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# On the deformation quantization of affine algebraic varieties

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**Abstract**

We compute an explicit algebraic deformation quantization for an affine Poisson variety described by an ideal in a polynomial ring, and inheriting its Poisson structure from the ambient space.

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# 1 Introduction

Since the fundamental work of Bayen et al [1] in the seventies, a lot of effort has been dedicated to show the existence of deformations of a Poisson manifold. Some landmarks in this way were the proof of the existence of differential star products for symplectic manifolds which was done independently by De Wilde and Lecomte [2] and Fedosov [3], using different constructions. It turned out that the star products on a symplectic manifold are classified, up to equivalence, by the de Rahm cohomology $H^2(M)$. Etingof and Kazhdan showed the existence of star products for another class of Poisson manifolds, the Poisson-Lie groups. Kontsevich gave the proof of existence and classification of star products on an arbitrary Poisson manifolds as a consequence of his formality theorem [5]. Tamarkin [6] gave another proof of the formality theorem that relates it to Deligne's conjecture on Hochschild's complexes.

More recently, there has grown an interest in translating all the results mentioned above (valid for $C^\infty$ manifolds and differential deformations) to the algebraic geometric setting [7, 8, 9]. We will comment on these approaches in Section 2. On the other hand, algebraic star products on the sphere appeared as soon as in Refs. [10], and later on, a more general construction appeared in Refs. [11, 12]. In these last references, the importance in physics of algebraic (and not necessarily differential) star products was stressed, because they are the physical choice in problems as fundamental as the quantization of angular momentum. As a consequence, they are also related to geometric quantization. This was not taken into account the original papers when the differentiability hypothesis was assumed through the whole process of deformation quantization.

The relation between algebraic and differential star products was intriguing, and it was studied in successive papers [13, 14]. The approach followed in these articles was restricted to coadjoint orbits of semisimple groups (so, to linear Poisson structures). Explicit algebras, defined by generators and relations, where considered and this allowed to find some new features in the algebraic case. One could easily see that there were non isomorphic deformations even in the simplest case.

It is our intention in this work to extend the approach of of Refs. [11, 12] to a wider case of affine Poisson algebraic varieties, where the degree of the Poisson structure is arbitrary. Although working with a restricted class of algebraic varieties, the main advantage of our approach is that the deformation is shown explicitly in terms of generators and relations, with no recursion to gluing procedures. We will construct a suitable non commutative algebra and then we will show that it is an algebraic deformation of the coordinate ring of the variety. The precise sense of this statement is explained in Section 2, where we fix the notation and briefly explore other approaches present in the literature. The starting point is the deformation of the affine space given by Kontsevich [5]. In Section 3 we present such algebra as a quotient of the tensor algebra by a certain ideal, very much in the way that the universal enveloping algebra is presented (which is the deformation of a linear Poisson structure [15]). In the case of the coadjoint orbits, the deformed algebra was obtained by quotienting the enveloping algebra by an ideal that it is related to the ideal of the variety. In Section 4 the same procedure is extended to a bigger class of algebraic varieties, with no particular restriction on the degree of the Poisson structure. Some assumptions, nevertheless, must be made in the course of the proof, but they are of technical nature and it is likely that they can be dropped. At this moment we do not know if this is possible.

Furthermore, one may think on gluing the deformations obtained for affine varieties to deform more general algebraic varieties, perhaps with a procedure à la Fedosov. This is out of the scope of the present paper but may be approached in other works.

# 2 Preliminaries

In this section we want to introduce some of the key definitions of the theory of deformation quantization. In particular we want to compare our definitions and approach with the ones appearing in the literature.

**Definition 2.1** *Let* $(\mathcal{A}, \{\ ,\ \})$ *be a Poisson algebra over a field* $k$*. We say that the associative algebra* $\mathcal{A}_{[h]}$ *over* $k[[h]]$ *is a formal deformation of* $\mathcal{A}$ *if*

1. *There exists an isomorphism of* $k[[h]]$*-modules* $\psi : \mathcal{A}[[h]] \longrightarrow \mathcal{A}_h$*;*
2. $\psi(f_1 f_2) = \psi(f_1)\psi(f_2) \operatorname{mod}(\mathrm{h}), \quad \forall f_1, f_2 \in A[[h]]$*;*
3. $\psi(f_1)\psi(f_2) - \psi(f_2)\psi(f_1) = h\psi(\{f_1, f_2\}) \operatorname{mod}(h^2), \quad \forall f_1, f_2 \in \mathcal{A}[[h]]$*.* ■

If $\mathcal{A}_{\mathbb{C}}$ is the complexification of a real Poisson algebra $\mathcal{A}$ we can give the definition of *formal deformation* of $\mathcal{A}_{\mathbb{C}}$ by replacing $\mathbb{R}$ with $\mathbb{C}$ in Definition 2.1. A reality condition on the star product may be required in order to have a star product defined also over $\mathbb{R}$.

The associative product in $\mathcal{A}[[h]]$ defined by

$$f \star g = \psi^{-1}(\psi(f) \cdot \psi(g)), \qquad f, g \in \mathcal{A}[[h]] \tag{1}$$

is called the *star product on* $\mathcal{A}[[h]]$ *induced by* $\psi$.

A star product on $\mathcal{A}[[h]]$ can be also defined as an associative $k[[h]]$-linear product given by the formula

$$f \star g = fg + B_1(f,g)h + B_2(f,g)h^2 + \cdots \in \mathcal{A}[[h]], \quad f, g \in \mathcal{A} \tag{2}$$

where the $B_i$'s are bilinear operators. The associativity of $\star$ implies that $\{f, g\} = B_1(f,g) - B_1(g,f)$ is a Poisson bracket on $\mathcal{A}$. So this definition is a special case of the previous one where $\mathcal{A}_h = \mathcal{A}[[h]]$ and $\star$ is induced by $\psi = \mathrm{Id}$.

Two star products on $\mathcal{A}[[h]]$, $\star$ and $\star'$ are said to be *equivalent* (or *gauge equivalent*) if there exists a linear map $T : \mathcal{A}[[h]] \to \mathcal{A}[[h]]$ of the form

$$T = \mathrm{Id} + \sum_{n>0} h^n T_n$$

with $T_n$ linear operators on $\mathcal{A}[[h]]$, such that

$$f \star g = T^{-1}(T(f) \star' T(g)).$$

Two star products that are equivalent are isomorphic and have the same first order term, so they are formal deformations of the same Poisson structure.

If $\mathcal{A} \subset C^\infty(M)$ and the operators $B_i$'s are bidifferential operators we say that the star product is *differential*. If in addition $\mathcal{A} = C^\infty(M)$ and $M$ is a real Poisson manifold, we will say that $\star$ is a *differential star product on* $M$. The set of (gauge) equivalence classes of differential star products on a manifold $M$ has been classified by Kontsevich in terms of equivalence classes of formal Poisson structures (modulo formal diffeomorphisms) [5].

Let $M_\mathbb{C}$ be a complex algebraic affine variety defined over $\mathbb{R}$, whose real points are a real algebraic Poisson variety $M$. We denote by $\mathcal{A}_\mathbb{C} = \mathbb{C}[M_\mathbb{C}]$ its coordinate ring, which is a Poisson algebra. We will say that a formal deformation of $\mathcal{A}_\mathbb{C}$ is a formal deformation of the algebraic variety $M_\mathbb{C}$.

We have an *algebraic star product on* $M$ (or $M_\mathbb{C}$) if the bilinear operators $B_i$ in (2) are algebraic operators.

A special example of algebraic Poisson varieties are the coadjoint orbits Lie groups. They are the symplectic leaves of the Kirillov Poisson structure in the dual of the Lie algebra, which is a linear Poisson structure. In Refs. [11, 12] the case of a compact semisimple group was considered and a a family of algebraic deformations of the coadjoint orbits was constructed. This was done using the embedding of the coadjoint orbit in the affine space as an affine algebraic, Poisson subvariety.

In [7] Kontsevich gives a definition of *semiformal deformations* of algebraic varieties. Basically, the deformed algebra must have a filtration and at each step it should be finitely generated. Adapted to our notation the definition is the following.

**Definition 2.2** *Let $(\mathcal{A}, \{,\})$ be a finitely generated Poisson algebra over a field $k$. We say that the associative algebra $\mathcal{A}_h$ over $k[[h]]$ is a semiformal deformation of $\mathcal{A}$ if*

*1. $\mathcal{A} \simeq \mathcal{A}_h \otimes_{k[[h]]} k$ (i.e. $\mathcal{A}_h/(h) \simeq \mathcal{A}$). We denote by $\pi : \mathcal{A}_h \rightarrow \mathcal{A}_h/(h) \cong \mathcal{A}$ the natural projection.*
*2. There exists on $\mathcal{A}_h$ an exhaustive increasing filtration, compatible with the product, and admitting a splitting as a filtration of $k[[h]]$-modules. In other words $\mathcal{A}_h = \cup_n \mathcal{A}^n$ with $\mathcal{A}^n \cdot \mathcal{A}^m \subset \mathcal{A}^{m+n}$, where $\mathcal{A}^n$ are finitely generated free $k[[h]]$-modules, each a direct addend of $\mathcal{A}_h$. This means that there exists a $k[[h]]$-module $B^n$ such that $\mathcal{A}_h = A^n \oplus B^n$.*
*3. $f_1 \star f_2 - f_2 \star f_1 = h\{\pi(f_1), \pi(f_2)\} \bmod(h^2), \quad \forall f_1, f_2 \in \mathcal{A}_h$.* ■

The concept of semiformal deformation differs from the formal one. Let $\mathcal{A}$ be a commutative graded algebra and let

$$\mathcal{A}_h = \mathcal{A} \otimes k[[h]], \text{ with } \mathcal{A}_n = \{f \in \mathcal{A} \otimes k[[h]] \quad | \quad \deg f \leq n\}.$$

Assume that we have a product in $\mathcal{A}_h$ compatible with the filtration. This is a prototype of a semiformal deformation. If $\mathcal{A}$ is not finite dimensional the $k[[h]]$-module $\mathcal{A} \otimes k[[h]]$ is strictly contained in $A[[h]]$, the $k[[h]]$-module underlying a formal deformation of $\mathcal{A}$.

In some cases, given a formal deformation of $\mathcal{A}$, it is possible to find a semiformal deformation sitting inside. This happens for example when the the $k[[h]]$-module $\mathcal{A} \otimes k[[h]]$ is closed under the star product.

In Ref.[7], §2.3, Kontsevich discusses this situation for the algebra $\mathcal{A} = \mathrm{Sym}(\mathrm{V}^*)$, where $V^*$ is a finite dimensional vector space over $k$. It appears that only Poisson structures with degree up to 2 admit semiformal deformations.

Kontsevich also proposes in §4.1 the problem of finding a semiformal deformation of regular coadjoint orbits of semisimple Lie groups. Remember that in this case the Poisson structure is linear. The problem was indeed already solved in [11, 12], where formal deformations of the polynomial algebra of coadjoint orbits were explicitly constructed. The deformations are also semiformal, in the sense that the subspace $\mathcal{A}\otimes\mathbb{C}[[h]]$ is closed under the star product.

Another approach to the deformation of algebraic varieties is taken in Ref. [8]. It is shown there that any smooth, Poisson algebraic variety (with some topological requirements) admits a deformation. Also, such deformations are classified in terms of formal Poisson structures (up to gauge equivalence), in the same way that deformations of differential manifolds where classified by Kontsevich [5]. The basic idea is to endow the smooth variety with a *differential trivialization* or *étale coordinates*. In each open set the local result of Ref. [5] can be reformulated in ring theoretic terms. Then, one can glue the star products in different open sets using a procedure analogous to the procedure that Fedosov used to show the existence of star products on symplectic manifolds [3]. This procedure was extended to general Poisson manifolds by Cattaneo, Felder and Tomassini [16] using the notions of *formal geometry* by Gelfand and Kazhdan [18]. The extension of these methods to the algebraic geometrical setting is non trivial.

The work by Bezrukavnikov and Kaledin [9] deals also with quantization in the algebro-geometric context. In particular, it deals with symplectic smooth varieties, and shows that the Fedosov quantization procedure can be translated into this context with appropriate cohomological assumptions

# 3 Deformation quantization of affine space with a Poisson structure

We consider an open domain in $\mathbb{R}^n$ with an arbitrary differential Poisson structure. Let us denote by $\{x_i\}_{i=1}^n$ the coordinates in such open domain and let

$$\alpha = \alpha^{ij}(x)\partial_i \otimes \partial_j$$

denote the Poisson structure. We want to briefly describe Kontsevich's local formula for the star product canonically associated to $\alpha$ (for a full description, we shall refer to the original paper [5]).

The star product is given in terms of certain admissible graphs, each of which has associated a bidifferential operator contributing to the sum (2) with an appropriate weight.

The bidifferential operators at order $n$ are constructed with the products of $n$ factors $\alpha$ acted by partial derivatives $\partial_i$ (up to 2n-2), and the indices contracted in appropriate way. For example, at order $n = 3$ one such operator could be

$$\sum_{\substack{i_1, i_2, i_3 \\ j_1, j_2, j_3}} (\partial_{i_2}\partial_{j_3}\alpha^{i_1 j_1})(\partial_{i_1}\alpha^{i_2 j_2})(\alpha^{i_3 j_3})\partial_{i_3} \otimes \partial_{j_2}\partial_{j_1}$$

We are interested in Poisson structures on the whole affine space $\mathbb{R}^n$ such that $\alpha^{ij}(x)$ are polynomial functions. Then, the Poisson structure is algebraic and the operators $B_n$ in (2) are bidifferential operators with polynomial coefficients. Moreover, denoting by $\deg(f)$ the maximum degree of the polynomial $f$ we have $\deg(B_m(f,g)) \leq \deg(f) + \deg(g) + (p-2)n$, being $p = \max_{(i,j)}(\deg(\alpha^{ij}))$. The star product of two polynomials will be an infinite series in $h$ with coefficients in $\mathbb{R}[x_1 \dots x_n]$, i.e. an element of $\mathbb{R}[x_1 \dots x_n][[h]]$. Then, Kontsevich's star product is an algebraic star product.

In the following we will take the Poisson structure fixed and Kontsevich's star product will be denoted simply by $\star$. We will always work with the complexifications of the Poisson structure and star product. In the affine space $\mathbb{A}^n$ we choose coordinates $\{x_i\}_{i=1}^n$ so $\mathbb{C}[\mathbb{A}^n] =_{\text{def}} \mathbb{C}[x_1 \dots x_n]$ (. The Poisson bracket is determined by its values on the generators of this algebra,

$$\{x_i, x_j\} = \alpha^{ij}(x_1, \dots, x_n)$$

and we have

$$\star : \mathbb{C}[\mathbb{A}^n][[h]] \times \mathbb{C}[\mathbb{A}^n][[h]] \longrightarrow \mathbb{C}[\mathbb{A}^n][[h]].$$

Our goal in this section is to give a presentation of the deformed algebra $(\mathbb{C}[\mathbb{A}^n], \star)$ in terms of generators and relations. That is, we want to present it as a quotient of the ring of formal power series in $h$ with coefficients in the full tensor algebra $T(X_1 \dots X_n)$ generated by $X_1 \dots X_n$ and a two-sided completed ideal $J$.

$$(\mathbb{C}[\mathbb{A}^n], \star) \simeq T(X_1 \dots X_n)[[h]]/J$$

When the Poisson structure is of degree 0, ($p = 0$, $\alpha_{ij}$ constant), then the star product is the Moyal star product.

When the Poisson structure is homogeneous of of degree 1 we have

$$\{x_i, x_j\} = c_{ij}^k x_k$$

and the star product algebra is isomorphic to the enveloping algebra over $\mathbb{C}[[h]]$ [15, 5] of the Lie algebra defined by the structure constants $c_{ij}^k$.

The case of a homogeneous quadratic Poisson structure is studied by Kontsevich in Ref.[7], pg 11. In that paper it is shown that the algebra $(\mathbb{C}[\mathbb{A}^n] \otimes \mathbb{C}[[h]], \star)$, which is strictly smaller than the one we are considering, is closed under the star product and that it can be given in terms of generators and (quadratic) relations. A similar presentation for Poisson structures of higher degree is not provided in that paper.

Our strategy in solving this problem proceeds as follows:

1. Let $I = (i_1, i_2, \dots, i_n)$ be a multiindex with $i_j = 1, \dots n$. We will prove that the ordered star monomials i.e. the monomials

$$x_{\star I} = x_{i_1} \star \cdots \star x_{i_m} \qquad i_1 \leq \cdots \leq i_m$$

are a basis for the $\mathbb{C}[[h]]$-module $\mathbb{C}[\mathbb{A}^n][[h]]$. (Notice that when we write $x_{i_1} \star \cdots \star x_{i_m}$ omitting the parenthesis we are making an implicit use of the associativity of $\star$).

2. Using part 1. we will find an algebra isomorphism

$$\begin{array}{ccc} \mathbb{C}[\mathbb{A}^n][[h]] & \longrightarrow & T(X_1 \dots X_n)[[h]]/J \\ x_{i_1} \star \cdots \star x_{i_m} & \longrightarrow & X_{i_1} \dots X_{i_m} \end{array} \qquad i_1 \leq \cdots \leq i_m$$

where $J$ is a two-sided completed ideal (completed in the $h$-adic topology). $J$ is generated by the relations obtained expressing the non ordered star monomials in terms of the ordered ones.

We start applying the procedure to the $\mathbb{C}[[h]]$-module $\mathbb{C}[\mathbb{A}^n][[h]]/(h^N) \simeq \mathbb{C}[\mathbb{A}^n][h]/(h^N)$. Then we will show that the inverse limit of the algebras obtained is a formal deformation of $\mathbb{C}[\mathbb{A}^n]$.

**Proposition 3.1** *Let $N \in \mathbb{N}$ be fixed. The ordered star monomials:*

$$x_{\star I} = x_{i_1} \star \cdots \star x_{i_m}, \qquad I = (i_1 \dots i_m), \qquad i_1 \leq \cdots \leq i_m$$

*form a basis for the $\mathbb{C}[[h]]$-module $\mathbb{C}[\mathbb{A}^n][h]/(h^N)$.*

Proof. The ordered commutative monomials $x_I = x_{i_1} \dots x_{i_m},\ \ i_1 \leq \dots \leq i_m$ form a basis for $\mathbb{C}[\mathbb{A}^n]$. By Kontsevich formula we can express any ordered star monomial as an infinite series in $h$ with coefficients in $\mathbb{C}[\mathbb{A}^n]$. Modulo $h^N$ the series becomes finite and can be rearranged as a finite $\mathbb{C}[h]/(h^N)$-linear combination of the commutative monomials. We denote it as

$$x_{\star I} = \sum_J A_I^J x_J,$$

where $A_I^J \in \mathbb{C}[h]/(h^N)$. Let us take an ordering in the set of multiindices $I = (i_1 \dots i_m)$, $i_1 \leq \dots \leq i_m$. With this ordering we denote by and $X_\star$ and $X$ the (infinite) column vectors of ordered star monomials and of commutative monomials respectively. Then we have an infinite $\mathbb{C}[h]/(h^N)$-linear system:

$$X_\star = AX$$

where $A$ is the infinite matrix with entries $A_I^J$. We notice two crucial facts:

(i) The matrix $A$ has only a finite number of entries different from zero in each row (as we stated before, this is because we are taking Kontsevich's formula modulo $h^N$).

(ii) $A = \mathrm{Id} + hB$, since by Kontsevich formula the term of order 0 in $h$ is the commutative product.

This implies that $A$ is invertible. Its inverse in $\mathbb{C}[\mathbb{A}^n][[h]]/(h^N)$ can be written as

$$A^{-1} = (\mathrm{Id} + hB)^{-1} = \sum_{m=0}^{N-1} (-1)^m h^m B^m$$

Notice that $B^m$ makes sense because of property 1. In fact all powers $B^m$ have only a finite number of entries different from zero in each row. We have $X = A^{-1}X_\star$, hence the ordered star monomials generate $\mathbb{C}[\mathbb{A}^n][[h]]$.

We want now to prove linear independence. Let us assume that there is a relation

$$\sum_I a^I x_{\star I} = 0 \mod (h^N), \qquad \text{i.e.} \qquad \sum a^I x_{\star I} = h^N q$$

with $a^I \in \mathbb{C}[h]/(h^N)$ and $q \in \mathbb{C}[\mathbb{A}^n][[h]]$.

We have that $\max_{\{I\}}(\deg(a^I)) \leq N$. Specializing $h$ to zero we have

$$\sum_I a^I(0)x_I = 0 \quad \Rightarrow \quad a^I(0) = 0$$

Hence we have that our relation can be written:

$$h \sum_I b^I x_{\star I} = h^N q$$

with $a^I = hb^I$, with $\max_{\{I\}}(\deg(b^I)) < \max_{\{I\}}(\deg(a^I))$. Since $\mathbb{C}[\mathbb{A}^n][[h]]$ is an integrity domain this implies

$$\sum b^I x_{\star I} = h^{N-1}q.$$

We can again specialize to $h = 0$, which will imply $b^I(0) = 0$, so $b^I = hc^I$, with $\max_{\{I\}}(\deg(c^I)) < \max_{\{I\}}(\deg(b^I))$. In each step we reduce the degree of the coefficients, so repeating the argument a sufficient number of times we obtain our result. ■

Let $\mathcal{R} \subset \mathbb{C}[\mathbb{A}^n][h]/(h^N)$ be the (infinite) set of linear relations expressing the non ordered star monomials in terms of the ordered ones.

$$x_{\star I} = \sum_J d_I^J x_{\star J}, \quad j_1 \leq \cdots \leq j_m \text{ and } i_1, \ldots i_m \text{ arbitrary}$$

Let $T(X_1, \ldots, X_n)[h]$ be the free tensor algebra generated over $\mathbb{C}[h]$ by $X_1 \ldots X_n$. We denote

$$T_N =_{\mathrm{def}} T(X_1, \ldots, X_n)[h]/(h^N)$$

Denote by $J_N$ the two-sided ideal generated in $T_N$ by the relations in $\mathcal{R}$ $\mathrm{mod}(h^N)$, where we replace $\star$ with the tensor multiplication.

**Proposition 3.2** *The $\mathbb{C}[h]/(h^N)$ linear morphism:*

$$\begin{array}{ccc} T_N/J_N & \xrightarrow{\psi_N} & \mathbb{C}[\mathbb{A}^n][h]/(h^N) \\ X_{i_1} \otimes \cdots \otimes X_{i_r} & \longrightarrow & x_{i_1} \star \cdots \star x_{i_r} \end{array} \qquad i_1 \leq \cdots \leq i_r$$

*is well defined and it is an algebra isomorphism.*

Proof. Consider the surjective linear map

$$\begin{array}{ccll} T_N & \xrightarrow{\phi_N} & \mathbb{C}[\mathbb{A}^n][h]/(h^N) & \\ X_{i_1} \otimes \cdots \otimes X_{i_r} & \longrightarrow & x_{i_1} \star \cdots \star x_{i_r} & i_1, \ldots i_r \quad \text{arbitrary.} \end{array}$$

Notice that it defines an algebra morphism. Moreover, $J_N \subset \ker(\phi_N)$. Hence the maps $\phi_N$ for all $N$ induce a family of surjective algebra homomorphisms

$$\begin{array}{ccc} T_N/J_N & \xrightarrow{\psi_N} & \mathbb{C}[x_1 \ldots x_n][h]/(h^n) \\ X_{i_1} \otimes \cdots \otimes X_{i_r} & \longrightarrow & x_{i_1} \star \cdots \star x_{i_r} \end{array}$$

It is easy to see that ordered monomials in $T_N/J_N$ form a basis. Because of the definition of $J_N$, it is clear that they are a system of generators. Moreover, they are linearly independent. In fact if there were a relation:

$$\sum a_I X_I = 0 \qquad \Rightarrow \qquad \sum a_I \psi_n(X_I) = 0 \qquad \Rightarrow \qquad a_I = 0$$

due to Proposition 3.1.

So we have obtained a surjective $\mathbb{C}[[h]]/(h^N)$-linear map $\psi_N$ which sends a basis into a basis. It is a linear isomorphism. Since it also preserves the product, it is also an algebra isomorphism. ■

We want to consider now the limit $N \to \infty$.

**Theorem 3.3** *Let $J_N$ be the family of ideals defined above, and let $J = \varprojlim J_N$ be its inverse limit. Then we have an algebra isomorphism*

$$(\mathbb{C}[\mathbb{A}^n][[h]], \star) \cong T(X_1, \ldots, X_n)[[h]]/J.$$

Proof. Consider the exact sequence:

$$0 \longrightarrow J_N \longrightarrow T_N \longrightarrow \mathbb{C}[\mathbb{A}^n] \longrightarrow 0$$

This is an exact sequence of inverse systems, i.e. it well behaves with respect to the sequences defining the inverse systems.

In general an exact sequence of inverse systems does not automatically give an exact sequence of the corresponding inverse limits. However in this case it happens, since $J_n$ is a surjective system i.e.

$$J_{N+1} \longrightarrow J_N$$

is surjective (see Ref. [20] pg 104).

So we have:

$$0 \longrightarrow \varprojlim J_N \longrightarrow \varprojlim T_N \longrightarrow \varprojlim \mathbb{C}[\mathbb{A}^n] \longrightarrow 0$$

which is what we wanted to prove. ■

In general, to obtain explicitly the generators of the ideal $J = \varprojlim J_n$ one needs to know the full star product series. In the particular case of a quadratic Poisson structure, they reduce to quadratic generators [7], but in the more general case we don't even know if the number of generators is finite.

Since

$$\mathbb{C}[x_1, \dots x_n] \simeq T(X_1, \dots, X_n)/(X_i \otimes X_j - X_j \otimes X_i)$$

we have

$$T(X_1, \dots, X_n)[[h]]/J \underset{h \to 0}{\longrightarrow} T(X_1, \dots, X_n)/(X_i \otimes X_j - X_j \otimes X_i), \qquad (3)$$

which is another way of expressing condition 2. in Definition 2.1.

# 4 Deformation quantization of regular affine Poisson varieties

We want to construct an explicit algebraic deformation quantization of the ring of polynomial functions $\mathbb{C}[X]$ of a regular affine Poisson variety. We assume that $X$ is a Poisson subvariety of some Poisson structure defined in the affine ambient space $\mathbb{A}^n$. We denote by $\mathcal{I}$ the ideal in $\mathbb{C}[\mathbb{A}^n]$ defining the variety $X$. By assumption, we have that $\mathcal{I}$ is also a Poisson ideal, that is,

$$\{\mathcal{I}, f\} \subset \mathcal{I}, \qquad \forall f \in \mathbb{C}[\mathbb{A}^n].$$

Let $\{p_1, \dots p_m\}$ be a basis of the ideal, so $\mathcal{I} = (p_1, \dots p_m)$. Let $\mathcal{A}_h = T(X_1, \dots, X_n)[[h]]/J$ be the quantization of the affine space as presented in Section 3. We want to construct an ideal $\mathcal{I}_h \in \mathcal{A}_h$ such that $\mathcal{A}_h/\mathcal{I}_h$ is a deformation quantization of the Poisson algebra $\mathbb{C}[\mathbb{A}^n]/\mathcal{I}$.

The general idea is the same than the one used for coadjoint orbits of semisimple groups in Refs. [11, 12]. In that case we had two advantages:

one is the fact that the Poisson structure in the ambient space is linear, which roughly allows to make inductions on the degree of the polynomials; the other is that the existence of the action of the semisimple group provides some tools that are not available in the general case. Nevertheless, assuming some technical conditions it is possible to overcome the difficulties.

The precise result is as follows:

**Theorem 4.1** *Let $X$ be an affine Poisson variety with ideal $\mathcal{I} = (p_1 \dots p_m)$ and Poisson structure induced from a Poisson structure in $\mathbb{C}[\mathbb{A}^n]$. Let $\mathcal{A}_h$ be an algebraic deformation of the the Poisson algebra $\mathbb{C}[\mathbb{A}^n]$, as constructed in Section 3. Assume that*
*1. The polynomials $\{p_1, \dots p_n\}$ are such that the matrix $(dp_1, \dots dp_n)$ has maximal rank on the points of $X$.*
*2. There exists liftings $P_1, \dots, P_m \in \mathcal{A}_h$ of $p_1, \dots p_m$,*

$$P_1, \dots, P_m \xrightarrow[h \to 0]{} p_1, \dots p_m$$

*such that the following left and right ideals coincide:*

$$\mathcal{I}_h = (P_1, \dots, P_2)_{\text{left}} = (P_1, \dots, P_m)_{\text{right}}.$$

*Then $\mathcal{A}_h/\mathcal{I}_h$ is an algebraic deformation quantization of $\mathbb{C}[X]$.* ■

Before going to the the proof, we want to make remarks on the hypothesis 2. For regular coadjoint orbits we can always fulfill this condition. It is enough to take $p_i$ invariant, which is always possible, and then to consider the Weyl map (or symmetrizer) from polynomials into the enveloping algebra,

$$W : \mathbb{C}[\mathbb{A}^n] \simeq \mathrm{Sym}(X_1, \dots, X_n) \to U_h.$$

Then $P_i = W(p_i)$ are in the center of the enveloping algebra and become adequate liftings. For non regular orbits $p_1, \dots p_m$ can be chosen spanning a finite dimensional representation of the group, which is always possible. Then, lifting with the Weyl map we obtain elements in the enveloping algebra satisfying condition 2. Even when condition 1. is not satisfied in this case, this lifting was used in Ref. [12] to construct a deformation quantization.

Another case where the lifting is available is when the generators $p_1, \dots, p_n$ are Casimirs of the Poisson structure, that is

$$\{p_i, f\} = 0, \qquad \forall f \in \mathbb{C}[\mathbb{A}^n], \quad i = 1, \dots n.$$

Indeed, it was found in Ref. [16] (Theorem 5.1) the explicit form of map $R : \mathbb{C}[x_1, \dots x_n][[h]] \to \mathbb{C}[x_1, \dots x_n][[h]]$ which is the identity when $h \to 0$ (a *quantization map*) that is an algebra isomorphism between the Casimirs of the Poisson structure and the center of the star product algebra. It is constructed in terms of the $L_\infty$-morphism which gives the formality theorem [5], and when applied to polynomials gives a formal series in $h$ with polynomial coefficients. Then it follows that $R(p_1), \dots, R(p_m)$ also satisfy condition 2. in Theorem 4.1.

Note that R is an algebra isomorphism but this is not necessary to fulfill condition 2. The Weyl map, for example, is not an algebra isomorphism.

Unfortunately, when the generators are not central, the same map does not give an appropriate lifting. Nevertheless, it is very likely that such lifting exists for every Poisson ideal, but we have not proved it in full generality.

Varieties defined by central elements are typically symplectic leaves of the Poisson structure in the ambient space, or stacks of such leaves. In particular, this extends our previous result on coadjoint orbits of semisimple Lie groups to regular coadjoint orbits of arbitrary (not necessarily semisimple) Lie groups.

We now return to the main result. We need only to prove that $\mathcal{A}_h/\mathcal{I}_h$ is isomorphic, as a $\mathbb{C}[[h]]$-module to $\mathcal{A}/\mathcal{I}[[h]]$. Then, properties 2. and 3. in Definition 2.1 are immediate. We need some lemmas.

**Lemma 4.2** *Let $p_1, \dots p_m \in \mathbb{C}[x_1 \dots x_n]$ be such that the matrix $(dp_1, \dots dp_m)$ has maximal rank on the points $p_1 = \cdots = p_m = 0$. Then, if*

$$\sum_\alpha a_\alpha p_\alpha = 0, \qquad a_\alpha \in \mathbb{C}[x_1 \dots x_n],$$

*there exist elements $b_{\alpha\beta} \in \mathbb{C}[x_1 \dots x_n]$ such that*

$$a_\alpha = \sum_j b_{\alpha\beta} p_\beta, \qquad \text{with} \quad b_{\alpha\beta} = -b_{\beta\alpha}$$

Proof. This result can be found for $C^\infty$ functions in Ref. [19]. We note that $b_{ij}$ are not unique, because one can always add a term $\tilde{b}_{ij}$ such that $\sum_j \tilde{b}_{\alpha\beta} p_\beta = 0$; that is, using the result in [19]

$$\tilde{b}_{\alpha\beta} = \sum_\gamma c_{\alpha\beta\gamma} p_\gamma, \qquad \text{with} \quad c_{\alpha\beta\gamma} = -c_{\alpha\gamma\beta}.$$

It is clear that if $p_\alpha$ and $a_\alpha$ are polynomials, $b_{\alpha\beta}$ can also be chosen polynomial functions ■

Next Lemma tells us that the $\mathbb{C}[[h]]$-module $\mathcal{A}_h/\mathcal{I}_h$ is torsion free.

**Lemma 4.3** *Let $X$, $p_1, \dots, p_m$ and $P_1, \dots, P_m$ be as in Theorem 4.1. Let $\mathcal{I}_h = (P_1, \dots, P_m) \subset \mathcal{A}_h$. Then, if $hA \in \mathcal{I}_h$ also $A \in \mathcal{I}_h$.*

Proof. Assume that $hA \in \mathcal{I}_h$. Since $I_h$ is two-sided we can write:

$$hA = \sum_\alpha A_\alpha P_\alpha. \tag{4}$$

Taking $h \to 0$ in this relation we get

$$\sum_i a_i p_i = 0 \quad \Rightarrow \quad a_\alpha = \sum_\beta b_{\alpha\beta} p_\beta, \quad \text{with} \quad b_{\alpha\beta} = -b_{\beta\alpha}$$

by Lemma 4.2. We can lift this relation,

$$\tilde{A}_\alpha = \sum_\beta B_{\alpha\beta} P_\beta, \quad \text{with} \quad B_{\alpha\beta} = -B_{\beta\alpha},$$

and it is clear that $A_\alpha - \tilde{A}_\alpha = hC_\alpha$. By substituting in (4) we get

$$hA = \sum_\alpha (\sum_\beta B_{\alpha\beta} P_\beta + hC_\alpha) P_\alpha = h \sum_\alpha C_\alpha P_\alpha,$$

and so $A \in \mathcal{I}_h$. ■

We now want to define a set of elements $\mathcal{B}^*$ in $A_h$ whose images in the quotient $A_h/I_h$ will turn out to be a topological basis.

We consider a monomial basis for $\mathbb{C}[x_1 \dots x_m]$,

$$\{x_J = x_{j_1} \cdots x_{j_k}\}, \qquad J = (j_1, \dots, j_k), \quad j_1 \leq \cdots \leq j_k,$$

and the associated topological basis of $\mathbb{C}[x_1 \dots x_m][[h]]$ of star monomials

$$\{x_{\star J} = x_{j_1} \star \cdots \star x_{j_k}\}, \qquad J = (j_1, \dots, j_k), \quad j_1 \leq \cdots \leq j_k,$$

(see Section 3).

Let $B$ be a set of multiindices with the property the images by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ of the elements in $\mathcal{B} = \{x_J\}_{J \in B}$ form a basis of $\mathcal{A}/\mathcal{I}$. Consider the set

$$\mathcal{B}^\star = \{x_{\star J}\}_{J \in B} \subset A_h.$$

Let $\pi_h : \mathcal{A}_h \longrightarrow \mathcal{A}_h/\mathcal{I}_h$. We want to show that the elements in $\pi_h(\mathcal{B}^\star)$ are a basis of $\mathcal{A}_h/\mathcal{I}_h$. We start by showing that they are linearly independent.

**Lemma 4.4** *The image under the projection map* $\pi_h : \mathcal{A}_h \longrightarrow \mathcal{A}_h/\mathcal{I}_h$ *of* $\mathcal{B}^\star = \{x_{\star J}\}_{J\in B}$ *is a linearly independent set in* $A_h/I_h$.

Proof. The argument of Proposition 3.11 in [11] works without changes, we repeat it here for completeness.

Suppose that there exists a linear relation among the elements of $\mathcal{B}^\star$ and let $G \in \mathcal{I}_h$ be such relation, $G = h^k F$ with

$$F \xrightarrow[h\to 0]{} f \neq 0$$

for some $k$. By Lemma 4 we have that $F \in \mathcal{I}_h$, so

$$F = \sum_\alpha A_\alpha P_\alpha, \qquad \Rightarrow \qquad \mathrm{f} = \sum_\alpha a_\alpha p_\alpha$$

by taking $h \to 0$. But $f$ is a non trivial relation between the ordinary monomials in $\mathcal{B}$, which is not possible since $\mathcal{B}$ is assumed to be a basis of $\mathcal{A}/\mathcal{I}$. So we have proven linear independence ■

To prove that the elements in Lemma $\pi_h(\mathcal{B}^\star)$ are a system of generators we cannot use the same argument that appears in Proposition 3.13 in Ref. [11]. The reason is that having a linear Poisson structure would allow us to use an induction on the degree of the polynomials, so to choose certain liftings in such way that the "correction" terms would have a degree strictly smaller than the largest degree appearing in the order $h^0$ term. Such induction argument is not possible for a Poisson structure with arbitrary degree. Nevertheless, we have the following

**Proposition 4.5** *Let the notation be as above.* $\pi_h(\mathcal{B}^*)$ *is a topological basis for* $\mathcal{A}_h/\mathcal{I}_h$.

Proof. We set

$$(\mathcal{A}_h/\mathcal{I}_h)_N = (\mathcal{A}_h/\mathcal{I}_h)/(h^N).$$

It is enough to show that $\pi(\mathcal{B}^*)$ is a basis for $(\mathcal{A}_h/\mathcal{I}_h)_N$ for all $N$.

Let $P$ be a complement of $B$ in the set of multiindices, and let

$$\mathcal{P} = \{x_J\}_{J\in P}, \qquad \mathcal{P}^\star = \{x_{\star J}\}_{J\in P}.$$

We will prove that any monomial in $\pi(\mathcal{P}^\star)$ is expressible in terms of monomials in $\pi_h(\mathcal{B}^\star)$. This will clearly suffice since we have proven that the monomials in $\pi_h(\mathcal{B}^\star)$ are linearly independent (also in $(A_h/I_h)_N$).

We consider total lexicographic orderings in the sets $B$ and $P$. Let $a \in \mathbb{N}$ be the position of certain multiindex $K \in B$ and let $\mu \in \mathbb{N}$ be the position of certain multiindex $J \in P$ with respect to the chosen orders. It will be convenient to denote the respective monomials as

$$\begin{aligned} e_a = x_K, \ \in K \in \mathcal{B}, \quad v_\mu = x_J \ J \in P \\ e_{\star a} = x_{\star K}, \ \in K \in \mathcal{B}, \quad v_{\star\mu} = x_{\star J} \ J \in P. \end{aligned}$$

Each monomial in $\mathcal{P}$ can be expressed as a linear combination of monomials in $\mathcal{B}$ modulo an element in $\mathcal{I}$

$$v_\mu = \sum_a b_{\mu a} e_a + \sum_i c_{\mu i}(x) p_i,$$

where $b_{\mu a} \in \mathbb{C}$ and $c_{\mu i}(x) \in \mathbb{C}[\mathbb{A}^n]$.

We can lift this relation to the deformed algebra $\mathcal{A}_h$ in many ways. Each star monomial $v_{\star\alpha} \in \pi(\mathcal{P}^\star)$ can be expressed as:

$$v_{\star\mu} = \sum_b B_{\mu a}(h) e_{\star a} + \sum_i C_{\mu i}(x,h) \star P_i + h \sum A_{\mu\nu}(h) v_{\star\nu}, \tag{5}$$

$B_{\mu a}(h), A_{\mu\nu}(h) \in \mathbb{C}[[h]]$ and $C_{\alpha\mu}(x,h) \in \mathbb{C}[\mathbb{A}^n][[h]]$. Modulo $\mathcal{I}_h$ we have then a linear system

$$\sum_\nu (\delta_{\mu\nu} - h A_{\mu\nu}(h)) v_{\star\nu} = \sum_a B_{\mu a}(h) e_{\star a}, \tag{6}$$

which in matrix form reads

$$D v_\star = B e_\star, \qquad D = \mathrm{Id} - hA.$$

The infinite matrix $D$, modulo $h^N$, has only a finite number of entries non zero for each row, and modulo $h$ it is the identity. Hence, by the same reasoning used in Section 3, we can invert $D$, and its inverse in $\mathbb{C}[\mathbb{A}^n][h]/(h^N)$ can be written as

$$D^{-1} = (\mathrm{Id} - hA)^{-1} = \sum_{m=0}^{N-1} h^m A^m.$$

$D^{-1}$ has again a finite number of entries different from zero in each row, so the multiplication $D^{-1}B$ is well defined and

$$v_\star = D^{-1} B e_\star$$

as we wanted to prove.

Making the inverse limit we have that $\pi(\mathcal{B}^\star)$ is a topological basis of $\mathcal{A}_h/\mathcal{I}_h$. ■

This concludes the proof of Theorem 4.1.


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