View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Repositori d'Objectes Digitals per a l'Ensenyament la Recerca i la Cultura

arXiv:math/0003142v1 [math.QA] 23 Mar 2000

Deformation Quantization of Coadjoint Orbits

M. A. Lledó

Dipartimento di Fisica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy, and
INFN, Sezione di Torino, Italy.
e-mail: lledo@athena.polito.it

Abstract

A method for the deformation quantization of coadjoint orbits of semisimple Lie groups is proposed. It is based on the algebraic structure of the orbit. Its relation to geometric quantization and differentiable deformations is explored.

Let G be a complex Lie group of dimension n and G_R a real form of G. Let $\mathcal{G}$ and $\mathcal{G}_R$ be their respective Lie algebras with Lie bracket $[\ ,\ ]$. As it is well known, $\mathcal{G}_R^*$ has a Poisson structure,

$$\{f_1, f_2\}(\lambda) =< [(df_1)_\lambda, (df_2)_\lambda], \lambda >, \qquad f_1, f_2 \in C^\infty(\mathcal{G}_R^*), \quad \lambda \in \mathcal{G}_R^*. \tag{1}$$

Choosing a basis $\{X_1, \dots X_n\}$ of $\mathcal{G}_R$ and its dual, $\{\xi^1, \dots \xi^n\}$, the Poisson bracket can be written as

$$\{f_1, f_2\}(x) = c_{ij}^k \lambda_k \frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial x_j}, \qquad x = \sum_{i=1}^n x_i \xi^i \in \mathcal{G}^*.$$

Notice that this Poisson bracket is never symplectic, in particular it is 0 at the origin. Under the action of $g \in G_R$, it satisfies $g^*\{f_1, f_2\} = \{g^* f_1, g^* f_2\}$, so G_R is a group of automorphisms of the Poisson algebra $C^\infty(\mathcal{G}_R^*)$. The action of G_R on $\mathcal{G}_R^*$ is not transitive, so $\mathcal{G}_R^*$ is foliated in orbits. This foliation coincides with the foliation given by the Hamiltonian vector fields of (1). So the orbits of the coadjoint action of a Lie group are symplectic manifolds.

We want to describe formal deformations of the Poisson algebra $C^\infty(\Theta)$ (Θ is a coadjoint orbit) or of some subalgebra of it. It is convenient to work with the complexification of the Poisson algebra.

An associative algebra $\mathcal{A}_h$ over $\mathbb{C}[[h]]$ is a formal deformation of a Poisson algebra $(\mathcal{A}, \{\ ,\ \})$ over $\mathbb{C}$ if there exists an isomorphism of $\mathbb{C}[[h]]$-modules $\psi : \mathcal{A}[[h]] \longrightarrow \mathcal{A}_h$ satisfying the following properties:

a. $\psi^{-1}(F_1F_2) = f_1f_2 \mod(h)$ where $F_i \in \mathcal{A}_h$ are such that $\psi^{-1}(F_i) = f_i \mod(h)$, $f_i \in \mathcal{A}$. (By $\mod(h)$ we mean that the projections $p : \mathcal{A}[[h]] \longrightarrow \mathcal{A}[[h]]/h\mathcal{A}[[h]]$ of both quantities coincide).
b. $\psi^{-1}(F_1F_2 - F_2F_1) = h\{f_1, f_2\} \mod(h^2)$.

An example of interest for our purposes is the polynomial algebra on $\mathcal{G}^*$. A formal deformation of $\mathrm{Pol}(\mathcal{G}^*)$ [1] is given by the algebra $U_h = T_{\mathbb{C}[[h]]}(\mathcal{G})/\mathcal{L}_h$, where $T_{\mathbb{C}[[h]]}(\mathcal{G})$ is the tensor algebra over $\mathbb{C}[[h]]$ and $\mathcal{L}_h$ is the proper two sided ideal

$$\mathcal{L}_h = \sum_{X,Y\in\mathcal{G}} T_{\mathbb{C}[[h]]}(\mathcal{G}) \otimes (X \otimes Y - Y \otimes X - h[X,Y]) \otimes T_{\mathbb{C}[[h]]}(\mathcal{G}) \subset T_{\mathbb{C}[[h]]}(\mathcal{G}).$$

The isomorphism $\psi : \mathrm{Pol}(\mathcal{G}^*) \longrightarrow U_h$ is not canonical. A possible choice is in terms of a Poincaré-Birkhoff-Witt basis,

$$\psi(x_{i_1}x_{i_2}\cdots x_{i_k}) = X_{i_1} \cdot X_{i_2} \cdots X_{i_k}, \qquad 1 \leq i_1 \leq \cdots \leq i_k \leq n. \tag{2}$$

Another choice is the symmetrizer map,

$$\mathrm{Sym}(x_{i_1}x_{i_2}\cdots x_{i_k}) = \frac{1}{k!}\sum_{\sigma\in S_k} X_{\sigma(i_1)} \cdot X_{\sigma(i_2)} \cdots X_{\sigma(i_k)}, \tag{3}$$

where S_k is the group of permutations of order k.

Given a choice for ψ one can define an associative product (star product) on $\mathcal{A}[[h]]$ by

$$a \star_\psi b = \psi^{-1}(\psi(a) \cdot \psi(b)).$$

Then, for any choice of ψ, $(\mathcal{A}_h, \star_\psi)$ is an algebra isomorphic to $\mathcal{A}_h$. With the star product we recover the semiclassical interpretation of the elements of the algebra as functions on the phase space. The star product can always be written as a formal series

$$a \star_\psi b = ab + \sum_{n>0} h^n C^n_\psi(a, b),$$

where C^n_ψ are some bilinear operators. Let $\star$ and $\star'$ be two isomorphic star products,

$$a \star b = T^{-1}(T(a) \star T(b)), \qquad T : \mathcal{A}[[h]] \longrightarrow A[[h]].$$

It is clear that T can be written as

$$T(a) = \sum_{n\geq 0} h^n T^n(a),$$

and because of property **a**, T^0 must be an automorphism of the commutative algebra $\mathcal{A}[[h]]$. If T_0 is the identity we say that $\star$ and $\star'$ are equivalent (or gauge equivalent) star products. (2) and (3) are two equivalent star products.

For $\mathcal{A}$ being the full algebra of C^∞ functions on the Poisson manifold, if the operators C^n_ψ are bidifferential operators we say that the star product is differentiable. Gauge equivalence can be restricted to the class of differentiable star products by considering only differentiable T^n. Notice that the differentiability is a property of the particular star product and not of the formal deformation. The star products (2) and (3) can be extended to $C^\infty(\Theta)$ as differentiable star products, but we will see later an example of a star product corresponding to the same formal deformation which is not differentiable [2].

We will consider only semisimple Lie groups. The semisimple coadjoint orbits of G on $\mathcal{G}$ are complex algebraic varieties defined over $\mathbb{R}$. They are given by the invariant polynomials. If l is the rank of G, we can choose l homogeneous polynomials $p_i(x)$, $i = 1, \dots, l$ generating the subalgebra of invariant polynomials, $\mathbb{C}[p_1, \dots p_n]$. Then the semisimple coadjoint orbits are given by the algebraic equations

$$p_i(x) = c_i. \tag{4}$$

(see for example [3]). The intersection of the complex orbit with $\mathcal{G}_R$ is a real algebraic variety consisting on a finite number of connected components, which are orbits of the real form of the group. For the compact real form there is only one connected component.

It is easy to check that the star products 2 and 3 do not restrict well to the orbit, that is, in general

$$a \star p_i|_\Theta \neq 0.$$

We want to know if there is some choice of ψ that gives a star product which restricts to Θ.

In the approach of geometric quantization, the algebra of quantum observables is given by the quotient of U_h by a certain ideal. This ideal, I_h, is prime and Ad_G-invariant, so there is a well defined action of G on U_h/I_h. We summarize here the results of [4], where the quantization of the coadjoint orbits are obtained in terms of the quotient of the enveloping algebra by a prime, Ad_G-invariant ideal. We consider the polynomial algebra over the real algebraic manifold (union of orbits) defined by (4),

$$\mathrm{Pol}(\Theta) = \mathrm{Pol}(\mathcal{G}^*)/\mathcal{I}_0, \qquad \mathcal{I}_0 = \{p \in \mathrm{Pol}(\mathcal{G}^*)/p|_\Theta = 0\}.$$

This is the Poisson algebra that we want to deform. We quote first a result from Varadarajan [5] that we need.

Lemma (1). *Let $x \in \mathcal{G}^*$ be a regular element of $\mathcal{G}^*$ (or equivalently, a point in which the centralizer has dimension equal to the rank of $\mathcal{G}^*$). Then $(dp_1)_x$, ..., $(dp_l)_x$ are linearly independent.*

From now on we will restrict to regular orbits only. It is clear that in this case $\mathcal{I}_0$ is generated by $p_1 - c_1, \dots p_l - c_l$. We consider now the elements in U_h that are the image of p_i by the symmetrizer, $P_i = \mathrm{Sym}(p_i)$, called Casimir operators. P_i are central elements in U_h and they are also Ad_G-invariant (the symmetrizer commutes with the action of G).

Let $\mathcal{I}_h$ be the ideal generated by $P_i - C_i(h)$, where $C_i(0) = c_i$. Then $U_h/\mathcal{I}_h$ is a formal deformation of $\mathrm{Pol}(\Theta)$. The technical assumption of regularity is needed to prove the existence of a $\mathbb{C}[[h]]$-module isomorphism $\psi : \mathrm{Pol}(\Theta)[[h]] \longrightarrow U_h/\mathcal{I}_h$, which is not obvious. The ideal $\mathcal{I}_h$ itself is Ad_G-invariant, so G has a natural action by automorphisms on the algebra $U_h/\mathcal{I}_h$. For special values of $C_i(h)$, $\mathcal{I}_h$ is in the kernel of an irreducible unitary representation of G_R.

This deformation of polynomials can be specialized for any value of h. For SU(2),

$$[H, X] = \hbar 2X, \quad [H, Y] = -\hbar 2Y, \quad [X, Y] = \hbar H,$$

the Casimir operator is

$$P = \frac{1}{2}(XY + YX + \frac{1}{2}H^2).$$

It was shown in [4] that with the choice

$$C = l(l + \hbar)), \qquad l = \hbar m/2,$$

the algebra obtained is the same than the one obtained in geometric quantization.

According to our definition, a star product in Pol(Θ) is given by a $\mathbb{C}[[h]]$-module isomorphism $\tilde{\psi} : \mathrm{Pol}(\Theta)[[h]] \longrightarrow U_h/\mathcal{I}_h$. In particular, to obtain a star product in Pol($\mathcal{G}^*$) which restricts to the orbit one should look for an isomorphism $\psi : \mathrm{Pol}(\mathcal{G}^*)[[h]] \longrightarrow U_h$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Pol}(\mathcal{G}^*)[[h]] & \xrightarrow{\psi} & U_h \\ \Big\downarrow \pi & & \Big\downarrow \hat{\pi} \\ \mathrm{Pol}(\Theta)[[h]] & \xrightarrow{\tilde{\psi}} & U_h/I_h. \end{array} \tag{5}$$

π and $\hat{\pi}$ are the natural projections. If $\psi(\mathcal{I}_0) = \mathcal{I}_h$, then $\tilde{\psi}$ is defined uniquely by the diagram (5). An example of such star product is given in [2], where it is also shown that it is not differentiable. Since $\hat{\psi}$ is not unique one could ask if there is some choice that renders it differentiable. This issue will be addressed in [6].

References

[1] J. Hoppe, *Quantum theory of a massless relativistic surface and a two dimensional bound state problem.* MIT PhD thesis, (1982).

[2] M. Cahen and S. Gutt, *C.R. Acad. Sc. Paris* **296** (1983) 821-823; in *Gravitation and Cosmology*, Monographs and Textbooks in Physical Sciences, Bibliopolis. Eds W. Rundler and A. Trautman, (1987); *Bull. Soc. Math. Belg.* **36 B**, 207-221, (1987).

[3] V. S. Varadarajan, *Harmonic Analysis on Real Reductive Groups.* Lecture Notes in Mathematics, no. 576. Springer-Verlag, (1977).

[4] R. Fioresi and M. A. Lledó, math.QA/9906104. To appear in the *Pacific Journal of Mathematics.*

[5] V. S. Varadarajan, *Amer. J. Math*, **90**, (1968).

[6] R. Fioresi, A. Levrero and M. A. Lledó. In preparation.