# Relativistic direct Urca processes in cooling neutron stars

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## Abstract

We derive a relativistic expression for neutrino energy losses caused by the direct Urca processes in degenerate baryon matter of neutron stars. We use two different ways to calculate the emissivity caused by the reactions to our interest. First we perform a standard calculation by Fermi's "golden" rule. The second calculation, resulting in the same expression, is performed with the aid of polarization functions of the medium. Our result for neutrino energy losses strongly differs from previous non-relativistic results. We also discuss nonconservation of the baryon vector current in reactions through weak charged currents in the medium, when the asymmetry between protons and neutrons is considered. The above effects, not discussed in the literature before, substantially modify the polarization functions responsible for the induced weak charged currents in baryon matter.

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In numerical simulations, neutrino cooling of the massive core of a neutron star, with the standard nuclear composition, is governed by the direct Urca processes $n \to p + e^- + \bar{\nu}_e$, $p + e^- \to n + \nu_e$. The important role of these reactions in the rapid cooling of neutron stars has been first pointed out by Boguta [1]. Later Lattimer et al. [2] have suggested a simple formula for neutrino energy losses caused by the above processes in degenerate nuclear matter under $\beta$-equilibrium, which exhibits a threshold of proton concentration necessary for the direct Urca processes to operate[1]. The above estimates have been made in a non-relativistic manner, assuming that the participating particles are free. Some improvement, which partially accounts for strong interactions, was made by replacing the baryon masses with their effective values.

Actually, the superthreshold proton fraction in the core of neutron stars appears at large densities, when Fermi momenta of participating nucleons are of the order, or larger, than their effective masses. The total four-momentum of the final lepton and antineutrino is time-like; therefore, in the free relativistic gas, the energy-momentum conservation requires a large difference in the effective masses of protons and neutrons $M_n^* - M_p^* \sim 10^2 MeV$ that is unlikely to appear. Thus, in the relativistic regime, the energy conservation can be fulfilled only by taking into account the difference in the potential energies of proton and neutron. A further simplification was made by neglecting the proton recoil. This is not justified in the relativistic regime, because the momentum of the final lepton is of the order of the proton effective mass. Both the proton recoil and the difference in the proton-neutron potential energies strongly modify the emissivity of the direct Urca reactions *even in the Mean Field Approximation*

In the present paper we derive a relativistic expression for neutrino energy losses caused by the direct Urca processes by taking into account the above effects. We use two different

[1] Prakash et al. [3] have found that an admixture of $\Lambda$ hyperons opens the direct Urca processes in baryon matter for an arbitrary proton concentration.

ways to calculate the emissivity caused by the reactions to our interest. First we perform a standard calculation by Fermi's "golden" rule. The second calculation, resulting in the same expression, is performed with the aid of polarization functions of the medium. In addition to allow us to check the result obtained by the previous method, this second method shows some important details, which must be taken into account in the next step (when RPA corrections are included).

Relativistic polarization functions of hot baryon matter have been investigated earlier by several authors (see e.g. [4], [5], [6] and references therein) having made analytic computations for the case when the particle-hole excitations correspond to identical baryons. The study of interactions through charged weak-currents requieres considering polarization functions of another type, in which the virtual particle and the hole correspond to baryons of different kinds. As we show below, the difference in potential energies of participating particles leads to non-conservation of the vector transition current of baryons in the direct Urca reactions. This effect dramatically modifies the corresponding polarization functions of the medium and must be taken into account in order to obtain the correct result.

According to mean-field theory, the medium-modified energy of a relativistic baryon of kind $B$ is similar to the energy of a single particle with effective Dirac mass $M_B^*$ in the effective potential $U_B$ of self-consistently generated meson fields[2]. The effective mass, as well as the effective potential, arises due to interactions of the fields. We denote by $\varepsilon_B(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M_B^{*2}}$ the baryon kinetic energy. The single-particle energy of a baryon with momentum $\mathbf{p}$ is $E_B(\mathbf{p}) = \varepsilon_B(\mathbf{p}) + U_B$, so that the individual Fermi distributions are of the form

$$f_B(\varepsilon_B) = \frac{1}{\exp\left(\left(\varepsilon_B + U_B - \mu_B\right)/T\right) + 1}, \tag{1}$$

where $\mu_B$ is the chemical potential of $B$-kind baryons.

The low-energy Lagrangian of baryon interaction with the lepton field can be written in

[2]In what follows we use the system of units $\hbar = c = 1$ and the Boltzmann constant $k_B = 1$.

a point-like approach (summation over repeated Greek indexes is assumed)

$$\mathcal{L}_{\text{weak}} = \frac{G_F C}{\sqrt{2}} j^{\alpha} J_{\alpha}, \tag{2}$$

where $G_F$ is the Fermi weak coupling constant, and the Cabibbo factor $C = \cos\theta_C = 0.973$ for change of strangeness $\Delta S = 0$ and $C = \sin\theta_C$ for $\Delta S = 1$. For the direct Urca process involving baryons $B_1$ and $B_2$

$$B_1 \rightarrow B_2 + l + \bar{\nu}_l, \tag{3}$$

the lepton and baryon weak charged currents are, respectively:

$$j^{\alpha} = \bar{u}_l \gamma^{\alpha} (1 - \gamma_5) \nu_l, \qquad J_{\alpha} = \bar{\psi}_2 \gamma_{\alpha} (C_V - C_A \gamma_5) \psi_1. \tag{4}$$

Here $\psi_1$ and $\psi_2$ stand for the initial and final baryon fields; $C_V$ and $C_A$ are the corresponding vector and axial-vector coupling constants, respectively. In what follows we consider massless neutrinos of energy and momentum $q_1 = (\omega_1, \mathbf{k}_1)$ with $\omega_1 = k_1$. The energy and momentum of the final lepton of mass $m_l$ is denoted as $q_2 = (\omega_2, \mathbf{k}_2)$ with $\omega_2 = \sqrt{k_2^2 + m_l^2}$.

We consider the total energy which is emitted into neutrino and antineutrino per unit volume and time. Within $\beta$-equilibrium, the inverse reaction $B_2 + l \rightarrow B_1 + \nu_l$ corresponding to the capture of a lepton $l$, gives the same emissivity as the decay (3), but in neutrinos. Thus, the total energy loss $Q$ for the Urca processes is twice more than that caused by $\beta$-decay (3). Taking this into account by Fermi's "golden" rule we have

$$\begin{aligned} Q = 2\frac{G_F^2 C^2}{2} \int & \frac{d^3k_2 d^3k_1 d^3p_2 d^3p_1}{2\omega_2 2\omega_1 2\varepsilon_2 2\varepsilon_1 (2\pi)^{12}} \left|\mathcal{M}_{fi}\right|^2 \omega_1 f_1 (1 - f_2) (1 - f_l) \\ & \times (2\pi)^4 \delta (\varepsilon_1 + U_1 - \varepsilon_2 - U_2 - \omega_1 - \omega_2) \, \delta (\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}_1 - \mathbf{k}_2), \end{aligned} \tag{5}$$

where the square of the matrix element of the reaction (3) summed over spins of initial and final particles has the following form

$$\begin{aligned} \left|\mathcal{M}_{fi}\right|^2 = 64 \Big[ & \left(C_A^2 - C_V^2\right) M_1^* M_2^* (q_1 q_2) + (C_A - C_V)^2 (q_1 P_2)(q_2 P_1) \\ & + (C_A + C_V)^2 (q_1 P_1)(q_2 P_2) \Big] \end{aligned} \tag{6}$$

with $P_1 = (\varepsilon_{1,}\mathbf{p}_1)$ and $P_2 = (\varepsilon_{2,}\mathbf{p}_2)$. Antineutrinos are assumed to be freely escaping. The distribution function of initial baryons $B_1$ as well as blocking of final states of the baryon $B_2$ and the lepton $l$ are taken into account by the Pauli blocking-factor $f_1(1-f_2)(1-f_l)$ with

$$f_l(\omega_2) = \frac{1}{\exp(\omega_2 - \mu_l)/T + 1} \tag{7}$$

being the Fermi-Dirac distribution function of leptons with the chemical potential $\mu_l$. By neglecting the chemical potential of escaping neutrinos, we can write the condition of chemical equilibrium as $\mu_l = \mu_1 - \mu_2$. We assume degenerate matter. Then by the use of the energy conservation equation $\varepsilon_1 + U_1 = \varepsilon_2 + U_2 + \omega_2 + \omega_1$, and taking the total energy of the final lepton and antineutrino as $\omega_2 + \omega_1 = \mu_l + \omega'$ we obtain the identity

$$\begin{aligned} & f_1(\varepsilon_1)(1 - f_2(\varepsilon_2))(1 - f_l(\omega_2)) \\ \equiv\, & f_1(\varepsilon_1)(1 - f_1(\varepsilon_1 - \omega'))(1 - f_l(\mu_l + \omega' - \omega_1)), \end{aligned} \tag{8}$$

where $\omega' \sim T$ because the energy exchange in the direct Urca reaction goes naturally on the temperature scale $\sim T$. Due to the strong degeneracy of the medium, the main contribution to the integral (5) comes from narrow regions of momentum space near the corresponding Fermi surfaces. Since the neutrino energy is $\omega_1 \sim T$, and the neutrino momentum $k_1 \sim T$ is much smaller than the momenta of other particles, we can neglect the neutrino contributions in the energy-momentum conserving delta-functions. Then

$$\begin{aligned} & \delta(\varepsilon_1 + U_1 - \varepsilon_2 - U_2 - \omega_1 - \omega_2)\,\delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}_1 - \mathbf{k}_2) \\ \simeq\, & \frac{\varepsilon_2}{p_1 k_2}\delta\left(\cos\theta - \frac{1}{2p_1k_2}\left(p_1^2 - p_2^2 + k_2^2\right)\right)\delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k}_2), \end{aligned} \tag{9}$$

where the $\theta$ is the angle between the momentum $\mathbf{p}_1$ of the initial baryon and the momentum $\mathbf{k}_2$ of the final lepton. Now integrations over $\mathbf{p}_2$ momenta and all solid angles can be easily done. As mentioned above, we can replace the momenta of particles by the corresponding Fermi momenta, i.e. $p_1 = p_{F1}$, $p_2 = p_{F2}$, $k_2 = p_{Fl}$ in all smooth functions of energy and momentum under the integral. Then the remaining integration is reduced to the following

$$\int d\omega_1 \omega_1^3 d\omega' d\varepsilon_1\, f_1(\varepsilon_1)(1 - f_1(\varepsilon_1 - \omega'))(1 - f_l(\mu_l + \omega' - \omega_1))$$
$$\simeq \int_{-\infty}^{\infty} d\omega' \frac{\omega'}{\exp\omega'/T - 1} \int_0^{\infty} d\omega_1 \frac{\omega_1^3}{1 + \exp(\omega_1 - \omega')/T} = \frac{457}{5040}\pi^6 T^6. \tag{10}$$

Finally, to the lowest order in $T/\mu_l$ we obtain the following neutrino emissivity[3]:

$$Q = \frac{457\pi}{10\,080} G_F^2 C^2 T^6 \Big[ C_V C_A \left( (\varepsilon_{F1} + \varepsilon_{F2})\, p_{Fl}^2 - (\varepsilon_{F1} - \varepsilon_{F2}) \left( p_{F1}^2 - p_{F2}^2 \right) \right)$$
$$+ 2C_A^2 \mu_l M_1^* M_2^* + \left( C_V^2 + C_A^2 \right) \Big( \mu_l \left( 2\varepsilon_{F1}\varepsilon_{F2} - M_1^* M_2^* \right) + \varepsilon_{F1} p_{Fl}^2$$
$$- \frac{1}{2} (\varepsilon_{F1} + \varepsilon_{F2}) \left( p_{F1}^2 - p_{F2}^2 + p_{Fl}^2 \right) \Big) \Big] \Theta(p_{Fl} + p_{F2} - p_{F1}) \tag{11}$$

with $\Theta(x) = 1$ if $x \geq 0$ and zero otherwise. When the baryon and lepton momenta are at their individual Fermi surfaces, the $\delta$- function (9) contributes only if $p_{F2} + p_{Fl} > p_{F1}$. This "triangle" condition required by the step-function in Eq. (11) defines the above mantioned threshold for the direct Urca reactions.

Eq.(11) can be obtained also by the use of the medium response functions. By the fluctuation-dissipation theorem, the total emissivity can be written as:

$$Q = 2\frac{G_F^2 C^2}{2} \int \frac{\omega_1 \;\, (1 - f_l(\omega_2))\; 2\,\mathrm{Im}\left[ W_R^{\alpha\beta} \mathrm{L}_{\alpha\beta} \right]}{\exp(q_0 - \mu_1 + \mu_2)/T - 1} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3}, \tag{12}$$

where the integration goes over the phase volume of the final lepton and antineutrino of total energy $q_0 = \omega_1 + \omega_2$ and total momentum $\mathbf{q} = \mathbf{k}_1 + \mathbf{k}_2$. The final-state blocking of the outgoing lepton is taken into account by the Pauli blocking-factor $(1 - f_l(\omega_2))$. The factor $[\exp(q_0 - \mu_1 + \mu_2)/T - 1]^{-1}$ in Eq. (12) arises due to the fact that the baryons $B_1$ and $B_2$ are in thermal equilibrium at temperature $T$ and in chemical equilibrium with chemical potentials $\mu_1$ and $\mu_2$, respectively. $\mathrm{L}^{\alpha\beta}$ is defined here by

$$\mathrm{L}^{\alpha\beta} = \frac{8}{2\omega_2 2\omega_1} \left( k_1^\alpha k_2^\beta + k_2^\alpha k_1^\beta - (k_1 \cdot k_2)\, g^{\alpha\beta} - i\epsilon^{\alpha\beta\gamma\delta} (k_1)_\gamma (k_2)_\delta \right), \tag{13}$$

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[3] We have corrected two misprints appearing in the published version of our paper (see Eq. (11) in ref. [9]).

and $W_R^{\alpha\beta}$ is the retarded weak-polarization tensor of the medium. The imaginary part of the retarded polarization is related to that of the causal (or time ordered) polarization $W_{\alpha\beta}$ as follows

$$\mathrm{Im}\, W_R^{\alpha\beta} = \tanh\left(\frac{q_0 - \mu_1 + \mu_2}{2T}\right) \mathrm{Im}\, W^{\alpha\beta}. \tag{14}$$

Since the baryon weak-current includes vector and axial-vector contributions, to the lowest order in weak interactions, the polarization tensor consists on those diagrams which have ends at the weak vertex $(C_V \gamma^\alpha - C_A \gamma^\alpha \gamma_5)$. Thus, we can take the weak polarization tensor as the sum of vector-vector, axial-axial, and mixed axial-vector pieces:

$$W^{\alpha\beta} = C_V^2 \Pi^{\alpha\beta} + C_A^2 \Pi_A^{\alpha\beta} - 2 C_V C_A \Pi_{VA}^{\alpha\beta}. \tag{15}$$

The vector-vector polarization tensor is of the following general form

$$\Pi^{\alpha\beta} = -i \int \frac{d^3p dP^0}{(2\pi)^4} \,\mathrm{Tr}\left[\, G_1\left(P^0 - U_1 + q_0, \mathbf{p} + \mathbf{q}\right) \gamma^\alpha G_2\left(P^0 - U_2, \mathbf{p}\right) \gamma^\beta\right], \tag{16}$$

were $Q^\mu = (q_0, \mathbf{q})$ is the total four-momentum transfer, and the baryon propagators $G_1$ and $G_2$ correspond to different baryons $B_1$ and $B_2$. In the following we consider a not too large momentum transfer and not too high temperatures, permitting us to neglect antibaryons in the system. Then the in-medium baryon propagator can be written as:

$$\hat{G}_B(P) = (\gamma P_B^* + M_B^*)\left[\frac{1}{P_B^{*2} - M_B^{*2} + i0} + \frac{\pi i}{\varepsilon_B}\delta\left(P_B^{*0} - \varepsilon_B\right) f_B\left(\varepsilon_B\right)\right]. \tag{17}$$

The first term in Eq. (17) describes propagation of free baryons, and the second term includes the Pauli principle restrictions[4]. Thus, the polarization tensors is the sum of a Feynman piece and an explicitly density-dependent piece. The energy-momentum transferred in the

[4]Notice that this form of the baryon propagator takes into account not only the Pauli principle, but also partially strong interactions, which are included by means of the effective mass of the baryon and the self-consistent nuclear field.

considered processes are not too large ($q_0^2 - \mathbf{q}^2 \simeq m_l^2$, with $m_l$ being the lepton mass), therefore we neglect the vacuum contribution and consider only the density-dependent part of polarization functions that are related to virtual excitations of a particle-hole type, with $P_B^* = (P^0 - U_B, \mathbf{p})$.

By a simple change of the integration variable $p_0 = P_0 - U_2$, Eq. (16) can be reduced to the standard form

$$\Pi^{\alpha\beta} = -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[ G_1(p+\tilde{Q})\, \gamma^\alpha\, G_2(p) \gamma^\beta \right], \tag{18}$$

with $p^\mu = (p_0, \mathbf{p})$ but with the effective momentum transfer given by

$$\tilde{Q}^\mu = (\tilde{q}_0, \mathbf{q})\,, \qquad \tilde{q}_0 = q_0 - U_1 + U_2. \tag{19}$$

Obviously, the polarization functions depend explicitly on $\tilde{q}_0$ instead of the total energy transfer $q_0$. The only known generalization (see refs. [5], [6]) of the polarization functions to the case of different baryons in the polarization loop have been made by the ansatz that the general form of the tensor (18) should be a sum of longitudinal and transverse components. Such an approach (which is valid in vacuum under the assumption of isotopic invariance of nucleons) is invalid for a system of interacting baryons, because in a medium the isovector current, caused by conversion of the baryon $B_1 \to B_2$, is not conserved.

Let's examine the nonconservation of the baryon current by contracting together the vector-vector polarization tensor and the effective four-momentum transfer

$$\tilde{Q}_\alpha \Pi^{\alpha\beta} = -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[ G_1(p+\tilde{Q}) \left(\gamma\tilde{Q}\right) G_2(p) \gamma^\beta \right]. \tag{20}$$

The baryon propagator (17) obeys the equation $(\gamma p - M_B^*)\, G_B(p) = 1$ with 1 being the identity matrix. By taking this into account and using the following identity

$$\begin{aligned} \tilde{q}_0\gamma_0 - \mathbf{q}\gamma = & [(p_0+\tilde{q}_0)\,\gamma_0 - (\mathbf{p}+\mathbf{q})\,\gamma - M_1^*] - [p_0\gamma_0 - \mathbf{p}\gamma - M_2^*] \\ & + (M_1^* - M_2^*) \end{aligned} \tag{21}$$

we obtain from Eq. (20)

$$\tilde{Q}_\alpha \Pi^{\alpha\beta} = -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\, G_2(p)\gamma^\beta + i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr} G_1(p+\tilde{Q})\gamma^\beta$$
$$+ (M_1^* - M_2^*)\, \Pi_M^\beta, \tag{22}$$

where $\Pi_M^\alpha$ is the mixed vector-scalar polarization, given by

$$\Pi_M^\beta \left(\tilde{Q}\right) = -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[ G_1(p+\tilde{Q})\; G_2(p)\gamma^\beta \right]. \tag{23}$$

We can make the change $p + \tilde{Q} \to p$ in the second integral of Eq.(22). Then, using the explicit form (17) of the baryon propagators, we obtain

$$\tilde{Q}_\alpha \Pi^{\alpha\beta} = \delta_{\beta 0}\ (n_2 - n_1\,) + (M_1^* - M_2^*)\ \Pi_M^\beta, \tag{24}$$

where $n_1$ and $n_2$ are the number densities of baryons $B_1$ and $B_2$, respectively, and $\delta_{\beta 0} = 1$ if $\beta = 0$ and zero otherwise. Eq. (24) can be equivalently written as

$$q^0 \Pi^{0\beta} - q^j \Pi^{j\beta} = (U_1 - U_2)\, \Pi^{0\beta} - \delta_{\beta 0}\ (n_1 - n_2\,) + (M_1^* - M_2^*)\ \Pi_M^\beta. \tag{25}$$

Thus, in a medium, the baryon vector current is conserved only in the case of symmetric nuclear matter ($n_1 = n_2$) and isotopic invariance of the baryons ($M_1^* = M_2^*$), which together with $U_1 = U_2$ provide $q^0\Pi^{0\beta} - q^j\Pi^{j\beta} = 0$. If at least one of the above conditions is not fulfilled, then $Q_\alpha \Pi^{\alpha\beta} \neq 0$. In particular, this means that the vector-vector polarization tensor for charged currents can not be written in terms of longitudinal and transverse components, as assumed in refs. ( [5], [6]). In a frame of reference where $Q = (q_0, 0, 0, q)$, the vector-vector polarization tensor (18) has four independent components $\Pi^{00}$, $\Pi^{03} = \Pi^{30}$, $\Pi^{33}$, and $\Pi^{11} = \Pi^{22}$.

The axial-axial tensor

$$\Pi_A^{\alpha\beta}(q_0, \mathbf{q}) = -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[ G_1(p+\tilde{Q})\gamma^\alpha \gamma_5\, G_2(p)\gamma^\beta \gamma_5 \right] \tag{26}$$

as well as the vector-axial

$$\Pi_{VA}^{\alpha\beta}(q_0, \mathbf{q}) = -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[ G_1(p+\tilde{Q})\gamma^\alpha\; G_2(p)\gamma^\beta \gamma_5 \right] \tag{27}$$

are of the standard form:

$$\Pi_A^{\mu\nu} = \Pi^{\mu\nu} + \Pi_A g^{\mu\nu}, \tag{28}$$

$$\Pi_{VA}^{\mu\nu} = \Pi_{VA}\frac{Q_\lambda}{q} i\epsilon^{\mu\nu\lambda 0}. \tag{29}$$

Our calculation of the integrals yields the following imaginary components of the retarded polarization tensors:

$$\mathrm{Im}\,\Pi_R^{00} = \frac{1}{2\pi q}\left[I_2 + \tilde{q}_0 I_1 + \frac{1}{4}\left(t - (M_1^* - M_2^*)^2\right) I_0\right], \tag{30}$$

$$\begin{aligned}\mathrm{Im}\,\Pi_R^{03} = \mathrm{Im}\,\Pi_R^{30} = \frac{1}{2\pi q^2}&\left[\tilde{q}_0 I_2 - \frac{1}{2}\left(M_1^{*2} - M_2^{*2} - 2\tilde{q}_0^2\right) I_1\right.\\ &\left.+\frac{\tilde{q}_0}{4}\left(t - M_1^{*2} + M_2^{*2}\right) I_0\right],\end{aligned} \tag{31}$$

$$\begin{aligned}\mathrm{Im}\,\Pi_R^{33} = \frac{1}{2\pi q^3}&\left[\tilde{q}_0^2 I_2 + \tilde{q}_0\left(\tilde{q}_0^2 - M_1^{*2} + M_2^{*2}\right) I_1 +\right.\\ &\left.\frac{1}{4}\left(q^2\left((M_1^* - M_2^*)^2 - \tilde{q}_0^2\right) + \left(\tilde{q}_0^2 - M_1^{*2} + M_2^{*2}\right)^2\right) I_0\right],\end{aligned} \tag{32}$$

$$\begin{aligned}\mathrm{Im}\,\Pi_R^{11} = \mathrm{Im}\,\Pi_R^{22} = -\frac{1}{4\pi q^3}&\left[t I_2 + \tilde{q}_0\left(t - M_1^{*2} + M_2^{*2}\right) I_1\right.\\ &\left.+\frac{1}{4}\left(4q^2 M_1^* M_2^* - q^4 + \left(\tilde{q}_0^2 - M_1^{*2} + M_2^{*2}\right)^2\right) I_0\right],\end{aligned} \tag{33}$$

$$\mathrm{Im}\,\Pi_R^{A} = -\frac{1}{2\pi q} M_1^* M_2^* I_0,$$

$$\mathrm{Im}\,\Pi_R^{VA} = \frac{1}{8\pi q^2}\left[2t I_1 + \left(t - M_1^{*2} + M_2^{*2}\right)\tilde{q}_0 I_0\right]. \tag{34}$$

Here

$$t \equiv \tilde{q}_0^2 - q^2, \tag{35}$$

and the functions $I_n$ are defined as

$$I_n = \tanh\left(\frac{q_0'}{2T}\right)\int_{\varepsilon_0}^{\infty} d\varepsilon_2\ \varepsilon_2^n \left[\left(1 - f_2\left(\varepsilon_2\right)\right) f_2\left(\varepsilon_2 + q_0'\right)\right.$$
$$\left. + f_2\left(\varepsilon_2\right)\left(1 - f_2\left(\varepsilon_2 + q_0'\right)\right)\right]. \tag{36}$$

with

$$q_0' = q_0 - \mu_1 + \mu_2. \tag{37}$$

The lower cut-off in Eq.(36) arises due to kinematical restrictions and reads

$$\varepsilon_0 = -\frac{\tilde{q}_0}{2t}\left(t - M_1^{*2} + M_2^{*2}\right)$$
$$-\frac{q}{2t}\sqrt{\left(t - \left(M_2^* + M_1^*\right)^2\right)\left(t - \left(M_2^* - M_1^*\right)^2\right)}. \tag{38}$$

This energy is real-valued if either $t > \left(M_2^* + M_1^*\right)^2$ or $t \leq \left(M_1^* - M_2^*\right)^2$. The first case corresponds to creation of a real baryon pair in the medium and requires inclusion of the vacuum (Feynman) piece of polarizations for a correct calculation. As mentioned above, we do not consider so large values of $t$. Thus, the imaginary part of the polarization functions does not vanish for $t \leq \left(M_1^* - M_2^*\right)^2$. In terms of $q_0$ and $q$, this can be written as

$$\left(q_0 - U_1 + U_2\right)^2 \leq q^2 + \left(M_1^* - M_2^*\right)^2. \tag{39}$$

Contrarily to the statement made in refs. ( [5], [6]) that the imaginary part of polarizations vanishes when the momentum transfer is time-like, our calculation shows some domain of $q_0 > q$ where the imaginary part does not vanish even if we assume $M_1^* = M_2^*$ (in fact, one can neglect the neutron-proton mass difference in nuclear matter). This result is obviously correct, otherwise, the neutron decay process $n \to p + e^- + \bar{\nu}$ , which requires $q_0^2 - q^2 > 0$ for final leptons, would not be possible in neutron stars.

The nuclear matter in the core of neutron stars becomes transparent for escaping neutrinos only at relatively low temperatures. Therefore we focus on the low-temperature limit ($\mu_B/T \gg 1$) and consider degenerate baryons and leptons under thermal and $\beta$-equilibrium. In this case the kinematical condition (38) restricts possible values of the momentum transfer. Obviously, in the degenerate case the cut-off (38) of the integral (36) should be lower than the baryon Fermi energy

$$\varepsilon_{F2} \geq -\frac{\tilde{q}_0}{2t}\left(t - M_1^{*2} + M_2^{*2}\right)$$
$$-\frac{q}{2t}\sqrt{\left(t - (M_2^* + M_1^*)^2\right)\left(t - (M_2^* - M_1^*)^2\right)}. \tag{40}$$

As mentioned above, the energy of the final lepton is close to its individual Fermi energy $\mu_l$, while the condition of chemical equilibrium is $\mu_l = \mu_1 - \mu_2$, where the baryon chemical potentials can be approximated by their individual Fermi energies. Thus, to the lowest order in $T/\mu_l$ we can take $\tilde{q}_0 = \varepsilon_{F1} - \varepsilon_{F2}$, and $t = (\varepsilon_{F1} - \varepsilon_{F2})^2 - q^2$ in all smooth functions. Inserting this into Eq.(40) we obtain the following domain of the momentum transfer

$$p_{F1} - p_{F2} < q < p_{F1} + p_{F2}, \tag{41}$$

where the imaginary part of the polarization functions does not vanish.

To the lowest accuracy in $T/\mu_l$, we obtain from (36):

$$I_0 \simeq q_0', \qquad I_1 \simeq q_0'\varepsilon_{F2}, \qquad I_2 = q_0'\varepsilon_{F2}^2, \tag{42}$$

where $q_0' \equiv \tilde{q}_0 - \varepsilon_{F1} + \varepsilon_{F2}$ is of the order of $T$.

By taking also $\tilde{q}_0 = \varepsilon_{F1} - \varepsilon_{F2}$ in all smooth functions of Eqs. (30 - 34), in the interval (41) we obtain

$$\mathrm{Im}\,\Pi_R^{\mu\nu} = \frac{q_0'}{8\pi q}\Phi^{\mu\nu}, \qquad \mathrm{Im}\,\Pi_R^{VA} = \frac{q_0'}{8\pi q}\Phi_{VA}, \qquad \mathrm{Im}\,\Pi_R^{A} = \frac{q_0'}{8\pi q}\Phi_{A}, \tag{43}$$

with

$$\Phi^{00} = \left[(\varepsilon_{F1} + \varepsilon_{F2})^2 - q^2 - (M_1^* - M_2^*)^2\right], \tag{44}$$

$$\Phi^{03} = \Phi^{30} = \frac{1}{q}\left[(\varepsilon_{F1} - \varepsilon_{F2})\left((\varepsilon_{F1} + \varepsilon_{F2})^2 - q^2\right)\right.$$
$$\left. - (\varepsilon_{F1} + \varepsilon_{F2})\left(M_1^{*2} - M_2^{*2}\right)\right], \tag{45}$$

$$\Phi^{33} = \frac{1}{q^2}\left[(\varepsilon_{F1} - \varepsilon_{F2})^2\left((\varepsilon_{F1} + \varepsilon_{F2})^2 - q^2\right)\right.$$
$$-2\left(\varepsilon_{F1}^2 - \varepsilon_{F2}^2\right)\left(M_1^{*2} - M_2^{*2}\right)$$
$$\left. + q^2 (M_1^* - M_2^*)^2 + \left(M_1^{*2} - M_2^{*2}\right)^2\right], \tag{46}$$

$$\Phi^{11}=\Phi^{22}=\frac{1}{2q^2}\left[q^4+4q^2\left(\varepsilon_{F1}\varepsilon_{F2}-M_1^*M_2^*\right)-\left(p_{F1}^2-p_{F2}^2\right)^2\right], \tag{47}$$

$$\Phi_A=-4M_1^*M_2^*, \tag{48}$$

$$\Phi_{VA}=\frac{1}{q}\left(\left(\varepsilon_{F1}-\varepsilon_{F2}\right)\left(p_{F1}^2-p_{F2}^2\right)-\left(\varepsilon_{F1}+\varepsilon_{F2}\right)q^2\right). \tag{49}$$

Contracting together the tensors, as indicated in Eq. (12), and calculating the imaginary part to the lowest order in $T/\mu_l$ we obtain

$$\begin{aligned}\operatorname{Im}\left(\mathrm{L}_{\alpha\beta}W^{\alpha\beta}\right)&=\frac{q_0'\omega_1}{\pi q}\left[\left(C_V^2+C_A^2\right)\left[q_0\Phi^{00}-2q\Phi^{03}+q_0\Phi^{33}\right.\right.\\&+2q_0\Phi^{11}+\left.\left(q\Phi^{00}-2q_0\Phi^{03}+q\Phi^{33}-2q\Phi^{11}\right)\cos\theta\right]\\&\left.-2C_A^2\left(q_0-q\cos\theta\right)\Phi_A-4C_VC_A\omega_1\left(q-q_0\cos\theta\right)\Phi_{VA}\right]\Psi(q_0,q),\end{aligned} \tag{50}$$

where $\theta$ is the angle between the $\mathbf{k}_1$ and $\mathbf{q}$ momenta. The step-function

$$\Psi(q_0,q)\equiv\Theta\left(q+p_{F2}-p_{F1}\right)\Theta\left(p_{F1}+p_{F2}-q\right),$$

with $\Theta\left(x\right)=1$ if $x\geq 0$ and zero otherwise, comes from the kinematical restrictions (41) and (39). By inserting expression (50) into Eq. (12), and taking $q_0\simeq\mu_l$, $q\simeq p_{Fl}$, we arrive to the integral over $dq_0'$ and the neutrino phase space. We omit this straightforward calculation, which results in the expression given by Eq. (11). We stress here the fact that agreement between these two methods is achieved only when we take into account non-conservation of the baryon vector current.

In a non-relativistic limit ($P_{FB}\ll M_B^*,\ \mu_l\ll M_B^*$), the leading terms of Eq. (11) give the following expression

$$Q_L=\frac{457\pi}{10080}G_F^2C^2\left(C_V^2+3C_A^2\right)M_1^*M_2^*\mu_lT^6\Theta\left(p_{Fl}+p_{F2}-p_{F1}\right), \tag{51}$$

which coincides with the known non-relativistic result of Lattimer et al. [2]. As in that case, our formula (11) exhibits a threshold dependence on the proton concentration. The "triangle" condition $p_{Fl}+p_{F2}\geq p_{F1}$, required by the step-function, is necessary for conservation of the total momentum in the reaction. We pay attention, however, that this condition should be supplemented by the one of chemical equilibrium. As mentioned in introduction, in the relativistic regime, the chemical equilibrium is possible only due to the fact that strong interactions create a gap between the proton and neutron energy spectrums, that is much larger than the mass difference of participating baryons.

The relative efficiency of the direct Urca processes involving different kinds of baryons depends essentially on the composition of the $\beta$-stable nuclear matter in the core of a neutron star. In order to quantify the relativistic effects in the direct Urca processes, we consider a simplified model for degenerate nuclear matter of the standard composition consisting on neutrons, protons and electrons under beta-equilibrium. We use a Walecka-type [7] self-consistent relativistic model of nuclear matter, by assuming that the interactions among nucleons are mediated by exchange of $\sigma, \omega$, and $\rho$ mesons. In this model the baryon effective mass $M^* = M - g_{\sigma B}\sigma_0$ as well as the potential energies $U_B = g_{\omega B}\omega_0 + g_{\rho B}t_{3B}b_0$ are calculated in a self-consistent way. Here $\omega_0$, $b_0$, and $\sigma_0$ are, respectively, the mean-field values of the $\omega$, $\rho$, and $\sigma$-mesons; $g_{\omega B}$, $g_{\rho B}$, and $g_{\sigma B}$ are the strong interaction coupling constants, and $t_{3B}$ is the third component of isospin for the baryons. Parameters of the model are chosen, as in ref. [6], to reproduce the nuclear matter equilibrium density, the binding energy per nucleon, the symmetry energy, the compression modulus, and the nucleon Dirac effective mass at saturation density $n_0 = 0.16 fm^{-3}$ . We have plotted in Fig. 1 the emissivity of the Urca process, as given by our formula Eq. (11) in comparison with the emissivity (51) given by Lattimer et al. Both magnitudes are normalized with respect to the emissivity given by our formula at threshold density. We can observe that relativistic effects dramatically modify the emissivity. The non-relativistic approximation approaches our result only at densities much smaller than the threshold density, indicated by an arrow on the density axis. Due to the decrease of the nucleon effective mass, the formula derived by Lattimer et al. predicts a decreasing of the emissivity as density increases above the threshold. In contrast, our formula shows a substantial increasing of the neutrino energy losses as we go to larger densities. Of course, the exact value depends on the underlying nuclear matter model.

Our Eq. (11) is obtained in the mean field approximation and does not take into account correlations among the baryons, which normally have a tendency to suppress neutrino energy losses. These calculations are out of the scope of this letter, and we leave these unwieldy calculations for a future work. At some densities neutrons may be superfluid and/or protons superconducting. The direct Urca rate is then reduced. This can be taken into account in the standard way (See e. g. [8]).

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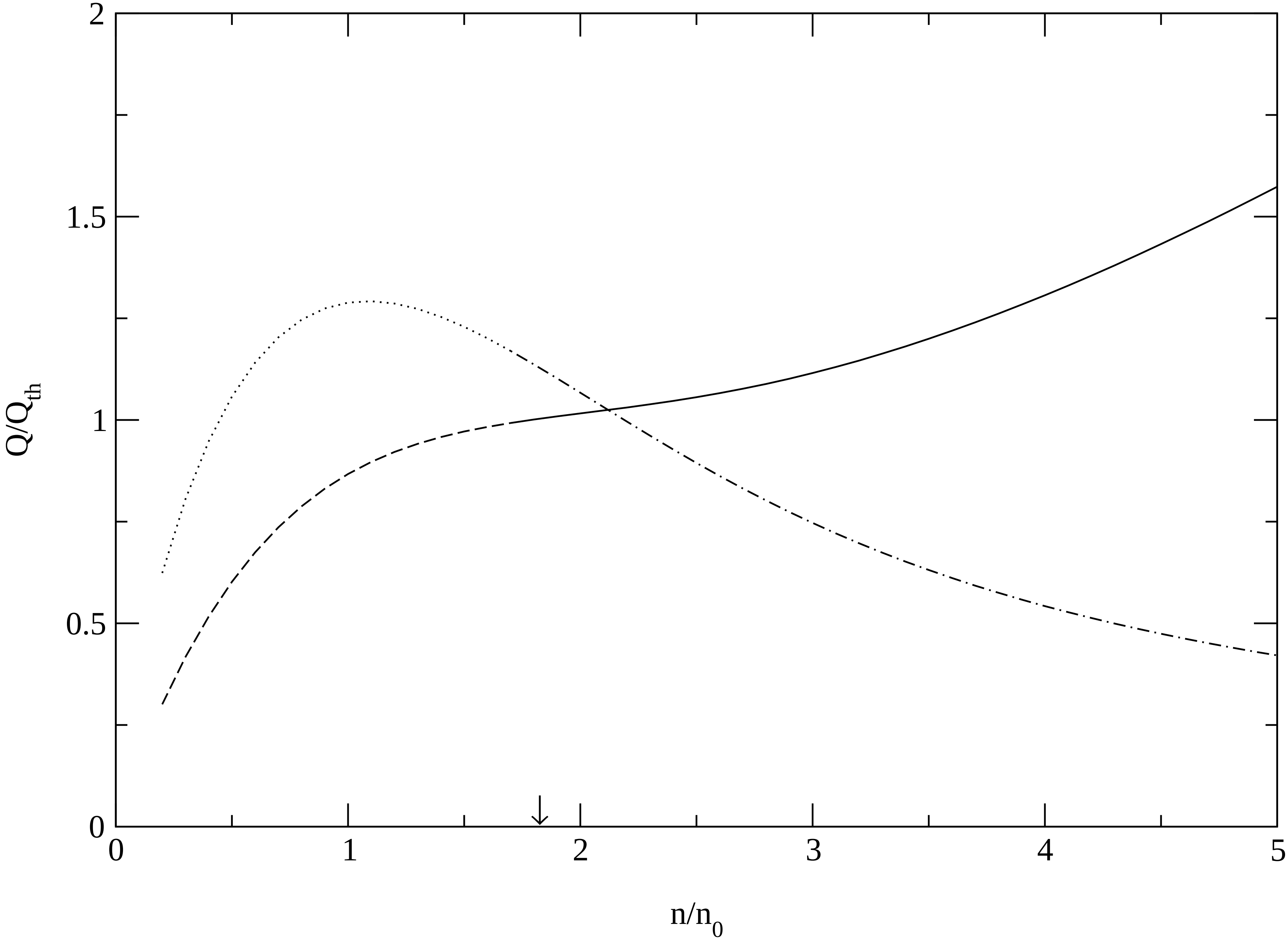
Q/Q$_{th}$
n/n$_0$
0
0.5
1
1.5
2
0
1
2
3
4
5

Fig 1. Neutrino emissivity due to the Urca processes. We plot, as a dashed line below the threshold density $n_{th} = 1.79 n_0$, and as a solid line above $n_{th}$ our result, given by Eq.(11). We also plot, for the sake of comparison, the emissivity given by Lattimer et al, as a dotted line below $n_{th}$, and as a dashed-dotted line above $n_{th}$. Both magnitueds are normalized with respect to the emissivity given by our formula at threshold density. We labeled the threshold density by an arrow on the abscisa axis, which shows the number density in units of saturation density $n_0 = 0.16 fm^{-3}$.