through [276, 277]

$$\text{DV}_w \equiv \frac{1}{2\pi i} \oint_{|s|=s_0} ds w(s) [\Pi_{LR}(s) - \Pi_{LR}^{\text{OPE}}(s)] = \int_{s_0}^{\infty} ds w(s) \rho(s). \quad (89)$$

This has been thoroughly studied in Refs. [287], using for the spectral function beyond $s_z \sim 2.1 \text{ GeV}$ the parametrization [276, 277]

$$\rho(s \geq s_z) = \kappa e^{-\gamma s} \sin(\beta(s - s_z)), \quad (90)$$

and finding the region in the 4-dimensional $(\kappa, \gamma, \beta, s_z)$ parameter space that is compatible with the most recent experimental ALEPH data [189], the first and second Weinberg sum rules [285] in (87) and the sum rule of Das et al. [288] that determines the pion electromagnetic mass difference:

$$\int_0^{\infty} ds s \log\left(\frac{s}{\Lambda^2}\right) \rho(s)|_{m_q=0} = (m_{\pi^0}^2 - m_{\pi^\pm}^2)_{\text{EM}} \frac{8\pi}{3\alpha} f^2. \quad (91)$$

The second Weinberg sum rule guarantees that this last integral does not depend on the arbitrary mass scale Λ inside the logarithm.

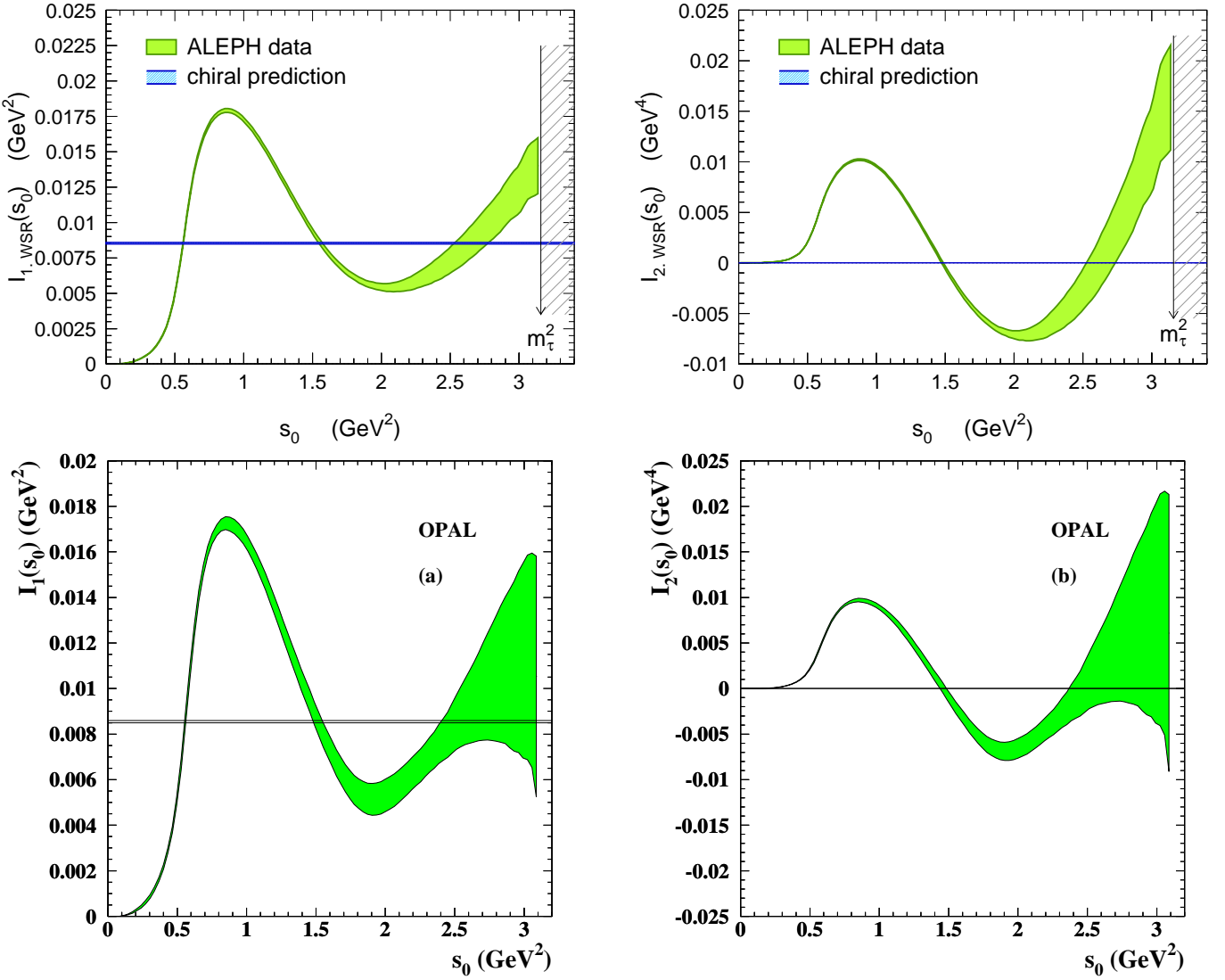


Figure 14: First (left) and second (right) Weinberg Sum Rules as a function of s_0 , using the ALEPH [17,189] (upper plots) and OPAL [271] (lower plots) spectral functions. The horizontal lines show the expected asymptotic result at $s_0 \rightarrow \infty$.

Refs. [287] perform a statistical analysis, scanning the parameter space $(\kappa, \gamma, \beta, s_z)$ and selecting those acceptable spectral functions which satisfy the experimental and theoretical constraints. The differences among them determine how much freedom is left for the behaviour of the physical spectral function beyond the kinematical end of the τ data. For each acceptable spectral function one calculates the parameters L_{10} , C_{87} , \mathcal{O}_6 and \mathcal{O}_8 , obtained through the dispersion relation (86) with appropriate weight functions, where \mathcal{O}_D is the coefficient of the $1/(-s)^{D/2}$ term in the OPE of $\Pi_{LR}(s)$. The resulting statistical distributions determine their finally estimated values; the dispersion of the numerical results provides a good quantitative assessment of the actual uncertainties. Pinched weight functions of the form $w(s) = s^n(s - s_z)^m$ ($m > 0$), which vanish at $s = s_z$, are found to minimize the uncertainties from duality-violation effects, allowing for a more precise determination of the hadronic parameters; one obtains [287]:

$$L_{10}^r(M_\rho) = -(4.06 \pm 0.39) \times 10^{-3}, \quad C_{87}^r(M_\rho) = (4.89 \pm 0.19) \times 10^{-3} \text{ GeV}^{-2}, \quad (92)$$

$$\mathcal{O}_6 = (-4.3^{+0.9}_{-0.7}) \times 10^{-3} \text{ GeV}^6, \quad \mathcal{O}_8 = (-7.2^{+4.2}_{-5.3}) \times 10^{-3} \text{ GeV}^8. \quad (93)$$

Duality-violation effects have very little impact on the determination of L_{10} and C_{87} because the

corresponding sum rules are dominated by the low-energy region where the data sits. Thus, one obtains basically the same results with pinched and non-pinched weight functions. Similar values of L_{10} and C_{87} have been obtained in a recent analysis of the OPAL data [289], using separate parametrizations of the type (90) for the vector and axial spectral functions (*i.e.*, 4 additional parameters); however, larger uncertainties are found because the OPAL distributions are less precise than the ALEPH ones.

The situation is different for \mathcal{O}_6 and \mathcal{O}_8 , which are sensitive to the high-energy behaviour of the spectral function; pinched-weights provide then a much better accuracy. This could explain the numerical differences among previous estimates [189, 271, 286, 290–301], where duality violation uncertainties were not properly assessed. Owing to the larger uncertainties in the higher s region, the results obtained from OPAL data are less robust [289]; nevertheless, they are consistent with the ALEPH values in Eq. (93).

The determination (92) of the two χ PT couplings is in good agreement with (but more precise than) recent theoretical calculations, using $R\chi$ T and large- N_C techniques at the next-to-leading order, which predict [158]: $L_{10}^r(M_\rho) = -(4.4 \pm 0.9) \times 10^{-3}$ and $C_{87}^r(M_\rho) = (3.6 \pm 1.3) \times 10^{-3} \text{ GeV}^{-2}$. It also agrees with present lattice estimates of $L_{10}^r(M_\rho)$ [302, 303]. The results (93) fix with accuracy the value of \mathcal{O}_6 and determine the sign of \mathcal{O}_8 . This information is needed to calculate the electromagnetic penguin contribution to the CP-violating ratio $\varepsilon'_K/\varepsilon_K$ in neutral kaon decays.

8 V_{us} Determination

The separate measurement of the $|\Delta S| = 1$ and $|\Delta S| = 0$ tau decay widths provides a very clean determination of V_{us} [304]. If quark masses are neglected, the experimental ratio of the two decay widths directly measures $|V_{us}/V_{ud}|^2$. Taking $V_{ud} = 0.97425 \pm 0.00022$ [164] and the HFAG values in Eq. (81), one obtains $|V_{us}|^{\text{SU}(3)} = 0.210 \pm 0.002$.

This rather remarkable determination is only slightly shifted by the small SU(3)-breaking contributions induced by the strange quark mass. These effects can be theoretically estimated through a careful QCD analysis of the difference [268, 269, 304–310]

$$\delta R_\tau \equiv \frac{R_{\tau,V+A}}{|V_{ud}|^2} - \frac{R_{\tau,S}}{|V_{us}|^2}. \quad (94)$$

Since the strong interactions are flavour blind, this quantity vanishes in the SU(3) limit. The only non-zero contributions are proportional to powers of the quark mass-squared difference $m_s^2 - m_d^2$ or to vacuum expectation values of SU(3)-breaking operators such as $\delta O_4 \equiv \langle 0 | m_s \bar{s}s - m_d \bar{d}d | 0 \rangle \approx (-1.4 \pm 0.4) \times 10^{-3} \text{ GeV}^4$ [268, 304]. The dimensions of these operators are compensated by corresponding powers of m_τ^2 , which implies a strong suppression of δR_τ [268]:

$$\delta R_\tau \approx 24 S_{\text{EW}} \left\{ \frac{m_s^2(m_\tau^2)}{m_\tau^2} (1 - \epsilon_d^2) \Delta_{00}(a_\tau) - 2\pi^2 \frac{\delta O_4}{m_\tau^4} Q_{00}(a_\tau) \right\}, \quad (95)$$

where $\epsilon_d \equiv m_d/m_s = 0.053 \pm 0.002$ [311].

The perturbative QCD corrections $\Delta_{00}(a_\tau)$ and $Q_{00}(a_\tau)$ are known to $O(\alpha_s^3)$ and $O(\alpha_s^2)$, respectively [268, 269]. Separating their longitudinal ($J = 0$) and transverse ($J = 0 + 1$) contributions, as indicated in Eq. (66),

$$\Delta_{00}(a_\tau) = \frac{3}{4} \Delta_{00}^{(0+1)}(a_\tau) + \frac{1}{4} \Delta_{00}^{(0)}(a_\tau), \quad (96)$$

one finds a rather pathological behaviour for the longitudinal component $\Delta_{00}^{(0)}(a_\tau)$, with clear signs of being a non-convergent perturbative series [$a_\tau = \alpha_s(m_\tau^2)/\pi \sim 0.1$]:

$$\begin{aligned} \Delta_{00}^{(0)}(a_\tau) &= 1 + 9.333 a_\tau + 109.989 a_\tau^2 + 1322.52 a_\tau^3 + \dots \\ \Delta_{00}^{(0+1)}(a_\tau) &= 1 + 4 a_\tau + 25.021 a_\tau^2 + 148.25 a_\tau^3 + \dots \end{aligned} \quad (97)$$

Doing a CIPT resummation does not improve the bad behaviour of the longitudinal series.

Fortunately, the longitudinal contribution to δR_τ can be estimated phenomenologically with good accuracy [304], because it is dominated by far by the well-known $\tau \rightarrow \nu_\tau \pi$ and $\tau \rightarrow \nu_\tau K$ contributions. To estimate the remaining transverse component, one needs an input value for the strange quark mass; we will adopt the lattice world average [163,312], but increasing conservatively its uncertainty to 6 MeV, *i.e.*, we take $\bar{m}_s \equiv m_s^{\overline{\text{MS}}}(2 \text{ GeV}) = (94 \pm 6) \text{ MeV}$. The numerical size of $\Delta_{00}^{(0+1)}(a_\tau)$ is estimated in a very conservative way, averaging the asymptotically summed CIPT and FOPT results and taking half of the difference as the uncertainty associated with the truncation of the series, giving the result [304,313]

$$\delta R_\tau = (0.1544 \pm 0.0037) + (9.3 \pm 3.4) \frac{\bar{m}_s^2}{1 \text{ GeV}^2} + (0.0034 \pm 0.0028) = 0.240 \pm 0.032. \quad (98)$$

The first term contains the phenomenologically estimated longitudinal contribution δR_τ^L , the second gives the transverse $D = 2$ contribution, while the third one accounts for the small higher-dimension corrections.

This SU(3)-breaking correction implies, through Eq. (94),

$$|V_{us}| = \left(\frac{R_{\tau,S}}{\frac{R_{\tau,V+A}}{|V_{ud}|^2} - \delta R_{\tau,\text{th}}} \right)^{1/2} = 0.2173 \pm 0.0020_{\text{exp}} \pm 0.0010_{\text{th}} = 0.2173 \pm 0.0022. \quad (99)$$

This result is lower than the most recent determination from $K_{\ell 3}$ decays, $|V_{us}| = 0.2238 \pm 0.0011$ [53,314,315].

The branching ratios measured by BaBar and Belle are smaller than previous world averages, which translates into smaller results for $R_{\tau,S}$ and $|V_{us}|$. Using Eq. (13) and the measured $K^- \rightarrow \mu^- \bar{\nu}_\mu$ decay width, one can determine the $\tau^- \rightarrow \nu_\tau K^-$ branching ratio with better accuracy than the present experimental world average; one finds $\text{Br}(\tau^- \rightarrow \nu_\tau K^-) = (0.713 \pm 0.003)\%$ [167], which is 1.7σ higher than the value quoted in table 10. In a similar way, combining the measured spectra in $\tau^- \rightarrow \nu_\tau (K\pi)^-$ decays with $K_{\ell 3}$ data [167], one predicts the $\tau^- \rightarrow \nu_\tau \bar{K}^0 \pi^-$ and $\tau^- \rightarrow \nu_\tau K^- \pi^0$ branching ratios in Eq. (55), which are also 1.0σ and 1.6σ higher, respectively, than the direct τ decay measurements in table 10. Since these branching ratios are the three largest contributions to the Cabibbo-suppressed τ decay width, their slight underestimate has a very significant effect on the V_{us} determination. Replacing the $\text{Br}(\tau \rightarrow \nu_\tau K)$ and $\text{Br}(\tau \rightarrow \nu_\tau K\pi)$ HFAG averages by these phenomenological estimates, and taking the remaining branching ratios from table 10, one gets the corrected inclusive results:

$$\text{Br}(\tau^- \rightarrow \nu_\tau X_s^-) = (2.967 \pm 0.060)\%, \quad R_{\tau,S} = 0.1665 \pm 0.0034. \quad (100)$$

The slight increase in the total branching fraction into modes with non-zero strangeness, $(0.092 \pm 0.078)\%$, is well within the experimentally allowed range for unaccounted modes in Eq. (59). Moreover, the sum of this higher value for $R_{\tau,S}$ with $R_{\tau,V+A}$ in (81), 3.6377 ± 0.0086 , fits better with R_τ^{uni} in (79). Using as input (100), one gets a determination of V_{us} in much better agreement with the kaon information:

$$|V_{us}| = 0.2207 \pm 0.0023_{\text{exp}} \pm 0.0011_{\text{th}} = 0.2207 \pm 0.0025. \quad (101)$$

Similar values are also obtained combining the measured Cabibbo-suppressed τ distribution with electroproduction data [316,317]. Contrary to $K_{\ell 3}$, the final error of the V_{us} determination from τ decay is dominated by the experimental uncertainties and, therefore, sizeable improvements can be expected. Progress on the theoretical side requires a better understanding of the perturbative QCD corrections included in δR_τ .

$|V_{us}|$ can also be obtained from exclusive modes, either from the ratio $\Gamma(\tau \rightarrow \nu_\tau K)/\Gamma(\tau \rightarrow \nu_\tau \pi)$ or from $\Gamma(\tau \rightarrow \nu_\tau K)$, using the appropriate hadronic inputs from lattice calculations (f_K/f_π , f_K). The first method gives the result in Eq. (49), while $|V_{us}| = 0.2214 \pm 0.0022$ [22] is obtained from $\text{Br}(\tau \rightarrow \nu_\tau K)$, taking $\sqrt{2}f_K = 156.1 \pm 1.1 \text{ MeV}$ [163,312]. These values are closer to the $K_{\ell 3}$ result.

9 Electromagnetic and Weak Dipole Moments

A general description of the electromagnetic coupling of an on-shell spin- $\frac{1}{2}$ charged lepton to the virtual photon involves three different form factors:

$$\mathcal{M}_{\ell\bar{\ell}\gamma^*} = e Q_\ell \varepsilon_\mu(q) \bar{u}_\ell(\vec{p}') \left[F_1(q^2) \gamma^\mu + i \frac{F_2(q^2)}{2m_\ell} \sigma^{\mu\nu} q_\nu + \frac{F_3(q^2)}{2m_\ell} \sigma^{\mu\nu} \gamma_5 q_\nu \right] u_\ell(\vec{p}), \quad (102)$$

where $q^\mu = (p' - p)^\mu$ is the incoming photon momentum and $Q_\ell = -1$. The only assumptions are Lorentz invariance and electromagnetic current conservation (required by gauge invariance). Owing to the conservation of the electric charge, $F_1(0) = 1$. At $q^2 = 0$, the other two form factors reduce to the lepton magnetic (μ_ℓ) and electric (d_ℓ) dipole moments:

$$\mu_\ell \equiv \frac{e}{2m_\ell} \frac{g_\ell^\gamma}{2} = \frac{e}{2m_\ell} [1 + F_2(0)], \quad d_\ell^\gamma = \frac{e}{2m_\ell} F_3(0). \quad (103)$$

Similar expressions are defined in section 9.3 for the $\ell\bar{\ell}$ coupling of a virtual Z .

The $F_i(q^2)$ form factors are sensitive quantities to a possible lepton substructure. Moreover, $F_3(q^2)$ violates T and P invariance; thus, d_ℓ^γ constitutes a good probe of CP violation. Owing to their chiral-changing structure, the dipole moments may provide important insights on the mechanism responsible for mass generation. In general, one expects that a fermion of mass m_f (generated by physics at some scale $M \gg m_f$) will have induced dipole moments proportional to some power of m_f/M . Therefore, heavy fermions such as the τ should be a good testing ground for this kind of effects.

9.1 Anomalous Magnetic Moments

The most stringent QED test [318–325] comes from the high-precision measurements of the electron [326, 327] and muon [328] anomalous magnetic moments, $a_\ell \equiv (g_\ell^\gamma - 2)/2$:

$$a_e = (1\,159\,652\,180.73 \pm 0.28) \times 10^{-12}, \quad (104)$$

$$a_\mu = (11\,659\,208.9 \pm 6.3) \times 10^{-10}. \quad (105)$$

The $O(\alpha^5)$ calculation has been completed in both cases [329], with an impressive agreement with the measured a_e value. The dominant QED uncertainty is the input value of α , therefore a_e provides the most accurate determination of the fine structure constant (0.25 parts per billion) [329],

$$\alpha^{-1} = 137.035\,999\,174 \pm 0.000\,000\,035, \quad (106)$$

in agreement with the next most precise value (0.66 ppb) $\alpha_{\text{Rb}}^{-1} = 137.035\,999\,037 \pm 0.000\,000\,091$ [330], deduced from a recent measurement of the ratio h/m_{Rb} between the Planck constant and the mass of the ^{87}Rb atom. The improved experimental accuracy on the electron anomalous magnetic moment is already sensitive to the hadronic contribution $\delta a_e^{\text{QCD}} = (1.685 \pm 0.033) \times 10^{-12}$, and approaching the level of the weak correction $\delta a_e^{\text{ew}} = (0.0297 \pm 0.0005) \times 10^{-12}$ [329].

The heavier muon mass makes a_μ much more sensitive to electroweak corrections from virtual heavier states [331–339]; compared to a_e , they scale as m_μ^2/m_e^2 . The main theoretical uncertainty on the muon anomalous magnetic moment has a QCD origin. Since quarks have electric charge, virtual quark-antiquark pairs induce *hadronic vacuum polarization* corrections to the photon propagator (Fig. 15.c). Owing to the non-perturbative character of QCD at low energies, the light-quark contribution cannot be calculated at present with the required precision. Fortunately, this effect can be extracted from the measurement of the cross section $\sigma(e^+e^- \rightarrow \text{hadrons})$ and from the invariant-mass distribution of the final hadrons in τ decays:

$$\delta a_\mu^{\text{hvp,LO}} = \frac{\alpha}{\pi} \int_0^\infty \frac{ds}{s} K(s/m_\mu^2) e^2 \frac{1}{\pi} \text{Im} \Pi_{\text{em}}(s), \quad (107)$$

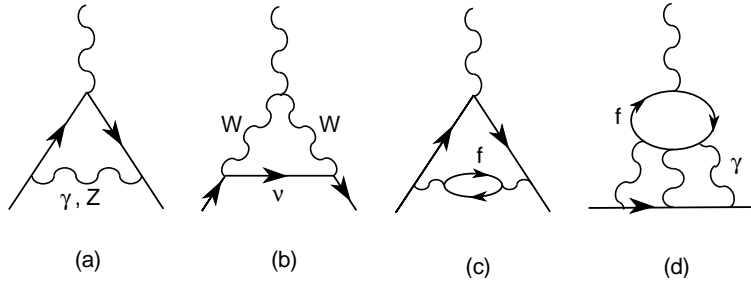


Figure 15: Feynman diagrams contributing to the lepton anomalous magnetic moment.

where $\Pi_{\text{em}}(s)$ is the QCD two-point correlation function of two electromagnetic currents. The kernel function [340, 341]

$$K(s/m_\mu^2) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + \frac{s}{m_\mu^2}(1-x)}, \quad (108)$$

is positive and monotonically decreasing as $K(s/m_\mu^2) \sim m_\mu^2/(3s)$ for increasing values s/m_μ^2 . Therefore, $\delta a_\mu^{\text{hvp,LO}}$ is dominated by the low-energy spectral region; the largest contribution being the 2π final state. As mentioned in section 6.3.1, there is still a slight discrepancy between the 2π spectral functions extracted from e^+e^- and τ data, which cannot be accounted for through isospin-breaking corrections [165, 166, 194, 342–345]. This translates into slightly different values for $\delta a_\mu^{\text{hvp,LO}}$ [194]. Additional disagreements among e^+e^- experiments remain in several hadronic final states [346], after the most recent BaBar [184, 347], CMD-2 [185, 348], CMD-3 [349], KLOE [187, 350] and SND [188, 351] analyses. The left plot in Fig. 16 [352] shows the 2π contribution to $\delta a_\mu^{\text{hvp,LO}}$ obtained from different τ -decay and e^+e^- experiments.

The SM prediction for a_μ can be decomposed in five types of contributions:

$$\begin{aligned} 10^{10} \times a_\mu^{\text{th}} &= 11\,658\,471.895\,1 \pm 0.008\,0 && \text{QED} \\ &+ 15.4 \pm 0.1 && \text{EW} \\ &+ 696.4 \pm 4.6 && \text{hvp}^{\text{LO}} \quad (701.5 \pm 4.7)_\tau \quad (692.3 \pm 4.2)_{e^+e^-} \\ &- 9.8 \pm 0.1 && \text{hvp}^{\text{NLO}} \\ &+ 10.5 \pm 2.6 && \text{lbl} \\ &= 11\,659\,184.4 \pm 5.3 && (11\,659\,189.5 \pm 5.4)_\tau \quad (11\,659\,180.3 \pm 4.9)_{e^+e^-}. \end{aligned} \quad (109)$$

The first line gives the very precise QED contribution [329, 353–366], including the recently computed $O(\alpha^5)$ corrections [329]. The quoted number adopts as input the value of α determined from the ^{87}Rb atom; using instead the a_e determination in (106), one gets the slightly more precise result $\delta a_\mu^{\text{QED}} = (11\,658\,471.884\,5 \pm 0.003\,7) \times 10^{-10}$ [329]. The pure electroweak correction is shown in the second line [331–339]. The leading-order hadronic-vacuum-polarization contribution in the third line is a weighted average of the τ and e^+e^- determinations, which are shown in parentheses [194]; the estimated next-to-leading correction is given in the fourth line [367, 368]. Additional QCD uncertainties stem from the smaller *light-by-light scattering* contribution (Fig. 15.d), given in the fifth line [369], which needs to be theoretically evaluated [369–381].

The final SM prediction differs from the experimental value by 3.0σ . The τ estimate of the hadronic vacuum polarization results in a smaller deviation of 2.3σ , while using e^+e^- data alone increases the discrepancy to 3.6σ . The right plot in Fig. 16 shows a recent compilation of SM predictions [382], using slightly different estimates of $\delta a_\mu^{\text{hvp,LO}}$ [194, 323, 367, 383, 384]. New precise e^+e^- and τ data sets are needed to settle the true value of a_μ^{th} . Improved predictions are needed to match the aimed 10^{-10} accuracy of the proposed muon experiments at Fermilab [385] and J-PARC [386].

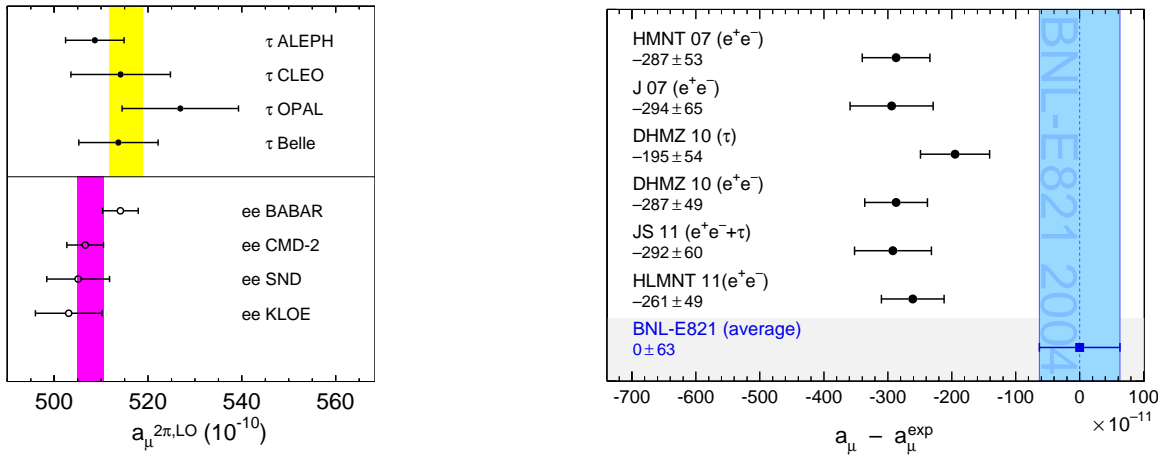


Figure 16: 2π contribution to $\delta a_\mu^{\text{hvp,LO}}$ (left), obtained from τ -decay and e^+e^- experiments [166,342,352]. The right plot shows a recent compilation [382] of SM results for a_μ [194,323,367,383,384], compared with the experimental measurement [328]. Figures taken from Refs. [352,382].

The τ anomalous magnetic moment has an enhanced sensitivity to new physics because of the large τ mass. In the SM, the interesting electroweak correction roughly scales as $(m_\tau/m_\mu)^2 = 283$. Using the same decomposition as before, the different contributions to a_τ take the following values [387]:

$$\begin{aligned}
10^8 \times a_\tau^{\text{th}} &= 117\,324 \pm 2 && \text{QED} \\
&+ 47.4 \pm 0.5 && \text{EW} \\
&+ 337.5 \pm 3.7 && \text{hvp}^{\text{LO}} \\
&+ 7.6 \pm 0.2 && \text{hvp}^{\text{NLO}} \\
&+ 5 \pm 3 && \text{lbl} \\
&= 117\,721 \pm 5. &&
\end{aligned} \tag{110}$$

The big enhancement of the electroweak contribution has another welcome feature: it is only a factor of seven smaller than the hadronic one, compared to a factor of 45 for a_μ . The electroweak contribution to a_τ is ten times larger than the present hadronic uncertainty, while this factor is only three for the muon case. This makes a_τ a much cleaner test of new physics contributions.

Unfortunately, the very short τ lifetime makes very difficult to measure a_τ with a precision good enough to perform a significative test. The spin-precession technique adopted in the muon $g - 2$ experiments is no-longer feasible. Instead, one measures the production of $\tau^+\tau^-$ pairs at different high-energy processes and compares the cross sections with the QED predictions [388–396]. The 95% CL limit quoted by the PDG [23] was derived by the DELPHI collaboration from measurements of $\sigma(e^+e^- \rightarrow e^+e^-\tau^+\tau^-)$ at \sqrt{s} between 183 and 208 GeV at LEP2 [388]:

$$-0.052 < a_\tau < 0.013. \tag{111}$$

The present sensitivity is one order of magnitude worse than the first QED contribution, $\delta a_\tau^{\text{QED,LO}} = \frac{\alpha}{2\pi}$ [397]. Thus, a_τ is still essentially unknown experimentally.

Several methods have been suggested to improve the measurement of a_τ . For instance, using precise $\tau^- \rightarrow \nu_\tau \ell^- \bar{\nu}_\ell \gamma$ ($\ell = e, \mu$) data from high-luminosity B factories [398], or taking advantage of the radiation amplitude zero which occurs in this decay at the high-energy end of the lepton distribution for $g_\tau^\gamma = 2$ [399]. This requires very good energy resolution, high statistics and a good control of systematic uncertainties [387]. A similar method to study a_τ using radiative $W^- \rightarrow \tau^- \bar{\nu}_\tau \gamma$ decays and potentially very high data samples at LHC was suggested in Ref. [400]. Some sensitivity to a_τ could also

be achieved through $\gamma\gamma \rightarrow \tau^+\tau^-$ collisions; *i.e.*, $e^+e^- \rightarrow e^+\gamma^*\gamma^*e^- \rightarrow e^+\tau^+\tau^-e^-$ [401, 402]. Another, more fancy, possibility could be to use polarized τ leptons from $B^+ \rightarrow \tau^+\nu_\tau$ decays, precess their spin in a bent crystal and measure the final τ polarization through the angular distribution of the daughter lepton in the $\tau^- \rightarrow \nu_\tau\ell^-\bar{\nu}_\ell$ decays [403]. The channelling of a short-lived polarized particle through a bent crystal [404], has been already tested successfully and used to measure the magnetic moment of the Σ^+ hyperon [405]. The feasibility of these ideas at a Super-B factory or at a hadron collider remains to be investigated [387].

A determination of the dipole form factor $F_2(s)$ can be performed in $e^+e^- \rightarrow \tau^+\tau^-$ by analysing the angular distribution of the τ decay products, which contains information on the $\tau^+\tau^-$ polarizations [406, 407]. With unpolarized electron beams, the imaginary part of the form factor could be accessed by measuring the normal (to the scattering plane) polarization of a single τ , while the real part would require measuring the spin correlations of both taus. Using polarized electron beams, the real part could also be obtained by measuring the transverse and longitudinal polarizations of a single τ . With 10^{12} $\tau^+\tau^-$ pairs at a future Super-B factory, the estimated sensitivity is of the order of 10^{-6} [406, 407].

Using an effective Lagrangian, invariant under the SM gauge group, and writing the lowest-dimension ($D = 6$) operators contributing to a_τ , it is possible to combine experimental information from τ production at LEP1, LEP2 and SLD with $W^- \rightarrow \tau^-\bar{\nu}_\tau$ data from LEP2 and $p\bar{p}$ colliders. This allows one to set a stronger model-independent bound on new-physics contributions to a_τ (95% CL) [408]:

$$-0.007 < a_\tau^{\text{New Phys}} < 0.005. \quad (112)$$

9.2 Electric Dipole Moments

The strong experimental limits on the electron [409] and muon [410] electric dipole moments

$$|d_e^\gamma| < 8.7 \times 10^{-29} \text{ e cm} \quad (90\% \text{ CL}), \quad (113)$$

$$d_\mu^\gamma = (-0.1 \pm 0.9) \times 10^{-19} \text{ e cm}, \quad (114)$$

provide stringent constraints on models of new physics beyond the SM [411–417].

The status is very different for the τ lepton, owing to the much more difficult experimental conditions. Several $\tau^+\tau^-$ production processes have been investigated: $e^+e^- \rightarrow \tau^+\tau^-$ [394, 418], $e^+e^- \rightarrow \tau^+\tau^-\gamma$ [390], $e^+e^- \rightarrow e^+e^-\tau^+\tau^-$ [388, 389, 401], $Z \rightarrow \tau^+\tau^-$ [392] and $Z \rightarrow \tau^+\tau^-\gamma$ [391, 393]. However, since the CP-violating form factor $F_3(s)$ does not interfere with the dominant CP-even amplitudes, the cross sections depend quadratically on d_τ^γ , suppressing strongly the sensitivity to the CP-odd component.

The Belle collaboration has searched for CP-violating effects in the process $e^+e^- \rightarrow \gamma^* \rightarrow \tau^+\tau^-$, using triple momentum and spin correlations which are linear in $F_3(s)$ [419, 420]. They obtain the 95% CL constraints [421]:

$$\begin{aligned} -0.22 < \text{Re}(d_\tau^\gamma) < 0.45 & \quad (10^{-16} \text{ e cm}), \\ -0.25 < \text{Im}(d_\tau^\gamma) < 0.08 & \quad (10^{-16} \text{ e cm}). \end{aligned} \quad (115)$$

Actually, these numbers refer to the real and imaginary parts of the corresponding form factor at the squared energy of the $\tau^+\tau^-$ system. This s dependence is expected to be small and has been ignored. $\text{Re}(d_\tau^\gamma)$ is obtained from a CP-odd and T-odd observable, while the measurement of $\text{Im}(d_\tau^\gamma)$ requires a CP-odd and T-even one.

With a longitudinally polarized electron beam, one could build a CP-odd asymmetry, associated with the normal (to the scattering plane) polarization of a single τ , proportional to $\text{Re}(d_\tau^\gamma)$ [422]. The estimated sensitivity at a Super-B factory ranges from 10^{-18} to 10^{-19} e cm, assuming integrated luminosities between 2 and 75 ab^{-1} [422].

9.3 Weak Dipole Moments

The anomalous weak dipole moments of the τ lepton are usually defined through the effective Z couplings [23]:

$$\mathcal{L}_{\text{wdm}}^Z = -\frac{1}{2 \sin \theta_W \cos \theta_W} Z_\mu \bar{\tau} \left[i \alpha_\tau^W \frac{e}{2m_\tau} \sigma^{\mu\nu} q_\nu + d_\tau^W \sigma^{\mu\nu} \gamma_5 q_\nu \right] \tau. \quad (116)$$

Both the weak magnetic term α_τ^W and the CP-violating weak electric term d_τ^W have been investigated with LEP data [423–425]. The SM predicts non-zero contributions to α_τ^W and d_τ^W , through quantum effects, which are well below the present experimental sensitivity [426, 427]. This opens the possibility to look for deviations from the SM.

Analysing the complete differential cross section for the production and decay of the τ leptons in $e^+e^- \rightarrow \tau^+\tau^-$, taking into account the correlations between the two τ spins, one is sensitive to the real and imaginary parts of α_τ^W and d_τ^W [419, 420, 426–430]. Assuming that the vector and axial-vector $Z\tau^+\tau^-$ couplings take their SM values and using optimal polarimeters to recover the τ spin information [83, 85, 431], the ALEPH collaboration has obtained the 95% CL limits [423]:

$$|\text{Re}(\alpha_\tau^W)| < 1.1 \times 10^{-3}, \quad |\text{Im}(\alpha_\tau^W)| < 2.7 \times 10^{-3}, \quad (117)$$

$$|\text{Re}(d_\tau^W)| < 0.50 \times 10^{-17} \text{ e cm}, \quad |\text{Im}(d_\tau^W)| < 1.1 \times 10^{-17} \text{ e cm}. \quad (118)$$

T-odd signals can be also generated through a relative phase between the vector and axial-vector couplings of the Z to the $\tau^+\tau^-$ pair [83]. This effect, which in the SM appears [432] at the one-loop level through absorptive parts in the electroweak amplitudes, gives rise [83] to a spin-spin correlation associated with the transverse (within the production plane) and normal (to the production plane) polarization components of the two taus. Ignoring possible contributions from weak dipole moments, the analysis of the ALEPH data gives the result [110]

$$C_{TN} = -\frac{2 \text{Im}(v_\tau a_\tau^*)}{v_\tau^2 + a_\tau^2} = 0.08 \pm 0.13 \pm 0.04. \quad (119)$$

A possible weak magnetic $W\tau\nu_\tau$ coupling of the form

$$\mathcal{L}_{\text{wdm}}^W = -\frac{g}{2\sqrt{2}} i \frac{\kappa_\tau^W}{2m_\tau} W_\mu [\bar{\tau} \sigma^{\mu\nu} q_\nu (1 - \gamma_5) \nu_\tau] + \text{h.c.} \quad (120)$$

was also searched for by the DELPHI collaboration, fitting the distribution of the final decay products in $e^+e^- \rightarrow \tau^+\tau^-$ events. The measured value of the anomalous tensor coupling κ_τ^W corresponds to the following 90% CL allowed interval [89]:

$$-0.096 < \kappa_\tau^W < 0.037. \quad (121)$$

Using the electroweak $SU(2)_L \otimes U(1)_Y$ gauge symmetry, the dipole interactions in Eqs. (102), (116) and (120) can all be written in terms of only two dimension-6 operators: $\mathcal{O}_B = g' B^{\mu\nu} (\bar{L}_\tau \phi \sigma_{\mu\nu} \tau_R)$ and $\mathcal{O}_W = g W_i^{\mu\nu} (\bar{L}_\tau \tau^i \phi \sigma_{\mu\nu} \tau_R)$, where $L_\tau^T = (\nu_\tau, \tau)_L$ and ϕ are the tau and Higgs $SU(2)_L$ doublets, respectively, $B^{\mu\nu}$ and $W_i^{\mu\nu}$ the $U(1)_Y$ and $SU(2)_L$ field strength tensors, g' and g the corresponding gauge couplings and τ^i the Pauli matrices. Analysing the experimental information in terms of these two operators, one gets the result in Eq. (112) and a stronger 95% CL bound on κ_τ^W [408]:

$$-0.003 < \kappa_\tau^W < 0.004. \quad (122)$$

10 CP violation

In the three-generation SM, the violation of the CP symmetry originates from the single phase naturally occurring in the quark mixing matrix [433]. Therefore, CP violation is predicted to be absent in the lepton sector (for massless neutrinos). However, the fundamental origin of the Kobayashi–Maskawa phase is still unknown. Obviously, CP violation could well be a sensitive probe for new physics.

The electroweak dipole moments d_τ^γ and d_τ^W test CP violation in τ production, but violations of the CP symmetry could also happen in the τ decay amplitude. In fact, one would naively expect CP-violating effects to be larger in τ decay than in $\tau^+\tau^-$ production. Since the decay of the τ proceeds through a weak interaction, these effects could be $\mathcal{O}(1)$ or $\mathcal{O}(10^{-3})$, if the leptonic CP violation is *weak* or *milliweak*. A variety of CP-violating observables (rate, angular and polarization asymmetries, triple products, Dalitz distributions, etc.) can be exploited to search for violations of the CP symmetry [434, 435].

With polarized electron (and/or positron) beams, one could use the longitudinal polarization vectors of the incident leptons to construct T-odd rotationally invariant products [436]. CP could be tested by comparing these T-odd products in τ^- and τ^+ decays. In the absence of beam polarization, CP violation can still be tested through $\tau^+\tau^-$ correlations [437]. Studying the final decays of the τ decay products, it is possible to separate possible CP-odd effects in the $\tau^+\tau^-$ production and in the τ decay [438]. For instance, one can study the chain process $e^+e^- \rightarrow \tau^+\tau^- \rightarrow (\rho^+\bar{\nu}_\tau)(\rho^-\nu_\tau) \rightarrow (\pi^+\pi^0\bar{\nu}_\tau)(\pi^-\pi^0\nu_\tau)$. The distribution of the final pions provides information on the ρ polarization, which allows to test for possible CP-violating effects in the $\tau \rightarrow \rho\nu_\tau$ decay.

CP violation can also be tested through rate asymmetries, *i.e.*, comparing the partial fractions $\Gamma(\tau^- \rightarrow X^-)$ and $\Gamma(\tau^+ \rightarrow X^+)$. However, this kind of signal requires the presence of strong final-state interactions in the decay amplitude. Another possibility is to study T-odd (CPT-even) asymmetries in the angular distributions of the final hadrons in semileptonic τ decays [439]. Explicit studies of the decay modes $\tau^- \rightarrow K^-\pi^-\pi^+\nu_\tau$ [440, 441], $\tau^- \rightarrow \pi^-K^-K^+\nu_\tau$ [440], $\tau^- \rightarrow (3\pi)^-\nu_\tau$ [442, 443] and $\tau^- \rightarrow (4\pi)^-\nu_\tau$ [442] show that sizeable CP-violating effects could be generated in some models of CP violation involving several Higgs doublets or left-right symmetry.

The $\tau^+ \rightarrow \pi^+K_S\bar{\nu}_\tau$ ($\geq 0\pi^0$) rate asymmetry recently measured by BaBar [444],

$$\mathcal{A}_\tau \equiv \frac{\Gamma(\tau^+ \rightarrow \pi^+K_S\bar{\nu}_\tau) - \Gamma(\tau^- \rightarrow \pi^-K_S\nu_\tau)}{\Gamma(\tau^+ \rightarrow \pi^+K_S\bar{\nu}_\tau) + \Gamma(\tau^- \rightarrow \pi^-K_S\nu_\tau)} = (-0.36 \pm 0.23 \pm 0.11)\%, \quad (123)$$

differs by 2.8σ from the expected value due to $K^0-\bar{K}^0$ mixing, $\mathcal{A}_\tau = (0.36 \pm 0.01)\%$ [445–447]. The Belle collaboration has also searched for a CP signal in this decay mode through a difference in the τ^\pm angular distributions. Specifically, they measure the difference between the mean values of $\cos\beta\cos\psi$ for τ^+ and τ^- events, evaluated in bins of the total invariant hadronic mass, where the angles β and ψ denote the directions of the K_S and τ , respectively, with respect to the one of the e^+e^- centre-of-mass system, as seen from the hadronic rest frame [439]. Belle finds a null result at the 0.2–0.3% level [448], improving the previous CLEO limits by one order of magnitude [449]. More precise experimental analyses are needed to clarify the compatibility between the BaBar and Belle results.

11 Tau Production in B and D Decays

Heavy meson decays into final states containing τ leptons are a good laboratory to look for new physics related to the fermion mass generation. Decays such as $B^- \rightarrow \tau^-\bar{\nu}_\tau$, $B \rightarrow D^{(*)}\tau^-\bar{\nu}_\tau$, $B_c^- \rightarrow \tau^-\bar{\nu}_\tau$ or $D_s^- \rightarrow \tau^-\bar{\nu}_\tau$ involve the heaviest elementary fermions that can be directly produced at flavour factories, providing important information about the underlying dynamics mediating these processes.

An excess of events in two $b \rightarrow c \tau^- \bar{\nu}_\tau$ transitions has been reported by BaBar [450]. Including the previous Belle measurements [451], the experimental ratios ($\ell = e, \mu$)

$$\begin{aligned}
 R(D) &\equiv \frac{\text{Br}(\bar{B} \rightarrow D \tau^- \bar{\nu}_\tau)}{\text{Br}(\bar{B} \rightarrow D \ell^- \bar{\nu}_\ell)} = 0.438 \pm 0.056, \\
 R(D^*) &\equiv \frac{\text{Br}(\bar{B} \rightarrow D^* \tau^- \bar{\nu}_\tau)}{\text{Br}(\bar{B} \rightarrow D^* \ell^- \bar{\nu}_\ell)} = 0.354 \pm 0.026,
 \end{aligned}
 \tag{124}$$

are significantly higher than the SM expectations, $R(D) = 0.296 \pm 0.017$ and $R(D^*) = 0.252 \pm 0.004$ [452–456]. If confirmed, this could signal new-physics contributions violating lepton-flavour universality.

A sizeable deviation from the SM was previously observed in the leptonic decay $B^- \rightarrow \tau^- \bar{\nu}_\tau$. However, Belle [457] finds now a much lower value in agreement with the SM; combined with the BaBar result [458], it gives the average

$$\text{Br}(B^- \rightarrow \tau^- \bar{\nu}_\tau) = (1.15 \pm 0.23) \times 10^{-4},
 \tag{125}$$

to be compared with the SM expectation $(0.733^{+0.121}_{-0.073}) \times 10^{-4}$ [459].

These results are intriguing enough to trigger the theoretical interest. The enhancement of τ production could be generated by new physics contributions with couplings proportional to fermion masses. In particular, it could be associated with the exchange of a charged scalar within two-Higgs-doublet models [58, 452, 454, 460–464]. Although the Babar data rules out the usually adopted “type II” scenario [450, 452], these measurements could be accommodated [460] within the more general framework of the “Aligned Two-Higgs-Doublet Model” (A2HDM) [465]. However, the fit to the B decay data suggests that the charged scalar should couple differently to the different quark families; *i.e.*, one needs a coupling proportional to the quark mass but with a family-dependent proportionality factor. Since charged scalar contributions to $R(D^*)$ are subdominant compared with the vector contributions, large Yukawa couplings are required to explain the observed excess in $R(D^*)$, thus generating a tension with present bounds from $D_{(s)}$ leptonic decays [58]. If $R(D^*)$ is not considered in the fit, all other observables can be consistently described within the family-universal A2HDM [460].

Using only measurements of branching fractions, it is not possible to disentangle charged scalar contributions in $B \rightarrow D^{(*)} \tau^- \bar{\nu}_\tau$ from other kinds of new physics [466, 467]. However, $B \rightarrow D^{(*)} \tau^- \bar{\nu}_\tau$ decays have a rich three-body kinematics and spin structure in the final state that has not been exploited so far. Several observables involving angular distributions, polarization fractions and momentum-transfer dependence would provide, if measured, crucial information to discriminate between different new-physics scenarios and clarify the possible role of charged scalar contributions in these processes [454, 460, 462–464, 468–473]. Another possibility would be to measure the leptonic decay $B_c^- \rightarrow \tau^- \bar{\nu}_\tau$. Large enhancements over the SM prediction, $\text{Br}(B_c^- \rightarrow \tau^- \bar{\nu}_\tau) = (1.94 \pm 0.18)\%$ [474], should be observed if the current excess in $R(D^*)$ is due to charged scalar contributions [460].

12 Lepton-Flavour Violation

We have clear experimental evidence that neutrinos are massive particles and there is mixing in the lepton sector. The solar, atmospheric, accelerator and reactor neutrino data lead to a consistent pattern of oscillation parameters [23]. The main recent advance is the establishment of a sizeable non-zero value of θ_{13} , both in accelerator (Minos [475], T2K [476]) and reactor experiments (Daya Bay [477], Double-Chooz [478], Reno [479]), with a statistical significance which reaches the 10σ at Daya Bay [477]. This increases the interest for a next-generation of long-baseline neutrino experiments to measure the CP-violating phase δ_{CP} and resolve the neutrino mass hierarchy.

A global analysis, combining the full set of neutrino data, leads to the following preferred ranges for the oscillation parameters [480]:

$$\begin{aligned}
\Delta m_{21}^2 &= (7.45^{+0.19}_{-0.16}) \cdot 10^{-5} \text{ eV}^2, & \delta_{\text{CP}} &= (265^{+56}_{-61})^\circ, \\
\Delta m_{31}^2 &= (2.417^{+0.013}_{-0.013}) \cdot 10^{-3} \text{ eV}^2, & \Delta m_{32}^2 &= (-2.410^{+0.062}_{-0.062}) \cdot 10^{-3} \text{ eV}^2, \\
\sin^2 \theta_{12} &= 0.306^{+0.012}_{-0.012}, & \sin^2 \theta_{13} &= 0.0229^{+0.0020}_{-0.0019}, \\
\sin^2 \theta_{23} &= 0.446^{+0.007}_{-0.007} \oplus 0.587^{+0.032}_{-0.037}, & & (126)
\end{aligned}$$

where $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ are the mass-squared differences between the neutrino mass eigenstates and θ_{ij} the corresponding angles in the standard three-flavour parametrization of the neutrino mixing matrix [23]. As atmospheric mass-squared splitting the fit uses Δm_{31}^2 for *normal ordering* and Δm_{32}^2 for *inverted ordering*. The statistical significance of the δ_{CP} determination is low, being compatible with any value between 0° and 360° at the 3σ level.

Non-zero neutrino masses constitute a clear indication of new physics beyond the SM. Right-handed neutrinos are an obvious possibility to incorporate Dirac neutrino masses; however, the ν_{iR} fields would be $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ singlets, without any SM interaction. If such objects do exist, it would seem natural to expect that they are able to communicate with the rest of the world through some still unknown dynamics. Moreover, the SM gauge symmetry would allow for a right-handed Majorana neutrino mass matrix of arbitrary size, because it is not related to the ordinary Higgs mechanism. A Majorana mass term mixes neutrinos and anti-neutrinos, violating lepton number by two units. Clearly, new physics is called for.

Adopting a more general effective field theory language, without any assumption about the existence of right-handed neutrinos or any other new particles, one can write the most general $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ invariant Lagrangian, in terms of the known low-energy fields (left-handed neutrinos only). The SM is the unique answer with dimension four. The first contributions from new physics appear through dimension-5 operators, and have also a unique form which violates lepton number by two units [481]:

$$\Delta \mathcal{L} = -\frac{c_{ij}}{\Lambda_{\text{NP}}} \bar{L}_i \tilde{\phi} \tilde{\phi}^t L_j^c + \text{h.c.}, \quad (127)$$

where L_i denotes the i -flavoured $SU(2)_L$ lepton doublet, $\tilde{\phi} \equiv i \tau_2 \phi^*$ is the \mathcal{C} -conjugate of the SM Higgs doublet ϕ and $L_i^c \equiv \mathcal{C} \bar{L}_i^T$. Similar operators with quark fields are forbidden, due to their different hypercharges, while higher-dimension operators would be suppressed by higher powers of the new-physics scale Λ_{NP} . After spontaneous symmetry breaking, $\langle \phi^{(0)} \rangle = v/\sqrt{2}$, $\Delta \mathcal{L}$ generates a Majorana mass term for the left-handed neutrinos, with² $M_{ij} = c_{ij} v^2 / \Lambda_{\text{NP}}$. Thus, Majorana neutrino masses should be expected on general symmetry grounds. Taking $m_\nu \gtrsim 0.05$ eV, as suggested by atmospheric neutrino data, one gets $\Lambda_{\text{NP}}/c_{ij} \lesssim 10^{15}$ GeV, amazingly close to the expected scale of Gran Unification.

With non-zero neutrino masses, the leptonic charged-current interactions involve a flavour mixing matrix V_L . The data on neutrino oscillations imply the following 3σ CL ranges for the magnitudes of the different V_L entries [480]:

$$|V_L|_{3\sigma} = \begin{pmatrix} 0.799 \rightarrow 0.844 & 0.515 \rightarrow 0.581 & 0.130 \rightarrow 0.170 \\ 0.214 \rightarrow 0.525 & 0.427 \rightarrow 0.706 & 0.598 \rightarrow 0.805 \\ 0.234 \rightarrow 0.536 & 0.452 \rightarrow 0.721 & 0.573 \rightarrow 0.787 \end{pmatrix}. \quad (128)$$

Therefore the mixing among leptons appears to be very different from the one in the quark sector. The number of relevant phases characterizing the matrix V_L depends on the Dirac or Majorana nature of neutrinos, because if one rotates a Majorana neutrino by a phase, this phase will appear in its mass

²This relation generalizes the well-known see-saw mechanism ($m_{\nu_L} \sim m^2/\Lambda_{\text{NP}}$) [482, 483].

Table 13: Best published limits on lepton-flavour-violating decays [523–533].

Br($\mu^- \rightarrow X^-$) · 10 ¹² (90% CL)									
$e^- \gamma$	0.57	$e^- 2\gamma$	72	$e^- e^- e^+$	1.0				
Br($\tau^- \rightarrow X^-$) · 10 ⁸ (90% CL)									
$e^- \gamma$	3.3	$e^- e^+ e^-$	2.7	$e^- \mu^+ \mu^-$	2.7	$e^- e^- \mu^+$	1.5	$e^- \pi^0$	8.0
$\mu^- \gamma$	4.4	$\mu^- e^+ e^-$	1.8	$\mu^- \mu^+ \mu^-$	2.1	$\mu^- \mu^- e^+$	1.7	$\mu^- \pi^0$	11
$e^- \eta$	9.2	$e^- \eta'$	16	$e^- \rho^0$	1.8	$e^- \omega$	4.8	$e^- \phi$	3.1
$\mu^- \eta$	6.5	$\mu^- \eta'$	13	$\mu^- \rho^0$	1.2	$\mu^- \omega$	4.7	$\mu^- \phi$	8.4
$e^- K_S$	2.6	$e^- K^{*0}$	3.2	$e^- \bar{K}^{*0}$	3.4	$e^- K^+ \pi^-$	3.1	$e^- \pi^+ K^-$	3.7
$\mu^- K_S$	2.3	$\mu^- K^{*0}$	5.9	$\mu^- \bar{K}^{*0}$	7.0	$\mu^- K^+ \pi^-$	4.5	$\mu^- \pi^+ K^-$	8.6
$e^- K_S K_S$	7.1	$e^- K^+ K^-$	3.4	$e^- \pi^+ \pi^-$	2.3				
$\mu^- K_S K_S$	8.0	$\mu^- K^+ K^-$	4.4	$\mu^- \pi^+ \pi^-$	2.1				
$e^- f_0(980) \rightarrow e^- \pi^+ \pi^-$			3.2	$\mu^- f_0(980) \rightarrow \mu^- \pi^+ \pi^-$			3.4		
Br($Z \rightarrow X^0$) · 10 ⁶ (95% CL)									
$e^\pm \mu^\mp$	1.7	$e^\pm \tau^\mp$	9.8	$\mu^\pm \tau^\mp$	12				
Br($B_{(s)}^0 \rightarrow X^0$) · 10 ⁸ (95% CL)									
$B^0 \rightarrow e^\pm \mu^\mp$	0.37	$B_s^0 \rightarrow e^\pm \mu^\mp$	1.4						
Br($\mu^- + N \rightarrow e^- + N$) · 10 ¹² (90% CL)									
Au	0.7	Ti	4.3	Pb	46				

term which will no longer be real. With only three Majorana (Dirac) neutrinos, the 3×3 matrix V_L involves six (four) independent parameters: three mixing angles and three (one) phases.

The smallness of neutrino masses implies a strong suppression of neutrinoless lepton-flavour-violating processes, which can be avoided in models with sources of lepton-flavour violation (LFV) not related to m_{ν_i} . LFV processes have the potential to probe physics at scales much higher than the TeV [413, 484–522]. The LFV scale can be constrained imposing the requirement of a viable leptogenesis. Recent studies within different new-physics scenarios find interesting correlations between μ and τ LFV decays, with $\mu \rightarrow e\gamma$ often expected to be close to the present exclusion limit.

Table 13 shows the best published limits on LFV decays of Z bosons [523–526], B mesons [527], muons [528–530] and taus [531, 532], together with the present experimental constraints on $\mu \rightarrow e$ conversions in muonic atoms [533]. The B factories are pushing the experimental limits on neutrinoless LFV τ decays to the 10^{-8} level, increasing in a drastic way the sensitivity to new physics scales. A rather competitive upper bound on $\tau \rightarrow 3\mu$ has been also obtained at LHCb, $\text{Br}(\tau^- \rightarrow \mu^- \mu^+ \mu^-) < 8.0 \times 10^{-8}$ [534], showing the potential of the high-statistics collider data in some particular decay modes. Future experiments could improve further some limits to the 10^{-9} level, allowing to explore interesting and totally unknown phenomena.

Complementary information is provided by the MEG experiment, which has recently set a very stringent limit on LFV in muon decay, $\text{Br}(\mu^+ \rightarrow e^+ \gamma) < 5.7 \times 10^{-13}$ (90% CL) [528], improving

Table 14: Best published limits on lepton-number-violating decays [531, 532, 534, 547–551].

Br($\tau^- \rightarrow X^-$) · 10 ⁸		(90% CL)							
$e^+\pi^-\pi^-$	2.0	$e^+K^-K^-$	3.3	$e^+\pi^-K^-$	3.2	$\pi^-\Lambda$	7.2	$\mu^-\mu^-p$	44
$\mu^+\pi^-\pi^-$	3.9	$\mu^+K^-K^-$	4.7	$\mu^+\pi^-K^-$	4.8	$\pi^-\bar{\Lambda}$	14	$\mu^+\mu^-\bar{p}$	33
Br($B^- \rightarrow X^-$) · 10 ⁷		(95% CL; * = 90% CL)							
$\pi^+e^-e^-$	0.23*	$\pi^+e^-\mu^-$	1.5*	$\pi^+\mu^-\mu^-$	0.13	$D^+\mu^-\mu^-$	6.9	$D^+e^-e^-$	26*
$K^+e^-e^-$	0.30*	$K^+e^-\mu^-$	1.6*	$K^+\mu^-\mu^-$	0.54	$D_s^+\mu^-\mu^-$	5.8	$D^+e^-\mu^-$	18*
$\rho^+e^-e^-$	1.7*	$\rho^+e^-\mu^-$	4.7*	$\rho^+\mu^-\mu^-$	4.2*	$D^{*+}\mu^-\mu^-$	24		
$K^{*+}e^-e^-$	4.0*	$K^{*+}e^-\mu^-$	3.0*	$K^{*+}\mu^-\mu^-$	5.9*	$D^0\pi^+\mu^-\mu^-$	15		
$e^-\Lambda$	0.32*	$\mu^-\Lambda$	0.6*	$e^-\bar{\Lambda}$	0.8*	$\mu^-\bar{\Lambda}$	0.6*		
Br($Z \rightarrow X^0$) · 10 ⁶		(95% CL)							
e^-p	1.8	μ^-p	1.8						

their previous best upper limit by a factor of four. With further data still being acquired (the final number of stopped muons is expected to double the sample analysed so far) and the upgrade program currently underway, MEG aims to enhance the sensitivity by one order of magnitude, down to the 6×10^{-14} level [535]. A possible 10^4 improvement in $\mu \rightarrow 3e$, reaching a sensitivity of 10^{-16} , is also under study at PSI [536], and ongoing projects at J-PARC [537] and FNAL [538] aim to study $\mu \rightarrow e$ conversions in muonic atoms at the 10^{-16} level, a factor 10^4 better than the current experimental limit from SINDRUM-II at PSI [533]. New proposals to reach further sensitivities around 10^{-18} are also being discussed [539, 540].

The upper limits on LFV decays of the μ and the τ provide also interesting constraints on possible LFV couplings of the Higgs-like boson discovered at the LHC. The present bounds on $\tau^- \rightarrow \ell^- \gamma$ and $\tau^- \rightarrow \ell^- \ell^+ \ell'^-$ ($\ell^{(\prime)} = e, \mu$) still allow for sizeable $H \rightarrow \tau^\pm \ell^\mp$ decay rates of $\mathcal{O}(10\%)$ [541–543]. Semileptonic transitions like $\tau \rightarrow \ell \pi \pi$ are sensitive to LFV scalar couplings while decays such as $\tau \rightarrow \ell \eta^{(\prime)}$ probe pseudoscalar couplings, thus providing a useful low-energy handle to disentangle possible Higgs LFV signals at the LHC [544–546].

Lepton-number violation has also been tested in τ [531, 532, 534, 547], Z [548] and meson [23, 549–552] decays with sensitivities approaching in some cases the 10^{-8} level. The strongest published limits are shown in table 14. These bounds provide useful constraints on models of new physics involving Majorana neutrinos with masses in the GeV range [553–564].

13 Tau Physics at the LHC

The study of processes with τ leptons in the final state is an important part of the LHC program. Owing to their high momenta, tightly collimated decay products and low multiplicity, τ leptons provide excellent signatures to probe new physics at high-energy colliders. Moreover, since τ decays are fully contained within the detector, the distribution of the τ decay products has precious polarization information.

The τ signal has been already exploited successfully at the LHC to measure W [565, 566], Z [567–569] and top [570, 571] production cross sections, through their subsequent decays to τ leptons ($W^- \rightarrow \tau^- \bar{\nu}_\tau$, $Z \rightarrow \tau^+ \tau^-$, $t \rightarrow b \tau^+ \nu_\tau$), in good agreement with the theoretical predictions [572–578]. These SM

processes have relatively large cross sections and are the largest sources of τ leptons at the LHC. They constitute important backgrounds on new-physics searches, which need to be carefully studied. Similar analyses were done previously at UA1 [579] and the Tevatron [580, 581].

ATLAS has reported the first τ polarization measurement ever made at hadron colliders, using the $\tau^- \rightarrow \rho^- \nu_\tau \rightarrow \pi^- \pi^0 \nu_\tau$ decay in $W^- \rightarrow \tau^- \bar{\nu}_\tau$ [582]. The angle between the τ flight direction and the hadronic decay products in the τ rest frame is sensitive to the τ polarization [583, 584], but its measurement requires the τ energy which is difficult to reconstruct in $W^- \rightarrow \tau^- \bar{\nu}_\tau$. ATLAS uses instead the angle between the flight directions of the ρ^- and the π^- , which is related to the kinematics of the two final pions. Using the correlation between the τ and ρ polarizations, this allows to determine $\mathcal{P}_\tau = -1.06 \pm 0.04_{-0.07}^{+0.05}$ [582], in agreement with the SM. This method is independent of the mode of τ production and could be applied to the characterization of new phenomena at the LHC. In particular, \mathcal{P}_τ may be used as a discriminating variable in searches for new particles.

The τ is the heaviest lepton coupling to the Higgs; with $M_H = 126$ GeV, the decay $H \rightarrow \tau^+ \tau^-$ has the fourth largest Higgs branching ratio. Thus, this decay mode plays a very important role to establish the nature of the recently discovered Higgs-like particle [113, 114]. The experimental analyses are quantified in terms of the signal-strength parameter, measuring the product of Higgs production cross section and branching ratio, normalized to the SM prediction. In the $H \rightarrow \tau^+ \tau^-$ mode ATLAS quotes $\mu_{\tau\tau} = 0.7_{-0.6}^{+0.7}$ [113], while CMS finds $\mu_{\tau\tau} = 1.10 \pm 0.41$ [114]. The ATLAS result is consistent with either the SM or the absence of a $H\tau^+\tau^-$ coupling, while the significance of the CMS value is still below 3σ . Adding all detected Higgs decay channels ($b\bar{b}$, W^*W , Z^*Z , $\tau^+\tau^-$ and $\gamma\gamma$), ATLAS finds $\mu = 1.23 \pm 0.18$ and CMS $\mu = 0.80 \pm 0.14$, which lead to an average signal strength $\langle\mu\rangle = 0.96 \pm 0.11$, in perfect agreement with the SM.

The new boson appears to couple to the known gauge bosons with the strength expected for the SM Higgs [585–589], although ATLAS observes an excess of events in the 2γ decay channel, compared with the SM expectation. Moreover, its fermionic couplings seem compatible with a linear dependence with the fermion mass, scaled by the electroweak scale $v \approx 246$ GeV. Thus, it has the properties expected for a Higgs-like particle, related with the spontaneous breaking of the electroweak symmetry. An obvious question to address is whether it corresponds to the unique Higgs boson incorporated in the SM, or it is just the first signal of a much richer scalar sector [590].

Present searches for new phenomena, taking advantage of the τ signal, include bounds on narrow resonances [591, 592], third generation leptoquarks [593, 594], supersymmetric neutral ($H \rightarrow \tau^+ \tau^-$) [595, 596] and charged ($H^+ \rightarrow \tau^+ \nu_\tau$) [597, 598] Higgses, other supersymmetric particles [599, 600] and the BaBar constraints on a light CP-odd neutral scalar ($\Upsilon \rightarrow \gamma A^0 \rightarrow \gamma \tau^+ \tau^-$) [601]. Significant improvements are to be expected with the increasing energy and luminosity of the LHC and the use of more refined tools to disentangle different new-physics scenarios.

14 Outlook

The flavour structure of the SM is one of the main pending questions in our understanding of weak interactions. Although we do not know the reason of the observed family replication, we have learned experimentally that the number of SM fermion generations is just three (and no more). Therefore, we must study as precisely as possible the few existing flavours to get some hints on the dynamics responsible for their observed structure.

The τ turns out to be an ideal laboratory to test the SM. It is a lepton, which means clean physics, and moreover it is heavy enough to produce a large variety of decay modes. Naively, one would expect the τ to be much more sensitive than the e or the μ to new physics related to the flavour and mass-generation problems. QCD studies can also benefit a lot from the existence of this heavy lepton, able to decay into hadrons. Owing to their semileptonic character, the hadronic τ decays provide a powerful

tool to investigate the low-energy effects of the strong interactions in rather simple conditions.

Our knowledge of the τ properties has been considerably improved with the clean $\tau^+\tau^-$ data samples collected at LEP and the large statistics accumulated at the B factories. Lepton universality has been tested to rather good accuracy, both in the charged and neutral current sectors. The Lorentz structure of the leptonic τ decays is not yet determined, but useful constraints on new-physics contributions have been established. The quality of the hadronic τ decay data has made possible to perform quantitative QCD tests and determine the strong coupling constant very accurately, providing a nice experimental verification of asymptotic freedom. A quite competitive determination of the Cabibbo mixing has also been obtained from τ decays into final states with strangeness, and significant improvements could be expected with the larger statistics of future flavour factories.

Searches for non-standard phenomena have been pushed to the limits that the existing data samples allow to investigate. The first hints of new physics beyond the SM have emerged recently, with convincing evidence of neutrino oscillations from solar, atmospheric, accelerator and reactor neutrino experiments. The existence of lepton flavour violation opens a very interesting window to unknown phenomena, which we are just starting to explore. It seems possible to push the present limits on neutrinoless τ decays beyond the 10^{-8} or even 10^{-9} level. At the same time, new neutrino oscillation experiments will investigate whether CP violating phases are also present in the lepton mixing matrix.

At present, all experimental results on the τ lepton seem consistent with the SM, with the exception of the anomalous excess of τ production observed in some B decays, the CP rate asymmetry in $\tau \rightarrow \pi K_S \nu_\tau$ and the slight violation of universality in $W^- \rightarrow \tau^- \bar{\nu}_\tau$. These effects are, however, not very significant statistically. There is large room for improvements, which require much larger samples of τ events and a better control of systematic uncertainties. Future τ experiments will probe the SM to a much deeper level of sensitivity and will explore the frontier of its possible extensions.

While the τ lepton continues being an increasingly precise laboratory to perform relevant tests of QCD and the electroweak theory, we are witnessing the opening of a new era with this heavy lepton becoming now a superb tool in searches for new phenomena at high-energy colliders. Being one of the fermions most strongly coupled to the scalar sector, the τ plays a very important role in testing the Higgs properties and the dynamics behind the electroweak symmetry breaking. The ongoing LHC program will be complemented with refined low-energy measurements at Belle-II, Bes-III and, perhaps, a future Super Tau-Charm Factory, and more precise muon experiments. There is an exciting future ahead of us and unexpected surprises may arise, probably establishing the existence of new physics beyond the SM and offering clues to the problems of mass generation, fermion mixing and family replication.

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References

- [1] M.L. Perl *et al* (MARK I), *Phys. Rev. Lett.* 35 (1975) 1489
- [2] M. Perl, *Ann. Rev. Nucl. Part. Sci.* 30 (1980) 299
- [3] K.G. Hayes and M.L. Perl, *Phys. Rev. D* 38 (1988) 3351
- [4] B.C. Barish and R. Stroynowski, *Phys. Rep.* 157 (1988) 1
- [5] K.K. Gan and M.L. Perl, *Int. J. Mod. Phys. A* 3 (1988) 531

- [6] C. Kiesling, in *High Energy Electron–Positron Physics*, eds. A. Ali and P. Söding, Advanced Series on Directions in High Energy Physics – Vol. 1 (World Scientific, Singapore, 1988) p. 177
- [7] A. Pich, *Mod. Phys. Lett. A* 5 (1990) 1995
- [8] A. Pich, in *Heavy Flavours*, eds. A.J. Buras and M. Lindner, Advanced Series on Directions in High Energy Physics – Vol. 10 (World Scientific, Singapore, 1992) p. 375
- [9] M. Perl, *Rep. Prog. Phys.* 55 (1992) 653
- [10] K. Riles, *Int. J. Mod. Phys. A* 7 (1992) 7647
- [11] A.J. Weinstein and R. Stroynowski, *Ann. Rev. Nucl. Part. Sci.* 43 (1993) 457
- [12] S. Gentile and M. Pohl, *Phys. Rep.* 274 (1997) 287
- [13] A. Pich, in *Heavy Flavours II*, eds. A.J. Buras and M. Lindner, Advanced Series on Directions in High Energy Physics, Vol. 15 (World Scientific, Singapore, 1998), p. 453
- [14] A. Pich, *Int. J. Mod. Phys. A* 15S1 (2000) 157
- [15] A. Stahl, *Physics with Tau Leptons*, Springer Tracts in Modern Physics 160 (Springer, Berlin, 2000)
- [16] A. Pich, *Int. J. Mod. Phys. A* 21 (2006) 5652
- [17] M. Davier, A. Hocker and Z. Zhang, *Rev. Mod. Phys.* 78 (2006) 1043
- [18] A. Pich, *Nucl. Phys. Proc. Suppl.* 169 (2007) 393
- [19] A. Pich, I. Boyko, D. Dedovich and I.I. Bigi, *Int. J. Mod. Phys. A* 24S1 (2009) 715
- [20] A. Pich, arXiv:1301.4474 [hep-ph]
- [21] Y.S. Tsai, *Phys. Rev. D* 4 (1971) 2821 [*Err*: D 13 (1976) 771]
- [22] Heavy Flavor Averaging Group, arXiv:1207.1158 [hep-ex]; <http://www.slac.stanford.edu/xorg/hfag/>
- [23] Particle Data Group, *Phys. Rev. D* 86 (2012) 010001; <http://pdg.lbl.gov/>
- [24] R.E. Behrends, R.J. Finkelstein and A. Sirlin, *Phys. Rev.* 101 (1956) 866
- [25] S.M. Berman, *Phys. Rev.* 112 (1958) 267
- [26] T. Kinoshita and A. Sirlin, *Phys. Rev.* 113 (1959) 1652
- [27] M. Roos and A. Sirlin, *Nucl. Phys. B* 29 (1971) 296
- [28] W.J. Marciano and A. Sirlin, *Phys. Rev. Lett.* 61 (1988) 1815
- [29] A. Ferroglia, G. Ossola and A. Sirlin, *Nucl. Phys. B* 560 (1999) 23
- [30] T. van Ritbergen and R.G. Stuart, *Phys. Rev. Lett.* 82 (1999) 488; *Nucl. Phys. B* 564 (2000) 343; *Phys. Lett. B* 437 (1998) 201
- [31] M. Steinhauser and T. Seidensticker, *Phys. Lett. B* 467 (1999) 271
- [32] A. Pak and A. Czarnecki, *Phys. Rev. Lett.* 100 (2008) 241807
- [33] A. Ferroglia, C. Greub, A. Sirlin and Z. Zhang, *Phys. Rev. D* 88 (2013) 033012
- [34] M. Fael, L. Mercolli and M. Passera, *Phys. Rev. D* 88 (2013) 093011
- [35] MuLan Collaboration, *Phys. Rev. Lett.* 106 (2011) 041803
- [36] Belle Collaboration, arXiv:1310.8503 [hep-ex]
- [37] BaBar Collaboration, *Phys. Rev. Lett.* 105 (2010) 051602
- [38] M.B. Voloshin, *Phys. Lett. B* 556 (2003) 153
- [39] B.H. Smith and M.B. Voloshin, *Phys. Lett. B* 324 (1994) 117 [*Erratum-ibid.* B 333 (1994) 564]
- [40] P. Ruiz-Femenia and A. Pich, *Phys. Rev. D* 64 (2001) 053001
- [41] D.M. Asner *et al.*, *Int. J. Mod. Phys. A* 24 (2009) S1
- [42] X.H. Mo, talk at the 12th International Workshop on Tau Lepton Physics (TAU2012), Nagoya, Japan, September 2012 [<http://tau2012.hepl.phys.nagoya-u.ac.jp/>]

- [43] W.J. Marciano and A. Sirlin, *Phys. Rev. Lett.* 71 (1993) 3629
- [44] M. Finkemeier, *Phys. Lett. B* 387 (1996) 391
- [45] V. Cirigliano and I. Rosell, *JHEP* 0710 (2007) 005; *Phys. Rev. Lett.* 99 (2007) 231801
- [46] G. Czapek *et al.*, *Phys. Rev. Lett.* 70 (1993) 17
- [47] D.I. Britton *et al.*, *Phys. Rev. Lett.* 68 (1992) 3000
- [48] D.A. Bryman *et al.*, *Phys. Rev. D* 33 (1986) 1211; *Phys. Rev. Lett.* 50 (1983) 7
- [49] NA62 Collaboration, *Phys. Lett. B* 719 (2013) 326, B 698 (2011) 105
- [50] KLOE Collaboration, *Eur. Phys. J. C* 64 (2009) 627 [Erratum-ibid. 65 (2010) 703]
- [51] PEN Collaboration, *AIP Conf. Proc.* 1182 (2009) 698
- [52] PIENU Collaboration, *AIP Conf. Proc.* 1441 (2012) 564
- [53] V. Cirigliano, G. Ecker, H. Neufeld, A. Pich and J. Portolés, *Rev. Mod. Phys.* 84 (2012) 399
- [54] M. Antonelli *et al.*, *Eur. Phys. J. C* 69 (2010) 399
- [55] R. Decker and M. Finkemeier, *Phys. Lett. B* 334 (1994) 199; *Nucl. Phys. B* 438 (1995) 17; *Nucl. Phys. B (Proc. Suppl.)* 40 (1995) 453
- [56] ALEPH, DELPHI, L3 and OPAL Collaborations and LEP Electroweak Working Group, *Phys. Rept.* 532 (2013) 119
- [57] A. Filipuzzi, J. Portolés and M. González-Alonso, *Phys. Rev. D* 85 (2012) 116010
- [58] M. Jung, A. Pich and P. Tuzón, *JHEP* 1011 (2010) 003
- [59] L. Michel, *Proc. Phys. Soc. A* 63 (1950) 514, 1371
- [60] C. Bouchiat and L. Michel, *Phys. Rev.* 106 (1957) 170
- [61] T. Kinoshita and A. Sirlin, *Phys. Rev.* 107 (1957) 593, 108 (1957) 844
- [62] F. Scheck, *Leptons, Hadrons and Nuclei* (North-Holland, Amsterdam, 1983); *Phys. Rep.* 44 (1978) 187
- [63] W. Fetscher, H.-J. Gerber and K.F. Johnson, *Phys. Lett. B* 173 (1986) 102
- [64] W. Fetscher and H.-J. Gerber, in *Precision Tests of the Standard Electroweak Model*, ed. P. Langacker, Advanced Series on Directions in High Energy Physics – Vol. 14 (World Scientific, Singapore, 1995), p. 657
- [65] A. Pich and J.P. Silva, *Phys. Rev. D* 52 (1995) 4006
- [66] A. Rouge, *Eur. Phys. J. C* 18 (2001) 491
- [67] C. Jarlskog, *Nucl. Phys.* 75 (1966) 659
- [68] L.Ph. Roesch *et al.*, *Helv. Phys. Acta* 55 (1982) 74
- [69] A. Jodidio *et al.*, *Phys. Rev. D* 37 (1988) 237, D 34 (1986) 1967
- [70] W. Fetscher, *Phys. Lett. B* 140 (1984) 117
- [71] K. Mursula, M. Roos and F. Scheck, *Nucl. Phys. B* 219 (1983) 321
- [72] TWIST Collaboration, *Phys. Rev. D* 85 (2012) 092013, D 84 (2011) 032005, D 78 (2008) 032010; *Phys. Rev. Lett.* 106 (2011) 041804
- [73] C.A. Gagliardi, R.E. Tribble and N.J. Williams, *Phys. Rev. D* 72 (2005) 073002
- [74] S.R. Mishra *et al.*, *Phys. Lett. B* 252 (1990) 170
- [75] CHARM-II Collaboration, *Phys. Lett. B* 364 (1995) 121
- [76] W. Fetscher and H.-J. Gerber, in [23]
- [77] S. Kawasaki, T. Shirafuji and S.Y. Tsai, *Prog. Theor. Phys.* 49 (1973) 1656
- [78] S.-Y. Pi and A.I. Sanda, *Ann. Phys. (N.Y.)* 106 (1977) 171

- [79] H.K. Kühn and F. Wagner, *Nucl. Phys. B* 236 (1984) 16
- [80] C.A. Nelson, *Phys. Rev. D* 43 (1991) 1465, *D* 40 (1989) 123 [*Err:* *D* 41 (1990) 2327]; *Phys. Rev. Lett.* 62 (1989) 1347
- [81] S. Goozovat and C.A. Nelson, *Phys. Rev. D* 44 (1991) 2818
- [82] W. Fetscher, *Phys. Rev. D* 42 (1990) 1544
- [83] J. Bernabéu, N. Rius and A. Pich, *Phys. Lett. B* 257 (1991) 219
- [84] R. Alemany, N. Rius, J. Bernabéu, J.J. Gómez-Cadenas and A. Pich, *Nucl. Phys. B* 379 (1992) 3
- [85] M. Davier, L. Duflot, F. Le Diberder and A. Rouge, *Phys. Lett. B* 306 (1993) 411
- [86] SLD Collaboration, *Phys. Rev. Lett.* 78 (1997) 4691
- [87] A. Stahl and H. Voss, *Z. Phys. C* 74 (1997) 73
- [88] ALEPH Collaboration, *Eur. Phys. J. C* 22 (2001) 217
- [89] DELPHI Collaboration, *Eur. Phys. J. C* 16 (2000) 229
- [90] OPAL Collaboration, *Eur. Phys. J. C* 8 (1999) 3
- [91] L3 Collaboration, *Phys. Lett. B* 438 (1998) 405, *B* 377 (1996) 313
- [92] ARGUS Collaboration, *Phys. Lett. B* 431 (1998) 179, *B* 349 (1995) 576, *B* 341 (1995) 441, *B* 316 (1993) 608, *B* 246 (1990) 278
- [93] CLEO Collaboration, *Phys. Rev. D* 56 (1997) 5320, *D* 32 (1985) 2468; *Phys. Rev. Lett.* 78 (1997) 4686
- [94] Crystal Ball Collaboration, *Phys. Lett. B* 228 (1989) 273
- [95] W.T. Ford *et al.*, *Phys. Rev. D* 36 (1987) 1971
- [96] N. Danneberg *et al.*, *Phys. Rev. Lett.* 94 (2005) 021802
- [97] B. Balke *et al.*, *Phys. Rev. D* 37 (1988) 587
- [98] I. Beltrami *et al.*, *Phys. Lett. B* 194 (1987) 326
- [99] H. Burkard *et al.*, *Phys. Lett. B* 160 (1985) 343, *B* 150 (1985) 242, *B* 129 (1983) 260
- [100] S.E. Derenzo, *Phys. Rev.* 181 (1969) 1854
- [101] A. Stahl, in [23]
- [102] A. Stahl, *Phys. Lett. B* 324 (1994) 121
- [103] K. Mursula and F. Scheck, *Nucl. Phys. B* 253 (1985) 189
- [104] H. Thurn and H. Kolanoski, *Z. Phys. C* 60 (1993) 277
- [105] CLEO Collaboration, *Phys. Rev. D* 61 (2000) 012002, *D* 55 (1997) 7291
- [106] OPAL Collaboration, *Z. Phys. C* 75 (1997) 593 *Z. Phys. C* 67 (1995) 45
- [107] ARGUS Collaboration, *Z. Phys. C* 58 (1993) 61; *Phys. Lett. B* 250 (1990) 164
- [108] A. Pich, arXiv:1201.0537 [hep-ph]
- [109] ALEPH, DELPHI, L3, OPAL, SLD, LEP Electroweak Working Group, SLD Electroweak Group and SLD Heavy Flavour Group Collaborations, *Phys. Rept.* 427 (2006) 257; <http://www.cern.ch/LEPEWWG/>
- [110] ALEPH Collaboration, *Phys. Lett. B* 405 (1997) 191
- [111] DELPHI Collaboration, *Phys. Lett. B* 404 (1997) 194
- [112] ALEPH, CDF, D0, DELPHI, L3, OPAL, SLD, LEP Electroweak Working Group, Tevatron Electroweak Working Group and SLD Electroweak and Heavy Flavour Groups Collaborations, arXiv:1012.2367 [hep-ex]
- [113] ATLAS Collaboration, *Phys. Lett. B* 716 (2012) 1, *B* 726 (2013) 88

- [114] CMS Collaboration, *Phys. Lett. B* 716 (2012) 30; *JHEP* 1306 (2013) 081; CMS-PAS-HIG-13-005 (2013)
- [115] ALEPH Collaboration, *Phys. Lett. B* 346 (1995) 379 [Erratum-ibid. B 363 (1995) 265], B 321 (1994) 168
- [116] CHARM-II Collaboration, *Phys. Lett. B* 320 (1994) 203
- [117] DONUT Collaboration, *Phys. Lett. B* 504 (2001) 218; *Phys. Rev. D* 78 (2008) 052002
- [118] A. Pich, in Proc. Tau-Charm Factory Workshop (SLAC, 1989), ed. L.V. Beers, SLAC-Report-343 (1989) p. 416
- [119] H.B. Thacker and J.J. Sakurai, *Phys. Lett. B* 36 (1971) 103
- [120] E. Braaten and C.-S. Li, *Phys. Rev. D* 42 (1990) 3888
- [121] J. Erler, *Rev. Mex. Fis.* 50 (2004) 200
- [122] F.J. Gilman and D.H. Miller, *Phys. Rev. D* 17 (1978) 1846
- [123] F.J. Gilman and S.H. Rhie, *Phys. Rev. D* 31 (1985) 1066
- [124] F.J. Gilman, *Phys. Rev. D* 35 (1987) 3541
- [125] J.H. Kühn and A. Santamaria, *Z. Phys. C* 48 (1990) 445
- [126] S.I. Eidelman and V.N. Ivanchenko, *Phys. Lett. B* 257 (1991) 437
- [127] S. Narison and A. Pich, *Phys. Lett. B* 304 (1993) 359
- [128] J.H. Kühn and E. Mirkes, *Z. Phys. C* 56 (1992) 661 [Erratum-ibid. C 67 (1995) 364]; *Phys. Lett. B* 286 (1992) 381
- [129] S. Weinberg, *Physica A* 96 (1979) 327
- [130] J. Gasser and H. Leutwyler, *Nucl. Phys. B* 250 (1985) 465, 517, 539
- [131] G. Ecker, *Prog. Part. Nucl. Phys.* 35 (1995) 1
- [132] A. Pich, *Rept. Prog. Phys.* 58 (1995) 563
- [133] V. Bernard and U.-G. Meissner, *Ann. Rev. Nucl. Part. Sci.* 57 (2007) 33
- [134] J. Wess and B. Zumino, *Phys. Lett. B* 37 (1971) 95
- [135] E. Witten, *Nucl. Phys. B* 223 (1983) 422
- [136] G. Colangelo, M. Finkemeier and R. Urech, *Phys. Rev. D* 54 (1996) 4403
- [137] G. Ecker and R. Unterdorfer, *Eur. Phys. J. C* 24 (2002) 535
- [138] R. Unterdorfer, *JHEP* 0207 (2002) 053
- [139] H. Davoudiasl and M.B. Wise, *Phys. Rev. D* 53 (1996) 2523
- [140] R. Fischer, J. Wess and F. Wagner, *Z. Phys. C* 3 (1979) 313
- [141] G. Kramer and W.F. Palmer, *Z. Phys. C* 25 (1984) 195, 39 (1988) 423
- [142] A. Pich, *Phys. Lett. B* 196 (1987) 561
- [143] E. Braaten, R.J. Oakes and S.-M. Tse, *Phys. Rev. D* 36 (1987) 2188; *Int. J. Mod. Phys. A* 5 (1990) 2737
- [144] J.J. Gómez-Cadenas, M.C. González-García and A. Pich, *Phys. Rev. D* 42 (1990) 3093
- [145] S. Jadach *et. al.*, *Comput. Phys. Commun.* 64 (1991) 275, 76 (1993) 361
- [146] R. Decker, E. Mirkes, R. Sauer and Z. Was, *Z. Phys. C* 58 (1993) 445
- [147] R. Decker and E. Mirkes, *Phys. Rev. D* 47 (1993) 4012; *Z. Phys. C* 57 (1993) 495
- [148] R. Decker, M. Finkemeier and E. Mirkes, *Phys. Rev. D* 50 (1994) 6863
- [149] R. Decker, M. Finkemeier, P. Heiliger and H.H. Jonsson, *Z. Phys. C* 70 (1996) 247
- [150] M. Finkemeier and E. Mirkes, *Z. Phys. C* 69 (1996) 243, C 72 (1996) 619

- [151] J. Portolés, *Nucl. Phys. Proc. Suppl.* 169 (2007) 3, 144 (2005) 3, 98 (2001) 210
- [152] G. Ecker, J. Gasser, A. Pich and E. de Rafael, *Nucl. Phys. B* 321 (1989) 311
- [153] G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, *Phys. Lett. B* 223 (1989) 425
- [154] A. Pich, in *Phenomenology of Large N_C QCD*, ed. R. Lebed, Proc. Inst for Nuclear Theory – Vol. 12 (World Scientific, Singapore, 2002), p.239
- [155] P.D. Ruiz-Femenía, A. Pich and J. Portolés, *JHEP* 0307 (2003) 003
- [156] V. Cirigliano *et al.*, *Nucl. Phys. B* 753 (2006) 139; *JHEP* 0504 (2005) 006; *Phys. Lett. B* 596 (2004) 96
- [157] A. Pich, I. Rosell and J.J. Sanz-Cillero, *JHEP* 1102 (2011) 109, 0408 (2004) 042
- [158] A. Pich, I. Rosell and J.J. Sanz-Cillero, *JHEP* 0807 (2008) 014, 0701 (2007) 039
- [159] I. Rosell, P. Ruiz-Femenía and J. Portolés, *JHEP* 0512 (2005) 020
- [160] K. Kampf and J. Novotny, *Phys. Rev. D* 84 (2011) 014036
- [161] V. Cirigliano and H. Neufeld, *Phys. Lett. B* 700 (2011) 7
- [162] W. J. Marciano, *Phys. Rev. Lett.* 93 (2004) 231803
- [163] G. Colangelo *et al.*, *Eur. Phys. J. C* 71 (2011) 1695
- [164] I.S. Towner and J.C. Hardy, *Rept. Prog. Phys.* 73 (2010) 046301
- [165] V. Cirigliano, G. Ecker and H. Neufeld, *JHEP* 0208 (2002) 002; *Phys. Lett. B* 513 (2001) 361
- [166] M. Davier *et al.*, *Eur. Phys. J. C* 66 (2010) 127
- [167] M. Antonelli, V. Cirigliano, A. Lusiani and E. Passemar, *JHEP* 1310 (2013) 070
- [168] F.V. Flores-Báez, J.R. Morones-Ibarra, G.C. Quiroz and D.R. Balbuena, *Phys. Rev. D* 88 (2013) 073009
- [169] F. Guerrero and A. Pich, *Phys. Lett. B* 412 (1997) 382
- [170] D. Gómez Dumm, A. Pich and J. Portolés, *Phys. Rev. D* 62 (2000) 054014
- [171] A. Pich and J. Portolés, *Phys. Rev. D* 63 (2001) 093005; *Nucl. Phys. Proc. Suppl.* 121 (2003) 179
- [172] J.J. Sanz-Cillero and A. Pich, *Eur. Phys. J. C* 27 (2003) 587
- [173] G. 't Hooft, *Nucl. Phys. B* 72 (1974) 461, 75 (1974) 461
- [174] E. Witten, *Nucl. Phys. B* 160 (1979) 57
- [175] C.A. Domínguez, *Phys. Lett. B* 512 (2001) 331
- [176] C. Bruch, A. Khodjamirian and J.H. Kühn, *Eur. Phys. J. C* 39 (2005) 41
- [177] R. Omnès, *Nuovo Cim.* 8 (1958) 316
- [178] ALEPH Collaboration, *Z. Phys. C* 76 (1997) 15
- [179] CLEO Collaboration, *Phys. Rev. D* 61 (2000) 112002
- [180] L.M. Barkov *et al.*, *Nucl. Phys. B* 256 (1985) 365
- [181] S.R. Amendolia *et al.*, *Nucl. Phys. B* 277 (1986) 168
- [182] Belle Collaboration, *Phys. Rev. D* 78 (2008) 072006
- [183] D. Gómez Dumm and P. Roig, *Eur. Phys. J. C* 73 (2013) 2528
- [184] BaBar Collaboration, *Phys. Rev. D* 86 (2012) 032013; *Phys. Rev. Lett.* 103 (2009) 231801
- [185] CMD-2 Collaboration, *Phys. Lett. B* 648 (2007) 28; *JETP Lett.* 84 (2006) 413, 82 (2005) 743
- [186] DM2 Collaboration, *Phys. Lett. B* 220 (1989) 321
- [187] KLOE Collaboration, *Phys. Lett. B* 720 (2013) 336, B 700 (2011) 102, B 670 (2009) 285, B 606 (2005) 12
- [188] SND Collaboration, *JETP* 103 (2006) 380, 101 (2005) 1053

- [189] ALEPH Collaboration, *Phys. Rept.* 421 (2005) 191
- [190] S. Binner, J.H. Kühn and K. Melnikov, *Phys. Lett. B* 459 (1999) 279
- [191] G. Rodrigo *et al.*, *Eur. Phys. J. C* 24 (2002) 71, C 27 (2003) 563, C 33 (2004) 333, C 39 (2005) 411, C 47 (2006) 617
- [192] S. Actis *et al.*, *Eur. Phys. J. C* 66 (2010) 585
- [193] V.P. Druzhinin, S.I. Eidelman, S.I. Serednyakov and E.P. Solodov, *Rev. Mod. Phys.* 83 (2011) 1545
- [194] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, *Eur. Phys. J. C* 71 (2011) 1515 [Erratum-ibid. C 72 (2012) 1874]
- [195] M. Jamin, A. Pich and J. Portolés, *Phys. Lett. B* 664 (2008) 78, B 640 (2006) 176
- [196] D.R. Boito, R. Escribano and M. Jamin, *JHEP* 1009 (2010) 031; *Eur. Phys. J. C* 59 (2009) 821
- [197] B. Moussallam, *Eur. Phys. J. C* 53 (2008) 401
- [198] M. Jamin, J.A. Oller and A. Pich, *Nucl. Phys. B* 622 (2002) 279, B 587 (2000) 331
- [199] M. Jamin, J.A. Oller and A. Pich, *Phys. Rev. D* 74 (2006) 074009; *JHEP* 0402 (2004) 047; *Eur. Phys. J. C* 24 (2002) 237
- [200] P. Buettiker, S. Descotes-Genon and B. Moussallam, *Eur. Phys. J. C* 33 (2004) 409
- [201] Belle Collaboration, *Phys. Lett. B* 654 (2007) 65
- [202] A. Adametz [BaBar Collaboration], *Nucl. Phys. Proc. Suppl.* 218 (2011) 134
- [203] R. Escribano, S. González-Solís and P. Roig, *JHEP* 1310 (2013) 039
- [204] BaBar Collaboration, *Phys. Rev. D* 83 (2011) 032002
- [205] Belle Collaboration, *Phys. Lett. B* 672 (2009) 209
- [206] BaBar Collaboration, *Phys. Rev. D* 86 (2012) 092010, D 77 (2008) 112002
- [207] A. Bramon, S. Narison and A. Pich, *Phys. Lett. B* 196 (1987) 543
- [208] S. Tisserant and T.N. Truong, *Phys. Lett. B* 115 (1982) 264
- [209] H. Neufeld and H. Rupertsberger, *Z. Phys. C* 68 (1995) 91
- [210] S. Descotes-Genon, E. Kou and B. Moussallam, arXiv:1303.2879 [hep-ph]
- [211] S. Nussinov and A. Soffer, *Phys. Rev. D* 78 (2008) 033006
- [212] N. Paver and Riazuddin, *Phys. Rev. D* 82 (2010) 057301
- [213] O. Shekhovtsova *et al.*, *Phys. Rev. D* 86 (2012) 113008; arXiv:1301.1964 [hep-ph]
- [214] D. Gómez Dumm, P. Roig, A. Pich and J. Portolés, *Phys. Lett. B* 685 (2010) 158
- [215] D. Gómez Dumm, A. Pich and J. Portolés, *Phys. Rev. D* 69 (2004) 073002
- [216] ALEPH Collaboration, *Eur. Phys. J. C* 4 (1998) 409
- [217] ALEPH Collaboration, *Phys. Lett. B* 307 (1993) 209
- [218] CLEO Collaboration, *Phys. Rev. D* 61 (2000) 052004, 012002
- [219] H.-K. Kühn and F. Wagner, *Nucl. Phys. B* 236 (1984) 16
- [220] M. Feindt, *Z. Phys. C* 48 (1990) 681
- [221] D. Gómez Dumm, P. Roig, A. Pich and J. Portolés, *Phys. Rev. D* 81 (2010) 034031
- [222] P. Roig, arXiv:1301.7626 [hep-ph]
- [223] D. Gómez Dumm and P. Roig, *Phys. Rev. D* 86 (2012) 076009
- [224] M. Finkemeier, J.H. Kühn and E. Mirkes, *Nucl. Phys. Proc. Suppl.* 55C (1997) 169
- [225] M. Davier *et al.*, *Eur. Phys. J. C* 56 (2008) 305
- [226] L.Y. Dai, J. Portolés and O. Shekhovtsova, *Phys. Rev. D* 88 (2013) 056001

- [227] Z.-H. Guo and P. Roig, *Phys. Rev. D* 82 (2010) 113016
- [228] A. Guevara, G. López Castro and P. Roig, *Phys. Rev. D* 88 (2013) 033007
- [229] BaBar Collaboration, *Phys. Rev. D* 86 (2012) 092013
- [230] S. Ryu [Belle Collaboration], arXiv:1302.4565 [hep-ex]
- [231] I. M. Nugent [BaBar Collaboration], arXiv:1301.7105 [hep-ex]
- [232] S. Narison and A. Pich, *Phys. Lett. B* 211 (1988) 183
- [233] E. Braaten, *Phys. Rev. Lett.* 60 (1988) 1606; *Phys. Rev. D* 39 (1989) 1458
- [234] E. Braaten, S. Narison and A. Pich, *Nucl. Phys. B* 373 (1992) 581
- [235] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys. B* 147 (1979) 385, 448, 519
- [236] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Phys. Rev. Lett.* 101 (2008) 012002
- [237] M. Beneke and M. Jamin, *JHEP* 0809 (2008) 044
- [238] M. Beneke, D. Boito and M. Jamin, *JHEP* 1301 (2013) 125
- [239] I. Caprini and J. Fischer, *Phys. Rev. D* 84 (2011) 054019; *Eur. Phys. J. C* 64 (2009) 35
- [240] G. Abbas, B. Ananthanarayan, I. Caprini and J. Fischer, *Phys. Rev. D* 87 (2013) 014008, D 88 (2013) 034026
- [241] G. Abbas, B. Ananthanarayan and I. Caprini, *Phys. Rev. D* 85 (2012) 094018
- [242] S. Groote, J.G. Korner and A.A. Pivovarov, *Phys. Part. Nucl.* 44 (2013) 285
- [243] K. Maltman and T. Yavin, *Phys. Rev. D* 78 (2008) 094020
- [244] D. Boito *et al.*, *Phys. Rev. D* 85 (2012) 093015, D 84 (2011) 113006
- [245] S. Menke, arXiv:0904.1796 [hep-ph]
- [246] S. Narison, *Phys. Lett. B* 673 (2009) 30
- [247] G. Cvetic, M. Loewe, C. Martínez and C. Valenzuela, *Phys. Rev. D* 82 (2010) 093007
- [248] A. Pich, arXiv:1107.1123 [hep-ph]; *Nucl. Phys. Proc. Suppl.* 218 (2011) 89, 39BC (1995) 326; *Acta Phys. Polon. Supp.* 3 (2010) 165
- [249] A. Pich, *PoS ConfinementX* (2012) 022
- [250] S.L. Adler, *Phys. Rev. D* 10 (1974) 3714
- [251] S.G. Gorishnii, A.L. Kataev and S.A. Larin, *Phys. Lett. B* 259 (1991) 144
- [252] L.R. Surguladze and M.A. Samuel, *Phys. Rev. Lett.* 66 (1991) 560 [Erratum-ibid. 66 (1991) 2416]
- [253] F. Le Diberder and A. Pich, *Phys. Lett. B* 286 (1992) 147
- [254] A.A. Pivovarov, *Z. Phys. C* 53 (1992) 461
- [255] A.L. Kataev and V.V. Starshenko, *Mod. Phys. Lett. A* 10 (1995) 235
- [256] G. Cvetic, C. Dib, T. Lee and I. Schmidt, *Phys. Rev. D* 64 (2001) 093016
- [257] P. Ball, M. Beneke and V.M. Braun, *Nucl. Phys. B* 452 (1995) 563
- [258] M. Neubert, *Nucl. Phys. B* 463 (1996) 511
- [259] G. Altarelli, P. Nason and G. Ridolfi, *Z. Phys. C* 68 (1995) 257
- [260] M. Jamin, *JHEP* 0509 (2005) 058
- [261] S. Descotes-Genon and B. Malaescu, arXiv:1002.2968 [hep-ph]
- [262] C.J. Maxwell and A. Mirjalili, *Nucl. Phys. B* 611 (2001) 423
- [263] C.J. Maxwell and D.G. Tonge, *Nucl. Phys. B* 535 (1998) 19, B 481 (1996) 681
- [264] C.N. Lovett-Turner and C.J. Maxwell, *Nucl. Phys. B* 452 (1995) 188, B 432 (1994) 147
- [265] C.J. Maxwell and J.A. Nicholls, *Phys. Lett. B* 236 (1990) 63

- [266] M. Beneke, *Phys. Rept.* 317 (1999) 1
- [267] K.G. Chetyrkin and A. Kwiatkowski, *Z. Phys. C* 59 (1993) 525
- [268] A. Pich and J. Prades, *JHEP* 9910 (1999) 004, 9806 (1998) 013
- [269] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Phys. Rev. Lett.* 95 (2005) 012003
- [270] F. Le Diberder and A. Pich, *Phys. Lett. B* 289 (1992) 165
- [271] OPAL Collaboration, *Eur. Phys. J. C* 7 (1999) 571
- [272] CLEO Collaboration, *Phys. Lett. B* 356 (1995) 580
- [273] M. Girone and M. Neubert, *Phys. Rev. Lett.* 76 (1996) 3061
- [274] P. Nason and M. Porrati, *Nucl. Phys. B* 421 (1994) 518
- [275] I.I. Balitsky, M. Beneke and V.M. Braun, *Phys. Lett. B* 318 (1993) 371
- [276] M.A. Shifman, *Quark hadron duality*, in *At the frontier of particle physics: Handbook of QCD*, ed. M.A. Shifman, Vol. 3 (World Scientific, Singapore, 2001) p. 1447
- [277] O. Cata, M. Golterman and S. Peris, *Phys. Rev. D* 79 (2009) 053002, *D* 77 (2008) 093006; *JHEP* 0508 (2005) 076
- [278] M. Jamin, *JHEP* 1109 (2011) 141
- [279] G. Rodrigo, A. Pich and A. Santamaria, *Phys. Lett. B* 424 (1998) 367
- [280] T. van Ritbergen, J.A. M. Vermaseren and S.A. Larin, *Phys. Lett. B* 400 (1997) 379
- [281] M. Czakon, *Nucl. Phys. B* 710 (2005) 485
- [282] Y. Schroder and M. Steinhauser, *JHEP* 0601 (2006) 051
- [283] K.G. Chetyrkin, J.H. Kühn and C. Sturm, *Nucl. Phys. B* 744 (2006) 121
- [284] G. Amoros, J. Bijnens and P. Talavera, *Nucl. Phys. B* 568 (2000) 319
- [285] S. Weinberg, *Phys. Rev. Lett.* 18 (1967) 507
- [286] M. Davier, L. Girlanda, A. Hoecker and J. Stern, *Phys. Rev. D* 58 (1998) 096014
- [287] M. González-Alonso, A. Pich and J. Prades, *Phys. Rev. D* 82 (2010) 014019, *D* 81 (2010) 074007, *D* 78 (2008) 116012
- [288] T. Das, G.S. Guralnik, V.S. Mathur, F.E. Low and J.E. Young, *Phys. Rev. Lett.* 18 (1967) 759
- [289] D. Boito, M. Golterman, M. Jamin, K. Maltman and S. Peris, *Phys. Rev. D* 87 (2013) 094008
- [290] J. Bijnens, E. Gámiz and J. Prades, *JHEP* 0110 (2001) 009
- [291] K.N. Zybalyuk, *Eur. Phys. J. C* 38 (2004) 215
- [292] J. Rojo and J.I. Latorre, *JHEP* 0401 (2004) 055
- [293] S. Narison, *Phys. Lett. B* 624 (2005) 223
- [294] S. Peris, B. Phily and E. de Rafael, *Phys. Rev. Lett.* 86 (2001) 14
- [295] S. Friot, D. Greynat and E. de Rafael, *JHEP* 0410 (2004) 043
- [296] A.A. Almasy, K. Schilcher and H. Spiesberger, *Eur. Phys. J. C* 55 (2008) 237; *Phys. Lett. B* 650 (2007) 179
- [297] P. Masjuan and S. Peris, *JHEP* 0705 (2007) 040
- [298] V. Cirigliano, J.F. Donoghue, E. Golowich and K. Maltman, *Phys. Lett. B* 555 (2003) 71
- [299] V. Cirigliano, E. Golowich and K. Maltman, *Phys. Rev. D* 68 (2003) 054013
- [300] C.A. Domínguez and K. Schilcher, *Phys. Lett. B* 581 (2004) 193
- [301] J. Bordes, C.A. Domínguez, J. Peñarrocha and K. Schilcher, *JHEP* 0602 (2006) 037
- [302] E. Shintani *et al.*, *Phys. Rev. Lett.* 101 (2008) 242001
- [303] P.A. Boyle *et al.*, *Phys. Rev. D* 81 (2010) 014504; arXiv:1311.0397 [hep-ph]

- [304] E. Gámiz *et al.*, *Phys. Rev. Lett.* 94 (2005) 011803; *JHEP* 0301 (2003) 060; *PoS KAON* (2008) 008; hep-ph/0610246
- [305] S. Chen *et al.*, *Eur. Phys. J. C* 22 (2001) 31; *Nucl. Phys. Proc. Suppl.* 98 (2001) 319
- [306] K.G. Chetyrkin, J.H. Kühn and A.A. Pivovarov, *Nucl. Phys. B* 533 (1998) 473
- [307] J.G. Korner, F. Krajewski and A.A. Pivovarov, *Eur. Phys. J. C* 20 (2001) 259
- [308] K. Maltman and C.E. Wolfe, *Phys. Lett. B* 639 (2006) 283
- [309] J. Kambor and K. Maltman, *Phys. Rev. D* 62 (2000) 093023, *D* 64 (2001) 093014
- [310] K. Maltman, *Phys. Rev. D* 58 (1998) 093015
- [311] H. Leutwyler, *Phys. Lett. B* 378 (1996) 313
- [312] J. Laiho, E. Lunghi and R.S. Van de Water, *Phys. Rev. D* 81 (2010) 034503; <http://latticeaverages.org/>
- [313] E. Gámiz, arXiv:1301.2206 [hep-ph]
- [314] A. Bazavov *et al.*, *Phys. Rev. D* 87 (2013) 073012
- [315] P.A. Boyle *et al.*, *JHEP* 1308 (2013) 132
- [316] K. Maltman, *Phys. Lett. B* 672 (2009) 257; *Nucl. Phys. Proc. Suppl.* 218 (2011) 146
- [317] K. Maltman *et al.*, *Nucl. Phys. Proc. Suppl.* 189 (2009) 175; *Int. J. Mod. Phys. A* 23 (2008) 3191
- [318] V.W. Hughes and T. Kinoshita, *Rev. Mod. Phys.* 71 (1999) S133
- [319] M. Davier and W.J. Marciano, *Ann. Rev. Nucl. Part. Sci.* 54 (2004) 115
- [320] M. Passera, *J. Phys. G* 31 (2005) R75
- [321] K. Melnikov and A. Vainshtein, *Theory of the muon anomalous magnetic moment*, Springer Tracts Mod. Phys. – Vol. 216 (2006) (Springer, Berlin - Heidelberg, 2006)
- [322] J.P. Miller, E. de Rafael and B.L. Roberts, *Rept. Prog. Phys.* 70 (2007) 795
- [323] F. Jegerlehner and A. Nyffeler, *Phys. Rept.* 477 (2009) 1
- [324] B.L. Roberts and W.J. Marciano (eds.), *Lepton dipole moments*, Advanced series on directions in high energy physics – Vol. 20 (World Scientific, Singapore, 2009)
- [325] J.P. Miller, E. de Rafael, B.L. Roberts and D. Stöckinger, *Ann. Rev. Nucl. Part. Sci.* 62 (2012) 237
- [326] D. Hanneke, S. Fogwell and G. Gabrielse, *Phys. Rev. Lett.* 100 (2008) 120801
- [327] B.C. Odom, D. Hanneke, B. D’Urso and G. Gabrielse, *Phys. Rev. Lett.* 97 (2006) 030801 [Erratum-*ibid.* 99 (2007) 039902]
- [328] Muon $g - 2$ Collaboration *Phys. Rev. D* 73 (2006) 072003; *Phys. Rev. Lett.* 92 (2004) 161802, 89 (2002) 101804, 86 (2001) 2227
- [329] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, *Phys. Rev. Lett.* 109 (2012) 111807, 111808; *PTEP* 2012 (2012) 01A107
- [330] R. Bouchendir *et al.*, *Phys. Rev. Lett.* 106 (2011) 080801
- [331] A. Czarnecki, B. Krause and W.J. Marciano, *Phys. Rev. D* 52 (1995) 2619; *Phys. Rev. Lett.* 76 (1996) 3267
- [332] A. Czarnecki and W.J. Marciano, *Phys. Rev. D* 64 (2001) 013014
- [333] A. Czarnecki, W.J. Marciano and A. Vainshtein, *Phys. Rev. D* 67 (2003) 073006 [Erratum-*ibid.* *D* 73 (2006) 119901]
- [334] T. Gribov and A. Czarnecki, *Phys. Rev. D* 72 (2005) 053016
- [335] M. Knecht, S. Peris, M. Perrottet and E. De Rafael, *JHEP* 0211 (2002) 003

- [336] S. Peris, M. Perrottet and E. de Rafael, *Phys. Lett. B* 355 (1995) 523
- [337] S. Heinemeyer, D. Stockinger and G. Weiglein, *Nucl. Phys. B* 699 (2004) 103, *B* 690 (2004) 62
- [338] T.V. Kukhto, E.A. Kuraev, Z.K. Silagadze and A. Schiller, *Nucl. Phys. B* 371 (1992) 567
- [339] C. Gnendiger, D. Stöckinger and H. Stöckinger-Kim, *Phys. Rev. D* 88 (2013) 053005
- [340] C. Bouchiat and L. Michel, *J. Phys. Radium* 22 (1961) 121
- [341] S.J. Brodsky and E. De Rafael, *Phys. Rev.* 168 (1968) 1620
- [342] M. Davier, A. Hoecker, B. Malaescu, C. Z. Yuan and Z. Zhang, *Eur. Phys. J. C* 66 (2010) 1
- [343] F.V. Flores-Báez, A. Flores-Tlalpa, G. López Castro and G. Toledo Sánchez, *Phys. Rev. D* 74 (2006) 071301; *Nucl. Phys. Proc. Suppl.* 169 (2007) 250
- [344] M. Davier, S. Eidelman, A. Hoecker and Z. Zhang, *Eur. Phys. J. C* 31 (2003) 503, *C* 27 (2003) 497
- [345] R. Alemany, M. Davier and A. Hoecker, *Eur. Phys. J. C* 2 (1998) 123
- [346] M. Davier, *Ann. Rev. Nucl. Part. Sci.* 63 (2013) 407
- [347] BaBar Collaboration, *Phys. Rev. D* 86 (2012) 012008, *D* 85 (2012) 112009, *D* 77 (2008) 092002, *D* 76 (2007) 012008, 092006, 092005, *D* 74 (2006) 091103, 111103, *D* 73 (2006) 052003, 012005, *D* 71 (2005) 052001, *D* 70 (2004) 072004
- [348] CMD-2 Collaboration, *Phys. Lett. B* 669 (2008) 217, *B* 595 (2004) 101, *B* 578 (2004) 285, *B* 562 (2003) 173, *B* 527 (2002) 161, *B* 509 (2001) 217, *B* 489 (2000) 125, *B* 475 (2000) 190
- [349] CMD-3 Collaboration, *Phys. Lett. B* 723 (2013) 82
- [350] KLOE Collaboration, *Phys. Lett. B* 669 (2008) 223
- [351] SND Collaboration, *JETP Lett.* 94 (2012) 2, 92 (2010) 80; *J. Exp. Theor. Phys.* 109 (2009) 379, 96 (2003) 789; *Phys. Rev. D* 88 (2013) 054013, *D* 76 (2007) 072012, *D* 74 (2006) 014016, *D* 68 (2003) 052006, *D* 66 (2002) 032001, *D* 65 (2002) 032002, *D* 63 (2001) 072002; *Phys. Lett. B* 559 (2003) 171, *B* 486 (2000) 29, *B* 462 (1999) 365
- [352] M. Davier, arXiv:1302.1907 [hep-ex]
- [353] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, *Phys. Rev. D* 85 (2012) 093013, 033007, *D* 84 (2011) 053003, *D* 83 (2011) 053002, 053003, *D* 82 (2010) 113004, *D* 78 (2008) 113006, *D* 77 (2008) 053012; *Nucl. Phys. B* 796 (2008) 184, *B* 740 (2006) 138; *Phys. Rev. Lett.* 99 (2007) 110406
- [354] T. Aoyama, K. Asano, M. Hayakawa, T. Kinoshita, M. Nio and N. Watanabe, *Phys. Rev. D* 81 (2010) 053009, *D* 78 (2008) 053005
- [355] T. Kinoshita and M. Nio, *Phys. Rev. D* 73 (2006) 053007, 013003, *D* 70 (2004) 113001; *Phys. Rev. Lett.* 90 (2003) 021803
- [356] R. Lee, P. Marquard, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *JHEP* 1303 (2013) 162
- [357] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn and C. Sturm, *Nucl. Phys. B* 867 (2013) 182
- [358] P.A. Baikov, K.G. Chetyrkin and C. Sturm, *Nucl. Phys. Proc. Suppl.* 183 (2008) 8
- [359] A.L. Kataev, *Phys. Rev. D* 86 (2012) 013010, *D* 74 (2006) 073011; *Phys. Lett. B* 284 (1992) 401 [Erratum-ibid. *B* 710 (2012) 710]
- [360] J.-P. Aguilar, D. Greynat and E. De Rafael, *Phys. Rev. D* 77 (2008) 093010
- [361] A. Czarnecki and M. Skrzypek, *Phys. Lett. B* 449 (1999) 354
- [362] S. Laporta and E. Remiddi, *Phys. Lett. B* 379 (1996) 283, *B* 301 (1993) 440
- [363] S. Laporta, *Phys. Lett. B* 328 (1994) 522, *B* 312 (1993) 495; *Nuovo Cim. A* 106 (1993) 675
- [364] G. Li, R. Mendel and M.A. Samuel, *Phys. Rev. D* 47 (1993) 1723
- [365] M.A. Samuel and G.-W. Li, *Phys. Rev. D* 44 (1991) 3935 [Erratum-ibid. *D* 48 (1993) 1879]

- [366] D.J. Broadhurst, A.L. Kataev and O.V. Tarasov, *Phys. Lett. B* 298 (1993) 445
- [367] K. Hagiwara, A. D. Martin, D. Nomura and T. Teubner, *Phys. Lett. B* 649 (2007) 173, *B* 557 (2003) 69; *Phys. Rev. D* 69 (2004) 093003
- [368] B. Krause, *Phys. Lett. B* 390 (1997) 392
- [369] J. Prades, E. de Rafael and A. Vainshtein, in Ref. [324]
- [370] J. Bijnens and J. Prades, *Mod. Phys. Lett. A* 22 (2007) 767
- [371] K. Melnikov and A. Vainshtein, *Phys. Rev. D* 70 (2004) 113006
- [372] M. Knecht, A. Nyffeler, M. Perrottet and E. de Rafael, *Phys. Rev. Lett.* 88 (2002) 071802
- [373] M. Knecht and A. Nyffeler, *Phys. Rev. D* 65 (2002) 073034
- [374] I.R. Blokland, A. Czarnecki and K. Melnikov, *Phys. Rev. Lett.* 88 (2002) 071803
- [375] J. Bijnens, E. Pallante and J. Prades, *Nucl. Phys. B* 626 (2002) 410, *B* 474 (1996) 379; *Phys. Rev. Lett.* 75 (1995) 1447 [Erratum-ibid. 75 (1995) 3781]
- [376] M. Hayakawa and T. Kinoshita, *Phys. Rev. D* 57 (1998) 465 [Erratum-ibid. D 66 (2002) 019902]; hep-ph/0112102
- [377] M.J. Ramsey-Musolf and M.B. Wise, *Phys. Rev. Lett.* 89 (2002) 041601
- [378] K.T. Engel and M.J. Ramsey-Musolf, arXiv:1309.2225 [hep-ph]
- [379] T. Goecke, C.S. Fischer and R. Williams, *Phys. Rev. D* 83 (2011) 094006 [Erratum-ibid. D 86 (2012) 099901]
- [380] T. Blum, M. Hayakawa and T. Izubuchi, *PoS LATTICE 2012* (2012) 022
- [381] M. Hoferichter, G. Colangelo, M. Procura and P. Stoffer, arXiv:1309.6877 [hep-ph]
- [382] Z. Zhang, arXiv:1302.1896 [hep-ph]
- [383] F. Jegerlehner and R. Szafron, *Eur. Phys. J. C* 71 (2011) 1632
- [384] K. Hagiwara, R. Liao, A.D. Martin, D. Nomura and T. Teubner, *J. Phys. G* 38 (2011) 085003
- [385] New Muon ($g - 2$) Collaboration, Fermilab Proposal 0989 (2009)
<http://lss.fnal.gov/archive/test-proposal/0000/fermilab-proposal-0989.pdf>
- [386] J-PARC $g - 2$ Collaboration, J-PARC-PAC2009-12 (2009)
http://j-parc.jp/researcher/Hadron/en/pac_1001/pdf/KEK_J-PARC-PAC2009-12.pdf
- [387] S. Eidelman and M. Passera, *Mod. Phys. Lett. A* 22 (2007) 159
- [388] DELPHI Collaboration, *Eur. Phys. J. C* 35 (2004) 159
- [389] L3 Collaboration, *Phys. Lett. B* 585 (2004) 53
- [390] L3 Collaboration, *Phys. Lett. B* 434 (1998) 169
- [391] OPAL Collaboration, *Phys. Lett. B* 431 (1998) 188
- [392] R. Escribano and E. Masso, *Phys. Lett. B* 395 (1997) 369, *B* 301 (1993) 419
- [393] J.A. Grifols and A. Mendez, *Phys. Lett. B* 255 (1991) 611 [Erratum-ibid. B 259 (1991) 512]
- [394] F. del Aguila and M. Sher, *Phys. Lett. B* 252 (1990) 116
- [395] G. Domokos, S. Kovesi-Domokos, C. Vaz and D. Wurmser, *Phys. Rev. D* 32 (1985) 247
- [396] D.J. Silverman and G.L. Shaw, *Phys. Rev. D* 27 (1983) 1196
- [397] J.S. Schwinger, *Phys. Rev.* 73 (1948) 416, 76 (1949) 790
- [398] M. Fael, L. Mercolli and M. Passera, arXiv:1301.5302 [hep-ph]
- [399] M.L. Laursen, M.A. Samuel and A. Sen, *Phys. Rev. D* 29 (1984) 2652 [Erratum-ibid. D 56 (1997) 3155]
- [400] M.A. Samuel and G. Li, *Int. J. Theor. Phys.* 33 (1994) 1471

- [401] F. Cornet and J.I. Illana, *Phys. Rev. D* 53 (1996) 1181
- [402] A.A. Billur and M. Koksals, arXiv:1306.5620 [hep-ph]
- [403] M.A. Samuel, G.-W. Li and R. Mendel, *Phys. Rev. Lett.* 67 (1991) 668 [Erratum-ibid. 69 (1992) 995]
- [404] I.J. Kim, *Nucl. Phys. B* 229 (1983) 251
- [405] E761 Collaboration, *Phys. Rev. Lett.* 69 (1992) 3286
- [406] J. Bernabéu, G.A. González-Sprinberg and J. Vidal, *JHEP* 0901 (2009) 062
- [407] J. Bernabéu, G.A. González-Sprinberg, J. Papavassiliou and J. Vidal, *Nucl. Phys. B* 790 (2008) 160
- [408] G.A. González-Sprinberg, A. Santamaria and J. Vidal, *Nucl. Phys. B* 582 (2000) 3
- [409] ACME Collaboration, arXiv:1310.7534 [physics.atom-ph]
- [410] Muon ($g - 2$) Collaboration, *Phys. Rev. D* 80 (2009) 052008
- [411] J.S.M. Ginges and V.V. Flambaum, *Phys. Rept.* 397 (2004) 63
- [412] M. Pospelov and A. Ritz, *Annals Phys.* 318 (2005) 119
- [413] M. Raidal *et al.*, *Eur. Phys. J. C* 57 (2008) 13
- [414] T. Fukuyama, *Int. J. Mod. Phys. A* 27 (2012) 1230015
- [415] J. Engel, M.J. Ramsey-Musolf and U. van Kolck, *Prog. Part. Nucl. Phys.* 71 (2013) 21
- [416] M. Jung, *JHEP* 1305 (2013) 168
- [417] M. Jung and A. Pich, arXiv:1308.6283 [hep-ph]
- [418] ARGUS Collaboration, *Phys. Lett. B* 485 (2000) 37
- [419] W. Bernreuther, O. Nachtmann and P. Overmann, *Phys. Rev. D* 48 (1993) 78
- [420] W. Bernreuther, A. Brandenburg and P. Overmann, *Phys. Lett. B* 391 (1997) 413
- [421] Belle Collaboration, *Phys. Lett. B* 551 (2003) 16
- [422] J. Bernabéu, G.A. González-Sprinberg and J. Vidal, *Nucl. Phys. B* 763 (2007) 283, B 701 (2004) 87
- [423] ALEPH Collaboration, *Eur. Phys. J. C* 30 (2003) 291; *Phys. Lett. B* 346 (1995) 371, B 297 (1992) 459
- [424] L3 Collaboration, *Phys. Lett. B* 426 (1998) 207
- [425] OPAL Collaboration, *Z. Phys. C* 74 (1997) 403, C 66 (1995) 31; *Phys. Lett. B* 281 (1992) 405
- [426] W. Bernreuther, U. Low, J.P. Ma and O. Nachtmann, *Z. Phys. C* 43 (1989) 117
- [427] J. Bernabéu, G.A. González-Sprinberg, M. Tung and J. Vidal, *Nucl. Phys. B* 436 (1995) 474
- [428] J. Bernabéu, G.A. González-Sprinberg and J. Vidal, *Phys. Lett. B* 326 (1994) 168
- [429] W. Bernreuther and O. Nachtmann, *Phys. Rev. Lett.* 63 (1989) 2787 [Erratum-ibid. 64 (1990) 1072]
- [430] W. Bernreuther, G.W. Botz, O. Nachtmann and P. Overmann, *Z. Phys. C* 52 (1991) 567
- [431] F. Sánchez, *Phys. Lett. B* 412 (1997) 137
- [432] J. Bernabéu and N. Rius, *Phys. Lett. B* 232 (1989) 127
- [433] M. Kobayashi and T. Maskawa, *Prog. Theor. Phys.* 49 (1973) 652
- [434] I.I. Bigi, arXiv:1210.2968 [hep-ph]
- [435] K. Kiers, arXiv:1212.6921 [hep-ph]
- [436] Y.S. Tsai, *Phys. Rev. D* 51 (1995) 3172
- [437] S. Goozovat and C. A. Nelson, *Phys. Lett. B* 267 (1991) 128 [Erratum-ibid. B 271 (1991) 468]

- [438] C.A. Nelson *et al.*, *Phys. Rev. D* 50 (1994) 4544
- [439] J. H. Kühn and E. Mirkes, *Phys. Lett. B* **398** (1997) 407
- [440] U. Kilian, J.G. Korner, K. Schilcher and Y.L. Wu, *Z. Phys. C* 62 (1994) 413
- [441] K. Kiers *et al.*, *Phys. Rev. D* 78 (2008) 113008
- [442] A. Datta, K. Kiers, D. London, P.J. O'Donnell and A. Szyrkman, *Phys. Rev. D* 75 (2007) 074007
[Erratum-ibid. *D* 76 (2007) 079902]
- [443] S.Y. Choi, K. Hagiwara and M. Tanabashi, *Phys. Rev. D* 52 (1995) 1614
- [444] BaBar Collaboration, *Phys. Rev. D* 85 (2012) 031102 [Erratum-ibid. *D* 85 (2012) 099904]
- [445] I.I. Bigi and A.I. Sanda, *Phys. Lett. B* 625 (2005) 47
- [446] G. Calderón, D. Delepine and G. López Castro, *Phys. Rev. D* 75 (2007) 076001
- [447] Y. Grossman and Y. Nir, *JHEP* 1204 (2012) 002
- [448] Belle Collaboration, *Phys. Rev. Lett.* 107 (2011) 131801
- [449] CLEO Collaboration, *Phys. Rev. Lett.* 88 (2002) 111803
- [450] BaBar Collaboration, *Phys. Rev. Lett.* 109 (2012) 101802
- [451] Belle Collaboration, *Phys. Rev. D* 82 (2010) 072005; arXiv:0910.4301 [hep-ex]
- [452] S. Fajfer, J.F. Kamenik, I. Nisandzic and J. Zupan, *Phys. Rev. Lett.* 109 (2012) 161801
- [453] J.F. Kamenik and F. Mescia, *Phys. Rev. D* 78 (2008) 014003
- [454] Y. Sakaki and H. Tanaka, *Phys. Rev. D* 87 (2013) 054002
- [455] D. Becirevic, N. Kosnik and A. Tayduganov, *Phys. Lett. B* 716 (2012) 208
- [456] J.A. Bailey *et al.*, *Phys. Rev. Lett.* 109 (2012) 071802
- [457] Belle Collaboration, *Phys. Rev. Lett.* 110 (2013) 131801, 97 (2006) 251802; *Phys. Rev. D* 82 (2010) 071101
- [458] BaBar Collaboration, *Phys. Rev. D* 88 (2013) 031102, 81 (2010) 051101
- [459] CKMfitter Group Collaboration, *Eur. Phys. J. C* 41 (2005) 1; <http://ckmfitter.in2p3.fr>
- [460] A. Celis, M. Jung, X.-Q. Li and A. Pich, *JHEP* 1301 (2013) 054
- [461] A. Crivellin, C. Greub and A. Kokulu, *Phys. Rev. D* 86 (2012) 054014
- [462] U. Nierste, S. Trine and S. Westhoff, *Phys. Rev. D* 78 (2008) 015006
- [463] M. Tanaka, *Z. Phys. C* 67 (1995) 321
- [464] M. Tanaka and R. Watanabe, *Phys. Rev. D* 87 (2013) 034028, *D* 82 (2010) 034027
- [465] A. Pich and P. Tuzón, *Phys. Rev. D* 80 (2009) 091702
- [466] N.G. Deshpande and A. Menon, *JHEP* 1301 (2013) 025
- [467] X.-G. He and G. Valencia, *Phys. Rev. D* 87 (2013) 014014
- [468] S. Fajfer, J.F. Kamenik and I. Nisandzic, *Phys. Rev. D* 85 (2012) 094025
- [469] A. Datta, M. Duraisamy and D. Ghosh, *Phys. Rev. D* 86 (2012) 034027
- [470] M. Duraisamy and A. Datta, *JHEP* 1309 (2013) 059
- [471] K. Hagiwara, A.D. Martin and M.F. Wade, *Phys. Lett. B* 228 (1989) 144; *Nucl. Phys. B* 327 (1989) 569
- [472] J.G. Korner and G.A. Schuler, *Z. Phys. C* 46 (1990) 93, *C* 38 (1988) 511 [Erratum-ibid. *C* 41 (1989) 690]
- [473] C.-H. Chen and C.-Q. Geng, *Phys. Rev. D* 71 (2005) 077501
- [474] C. McNeile, C.T.H. Davies, E. Follana, K. Hornbostel and G.P. Lepage, *Phys. Rev. D* 86 (2012) 074503

- [475] Minos Collaboration, *Phys. Rev. Lett.* 110 (2013) 171801, 107 (2011) 181802
- [476] T2K Collaboration, *Phys. Rev. Lett.* 107 (2011) 041801; *Phys. Rev. D* 88 (2013) 032002
- [477] Daya Bay Collaboration, *Phys. Rev. Lett.* 108 (2012) 171803; *Chin. Phys. C* 37 (2013) 011001; arXiv:1310.6732 [hep-ex]
- [478] Double-Chooz Collaboration, *Phys. Rev. Lett.* 108 (2012) 131801
- [479] Reno Collaboration, *Phys. Rev. Lett.* 108 (2012) 191802
- [480] M.C. González-García, M. Maltoni, J. Salvado and T. Schwetz, *JHEP* 1212 (2012) 123; <http://www.nu-fit.org>
- [481] S. Weinberg, *Phys. Rev. Lett.* 43 (1979) 1566
- [482] M. Gell-Mann, P. Ramond and R. Slansky, *Complex Spinors and Unified Theories*, in *Supergravity*, eds. P. van Nieuwenhuizen and D.Z. Freedman (North Holland, Stony Brook 1979) p. 315
- [483] T. Yanagida, in Proc. *Workshop on Unified Theory and Baryon Number in the Universe*, eds. O. Sawada and A. Sugamoto (KEK, 1979) p. 95
- [484] W.J. Marciano, T. Mori and J.M. Roney, *Ann. Rev. Nucl. Part. Sci.* 58 (2008) 315
- [485] R. Barbieri, L.J. Hall and A. Strumia, *Nucl. Phys. B* 445 (1995) 219
- [486] J. Hisano, D. Nomura and T. Yanagida, *Phys. Lett. B* 437 (1998) 351
- [487] J. Hisano, D. Nomura, Y. Okada, Y. Shimizu and M. Tanaka, *Phys. Rev. D* 58 (1998) 116010
- [488] J. Hisano and D. Nomura, *Phys. Rev. D* 59 (1999) 116005
- [489] K. Huitu, J. Maalampi, M. Raidal and A. Santamaria, *Phys. Lett. B* 430 (1998) 355
- [490] M. Raidal and A. Santamaria, *Phys. Lett. B* 421 (1998) 250
- [491] A. Ilakovac, *Phys. Rev. D* 62 (2000) 036010
- [492] Y. Kuno and Y. Okada, *Rev. Mod. Phys.* 73 (2001) 151
- [493] J. Sato, K. Tobe and T. Yanagida, *Phys. Lett. B* 498 (2001) 189
- [494] J. Sato and K. Tobe, *Phys. Rev. D* 63 (2001) 116010
- [495] D. Black, T. Han, H.-J. He and M. Sher, *Phys. Rev. D* 66 (2002) 053002
- [496] G. Cvetič, C. Dib, C.S. Kim and J.D. Kim, *Phys. Rev. D* 66 (2002) 034008 [Erratum-ibid. D 68 (2003) 059901]
- [497] A. Masiero, S.K. Vempati and O. Vives, *Nucl. Phys. B* 649 (2003) 189
- [498] J.I. Illana and M. Masip, *Eur. Phys. J. C* 35 (2004) 365; *Phys. Rev. D* 67 (2003) 035004
- [499] V. Cirigliano, A. Kurylov, M.J. Ramsey-Musolf and P. Vogel, *Phys. Rev. Lett.* 93 (2004) 231802; *Phys. Rev. D* 70 (2004) 075007
- [500] V. Cirigliano, B. Grinstein, G. Isidori and M.B. Wise, *Nucl. Phys. B* 728 (2005) 121
- [501] S.R. Choudhury, N. Gaur and A. Goyal, *Phys. Rev. D* 72 (2005) 097702
- [502] E. Arganda and M.J. Herrero, *Phys. Rev. D* 73 (2006) 055003
- [503] C.E. Yaguna, *Int. J. Mod. Phys. A* 21 (2006) 1283
- [504] C.-H. Chen and C.-Q. Geng, *Phys. Rev. D* 74 (2006) 035010
- [505] V. Cirigliano and B. Grinstein, *Nucl. Phys. B* 752 (2006) 18
- [506] V. Cirigliano, G. Isidori and V. Porretti, *Nucl. Phys. B* 763 (2007) 228
- [507] K. Agashe, A.E. Blechman and F. Petriello, *Phys. Rev. D* 74 (2006) 053011
- [508] S. Antusch, E. Arganda, M.J. Herrero and A.M. Teixeira, *JHEP* 0611 (2006) 090
- [509] S.R. Choudhury, A.S. Cornell, A. Deandrea, N. Gaur and A. Goyal, *Phys. Rev. D* 75 (2007) 055011

- [510] M. Blanke, A.J. Buras, B. Duling, A. Poschenrieder and C. Tarantino, *JHEP* 0705 (2007) 013
- [511] E. Arganda, M.J. Herrero and A.M. Teixeira, *JHEP* 0710 (2007) 104
- [512] E. Arganda, M.J. Herrero and J. Portolés, *JHEP* 0806 (2008) 079
- [513] M.J. Herrero, J. Portolés and A.M. Rodríguez-Sánchez, *Phys. Rev. D* 80 (2009) 015023
- [514] V. Cirigliano, R. Kitano, Y. Okada and P. Tuzón, *Phys. Rev. D* 80 (2009) 013002
- [515] Z.-H. Li, Y. Li and H.-X. Xu, *Phys. Lett. B* 677 (2009) 150
- [516] R. Benbrik and C.-H. Chen, *Phys. Lett. B* 672 (2009) 172
- [517] A. Arhrib, R. Benbrik and C.-H. Chen, *Phys. Rev. D* 81 (2010) 113003
- [518] W. Liu, C.-X. Yue and J. Zhang, *Eur. Phys. J. C* 68 (2010) 197
- [519] W.-J. Li, Y.-Y. Fan, G.-W. Liu and L.-X. Lu, *Int. J. Mod. Phys. A* 25 (2010) 4827
- [520] J. Hisano, S. Sugiyama, M. Yamanaka and M.J.S. Yang, *Phys. Lett. B* 694 (2011) 380
- [521] F. del Aguila, J.I. Illana and M.D. Jenkins, *JHEP* 1103 (2011) 080, 1009 (2010) 040, 0901 (2009) 080
- [522] S. Kaneko *et al.*, *Phys. Rev. D* 83 (2011) 115005
- [523] ALEPH Collaboration, *Phys. Rept.* 216 (1992) 253
- [524] DELPHI Collaboration, *Z. Phys. C* 73 (1997) 243
- [525] L3 Collaboration, *Phys. Lett. B* 316 (1993) 427
- [526] OPAL Collaboration, *Z. Phys. C* 67 (1995) 555
- [527] LHCb Collaboration, *Phys. Rev. Lett.* 111 (2013) 141801
- [528] MEG Collaboration, *Phys. Rev. Lett.* 110 (2013) 201801, 107 (2011) 171801
- [529] SINDRUM Collaboration, *Nucl. Phys. B* 299 (1988) 1
- [530] R.D. Bolton *et al.*, *Phys. Rev. D* 38 (1988) 2077
- [531] BaBar Collaboration, *Phys. Rev. D* 81 (2010) 111101, 79 (2009) 012004; *Phys. Rev. Lett.* 104 (2010) 021802, 103 (2009) 021801, 100 (2008) 071802, 99 (2007) 251803, 98 (2007) 061803, 96 (2006) 041801, 95 (2005) 191801, 041802, 92 (2004) 121801
- [532] Belle Collaboration, *Phys. Lett. B* 719 (2013) 346, B 699 (2011) 251, B 692 (2010) 4, B 682 (2010) 355, B 672 (2009) 317, B 666 (2008) 16, B 664 (2008) 35, B 660 (2008) 154, B 648 (2007) 341, B 640 (2006) 138, B 639 (2006) 159, B 622 (2005) 218, B 613 (2005) 20, B 589 (2004) 103; *Phys. Rev. Lett.* 93 (2004) 081803, 92 (2004) 171802
- [533] SINDRUM II Collaboration, *Eur. Phys. J. C* 47 (2006) 337; *Phys. Rev. Lett.* 76 (1996) 200; *Phys. Lett. B* 317 (1993) 631
- [534] LHCb Collaboration, *Phys. Lett. B* 724 (2013) 36
- [535] MEG Collaboration, arXiv:1301.7225 [physics.ins-det]
- [536] A. Blondel *et al.*, arXiv:1301.6113 [physics.ins-det]
- [537] Y. Kuno [COMET Collaboration], *PTEP* 2013 (2013) 022C01
- [538] Mu2e Collaboration, arXiv:1211.7019 [physics.ins-det]
- [539] Mu2e Collaboration, arXiv:1307.1168 [physics.ins-det]
- [540] R.J. Barlow, *Nucl. Phys. Proc. Suppl.* 218 (2011) 44
- [541] G. Blankenburg, J. Ellis and G. Isidori, *Phys. Lett. B* 712 (2012) 386
- [542] R. Harnik, J. Kopp and J. Zupan, *JHEP* 1303 (2013) 026
- [543] S. Davidson and P. Verdier, *Phys. Rev. D* 86 (2012) 111701
- [544] A. Celis, V. Cirigliano and E. Passemar, arXiv:1309.3564 [hep-ph]

- [545] A.A. Petrov and D.V. Zhuridov, arXiv:1308.6561 [hep-ph]
- [546] J.T. Daub, H.K. Dreiner, C. Hanhart, B. Kubis and U.G. Meissner, *JHEP* 1301 (2013) 179
- [547] Belle Collaboration, *Phys. Lett. B* 632 (2006) 51
- [548] OPAL Collaboration, *Phys. Lett. B* 447 (1999) 157
- [549] BaBar Collaboration, *Phys. Rev. D* 85 (2012) 071103, *D* 83 (2011) 091101; arXiv:1310.8238 [hep-ex]
- [550] LHCb Collaboration, *Phys. Rev. Lett.* 108 (2012) 101601; *Phys. Rev. D* 85 (2012) 112004
- [551] Belle Collaboration, *Phys. Rev. D* 84 (2011) 071106
- [552] CLEO Collaboration, *Phys. Rev. D* 65 (2002) 111102
- [553] A. Ali, A.V. Borisov and N.B. Zamorin, *Eur. Phys. J C* 21 (2001) 123
- [554] A. Atre, V. Barger and T. Han, *Phys. Rev. D* 71 (2005) 113014
- [555] M.A. Ivanov and S.G. Kovalenko, *Phys. Rev. D* 71 (2005) 053004
- [556] A. Atre, T. Han, S. Pascoli and B. Zhang, *JHEP* 0905 (2009) 030
- [557] J.C. Helo, S. Kovalenko and I. Schmidt, *Nucl. Phys. B* 853 (2011) 80
- [558] J.-M. Zhang and G.-L. Wang, *Eur. Phys. J. C* 71 (2011) 1715
- [559] S.-S. Bao, H.-L. Li, Z.-G. Si and Y.-B. Yang, *Commun. Theor. Phys.* 59 (2013) 472
- [560] G. Cvetič, C. Dib, S.K. Kang and C.S. Kim, *Phys. Rev. D* 82 (2010) 053010
- [561] G. Cvetič, C. Dib and C.S. Kim, *JHEP* 1206 (2012) 149
- [562] N. Quintero, G. López Castro and D. Delepine, *Phys. Rev. D* 84 (2011) 096011 [Erratum-ibid. *D* 86 (2012) 079905]
- [563] N. Quintero, *Phys. Rev. D* 87 (2013) 056005
- [564] G. López Castro and N. Quintero, *Phys. Rev. D* 87 (2013) 077901, *D* 85 (2012) 076006 [Erratum-ibid. *D* 86 (2012) 079904]
- [565] ATLAS Collaboration, *Phys. Lett. B* 706 (2012) 276
- [566] CMS Collaboration, CMS-PAS-EWK-11-002 (2011)
- [567] ATLAS Collaboration, *Phys. Rev. D* 84 (2011) 112006; ATLAS-CONF-2012-006 (2012)
- [568] CMS Collaboration, *JHEP* 1108 (2011) 117
- [569] LHCb Collaboration, *JHEP* 1301 (2013) 111
- [570] ATLAS Collaboration, *Eur. Phys. J. C* 73 (2013) 2328; *Phys. Lett. B* 717 (2012) 89
- [571] CMS Collaboration, *Eur. Phys. J. C* 73 (2013) 2386; *Phys. Rev. D* 85 (2012) 112007
- [572] K. Melnikov and F. Petriello, *Phys. Rev. D* 74 (2006) 114017; *Phys. Rev. Lett.* 96 (2006) 231803
- [573] C. Anastasiou, L.J. Dixon, K. Melnikov and F. Petriello, *Phys. Rev. D* 69 (2004) 094008
- [574] S. Catani, L. Cieri, D. de Florian, G. Ferrera and M. Grazzini, *Eur. Phys. J C* 72 (2012) 2195; *Phys. Rev. Lett.* 103 (2009) 082001
- [575] A.D. Martin, W.J. Stirling, R.S. Thorne and G. Watt, *Eur. Phys. J. C* 63 (2009) 189
- [576] S. Moch, P. Uwer and A. Vogt, *Phys. Lett. B* 714 (2012) 48
- [577] S. Moch and P. Uwer, *Phys. Rev. D* 78 (2008) 034003; *Nucl. Phys. Proc. Suppl.* 183 (2008) 75
- [578] N. Kidonakis, *Phys. Rev. D* 82 (2010) 114030
- [579] UA1 Collaboration, *Phys. Lett. B* 185 (1987) 233 [Addendum-ibid. *B* 191 (1987) 462]
- [580] CDF Collaboration, *Phys. Rev. Lett.* 68 (1992) 3398, 79 (1997) 3585, 109 (2012) 192001; *Phys. Lett. B* 639 (2006) 172; *Phys. Rev. D* 75 (2007) 092004

- [581] D0 Collaboration, *Phys. Rev. Lett.* 84 (2000) 5710; *Phys. Lett. B* 670 (2009) 292, B 679 (2009) 177; *Phys. Rev. D* 82 (2010) 071102
- [582] ATLAS Collaboration, *Eur. Phys. J. C* 72 (2012) 2062
- [583] A. Rouge, *Z. Phys. C* 48 (1990) 75
- [584] K. Hagiwara, A.D. Martin and D. Zeppenfeld, *Phys. Lett. B* 235 (1990) 198
- [585] A. Pich, arXiv:1307.7700 [hep-ph]
- [586] J. Ellis and T. You, *JHEP* 1306 (2013) 103
- [587] P.P. Giardino *et al.*, arXiv:1303.3570 [hep-ph]
- [588] K. Cheung, J.S. Lee and P.-Y. Tseng, *JHEP* 1305 (2013) 134
- [589] A. Falkowski, F. Riva and A. Urbano, *JHEP* 1311 (2013) 111
- [590] A. Celis, V. Ilisie and A. Pich, *JHEP* 1307 (2013) 053; arXiv:1310.7941 [hep-ph]
- [591] ATLAS Collaboration, *Phys. Lett. B* 723 (2013) 15, B 719 (2013) 242; ATLAS-CONF-2013-066 (2013)
- [592] CMS Collaboration, *Phys. Lett. B* 716 (2012) 82
- [593] ATLAS Collaboration, *JHEP* 1306 (2013) 033
- [594] CMS Collaboration, *Phys. Rev. Lett.* 110 (2013) 081801
- [595] ATLAS Collaboration, ATLAS-CONF-2011-132 (2011)
- [596] CMS Collaboration, *Phys. Lett. B* 713 (2012) 68; CMS-PAS-HIG-12-050 (2012); CMS-PAS-HIG-13-021 (2013)
- [597] ATLAS Collaboration, *JHEP* 1206 (2012) 039
- [598] CMS Collaboration, *JHEP* 1207 (2012) 143
- [599] ATLAS Collaboration, *Eur. Phys. J. C* 72 (2012) 2215; *Phys. Lett. B* 714 (2012) 197, B 714 (2012) 180; ATLAS-CONF-2013-026 (2013), ATLAS-CONF-2013-028 (2013)
- [600] CMS Collaboration, *Eur. Phys. J. C* 73 (2013) 2493; CMS-PAS-SUS-11-029 (2011)
- [601] BaBar Collaboration, *Phys. Rev. D* 88 (2013) 071102; *Phys. Rev. Lett.* 103 (2009) 181801