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Generalized Virasoro anomaly and stress tensor for dilaton coupled theories

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Abstract

We derive the anomalous transformation law of the quantum stress tensor for a 2D massless scalar field coupled to an external dilaton. This provides a generalization of the Virasoro anomaly which turns out to be consistent with the trace anomaly. We apply it together with the equivalence principle to compute the expectation values of the covariant quantum stress tensor on a curved background. Finally we briefly illustrate how to evaluate vacuum polarization and Hawking radiation effects from these results.

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Two-dimensional conformal invariance is a key ingredient to understand critical behaviour of certain planar statistical mechanical systems [1]. It also plays a pivotal role in the formulation of superstring theory [2] and in the quantum mechanics of black holes. The Bekenstein-Hawking area law is derived in many different ways by applying Cardy's formula for conformal field theories living in the black hole horizon (see for instance [3] and references therein). The universal thermal character of black hole radiation is also related to the fact that matter fields exhibit two-dimensional conformal invariance in the vicinity of the horizon.

Many of the basic properties of 2d conformal field theories can be obtained by studying a simple model, namely a massless scalar field

$$S = -\frac{1}{2}\int d^2x\, (\nabla f)^2 \ . \tag{1}$$

Standard canonical quantization and Wick theorem lead to the well-known operator product expansion of the quantum (normal ordered) stress tensor

$$\begin{aligned} T_{\pm\pm}(x^\pm)T_{\pm\pm}(x'^\pm) &= \frac{1}{8\pi^2(x^\pm - x'^\pm)^4} - \frac{1}{\pi(x^\pm - x'^\pm)^2}T_{\pm\pm}(x'^\pm) \\ &- \frac{1}{2\pi(x^\pm - x'^\pm)}\partial_\pm T_{\pm\pm}(x'^\pm) + \ \dots \ , \end{aligned} \tag{2}$$

where $x^\pm = x^0 \pm x^1$ are null Minkowskian coordinates. The above expansion leads to the Lie algebra

$$\begin{aligned} [T_{\pm\pm}(x^\pm), T_{\pm\pm}(x'^\pm)] &= \frac{1}{2\pi}\partial_{x^\pm}\delta(x^\pm - x'^\pm)T_{\pm\pm}(x'^\pm) - \frac{1}{96\pi^2}\partial^3_{x^\pm}\delta(x^\pm - x'^\pm) \\ &- (x^\pm \leftrightarrow x'^\pm) \ . \end{aligned} \tag{3}$$

Since $T_{\pm\pm}(x^\pm)$, up to normalization, are the generators of infinitesimal conformal transformations $x^\pm \to x^\pm + \epsilon^\pm(x^\pm)$, this implies the following infinitesimal transformation law for the stress tensor

$$\delta_{\epsilon^\pm}T_{\pm\pm} = \epsilon^\pm\partial_\pm T_{\pm\pm} + 2\partial_\pm\epsilon^\pm T_{\pm\pm} - \frac{1}{24\pi}\partial^3_\pm\epsilon^\pm \ . \tag{4}$$

Exponentiating the action (4) one gets, under the conformal transformation $x^\pm \to y^\pm(x^\pm)$, the following anomalous transformation law

$$T_{\pm\pm}(y^\pm) = \left(\frac{dx^\pm}{dy^\pm}\right)^2 T_{\pm\pm}(x^\pm) - \frac{1}{24\pi}\{x^\pm, y^\pm\} \ , \tag{5}$$

where $\{x^\pm, y^\pm\} = \frac{\partial^3 x^\pm}{\partial y^{\pm 3}} / \frac{\partial x^\pm}{\partial y^\pm} - \frac{3}{2}\left(\frac{\partial^2 x^\pm}{\partial y^{\pm 2}} / \frac{\partial x^\pm}{\partial y^\pm}\right)^2$ is the Schwarzian derivative. All these expressions can be regarded as different realizations of the so-called Virasoro anomaly. For a generic conformal field theory the above results are valid provided we multiply the c-number terms of the above equations by the central charge c characterizing the particular theory [1].

The first aim of this work is to study the modification of the transformation law (5), when a dilaton field ϕ is present and (1) is replaced by

$$S = -\frac{1}{2}\int d^2 x e^{-2\phi} (\nabla f)^2 \ . \tag{6}$$

A nice justification of the form of the dilaton coupling comes from General Relativity. If a scalar field f is minimally coupled to a 4D spherically symmetric metric

$$ds^2_{(4)} = ds^2_{(2)} + e^{-2\phi} d\Omega^2 \ , \tag{7}$$

and we perform dimensional reduction from $-\frac{1}{8\pi}\int d^4 x\sqrt{-g}(\nabla f)^2$, we obtain the above action (6) in case of flat 2d space. We shall also study the quantum stress tensor of the theory (6) in a generic two-dimensional curved background.

Let us now consider a simple case, namely the one associated to the four-dimensional Minkowski space. In this situation it is $ds^2_{(2)} = -dx^+ dx^-$, where $x^\pm = t \pm r$, and $e^{-2\phi} = r^2$. The mode expansion of the field f living in the $t-r$ plane (with the condition $f(r=0)=0$) is

$$f = \int_0^\infty \frac{dw}{\sqrt{4\pi w}} \left[a_w (e^{-iwx^+} - e^{-iwx^-}) + a^\dagger_w (e^{iwx^+} - e^{iwx^-})\right] e^{\phi} \ . \tag{8}$$

The null components of the stress tensor are given by

$$T_{\pm\pm}(x^+, x^-) = e^{-2\phi}(\partial_\pm f)^2, \tag{9}$$

and the corresponding normal ordered operators can be defined, as usual, by point-splitting (from now on we shall use an explicit notation for the normal ordered stress tensor)

$$:T_{\pm\pm}(x^+, x^-) := \lim_{x^\pm \to x'^\pm} e^{-(\phi(x)+\phi(x'))} \frac{\partial}{\partial x^\pm}\frac{\partial}{\partial x'^\pm}(f(x)f(x') - \langle f(x)f(x')\rangle), \tag{10}$$

where the two-point function is

$$\langle f(x)f(x')\rangle = -\frac{1}{4\pi}e^{\phi(x)+\phi(x')}\ln\frac{(x^+-x'^+)(x^--x'^-)}{(x^+-x'^-)(x^--x'^+)}\ . \tag{11}$$

Under a conformal transformation $x^\pm \to y^\pm(x^\pm)$ normal ordering breaks covariance and the transformed stress tensor picks up the following anomalous non-tensorial contributions

$$\begin{aligned} :T_{\pm\pm}(y^+,y^-): \ &= \left(\frac{dx^\pm}{dy^\pm}\right)^2 :T_{\pm\pm}(x^+,x^-): -\frac{1}{24\pi}\{x^\pm,y^\pm\} \\ &- \frac{1}{4\pi}\left[\frac{d^2x^\pm}{dy^{\pm 2}}\left(\frac{dx^\pm}{dy^\pm}\right)^{-1}\frac{\partial\phi}{\partial y^\pm} + \ln\left(\frac{dx^+}{dy^+}\frac{dx^-}{dy^-}\right)\left(\frac{\partial\phi}{\partial y^\pm}\right)^2\right] \end{aligned} \tag{12}$$

This expression generalizes the Virasoro-type transformation law (5) by adding terms depending on the derivatives of ϕ. At this point we would like to remark that the above expression has been obtained for a particular form of ϕ in terms of the null coordinates $x^\pm$, namely

$$\phi = -\ln\frac{x^+-x^-}{2}. \tag{13}$$

However we want to stress that the result has general validity, irrespective of the particular form of the external dilaton field. We shall prove this in two different ways:

i) the short-distance behaviour for the Hadamard function does not depend on the specific model;

ii) we shall show that eq. (12) is the only local expression which is consistent with the trace anomaly derived in the context of gravitational physics [4].

We point out that the conformal symmetry can be recovered in regions where $\partial_\pm\phi \to 0$. This happens typically when r approaches infinity and also, in the context of curved spacetime, at the black hole horizons.

The equation of motion for the field f, derived from the action (6), is

$$\partial_+(e^{-2\phi}\partial_- f) + \partial_-(e^{-2\phi}\partial_+ f) = 0. \tag{14}$$

In general this can be solved only for particular forms of ϕ, for instance in the situation where it is given by (13) or

$$\phi = -\frac{1}{2}\ln\frac{(x^+ - x^-)}{2} \ . \tag{15}$$

In the latter case the equation of motion for f (14) coincides with the equation of a minimal scalar field in a three-dimensional spacetime, described by the action

$$S = -\frac{1}{4\pi}\int d^3x\sqrt{-g}(\nabla f)^2 \ , \tag{16}$$

under the assumption of axi-symmetry for the field f and the metric $ds^2_{(3)} = ds^2_{(2)} + r^2 d\varphi^2$, where the radial function is given by $r = e^{-2\phi}$. This equation turns out to be equivalent to one equation of the Einstein-Rosen subsector of pure General Relativity. The system is exactly solvable both classically and quantum-mechanically (details can be found in [5], [6], [7], [8]) and, therefore, it can provide a nontrivial test of the formula (12). The field f can be expanded in modes as follows

$$f = \int_0^\infty \frac{dw}{\sqrt{2}} J_0(rw)\left[a_w e^{-iwt} + a_w^\dagger e^{iwt}\right] \tag{17}$$

where J_0 is the zero order Bessel function. At the quantum level the coefficients a_w and $a_w^\dagger$ are converted into annihilation and creation operators obeying the commutation relation $[a_w, a_{w'}^\dagger] = \delta(w - w')$. To work out the quantum behaviour of the stress tensor we need to evaluate the Hadamard function $G^{(1)}(x,x') \equiv \frac{1}{2}\langle 0|\{f(x), f(x')\}|0\rangle$. This turns out to be equal to [9], [7]

i) for $0 < |t' - t| < |r' - r|$

$$G^{(1)}(x,x') = \frac{1}{\pi\sqrt{[(r'+r)^2 - (t'-t)^2]}} K\left(\sqrt{\frac{4rr'}{(r'+r)^2 - (t'-t)^2}}\right) ;$$

ii) for $|r' - r| < |t' - t| < r' + r$

$$G^{(1)}(x,x') = \frac{1}{2\pi}\frac{1}{\sqrt{rr'}} K\left(\sqrt{\frac{(r'+r)^2 - (t'-t)^2}{4rr'}}\right) .$$

iii) for $r + r' < |t' - t|$ it is $G^{(1)}(x,x') = 0$,

where $K(k) = \int_0^{\pi/2} d\theta/\sqrt{1 - k^2\sin^2(\theta)}$ is the complete elliptic integral. Using the expansion [10]

$$K(k') = \ln\frac{4}{k'} + (\frac{1}{2})^2\left(\ln\frac{4}{k'} - 1\right)k'^2 + O(k'^4\ln\frac{4}{k'}), \tag{18}$$

where $k' = \sqrt{1-k^2}$, we obtain

$$\begin{aligned} G^{(1)}(x,x') &= -\frac{e^{\phi(x)+\phi(x')}}{4\pi}[\ln(x^+ - x'^+)(x^- - x'^-) + const. \\ &+ O\left((x^+ - x'^+)(x^- - x'^-)\ln(x^+ - x'^+)(x^- - x'^-)\right)\right] . \end{aligned} \tag{19}$$

In the computation of the transformation law of the stress tensor, via point-splitting, only the leading term in (19) produces a nontrivial contribution. Therefore it is easy to see that the final result is (12). Moreover, the above expression agrees with the De Witt-Schwinger expansion of $G^{(1)}(x,x')$, restricted to flat space-time, given in [11], [12]

$$G^{(1)}(x,x') = \frac{e^{\phi(x)+\phi(x')}}{2\pi}\left[-(\gamma + \frac{1}{2}\ln\frac{m^2\sigma}{2}) + O(\sigma\ln\sigma)\right] , \tag{20}$$

where γ is the Euler constant, m^2 is an infrared cutoff and σ is one half the square of the distance between the points x and x'.

Due to presence of ϕ the classical conservation laws $\partial_\mp T_{\pm\pm} = 0$ get modified to (see [13], [14] for a higher-dimensional interpretation)

$$\partial_\mp T_{\pm\pm} + \partial_\pm\phi\frac{\delta S}{\delta\phi} = 0, \tag{21}$$

where

$$\frac{\delta S}{\delta\phi} = -2e^{-2\phi}\partial_+ f\partial_- f \ . \tag{22}$$

Let us analyze the quantum analogous of these equations. The transformation law for $\langle : T_{\pm\pm} :\rangle$ is given by eq. (12) and the corresponding one for $\left\langle\frac{\delta S}{\delta\phi}\right\rangle$ should be, on general grounds, of the form

$$\left\langle\frac{\delta S}{\delta\phi}(y^\pm)\right\rangle = \frac{dx^+}{dy^+}\frac{dx^-}{dy^-}\left\langle\frac{\delta S}{\delta\phi}(x^\pm)\right\rangle + \Delta(\phi; x^\pm, y^\pm). \tag{23}$$

Let us suppose that

$$\partial_\mp\langle : T_{\pm\pm} :\rangle + \partial_\pm\phi\left\langle\frac{\delta S}{\delta\phi}\right\rangle = 0. \tag{24}$$

If we transform this relation according to (12) and (23) we get, by consistency,

$$
\begin{aligned}
&- \frac{1}{4\pi} \frac{\frac{\partial^2 x^\pm}{\partial y^{\pm 2}}}{\frac{\partial x^\pm}{\partial y^\pm}} \frac{\partial}{\partial y^+} \frac{\partial}{\partial y^-}\phi - \frac{1}{2\pi} \ln\left(\frac{\partial x^+}{\partial y^+}\frac{\partial x^-}{\partial y^-}\right) (\frac{\partial \phi}{\partial y^\pm}) \frac{\partial}{\partial y^+}\frac{\partial}{\partial y^-}\phi \\
&- \frac{1}{4\pi}\left(\frac{\partial\phi}{\partial y^\pm}\right)^2 \frac{\frac{\partial^2 x^\mp}{\partial y^{\mp 2}}}{\frac{\partial x^\mp}{\partial y^\mp}} + \frac{\partial \phi}{\partial y^\pm}\Delta(\phi; x^\pm, y^\pm) = 0. \qquad (25)
\end{aligned}
$$

These two equations are compatible with the uniqueness of $\Delta(\phi; x^\pm, y^\pm)$ only if

$$\Box\phi = (\nabla\phi)^2. \qquad (26)$$

If ϕ does not obey (26) the quantum conservation law (24) must be modified. We find that the only possibility to maintain consistency with the transformation law (12) is by adding a nontrivial trace $\langle T_{+-}\rangle$ just of the form

$$\langle T_{+-}\rangle = -\frac{1}{4\pi}\left(\partial_+\phi\partial_-\phi - \partial_+\partial_-\phi\right). \qquad (27)$$

Then for $\Delta(\phi; x^\pm, y^\pm)$ we obtain

$$
\begin{aligned}
\Delta &= \frac{1}{2\pi}\ln\left(\frac{dx^+}{dy^+}\frac{dx^-}{dy^-}\right)\frac{d^2\phi}{dy^+dy^-} \\
&+ \frac{1}{4\pi}\left[\frac{d^2x^-}{dy^{-2}}\left(\frac{dx^-}{dy^-}\right)^{-1}\frac{d\phi}{dy^+} + \frac{d^2x^+}{dy^{+2}}\left(\frac{dx^+}{dy^+}\right)^{-1}\frac{d\phi}{dy^-}\right]. \qquad (28)
\end{aligned}
$$

Finally, the quantum conservation law, invariant under conformal transformations, reads

$$\partial_\mp\langle : T_{\pm\pm} :\rangle + \partial_\pm\langle T_{+-}\rangle + \partial_\pm\phi\left\langle\frac{\delta S}{\delta\phi}\right\rangle = 0. \qquad (29)$$

We have to point out that the anomalous trace derived in this approach agrees with the one derived in curved space-time (first derived in [4]) . For the dilaton-coupled theory the trace anomaly, obtained in a covariant quantization scheme, is

$$\langle T\rangle = \frac{1}{24\pi}\left(R - 6(\nabla\phi)^2 + 6\Box\phi\right). \qquad (30)$$

If we restrict to flat space-time we obtain (27). We mention that in [15] the above trace anomaly was derived with a different numerical coefficient for the $\Box\phi$ term. This coefficient was then corrected, according to (30) , in [16] (the same result was obtained in [17]) . So our derivation can be seen, as a by-product, as an alternative and simple way to get the dilaton contribution to the trace anomaly. Moreover, the argument can be applied the other way around: assuming (27), (29) and locality one gets the ϕ dependent terms of (12).

Now we shall apply the above results to gravitational physics. We shall work out an expression for the expectation values of the covariant stress tensor using the anomalous transformation law (12) and the help of the equivalence principle to deal with curved space. In a generic point X of the space-time one can always introduce locally inertial coordinates ξ^α_X. Restricting our attention to the $(t-r)$-sector we can then construct the corresponding null coordinates $\xi^\pm_X$. Since normal ordering breaks general covariance we need a different prescription to construct a quantum stress tensor compatible with diffeomorphism invariance. One can do it starting from the expectation value of the normal ordered stress tensor $\langle\Psi|T_{\pm\pm}(\xi^\pm(X))|\Psi\rangle$ in the locally inertial frame $\{\xi^\pm_X\}$ with respect to some generic state $|\Psi\rangle$. The corresponding expectation values in the curved background, at the generic point X in the coordinates $\{x^\pm\}$, can be naturally defined as

$$\langle\Psi|T_{\pm\pm}(x^+(X),x^-(X))|\Psi\rangle \equiv \left(\frac{d\xi^\pm_X}{dx^\pm}(X)\right)^2 \langle\Psi| :T_{\pm\pm}(\xi^+_X(X),\xi^-_X(X):|\Psi\rangle\ , \tag{31}$$

this way we get the desired covariant property

$$\langle\Psi|T_{\pm\pm}(y^+(X),y^-(X))|\Psi\rangle = \left(\frac{dx^\pm}{dy^\pm}(X)\right)^2 \langle\Psi|T_{\pm\pm}(x^+(X),x^-(X))|\Psi\rangle \tag{32}$$

where $\{y^\pm\}$ and $\{x^\pm\}$ are arbitrary coordinate systems around the generic point X.

Now the relation between $:T_{\pm\pm}(x^+(X),x^-(X)):$ and $:T_{\pm\pm}(\xi^+_X(X),\xi^-_X(X)):$ is given by (using (12)):

$$\begin{aligned} :T_{\pm\pm}(x^+(X),x^-(X)):&= \left(\frac{d\xi^\pm_X}{dx^\pm}(X)\right)^2 :T_{\pm\pm}(\xi^+_X(X),\xi^-_X(X)): \\ -\ & \frac{1}{24\pi}\{\xi^\pm_X,x^\pm\}|_X - \frac{1}{4\pi}[\frac{d^2\xi^\pm_X}{dx^{\pm 2}}(X)\left(\frac{d\xi^\pm_X}{dx^\pm}(X)\right)^{-1}\frac{d\phi}{dx^\pm}(X) \\ +\ & \ln\frac{d\xi^+_X}{dx^+}(X)\frac{d\xi^-_X}{dx^-}(X)\left(\frac{d\phi}{dx^\pm}(X)\right)^2]\ . \end{aligned} \tag{33}$$

Inserting (33) into (31) we finally obtain

$$
\begin{aligned}
&\langle\Psi|T_{\pm\pm}(x^+(X),x^-(X))|\Psi\rangle = \langle\Psi| : T_{\pm\pm}(x^+(X),x^-(X)) : |\Psi\rangle \\
+ \quad & \frac{1}{24\pi}\{\xi_X^\pm, x^\pm\}|_X + \frac{1}{4\pi}[\frac{d^2\xi_X^\pm}{dx^{\pm 2}}(X)\left(\frac{d\xi_X^\pm}{dx^\pm}(X)\right)^{-1}\frac{d\phi}{dx^\pm}(X) \\
+ \quad & \ln\frac{d\xi_X^+}{dx^+}(X)\frac{d\xi_X^-}{dx^-}(X)\left(\frac{d\phi}{dx^\pm}(X)\right)^2] \ .
\end{aligned}
\tag{34}
$$

To go further we need the relations between $\{\xi_X^\pm\}$ and $\{x^\pm\}$. Up to second order and Poincaré transformations they are unambiguous and can be chosen to be conformal [18]

$$
\xi_X^\pm = b_\pm^\pm\left[(x^\pm - x^\pm(X)) + \frac{\Gamma_{\pm\pm}^\pm}{2}(x^\pm - x^\pm(X))^2 + F_\pm(x^\pm - x^\pm(X))^3 + ...\right] \ . \tag{35}
$$

In a conformal frame $ds^2 = -e^{2\rho}dx^+dx^-$ the constants $b_\pm^\pm$ satisfy the constraint $b_+^+b_-^- = e^{2\rho(X)}$ and $\Gamma_{\pm\pm}^\pm = 2\partial_\pm\rho$. Note that the Schwarzian derivative requires the third order as well, which is not determined by the requirement that $\xi_X^\pm$ are locally inertial. We naturally fix it by imposing that, for a flat metric, $\xi^\pm(X)$ are the global null minkowskian coordinates. This leads to

$$
F_\pm = \frac{1}{3}\partial_\pm^2\rho(X) + \frac{2}{3}\left(\partial_\pm\rho(X)\right)^2 \ . \tag{36}
$$

Using now the above expressions a straightforward computation leads to the following form for the stress tensor, for an arbitrary point X,

$$
\begin{aligned}
\langle\Psi|T_{\pm\pm}(x^+,x^-)|\Psi\rangle \quad &= \quad \langle\Psi| : T_{\pm\pm}(x^+,x^-) : |\Psi\rangle - \frac{1}{12\pi}(\partial_\pm\rho\partial_\pm\rho - \partial_\pm^2\rho) \\
&+ \quad \frac{1}{2\pi}\left[\partial_\pm\rho\partial_\pm\phi + \rho(\partial_\pm\phi)^2\right] \ .
\end{aligned}
\tag{37}
$$

We remark that neglecting the terms containing the dilaton these are the null components of the stress tensor derived from the Polyakov effective action [19].

We want to compute now a covariant expression for $\langle\frac{\delta S}{\delta\phi}\rangle$. To this end we shall impose the quantum covariant conservation laws

$$
\nabla^\mu\langle T_{\mu\nu}\rangle = \nabla_\nu\phi\frac{1}{\sqrt{-g}}\langle\frac{\delta S}{\delta\phi}\rangle \ , \tag{38}
$$

which in the conformal frame are translated into

$$
\partial_\mp\langle T_{\pm\pm}\rangle + \partial_\pm\langle T_{+-}\rangle - 2\partial_\pm\rho\langle T_{+-}\rangle + \partial_\pm\phi\left\langle\frac{\delta S}{\delta\phi}\right\rangle = 0. \tag{39}
$$

The $\langle T_{+-}\rangle$ component is, as usual, fixed by the trace anomaly:

$$\langle T_{+-}\rangle = -\frac{1}{12\pi}\left(\partial_+\partial_-\rho + 3\partial_+\phi\partial_-\phi - 3\partial_+\partial_-\phi\right). \tag{40}$$

Combining (40), (37) and (39) the final result is

$$\langle\Psi|\frac{\delta S}{\delta\phi}|\Psi\rangle = \langle\Psi|\frac{\delta S}{\delta\phi}|\Psi\rangle_{\rho=0} - \frac{1}{2\pi}(\partial_+\partial_-\rho + \partial_+\rho\partial_-\phi + \partial_-\rho\partial_+\phi + 2\rho\partial_+\partial_-\phi)\ . \tag{41}$$

The last three terms can be obtained from the anomalous transformation law for $\langle\frac{\delta S}{\delta\phi}\rangle_{\rho=0}$ (eqs.(23) and (28)), while the term $\partial_+\partial_-\rho$ comes directly from the imposition of the conservation equations (39). The state dependent quantities in (37) and (41) are conserved, namely they satisfy

$$\partial_\mp\langle\Psi| : T_{\pm\pm}(x^+,x^-) : |\Psi\rangle + \partial_\pm\langle T_{+-}\rangle|_{\rho=0} + \partial_\pm\phi\langle\Psi|\frac{\delta S}{\delta\phi}|\Psi\rangle_{\rho=0} = 0\ . \tag{42}$$

This is a crucial ingredient in order to fulfill equations (39). To match with the standard notation of 2D dilaton gravity [20] we define the following functions $t_\pm(x^+,x^-)$ and $t(x^+,x^-)$:

$$\begin{aligned} -\frac{1}{12\pi}t_\pm(x^+,x^-) &\equiv \langle\Psi| : T_{\pm\pm}(x^+,x^-) : |\Psi\rangle\ , \\ -\frac{1}{2\pi}t(x^+,x^-) &\equiv \langle\Psi|\frac{\delta S}{\delta\phi}|\Psi\rangle_{\rho=0} \end{aligned} \tag{43}$$

characterizing the quantum state $|\Psi\rangle$. Notice that now, in contrast with the minimally coupled case, the functions $t_\pm$ are no more chiral (the same is true for the new function t) and satisfy a more involved set of equations reflecting the nontriviality of the theory even in flat 2d space.

As an application of these equations we shall perform a brief analysis of the different choices of quantum states. To this end let us consider the eternal Schwarzschild spacetime, described by the 2d metric:

$$ds^2_{(2)} = -(1-2M/r)dudv\ , \tag{44}$$

where $v = t + r^*$ and $u = t - r^*$, $r^* = r + 2M\ln(\frac{r}{2M} - 1)$, and the dilaton field given by

$$e^{-2\phi} = r(u,v)^2\ . \tag{45}$$

We can naturally choose the state such that $(\{x^+ = v, x^- = u\})$

$$t_\pm = 0 \tag{46}$$

and consequently, because of (42),

$$-\frac{1}{2\pi}t \equiv \langle\Psi|\frac{\delta S}{\delta\phi}|\Psi\rangle_{\rho=0} = -\frac{3}{8\pi}\frac{M}{r^3} + \frac{1}{\pi}\frac{M^2}{r^4} \ . \tag{47}$$

This corresponds to (or at least is a good approximation of) the Boulware vacuum state $|B\rangle$ [21], describing the vacuum polarization outside a static (not collapsed) star. Applying expressions (37) we get

$$\langle B|T_{\pm\pm}|B\rangle = \frac{1}{24\pi}\left(-\frac{4M}{r^3} + \frac{15}{2}\frac{M^2}{r^4}\right) + \frac{1}{16\pi r^2}(1-\frac{2M}{r})^2\ln(1-\frac{2M}{r}) \ . \tag{48}$$

The $+-$ component is state independent and fixed by the trace anomaly (30)

$$\langle T_{+-}\rangle = \frac{1}{12\pi}(1-\frac{2M}{r})\frac{M}{r^3} \ . \tag{49}$$

Finally, we also have, from equations (41) and (47),

$$\langle B|\frac{\delta S}{\delta\phi}|B\rangle = -\frac{7}{8\pi}\frac{M}{r^3} + \frac{2}{\pi}\frac{M^2}{R^4} + \frac{1}{8\pi r^2}(1-\frac{4M}{r})(1-\frac{2M}{r})\ln(1-\frac{2M}{r}) \ . \tag{50}$$

Similar results, based on exact properties of the effective action under Weyl transformations, were derived in [22]. It is worthwhile to remark that in the horizon limit and at infinity they are in agreement with the results derived from canonical quantization [12].

A physically more interesting case is the one leading to black hole evaporation. For it a natural choice is

$$-\frac{1}{12\pi}t_-(x^+,x^-)_{v\to-\infty} \sim \frac{1}{768\pi M^2} \tag{51}$$

at the past horizon and

$$-\frac{1}{12\pi}t_+(x^+,x^-)_{u\to-\infty} \sim 0 \tag{52}$$

at past null infinity. These conditions define the Unruh vacuum state [23]. In the absence of dilaton (minimally coupled theory), the $t_\pm$ functions are chiral and then $t_- = -\frac{1}{64M^2}$ and $t_+ = 0$ everywhere. In terms of the Fock space these conditions are related to the following density matrix

$$\rho_U = \prod_w \left(1-e^{-2\pi w\kappa^{-1}}\right)\sum_{\vec{n}} e^{-2\pi\vec{n}w\kappa^{-1}}|\vec{n}_w><\vec{n}_w| \ , \tag{53}$$

where $|\vec{n}_w>$ is the state in the Fock space with $\vec{n}_w$ outgoing particles of frequency w. Without dilaton the corresponding modes are plane waves and one can see immediately that this state reproduces the above value for the function t_- and, at future null infinity, leads to the Hawking flux

$$\langle U|T_{uu}|U\rangle_{r\to+\infty} \sim \frac{1}{2\pi}\int_0^\infty \frac{wdw}{e^{8\pi Mw}-1} = \frac{\pi}{6}T_H^2 \ , \tag{54}$$

where $T_H = \frac{1}{8\pi M}$ is the Hawking temperature. In the presence of the dilaton the modes are no longer planewaves because they are affected by the potential barrier [12], [24]. In this case the result is

$$\langle U|T_{uu}|U\rangle_{r\to+\infty} \sim \frac{1}{2\pi}\int_0^\infty \frac{wdw}{e^{8\pi Mw}-1}|B(w)|^2 = \xi\frac{\pi}{6}T_H^2 \ , \tag{55}$$

where $B(w)$ is the transmission coefficient [24] and ξ the greybody factor. The greybody factor ξ, related to $|B(w)|^2$ from the above equation, produces a damping of the Hawking flux with respect to that obtained without the dilaton coupling (for the present theory it is $\xi \simeq 1.62/10$, see [12]). For the massless minimally coupled 2d scalar field there is no potential barrier, hence $\xi = 1$ and the Hawking flux is therefore given by (54). The evaluation of the expectation value of the stress tensor at the future horizon also provides the expected result. For the normal ordered operator we have

$$\langle U| : T_{vv} : |U\rangle_{r\to 2M} \sim \frac{1}{2\pi}\int_0^\infty \frac{wdw}{e^{8\pi Mw}-1}|A(w)|^2 \ , \tag{56}$$

where $A(w)$ is the reflection coefficient. Now taking into account that $|A(w)|^2 + |B(w)|^2 = 1$ and (37) we get

$$\langle U|T_{vv}|U\rangle_{r\to 2M} \sim -\frac{1}{2\pi}\int_0^\infty \frac{wdw}{e^{8\pi Mw}-1}|B(w)|^2 \ . \tag{57}$$

We see that this is the negative flux entering the black hole horizon which compensates the Hawking radiation at infinity. A similar study can also be performed for the Hartle-Hawking thermal state.

With the above analysis concerning the choice of states we have checked again the physical consistency of the proposed expression for the covariant quantum stress tensor for dilaton coupled theories. However we have to point out that the advantage of having the entire expression for the quantum stress tensor is that it allows to properly consider the one-loop semiclassical equations and to attack the interesting and difficult problem of backreaction.

To end the paper, we would like to remark that the fact that (12) is the exact transformation law of the quantum stress tensor for a generic dilaton field ϕ should not be a surprise at all. One of the main features of 2d conformal field theories is the existence of universal behaviours, irrespective of the particular model considered. Therefore one could be tempted to conjecture that (12) is also valid for an arbitrary conformal field theory coupled to a dilaton, up to numerical coefficients in the c-numer terms.

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References

[1] A.A. Belavin, A.M. Polyakov and A.B, Zamolodchikov, *Nucl. Phys.* B241 (1984) 333

[2] J.Polchinski, *String Theory*, Cambridge University Press (1998)

[3] S. Carlip, *Phys. Rel. Lett.* 88 (2002) 241301

[4] V. Mukhanov, A. Wipf and A. Zelnikov, *Phys. Lett.* B332 (1994) 283.

[5] K. Kuchar, *Phys. Rev.* D4 (1971) 955

[6] A. Ashtekar, *Phys. Rev. Lett.* 77 (1996) 4864; J. Cruz, A. Mikovic and J. Navarro-Salas, *Phys. Lett.* B437 (1998) 273; M.E. Angullo and G.A. Mena Marugan, *Int. J. Mod. Phys.* D9 (2000) 669

[7] M. Niedermaier, *Phys. Lett.* B498 (2001) 83

[8] J.F. Barbero, G.A. Mena Marugan and E.J.S. Villaseñor, *Phys. Rev.* D67 (2003) 124006

[9] J.F. Barbero, private communication

[10] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, London (1994)

[11] T.S. Bunch, S.M. Chistensen and S.A. Fulling, *Phys. Rev.* D18 (1978) 4435

[12] R. Balbinot, A. Fabbri, P. Nicolini, V. Frolov, P. Sutton and A. Zelnikov, *Phys. Rev.* D63 (2001) 084029

[13] R. Balbinot and A. Fabbri, *Phy. Rev.* D59 (1999) 044031; *Phys. Lett.* B459 (1999) 112

[14] W. Kummer and D.V. Vassilevich, *Phys. Rev.* D60 (1999) 084021; *Annalen Phys.* 8 (1999) 801

[15] R. Bousso and S.W. Hawking, *Phys. Rev.* D56 (1997) 7788

[16] W. Kummer, H. Liebl and D.V. Vassilevich, *Phys. Rev.* D58 (1998) 108501; J.S. Dowker, *Class. Quant. Grav.* 15 (1998) 1881

[17] T. Chiba and M. Siino, *Mod. Phys. Lett.* A12 (1997) 709; S. Nojiri and S.D. Odintsov, *Mod. Phys. Lett.* A12 (1997) 2083; W. Kummer, H. Liebl and D.V. Vassilevich, *Mod. Phys. Lett.* A12 (1997) 2683; A. Mikovic and V. Radovanovic, *Class. Quant. Grav.* 15 (1998) 827; S. Ichinose, *Phys. Rev.* D57 (1998) 6224

[18] S. Weinberg, *Gravitation and Cosmology*, J. Wiley and Sons , New York (1972)

[19] A.M. Polyakov, *Phys. Lett.* B103 (1981) 207

[20] L. Thorlacious, *Nucl. Phys. Proc. Suppl.* 41 (1995) 245; S.G. Giddings, *"Quantum mechanics of black holes"* hep-th/9412138 (1994); A. Strominger, *"Les Houches lectures on black holes"* hep-th/9501071 (1995)

[21] D.G. Boulware, *Phys. Rev.* D11 (1975) 1404

[22] R. Balbinot and A. Fabbri, hep-th/0012140v2 (2003)

[23] W.G. Unruh, *Phys. Rev.* D14 (1976) 870

[24] B.S. DeWitt, *Phys. Rep.* 19C (1976) 198