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Memory effects in fractional Brownian motion with Hurst exponent $H<1/3$

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We study the regression to the origin of a walker driven by dynamically generated fractional Brownian motion (FBM) and we prove that when the FBM scaling, i.e., the Hurst exponent $H<1/3$, the emerging inverse power law is characterized by a power index that is a compelling signature of the infinitely extended memory of the system. Strong memory effects leads to the relation $H=\theta/2$ between the Hurst exponent and the persistent exponent θ, which is different from the widely used relation $H=1-\theta$. The latter is valid for $1/3<H<1$ and is known to be compatible with the renewal assumption.

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Fractional Brownian motion (FBM) is a generalization of ordinary Brownian motion that since the publication [1] has been the subject of active research because the anomalous scaling and the memory properties of many diffusion processes are considered to be the attributes of FBM [2–8]. There are two important parameters associated with FBM: the Hurst exponent H and the persistent exponent θ. The former shows scaling of the mean-square displacement with time

$$\langle x^2(t)\rangle \propto t^{2H}, \quad 0<H<1, \tag{1}$$

and the latter characterizes the power-law tail of the distribution function of time intervals τ between two consecutive returns to the origin $x=0$

$$\psi_R(\tau) \propto 1/\tau^{1+\theta}, \quad \tau\to\infty. \tag{2}$$

It turns out that these two exponents are related

$$H = 1-\theta. \tag{3}$$

There are several ways of deriving Eq. (3). Ding and Yang [9] obtained Eq. (3) using the fractal dimension of the trajectory $x(t)$. In 2000 Rangarajan and Ding [10] revisited the same issue by adopting dynamical rather than fractal-dimension based theoretical arguments. Using computer calculations they confirmed the relation (3), which, on the other hand, coincides with the earlier theoretical prediction of Molchan [11].

In 1996 Krug and Dobbs [12] and more recently, Failla *et al.* [13] showed that Eq. (3) can be easily derived by making the assumption that the origin recrossing is a renewal, i.e., memoryless, process. The derivation [12] based on the renewal assumption was criticized by the authors of Ref. [14] as conflicting with the infinitely extended FBM memory, which is usually illustrated by the individual trajectory correlator $\langle x(t)x(-t)\rangle = \langle x^2(t)\rangle(1-2^{2H-1})$, which does not vanish for $H\neq 1/2$.

The authors of Ref. [15] made the conjecture that the origin recrossing may, nevertheless, be renewal in spite of the infinitely extended memory of FBM. To verify this conjecture they generated a sequence of time distances τ_i between two consecutive axis recrossings. It was shown that the binary correlation function of these time intervals is δ-correlated, i.e., $\langle(\tau_i-\bar{\tau})(\tau_k-\bar{\tau})\rangle \propto \delta_{ik}$. One more support of the renewal assumption was obtained from the aging effect generated by the intervals τ_i. The magnitude of aging coincides (within the limits of numerical accuracy) with the results of renewal approach. Thus, the results obtained in Ref. [15] suggest that the origin recrossings are renewal events.

However, in a recent publication [8] this conclusion was criticized and numerical evidence of the correlations in the zero-crossing events was given. The authors of Ref. [8] study the statistics of so-called longest excursion up to time t. Asymptotical behavior of the probability $Q(t)$ that the last (unfinished) excursion of length $A(t)$ is the longest of the intervals $\tau_1,\tau_2,\ldots,\tau_N,A(t)$ affords a criterion to establish the statistics of the intervals τ_i. When the return to the origin is a renewal process, the limit $Q_\infty = Q(t\to\infty)$ gets a well defined analytical form, called $Q_\infty^R(\theta)$ [7]. Any deviation of $Q(t)$ at $t\to\infty$ from the analytical result $Q_\infty^R(\theta)$ is evidence of memory of zero-crossing events. The main result reported in Ref. [8] is that such deviations have been found numerically for all values of H, but $H=0.5$. Would it be true to conclude now that any FBM trajectory exhibits memory for the origin recrossing events? In this Rapid Communication we show that the answer to this question depends on H and that Eq. (3) is not always true and that within the interval $0<H<1/3$ it is replaced by

$$H = \theta/2. \tag{4}$$

Our calculations are based on the well-known formula [16] for the first-passage time distribution density $\psi_{x_0}(t)$ for a random walker to arrive (for the first time) at $x_0\neq 0$ at time t. Let us consider a set of random walkers moving from $x=0$ at $t=0$. Thus the probability distribution density $p(x,t)$ fulfills the condition $p(x,0)=\delta(x)$. The first-passage time t' is defined as the interval between the departure from the origin at time $t-t'$ and the arrival at the final point $x_0\neq 0$ at time t, regardless of the number of origin recrossing may occur prior to the arrival. The density $p(x_0,t)$ is obtained via integration over all possible t' [16]

$$p(x_0,t) = \int_0^t p(0,t-t')\psi_{x_0}(t')dt'. \tag{5}$$

The information about the possible memory of the origin recrossings prior to the arrival is hidden in the diffusion process turning $\delta(x)$ into $p(x,t)$. We apply Laplace transformation $\hat{f}(u) \equiv \int_0^\infty \exp(-ut) f(t) dt$ to Eq. (5) and obtain

$$\hat{\psi}_{x_0}(u) = \hat{p}(x_0,u)/\hat{p}(0,u). \tag{6}$$

The main result of this paper is based on Eq. (6) which is free from any assumption about memory. Unlike this, the conventional derivation [16,17] for the distribution density of the returns to the origin, $\psi_R(t)$, rests on the tacit assumption that they are renewal. We note that in this case,

$$p(0,t) = \delta(t) + \int_0^t p(0,t-t')\psi_R(t')dt', \tag{7}$$

with the time integration running over consecutive returns to the origin. The Laplace transform of Eq. (7) yields

$$\hat{p}(0,u) = [1 - \hat{\psi}_R(u)]^{-1}. \tag{8}$$

For practical purposes it is convenient to discretize space and time, introducing small intervals Δx and Δt. Then Eq. (7) is replaced by the following relation [13,15]:

$$p(0,t)\Delta x = \sum_{N=0}^{\infty} \psi_R^{(N)}(t)\Delta t. \tag{9}$$

Here the probability density $\psi_R(t')$ is converted into probability $\psi_R^{(N)}(t)$ for the particle to return to the origin N times, providing that the last return occurs exactly at time t. Similar relation is valid for the Laplace transforms

$$\hat{p}(0,u)\Delta x = \sum_{N=0}^{\infty} \hat{\psi}_R^{(N)}(u)\Delta t. \tag{10}$$

It is easy to show that if the returns are renewal the following relation holds [18]:

$$\hat{\psi}_R^{(N)}(u) = [\hat{\psi}_R(u)]^N. \tag{11}$$

We thus obtain from Eqs. (10) and (11)

$$\hat{p}(0,u)\left(\frac{\Delta x}{\Delta t}\right) = \sum_{N=0}^{\infty} (\hat{\psi}_R(u))^N = \frac{1}{1-\hat{\psi}_R(u)}, \tag{12}$$

which, multiplicative factor apart, coincides with Eq. (8). We then conclude that Eq. (7) implies the renewal nature of the process. For this reason we continue with Eqs. (5) and (6).

To study the distribution function $\psi_{x_0}(t)$ we adopt a dynamical approach to FBM [15] which in the asymptotic limit $t \to \infty$ coincides with the original FBM [1,2]. One of the reasons of this choice is that the FBM algorithms create trajectories that in one time step may overshoot the arrival point, thereby creating technical problems that the theory [7] wisely bypasses by focusing on the quantity $A(t)$, which is well defined even when in a single time step the trajectory $x(t)$ overshoots the origin. In the dynamical approach the FBM trajectories are obtained from the stochastic equation

$$\dot{x} \approx \Delta x/\Delta t = \xi(t), \tag{13}$$

where $\xi(t)$ is not the white noise, as it is for the case of Brownian diffusion. We assume that the binary correlation function $\Phi_\xi(t)$ of $\xi(t)$ has a power-law tail

$$\Phi_\xi(t) \propto \frac{\text{sgn}(1-\delta)}{t^\delta}, \tag{14}$$

with $0 < \delta < 2$. It is straightforward to prove [15] that in this case the Hurst exponent is related to δ

$$H = 1 - \delta/2. \tag{15}$$

For the well-developed stages of anomalous diffusion the probability density $p(x,t)$ is defined as follows

$$p(x,t) = \frac{1}{\sqrt{2\pi D t^{2H}}} \exp\left(-\frac{x^2}{2Dt^{2H}}\right). \tag{16}$$

This formula together with Eq. (6) will be used to calculate the distribution of the first-passage time $\psi_{x_0}(t)$. Although, the analytical formula for the Laplace transform of Eq. (16) is not known in general case, it is sufficient for us to study its behavior for small values of the parameter u. Taking into account that $\hat{p}(0,u) = \Gamma(1-H)u^{H-1}$, after some algebra the following expansion is obtained

$$\hat{\psi}_{x_0}(u) \approx 1 + c_1 u^{1-H} + c_2 u^{2H}. \tag{17}$$

Asymptotical behavior of $\psi(t)$ at $t \to \infty$ is obtained from Eq. (17) by taking its anti-Laplace transform

$$\psi_{x_0}(t) = \frac{C_1}{t^{2-H}} + \frac{C_2}{t^{1+2H}}. \tag{18}$$

Depending on the Hurst exponent, only one term survives in Eq. (18) at $t \to \infty$. For $H > 1/3$ it is the term $\sim 1/t^{2-H}$, and for $H < 1/3$ it is the term $\sim 1/t^{1+2H}$.

The coefficients c_1, c_2, C_1, C_2 in Eqs. (17) and (18) can be calculated for each value of H. For example, for $H = 3/4$ we obtain the following asymptotics:

$$\hat{\psi}_{x_0}(u) \approx 1 - \frac{4\Gamma(5/6)}{\Gamma(1/4)}\left(\frac{x_0^2}{2D}\right)^{1/6} u^{1/4},$$

$$\psi_{x_0}(t) \approx \frac{\Gamma(5/6)}{\sqrt{2\pi}}\left(\frac{x_0^2}{2D}\right)^{1/6} \frac{1}{t^{5/4}}. \tag{19}$$

As a relevant example of $H < 1/3$, let us consider $H = 1/4$.

$$\hat{\psi}_{x_0}(u) \approx 1 - \frac{\Gamma^2(1/4)}{\pi\sqrt{2}} \frac{x_0^2\sqrt{u}}{D}\left[1 - \frac{8\sqrt{\pi}}{3\Gamma(1/4)}\left(\frac{x_0^2\sqrt{u}}{2D}\right)^{1/2}\right],$$

$$\psi_{x_0}(t) \approx \frac{\Gamma^2(1/4)}{\pi\sqrt{2\pi}} \frac{x_0^2}{2Dt^{3/2}}\left[1 - \frac{2\pi}{\Gamma^2(1/4)}\left(\frac{x_0^2}{2D\sqrt{t}}\right)^{1/2}\right]. \tag{20}$$

For this case we keep two terms in the power-law tail of $\psi_{x_0}(t)$. Although the principal contribution at $t \to \infty$ comes from the term that decays as $t^{-3/2}$, the next correction $(\sim t^{-7/4})$ gives a considerable contribution for finite times.

Let us now study $\psi_R(t)$ without involving the renewal assumption. We set the initial condition $x = 0$. After the first

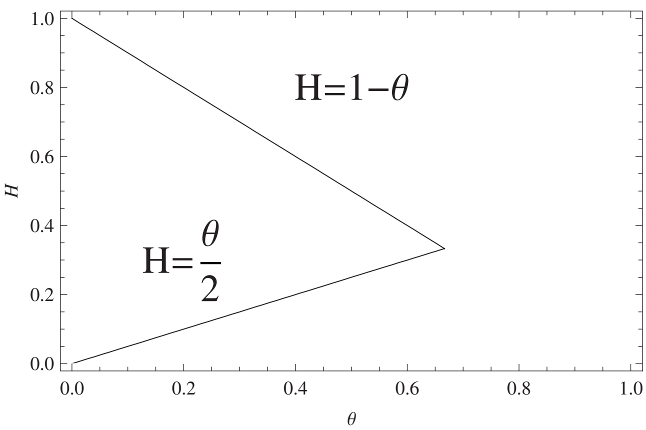

FIG. 1. The dependence H vs θ. Note that θ varies within the interval [0,2/3].

time step Δt we have $x_1 = \xi \Delta t$, where velocity ξ is randomly selected from the Gaussian distribution $\pi(\xi)$. Each of these x_1's is the initial value for infinitely many trajectories that sooner or later will go back to $x=0$. In other words, we divide the set of trajectories moving back to the origin into infinitely many subsets of trajectories, each subset with the origin at a given x_1. For each of these subsets we can use the theory that led us to $\psi(t)$. We, thus, obtain

$$\psi_R(t) \propto \int_{-\infty}^{\infty} \pi(\xi) \psi_{x_1}(t - \Delta t) d\xi, \qquad (21)$$

where $\psi_{x_1}(t)$ is the first-passage time distribution density for the random walker to move from $x_1 = \xi \Delta$ to the origin $x=0$. Note that we can use the earlier results by adopting for any ξ a reference system with the origin in x_1 and $x_0 = -x_1$. Since the inverse power law in the long-time limit contains the parameter x_0 as a factor, we conclude that the functions $\psi_R(t)$ and $\psi_{x_0}(t)$ have the same asymptotic behavior at $t \to \infty$. Thus, according to Eq. (18) the well-known relation (3) is replaced by formula (4) for $H < 1/3$.

This finding sheds light into [8]. In Fig. 1 we plot the dependence $H(\theta)$ given by Eqs. (3) and (4). There are two different values of the Hurst exponent for each value of θ. They correspond to two terms in Eq. (18). It is commonly believed that for the recrossing events the exponent θ takes the values within the interval $0 < \theta < 1$. Now we conclude that for the FBM the persistent exponent θ cannot exceed 2/3, with a significant consequence on the results of Ref. [8], where the departure of the quantity $Q_\infty^R - Q_\infty$ from 0 is a measure of the memory of the FBM generated origin recrossings. Their numerical results yield for $\theta = 0.9$ a deviation from the renewal prediction only slightly larger than for $\theta = 0.1$. Now, it is clear that the value of $\theta = 0.9$, which in Ref. [8] was associated with the FBM with $H = 1 - \theta = 0.1$, must be replaced by $\theta = 2H = 0.2$. The renewal theory gives $Q_\infty^R(\theta = 0.2) \approx 0.87$ and $Q_\infty^R(\theta = 0.9) \approx 0.18$ [7]. The numerical result extracted from the FBM with $H = 0.1$ converges at $t \to \infty$ to $Q_\infty \approx 0.1$. Since $Q_\infty^R(\theta = 0.2) - Q_\infty > Q_\infty^R(\theta = 0.9) - Q_\infty$, the departure from the results of the renewal theory turns out to be much stronger in the case of $H = 0.1$ than in the case of $H = 0.9$, thereby leading us to reinforce the interesting results Ref. [8] as follows: the memory effects in the sequence of the recrossing events are much stronger for the trajectories of FBM with $0 < H < 1/3$ than for $1/3 < H < 1$. The fact that the relation $H = 1 - \theta$ is compatible with the renewal assumption and white-noise-like behavior of the correlator $\langle (\tau_i - \bar{\tau})(\tau_k - \bar{\tau}) \rangle$ for $H \geq 1/3$ reported in Ref. [15] also supports this conclusion. Using the algorithm proposed in Ref. [19], we have numerically generated four ensembles of correlated sequences of stochastic kicks $\xi(t)$ with correlated function Φ_ξ decaying according to Eq. (14) with $\delta = 1.1$, 1.2, 1.5, and 1.8 (see the details in Ref. [15]). Each ensemble consists of about 10^3 sequences, each of the length $N \sim 10^4$. The corresponding FBM trajectories with $H = 0.45$, 0.40, 0.25, and 0.10 were obtained from Eq. (13) and for each ensemble of the trajectories the waiting time distribution function $\psi_R(t)$ was calculated. In Figs. 2 and 3, we plot the survival probability $\Psi(\tau) = \int_\tau^\infty \psi_R(t) dt \approx \mathrm{const}/\tau^\theta$, which, being smoother than $\psi_R(t)$ behavior, is more convenient for the evaluation of θ. The values of θ obtained from Fig. 2 are in excellent agreement with the standard relation $H = 1 - \theta$. For the plots of Fig. 3, where $H < 1/3$, the standard relation Eq. (3) is not valid at all. Here the new relation $H = \theta/2$ is confirmed, although the agreement with the theoretical prediction is not as good as for the case $H > 1/3$. The reason is the competition between the two terms of Eq. (18), generating the transient period of Fig. 3, which makes the true asymptotical behavior with $\theta = 2H$ emerge only at very long times. To get better agreement one should run the numerical experi-

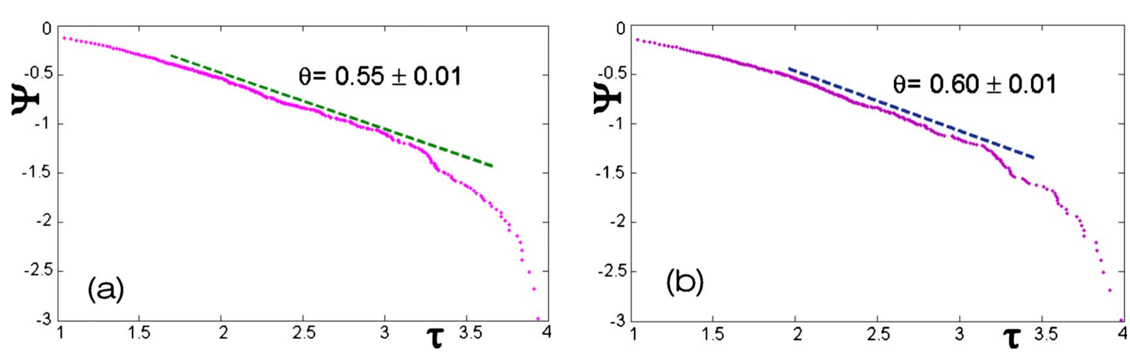

FIG. 2. (Color online) Log-log plot for the survival probability $\Psi(t)$ which has power-law asymptotics $\Psi(t) \propto 1/t^\theta$ for the FBM trajectories with (a) $H = 0.45$, and (b) $H = 0.40$. The extracted from the slope values of θ are in agreement with Eq. (3).

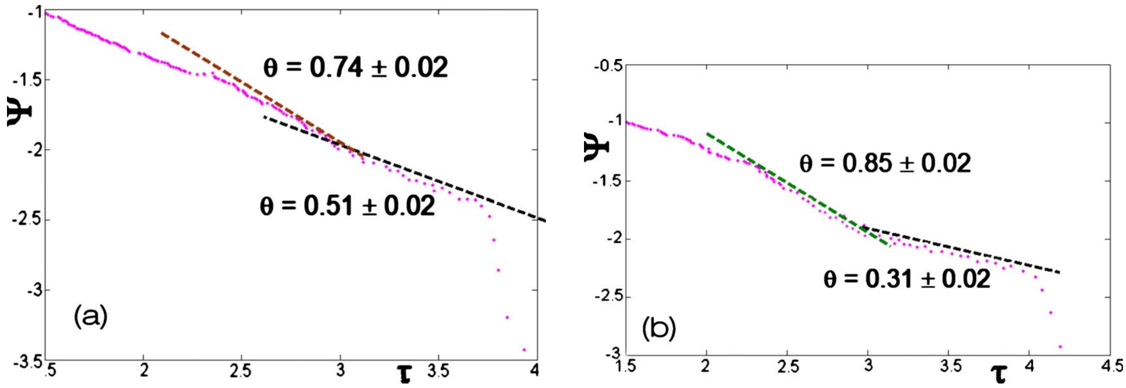

FIG. 3. (Color online) The same as in Fig. 2 but for (a) $H=0.25$ and (b) $H=0.10$. The extracted from the slope values of θ are in agreement with Eq. (3) for the shorter times and with Eq. (4) for the longer times. The transient period is also clearly seen around $\tau=3$.

ment for much longer times. This probably explains why this regime has been never observed in previous numerical experiments.

In conclusion, we predict analytically and confirm numerically new regime of FBM which is characterized by strong memory effects in the sequence of zero-crossing events. This new regime affects the return to the origin of the trajectories with $H<1/3$.

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[18] This relation originates from the following recursion which is true for the renewal events, $\psi_N(t)=\int_0^t \psi_{N-1}(t')\psi_1(t-t')dt'$. Its Laplace transform reads $\hat{\psi}_N(u)=\hat{\psi}_{N-1}(u)\hat{\psi}_1(u)=\hat{\psi}_{N-2}(u)\hat{\psi}_1^2(u)=\cdots=\hat{\psi}_0(u)\hat{\psi}_1^N(u)$. Taking into account that $\hat{\psi}_1(u)\equiv\hat{\psi}(u)$ and also that $\hat{\psi}_0(t)=\delta(t)$, one obtains Eq. (11).
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