Deformation Quantization of non Regular Orbits of Compact Lie Groups

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Abstract

In this paper we construct a deformation quantization of the algebra of polynomials of an arbitrary (regular and non regular) coadjoint orbit of a compact semisimple Lie group. The deformed algebra is given as a quotient of the enveloping algebra by a suitable ideal.

1 Introduction

Coadjoint orbits of Lie groups can model phase spaces of physical systems since they are symplectic manifolds with a Hamiltonian symmetry group, the original Lie group G itself. Not only do they enjoy nice properties at the classical level, but they are extremely interesting as quantum systems. The Kirillov-Kostant principle associates in many cases a unitary representation of G to the orbit, and this representation is the starting point of the quantization of the system. The algebra of quantum observables is related to the enveloping algebra U of of the Lie algebra $\mathcal{G}$ of G. In fact, it can be expressed as a quotient of U by a suitable ideal $\mathcal{I}$ [26]. This ideal belongs to the kernel of the associated representation.

Given such algebra of observables, one may wonder if the approach of deformation quantization, which does not make reference to the Hilbert space, can give an isomorphic algebra. There are at least two paths that one can follow at this point. One is to make a differential deformation in the sense of Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3], De Wilde and Lecomte [11], Fedosov [12] and Kontsevich [18]. Progress were made in the study of tangential differential deformations in [16, 7, 2, 8, 19]. The other approach is to see the algebra $U/\mathcal{I}$ as some kind of deformation of the polynomials on an algebraic manifold (the coadjoint orbit itself for the case of compact groups).

In [14] the second approach was taken and a family deformations was constructed in this way for regular orbits of compact Lie groups. The family contained as a particular case the star product found in Ref.[6]. It was also shown in [6] that such star product is not differential, so it seemed that there was some kind of incompatibility among the two approaches, the differential and the algebraic one, the last one making use of the structure of the enveloping algebra. These star products were further studied in Ref.[13], and the problem of the compatibility or equivalence among these two kinds of deformations is studied in Ref.[15]. There, it was proven that there is an injective homomorphism from one algebraic star product into the differential star product algebra.

All these constructions rely heavily on the regularity of the orbit. Nevertheless, the structure of the coadjoint representation is much richer. There are many interesting cases of symplectic manifolds with symmetries which are diffeomorphic to non regular orbits (or non generic orbits, with dimension less than the maximal one). So it is of great interest to see their quantization in some way. Given a star product, one can define the star exponential and then a star representation of the original group. Much work has been done in star representation theory of semisimple groups (See for example Refs.[1, 20, 21, 5]).

Also, some precursor of this result may be found in Ref.[17] and for the differential version in Refs.[9, 22].

In this paper we solve this problem by generalizing the algebraic approach of [14, 13] to the case of non regular orbits. In Section 2 we make a short summary of the properties of the coadjoint orbits and present the non regular orbits in the appropriate way for our purposes. As an example, we give the coadjoint orbits of SU(n). In Section 3 the existence of the deformation quantization is proven by showing that a certain algebra $U_h/\mathcal{I}_h$ has the right properties, in particular the one of being isomorphic as a $\mathbb{C}[h]$-module to the polynomials on the orbit.

Finally we want to note that the construction could be extended to semisimple orbits of non compact groups, by quantizing the real form of the complex orbit, which is in fact a union of connected components each of them a real orbit. The case of nilpotent orbits is for the moment not considered in this approach.

2 Regular and non regular orbits as algebraic varieties

Let G be a compact semisimple group of dimension n and rank m and $\mathcal{G}$ its Lie algebra. Let $\mathcal{G}^*$ be the dual space to $\mathcal{G}$. On $C^\infty(\mathcal{G}^*)$ we have the Kirillov Poisson structure

$$\{f_1, f_2\}(\lambda) = \langle [(df_1)_\lambda, (df_2)_\lambda], \lambda \rangle, \qquad f_1, f_2 \in C^\infty(\mathcal{G}^*), \quad \lambda \in \mathcal{G}^*.$$

$df_\lambda : \mathcal{G}^* \to \mathbb{R}$ can be considered as an element of $\mathcal{G}$, and [,] is the Lie bracket on $\mathcal{G}$. Let $\{X_1 \dots X_n\}$ be a basis of $\mathcal{G}$ and $\{x^1, \dots x^m\}$ the coordinates on $\mathcal{G}^*$ in the dual basis. We have that

$$\{f_1, f_2\}(x^1, \dots x^n) = \sum_{ijk} c_{ij}^k x^k \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j}, \qquad \text{where} \qquad [X_i, X_j] = \sum_k c_{ij}^k X_k.$$

Notice that the ring of polynomials of $\mathcal{G}^*$, Pol$(\mathcal{G}^*)$, is closed under the Poisson bracket.

The Kirillov Poisson structure is not symplectic nor regular. As any Poisson manifold, $\mathcal{G}^*$ can be foliated in symplectic leaves, the Poisson bracket restricting to a symplectic Poisson bracket on the leaves of the foliation. The symplectic leaves support a hamiltonian action of G. Indeed, they are the orbits of the coadjoint action of G on $\mathcal{G}^*$.

The coadjoint action of G on $\mathcal{G}^*$, Ad^*, is defined by

$$\langle \mathrm{Ad}^*_g \lambda, Y\rangle = \langle \lambda, \mathrm{Ad}_{g^{-1}} Y\rangle \quad \forall\, g \in G, \quad \lambda \in \mathcal{G}^*, \quad Y \in \mathcal{G}.$$

We denote by C_λ the orbit of an element $\lambda \in \mathcal{G}^*$ under the coadjoint action. The coadjoint orbits are real irreducible algebraic varieties (see for example [4, 23]). Let $\mathrm{Inv}(\mathcal{G}^*) \subset \mathrm{Pol}(\mathcal{G}^*)$ the subalgebra of polynomials on $\mathcal{G}^*$ invariant under the coadjoint action. Then $\mathrm{Inv}(\mathcal{G}^*) = \mathbb{R}[p_1, \dots p_m]$, where $\{p_1 \dots p_m\}$ is a system of algebraically independent homogeneous invariant polynomials, and m is the rank of $\mathcal{G}$ (Chevalley's theorem).

Since G is a semisimple Lie group, we can identify $\mathcal{G} \simeq \mathcal{G}^*$ by means of the invariant Cartan-Killing form, the isomorphism intertwining the adjoint and coadjoint representations. From now on we will assume that such identification has been made. The set $\mathrm{Inv}(\mathcal{G}^*)$ is in one to one correspondence with the set of polynomials on the Cartan subalgebra that are invariant under the Weyl group, the isomorphism being given by the restriction.

Consider the adjoint action of $\mathcal{G}$ on $\mathcal{G}$, ad. The characteristic polynomial of $X \in \mathcal{G}$ in the indeterminate t is

$$\det(t \cdot \mathbf{1} - \mathrm{ad}_X)) = \sum_{i \geq m} q_i(X) t^i.$$

The q_i's are invariant polynomials. An element $X \in \mathcal{G}$ is regular if $q_m(X) \neq 0$. When restricting to the Cartan subalgebra $\mathcal{H}$, q_m is of the form

$$q_m(H) = \prod_{\alpha \in \Delta} \alpha(H), \qquad H \in \mathcal{H},$$

with Δ the set of roots. It is clear that an element H is regular if and only if it belongs to the interior of a Weyl chamber. Any orbit of the adjoint action intersects the Weyl chamber in one and only one point. If H is regular, $\mathrm{Ad}_g = gHg^{-1}$ is also regular, so the orbit C_H is a regular orbit. The differentials of Chevalley's generators dp_i, $i = 1, \dots m$ are linearly independent only on the regular elements [24]. Moreover, the regular orbits are defined as algebraic varieties by the polynomial equations

$$p_i = c_i^0, \qquad c_i^0 \in \mathbb{R} \qquad i = 1, \dots m.$$

The ideal of polynomials vanishing on a regular orbit is given by

$$\mathcal{I}_0 = (p_i - c_i,\ i = 1, \dots m),$$

and the coordinate ring of C_X is $\mathrm{Pol}(C_X) \simeq \mathrm{Pol}(\mathcal{G}^*)/\mathcal{I}_0$.

Example 2.1 *Coadjoint orbits of SU(n)*

We will consider the compact Lie group SU(n), with (complexified) Lie algebra $\mathcal{A}_m = \mathrm{sl}(m+1, \mathbb{C})$, $n = m+1$. A Cartan subalgebra of $\mathcal{A}_m$ is given by the diagonal matrices

$$\mathcal{H} = \{\mathrm{diag}(a_1, a_2, \dots a_{m+1}), \quad a_1 + a_2 + \cdots + a_{m+1} = 0 \quad a_i \in \mathbb{C}\}.$$

Denoting by

$$\lambda_i(\mathrm{diag}(a_1, a_2, \dots a_{m+1}) = a_i,$$

then the roots are given by $\alpha_{ij} = \lambda_i - \lambda_j$. We will denote the simple roots as $\alpha_i = \lambda_i - \lambda_{i+1}$. The *root vectors* are defined by

$$\alpha_{ij}(H) = \langle H, H_{ij} \rangle,$$

where $\langle\ ,\ \rangle$ denotes the Cartan-Killing form on $\mathcal{H}$. One normalizes the root vectors as

$$\bar{H}_{ij} = \frac{2}{\langle H_{ij}, H_{i,j} \rangle} H_{ij}.$$

The Weyl group is the group of reflections of $\mathcal{H}$ generated by

$$\omega_{ij}(H) = H - \alpha_{ij}(H)\bar{H}_{ij}.$$

We have

$$\omega_{ij}(\mathrm{diag}(a_1, \dots a_i, \dots a_j, \dots a_{m+1})) = \mathrm{diag}(a_1, \dots a_j, \dots a_i, \dots a_{m+1}),$$

so an element of the Weyl group is

$$\omega_s(\mathrm{diag}(a_1, a_2, \dots a_{m+1})) = (\mathrm{diag}(a_{s^{-1}(1)}, a_{s^{-1}(2)} \dots a_i, \dots a_{s^{-1}(m+1)})), \quad s \in \Pi_{m+1},$$

with Π_{m+1} the group of permutations of order $m+1$.

We take the real span $\mathcal{H}^{\mathbb{R}} = \oplus_{i=1}^{l} \mathbb{R}\bar{H}_i$. Any point $(a_1, a_2, \dots a_{m+1}) \in \mathcal{H}^R$ can be brought to the form

$$(a_1, a_2, \dots a_{m+1}), \qquad (a_1 \geq a_2 \cdots \geq a_{m+1})$$

by applying a suitable permutation. The intersection of this subset of $\mathbb{R}^n$ with the solution of $a_1 + a_2 + \cdots + a_{m+1} = 0$ is the closed principal Weyl chamber since

$$\alpha_i(a_1, a_2, \dots a_{m+1}) \geq 0 \qquad \forall i.$$

We go now to the compact real form. By means of the map

$$A + iB \longrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{SO(2n)}, \qquad A + iB \in \mathrm{SU(n)},$$

(real representation of $\mathrm{SU}(n)$, $n = m + 1$) we obtain an isomorphism

$$\mathrm{SU(n)} \simeq \mathrm{SO(2n)} \cap \mathrm{Sp(2n)}.$$

The Cartan subalgebra is represented by matrices of the form

$$\mathrm{diag}(a_1\Omega, a_2\Omega, \dots a_n\Omega), \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad a_1 + a_2 + \cdots + a_n = 0, \quad a_i \in \mathbb{R}.$$

To compute the orbits of the Weyl group it is enough to restrict to one Weyl chamber. Consider the partition $n = p_1 + \cdots + p_r$, p_i positive integers, and consider a point $H = (a_1, a_2, \dots a_n)$ in the Weyl chamber such that

$$a_1 = \cdots = a_{p_1} \geq a_{p_1+1} = \cdots = a_{p_1+p_2} \geq \cdots = a_n.$$

The group of matrices leaving H invariant by the adjoint action, gHg^{-1} is isomorphic to $\mathrm{Sp}(2p_1) \times \mathrm{Sp}(2p_2) \times \mathrm{Sp}(2p_r) \subset \mathrm{Sp}(2n)$. So the isotropy group of H in SU(n) is

$$\mathrm{SO}(2n) \cap \mathrm{Sp}(2p_1) \times \mathrm{Sp}(2p_r) = \mathrm{S}(\mathrm{U}(p_1) \times \cdots \mathrm{U}(p_r)). \tag{1}$$

The coadjoint orbits are spaces of the type G/H with $G = \mathrm{SU}(n)$ and H one of the isotropy groups in (1). The regular orbits are diffeomorphic to $\mathrm{SU(n)}/\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$ (with $n - 1$ U(1) factors). The non regular orbits correspond obviously to the border of the Weyl chamber, where at least one of the roots has value zero, $\alpha_i(H) = 0$. ■

Non regular orbits are also algebraic varieties, but unlike the regular orbits, the ideal $\mathcal{I}_0$ of a non regular orbit is not generated by the invariant polynomials $(p_i - c_i,\ i = 1, \dots m)$. The next proposition shows that there is a special set of generators that are invariant as a set.

Proposition 2.1 *If C_λ is a semisimple coadjoint orbit of a semisimple Lie group, then the ideal of C_λ, $\mathcal{I}_0$, is generated by polynomials $r_1(x), \dots r_l(x)$ such that*

$$r_\alpha(g \cdot x) = \sum_{\beta=1}^{l} T(g)_{\alpha\beta} r_\beta(x), \qquad \alpha = 1, \dots l, \qquad g \in G. \tag{2}$$

where T is a representation of G (and of $\mathcal{G}$).

Proof. By the Hilbert basis theorem, every ideal in $\mathbb{C}[x_1 \dots x_n]$ has a finite generating set. Let $\{q_1(x), \dots q_r(x)\}$ be an arbitrary finite set of generators of $\mathcal{I}_0$. Let $q(x) \in \mathcal{I}_0$. G is an algebraic group and the action of G on $\mathcal{G}$ sends polynomials into polynomials. Then $q^g(x) = q(g^{-1}x) \in \mathcal{I}_0$. We consider the set

$$\{q_i^g(x), \quad i = 1, \dots r, \quad g \in G\}$$

which obviously generates $\mathcal{I}_0$. We consider the $\mathbb{C}$-linear span of $\{q_i^g\}$. Notice that it is a finite dimensional vector space since the degree of q_i doesn't change under the action of the group. We take a $\mathbb{C}$-basis of it denoted by $r_1(x), \dots r_l(x)$. Then

$$r_j^g(x) = r_j(g^{-1}x) = \sum_{k=1}^{l} T_{jk}(g^{-1}) r_k(x), \qquad j = 1, \dots l.$$

It is immediate to see that the matrices $T(g)$ form a representation of G, as we wanted to show. ■

In terms of the Lie algebra, equation (2) can be written as

$$X.r_i(x) = \sum_k T(X)_{ik} r_k(x), \qquad X \in \mathcal{G} \tag{3}$$

where X acts as a derivation on $\mathbb{C}[\mathcal{G}^*]$.

3 Deformation quantization of non regular orbits

In this section we consider algebras over $\mathbb{C}[h]$ (polynomials on the indeterminate h). We will consider the complexification of the polynomial ring of the orbit. The deformation quantization that we will obtain is an algebra over $\mathbb{C}[h]$, so it can be evaluated at any particular value $h = h_0 \in \mathbb{R}$.

We consider the tensor algebra $T_{\mathbb{C}}(\mathcal{G})[h]$ and its proper two sided ideal

$$\mathcal{L}_h = \sum_{X,Y\in\mathcal{G}} T_{\mathbb{C}}(\mathcal{G})[h] \otimes (X\otimes Y - Y\otimes X - h[X,Y]) \otimes T_{\mathbb{C}}(\mathcal{G})[h]. \qquad (4)$$

We define $U_h = T_{\mathbb{C}}(\mathcal{G})[h]/\mathcal{L}_h$. It is well known that U_h is a deformation quantization of $\mathbb{C}[\mathcal{G}^*] = \mathrm{Pol}(\mathcal{G}^*)$. In [14] it was shown that if C_λ is a regular orbit, then there exists a deformation quantization $\mathrm{Pol}(C_\lambda)$ as $U_h/\mathcal{I}_h$ where $\mathcal{I}_h \to \mathcal{I}_0$ when $h \to 0$. In this section we want to generalize that construction to the case of non regular orbits. First we briefly review how the deformation is obtained in the regular case.

Let $\{X_1,\dots,X_n\}$ be a basis for $\mathcal{G}$ and let $\{x_1,\dots,x_n\}$ be the corresponding generators of $\mathbb{C}[\mathcal{G}^*]$. There is a natural isomorphism of $\mathbb{C}[\mathcal{G}^*]$ with the symmetric tensors $ST_{\mathbb{C}}(\mathcal{G}) \subset T_{\mathbb{C}}(\mathcal{G})$, $\mathrm{Sym} : \mathbb{C}[h][\mathcal{G}^*] \longrightarrow ST_{\mathbb{C}[h]}(\mathcal{G})$ by

$$\mathrm{Sym}(x_1\cdots x_p) = \frac{1}{p!}\sum_{s\in S_p} X_{s(1)}\otimes\cdots\otimes X_{s(p)} \qquad (5)$$

where S_p is the group of permutations of order p. The composition of the symmetrizer with the natural projection $T_{\mathbb{C}[h]}(\mathcal{G}) \longrightarrow U_h$ is a linear isomorphism $W : \mathbb{C}[h][\mathcal{G}^*] \to U_h$ called the Weyl map. This map has the following interesting property (see for example Ref. [25] for a proof). Let A be an automorphism (derivation) of $\mathcal{G}$. It extends to an automorphism (derivation) of U_h denoted by $\tilde{A}$. It also extends to an automorphism (derivation) $\bar{A}$ of $ST_{\mathbb{C}}(\mathcal{G}) \simeq \mathbb{C}[\mathcal{G}^*]$. Then

$$W \circ \bar{A} = \tilde{A} \circ W. \qquad (6)$$

Taking $A = \mathrm{ad}_X$, $X \in \mathcal{G}$, (6) implies that the images of the invariant polynomials $P_i = W(p_i)$, $i = 1,\dots,m$ belong to the center of U_h. The two sided ideal $\mathcal{I}_h = (P_i - c_i(h))$, $c_i(h) \in \mathbb{C}[h]$ with

$$\mathcal{I}_h \xrightarrow[h\mapsto 0]{} \mathcal{I}_0$$

is equal to the right and left ideals with the same generators, $I_h = \mathcal{I}_h^L = \mathcal{I}_h^R$. We have that $U_h/\mathcal{I}_h$ is a deformation quantization of $\mathrm{Pol}(C_\lambda)$ for C_λ a regular orbit [14, 13]. The subtle point in the proof was to show that there is a $\mathbb{C}[h]$-module isomorphism $\mathbb{C}[\mathcal{G}^*][h]/\mathcal{I}_0 \simeq U_h/\mathcal{I}_h$. This was done by choosing a basis in $\mathbb{C}[\mathcal{G}^*][h]/\mathcal{I}_0$, mapping it to $U_h/\mathcal{I}_h$ and showing that the image is a basis of $U_h/\mathcal{I}_h$. Proving the linear independence of the images made use of the regularity hypothesis. Here we will give a proof of the independence that does not use the regularity condition, so it applies for non regular orbits as well.

Let C_λ be a non regular orbit, and let its ideal $\mathcal{I}_0$ be generated by r_α, $\alpha = 1, \dots l$ satisfying the condition (2). We consider the images under the Weyl map of the generators, $R_\alpha = W(r_\alpha)$. We have the following

Lemma 3.1 *The left and right ideals $\mathcal{I}_h^{L,R}$ generated by R_α, $\alpha = 1, \dots l$ are equal and then equal to the two sided ideal $\mathcal{I}_h$.*

Proof. It is enough to prove that $[X, R_\alpha] = \sum_\beta C_{\alpha\beta} R_\beta$ for any $X \in \mathcal{G}$ and some $C_{\alpha\beta} \in \mathbb{C}$. Using (6) and (3) for $A = \mathrm{ad}_X$, we have

$$\begin{aligned}[X, R_\alpha] = [X, W(r_\alpha)] = W(X.r_\alpha) = W(T(X)_{\alpha\beta} r_\beta) = \\ T(X)_{\alpha\beta} W(r_\beta) = T(X)_{\alpha\beta} R_\beta.\end{aligned}$$

■

In order to show that $U_h/\mathcal{I}_h$ is a deformation quantization of $\mathrm{Pol}(C_\lambda)$ we have to show that $\mathrm{Pol}(C_\lambda)[[h]]$ isomorphic as a $C[[h]]$-module to $U_h/\mathcal{I}_h$. We will do it in several steps. We need to introduce the evaluation map $\mathrm{ev}_{h_0} : U_h \to U_h/(h - h_0) \simeq U_{h_0}$ where U_{h_0} is the enveloping algebra of $\mathcal{G}$ with bracket $h_0[\ ,\ ]$. As it is well known for any enveloping algebra, U_{h_0} is a filtered algebra,

$$\mathcal{G} \simeq U_{h_0}^{(1)} \subset U_{h_0}^{(1)} \subset \cdots \subset U_{h_0}^{(n)} \subset \cdots$$

where

$$U_{h_0}^{(n)}/U_{h_0}^{(n-1)} \simeq ST_{\mathbb{C}}^{(n)}(\mathcal{G}) \simeq \mathbb{C}[\mathcal{G}^*]^{(n)}.$$

The graded algebra associated to the filtered algebra U_{h_0} is then the algebra of symmetric tensors on $\mathcal{G}$ (or polynomials on $\mathcal{G}^*$), so there exists a natural projection

$$\pi : U_{h_0} \to \mathbb{C}[\mathcal{G}^*]. \tag{7}$$

We introduce the graded lexicographic ordering in $\mathbb{C}[\mathcal{G}^*]$. We consider the ideal generated by the leading terms of r_α, $(LT(r_\alpha))$. Any equivalence class in $\mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0$ has a unique representative as a linear combination of elements in the set $\mathcal{B} = \{x_{i_1}x_{i_2}\cdots x_{i_k},\ x_{i_1}x_{i_2}\cdots x_{i_k} \notin (LT(r_\alpha))\}$. (The basis $\{r_\alpha\}$ can be chosen as a Groebner basis). In fact, the elements of $\mathcal{B}$ are linearly independent over $\mathbb{C}$ and form a basis of $\mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0$ (see for example [10] for a proof). We will denote by J the set of indices $(i_1,\dots i_k)$ of elements of $\mathcal{B}$.

We have the following

Lemma 3.2 *The standard monomials*

$$\{X_{i_1}\cdots X_{i_k}, \quad (i_1,\dots i_k)\in J\} \tag{8}$$

are linearly independent in $U_{h_0}/\mathcal{I}_{h_0}$.

Proof. Suppose that there exists a linear relation among them

$$\sum_{(i_1\dots i_k)\in J} c^{i_1\dots i_k} X_{i_1}\cdots X_{i_k} = \sum_\alpha A^\alpha R_\alpha, \qquad A^\alpha \in U_{h_0}. \tag{9}$$

We project (9) onto $\mathbb{C}[\mathcal{G}^*]$ as in (7),

$$\begin{aligned} \pi(\sum_{(i_1\dots i_k)\in J} X_{i_1}\cdots X_{i_k}) &= \sum_{(i_1\dots i_k)\in J_0} c^{i_1\dots i_k} x_{i_1}\cdots x_{i_k}, \\ \pi(\sum_\alpha A^\alpha R_\alpha) &= \sum_\alpha b^\alpha r_\alpha + \text{terms with degree } < l_0 \end{aligned} \tag{10}$$

where $J_0 \subset J$ corresponds to the monomials with highest degree (l_0) in the left hand side of (9). The second equation in (10) expresses the fact that the right hand side of (9) must project to a linear combination of r_α modulo terms of lower degree. So we have

$$\sum_{(i_1\dots i_k)\in J_0} c^{i_1\dots i_k} x_{i_1}\cdots x_{i_k} = \sum_\alpha b^\alpha r_\alpha + \text{terms with degree } < l_0. \tag{11}$$

Taking the leading term of both sides of the equation (11), and taking into account that $\{r_\alpha\}$ is a Groebner basis, we obtain that the leading term of $\sum_{(i_1\dots i_k)\in J_0} c^{i_1\dots i_k} x_{i_1}\cdots x_{i_k}$ must be proportional to one of the leading terms $LT(r_\alpha)$, which is not possible by the construction of the basis $\mathcal{B}$. ∎

We can prove now the independence of the monomials (8) on $U_h/\mathcal{I}_h$.

Proposition 3.1 *The standard monomials*

$$\{X_{i_1}\cdots X_{i_k}, \quad (i_1,\dots i_k)\in J\}$$

are linearly independent in $U_h/\mathcal{I}_h$.

Proof. Suppose that there is a linear combination of them equal to zero,

$$\sum_{(i_1 \dots i_k) \in J} c^{i_1 \dots i_k}(h) X_{i_1} \cdots X_{i_k} = \sum_\alpha A^\alpha R_\alpha, \qquad A^\alpha \in U_h. \tag{12}$$

Applying the evaluation map ev_{h_0} to (12), we have

$$\sum_{(i_1 \dots i_k) \in J} c^{i_1 \dots i_k}(h_0) X_{i_1} \cdots X_{i_k} = \sum_\alpha A_0^\alpha R_\alpha, \qquad \mathrm{ev}_{h_0} A^\alpha = A_0,$$

which implies, by Lemma 3.2 that $c^{i_1 \dots i_k}(h_0) = 0$. Since this is true for infinitely many h_0 and $c^{i_1 \dots i_k}(h)$ is a polynomial, we have that $c^{i_1 \dots i_k}(h) = 0$. ■

We want now to prove that the monomials (8) generate $U_h/\mathcal{I}_h$, so they form a basis of the $\mathbb{C}[h]$-module which is then free and isomorphic to $\mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0[h]$. The proof is the same than the one in Ref. [14] for regular orbits, so we do not repeat it here

Proposition 3.2 *The standard monomials $\{X_{i_1} \cdots X_{i_k}\}$ with $(i_1, \dots, i_k) \in J$ generate U_h/I_h as $\mathbb{C}[h]$-module.*

Summing up, we have the following

Theorem 3.1 *Let Θ be a (possibly non regular) coadjoint orbit of a compact semisimple group. In the same notation as above, U_h/I_h is a deformation quantization of* $\mathrm{Pol}(\Theta) = \mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0$*. It has the following properties:*
1. U_h/I_h is isomorphic to $\mathbb{C}[\mathcal{G}^*]/I_0[h]$ as a $\mathbb{C}[h]$-module.
2. The multiplication in U_h/I_h reduces $\mathrm{mod}(h)$ *to the one in $\mathbb{C}[\mathcal{G}^*]/I_0$.*
3. The bracket $[F, G] = FG - GF$, in U_h/I_h, reduces $\mathrm{mod}(h^2)$ *to (h times) the Poisson bracket on the orbit.*

Proof. 1 is a consequence of Propositions 3.1 and 3.2. 2 and 3 are trivial.

Remark 3.1 *Extension to* $\mathbb{C}[[h]]$

The extension to $\mathbb{C}[[h]]$ can be made by taking the inverse limits of the families $U_h/h^n U_h$ and $(U_h/\mathcal{I}_h)/h^n(U_h/\mathcal{I}_h)$. The elements $\{X_{i_1} \cdots X_{i_k}\}$ with $(i_1, \dots, i_k) \in J$ are linearly independent in the inverse limit since they are so in each of the projections to $(U_h/\mathcal{I}_h)/h^n(U_h/\mathcal{I}_h)$. Then one can show that they form a basis.

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