

OPTIMAL DESIGN OF DUTCH AUCTIONS WITH DISCRETE BID LEVELS

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The theory of auction has become an active research area spanning multiple disciplines such as economics, finance, marketing and management science. But a close examination of it reveals that most of the existing studies deal with ascending (i.e., English) auctions in which it is assumed that the bid increments are continuous. There is a clear lack of research on optimal descending (i.e., Dutch) auction design with discrete bid levels. This dissertation aims to fill this void by considering single-unit, open-bid, first price Dutch auctions in which the bid levels are restricted to a finite set of values, the number of bidders may be certain or uncertain, and a secret reserve price may be present or absent. These types of auctions are most attractive for selling products that are perishable (e.g., flowers) or whose value decreases with time (e.g., air flight seats and concert tickets) (Carare and Rothkopf, 2005).

I began by conducting a comprehensive survey of the current literature to identify the key dimensions of an auction model. I then zeroed in on the particular combination of parameters that characterize the Dutch auctions of interest. As a significant departure from the traditional methods employed by applied economists and game theorists, a novel approach is taken by formulating the auctioning problem as a constrained mathematical program and applying standard nonlinear optimization techniques to solve it. In each of the basic Dutch auction model and its two extensions, interesting properties possessed by the optimal bid levels and the auctioneer's maximum expected revenue are uncovered. Numerical examples are provided to illustrate the major propositions where appropriate. The superiority of the optimal strategy recommended in this study over two commonly-used heuristic procedures for setting bid levels is

also demonstrated both theoretically and empirically. Finally, economic as well as managerial implications of the findings reported in this dissertation research are discussed.

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## CHAPTER 1

### INTRODUCTION

#### 1.1 Auction, Auctioneer and Auction House

An auction is a competitive bidding process where objects of uncertain value are sold to those who are willing to pay the highest prices for them. It is a unique system to allocate properties based on price making by competition among bidders for the right to purchase (Cassady, 1967). An auction is usually used when the owner of the object is not certain how much it is worth and tries to sell it at the highest possible price. Thus, uncertainty about value is an inherent feature of auctions (Krishna, 2002).

As an agent of the seller, the auctioneer generally does not own the object but merely acts in his client's interest. His major role is to determine the conditions of sale acceptable to both the seller and the would-be buyer aside from striving to seek the maximum price for the object offered (Cassady, 1967). The auctioneer has the freedom to design the auction and perform behind-the-scene activities in order to ensure the success of the event.

An auction house is a commercial establishment where goods or services are auctioned off. Founded in 1766, Christie's is the world's largest auction house, which conducts more than 450 auctions and private sales annually in more than 80 categories including fine/decorative arts, collectibles, jewelry, photographs, and more with prices ranging from \$200 to over \$80 million. According to its website ([www.christies.com](http://www.christies.com)), Christie's sales of \$6.3 billion in 2007 made the highest total in art auction history. For the first half of 2008 alone, its global art sales totaled \$3.5 billion. The second largest auction house is Sotheby's, which was established in 1744 and

has extended its reach to 58 countries. Its annual financial report showed a gross revenue of \$918 million and a net income of \$213 million in 2007 ([www.sothebys.com](http://www.sothebys.com)).

## 1.2 Historical Background

The history of auction can be traced back to 500 B.C. when women were auctioned off as wives in Babylon (Cassady, 1967). During the Roman Empire, auctions were popular for family estates and war plunders. The auctioning process grew to be much more organized with a formal structure as time went on. In the European Middle Ages between the 5<sup>th</sup> and the 15<sup>th</sup> centuries, the earliest auction law (including auction licenses) was instituted in England (Rosenkrans, 2005). In the 17<sup>th</sup> and the 18<sup>th</sup> centuries, auction catalogs in English and French were formally published that included detailed information about the items being for sale. Since then, auctions have been widely used to sell unique and collectible items as well as general merchandises.

Today, auctions are held to conduct a huge volume of business transactions in the private sector of the economy such as the sale of houses, cars, stamps, coins and agricultural products. Even in the public sector, it is not uncommon for governments at different levels to sell mineral rights, foreign exchange and other assets through various competitive bidding processes. A notable example is that President Clinton granted the Federal Communications Commission the authority to auction off spectrum licenses in 1993 (Sengupta and Chatterjee, 2008). Over the last few years, the range of items sold at auction has been considerably widened by e-commerce (Klemperer, 2004).

Traditional auctions have evolved for thousands of years, but the emergence of the Internet has led to a proliferation of Web-based auctions only more recently. The late 1990s marked the coming of the age of electronic commerce as information technologies provided opportunities for entrepreneurs to extract maximum performance from participation in network

constellations (Reedy and Schullo, 2004; Roberts, 2003). It is generally accepted that online auctions began in the mid 1990s with the earliest websites being hosted by Onsale and eBay, which were founded in May and September of 1995, respectively (Lucking-Reiley, 2000).

Internet auctions have experienced exponential growth since 1999. In particular, eBay has maintained a dominant position in the customer-oriented market due to its rapid development over the past decade. In 1998, about \$700 million worth of items were sold on eBay with a total revenue of \$86 million. In 2003, the company conducted \$23,800 million worth of auctions and achieved a profitability of approximately \$2,165 million (Ellison and Ellison, 2005). By 2006, it had 180 million registered users and 50 million items listed in over 50,000 categories on any given day with an annual sale of \$43 billion (Ellison and Ellison, 2005; Ho, Chu and Lam, 2007). In the fourth quarter of 2007 alone, more than 100 million users made transactions with a sales volume of more than \$16 billion (Angst, Agarwal and Kuruzovich, 2008).

### 1.3 Importance of Auction

Auction joins the posted price and the negotiation process as the third fundamental trading mechanism in today's marketplace (Pinter, Seidmann and Vakrat, 2003; Subramanian and Zeckhauser, 2004). It has become one of the most successful business models due to its economic value (Chua, Wareham and Robey, 2007). First and foremost, an auction provides valuable to all parties involved in the event. For the auctioneer, he may benefit from having first-hand knowledge about the bidders and their financial goals since anyone interested in the object for sale must register with the auction house and set up an account, which contains useful information about the participant's cumulative bidding records and buying behavior. This helps auctioneers better manage excess capacities and exploit business opportunities from prospective customers by, for instance, making special offers to a target bidder or introducing new products

into the market (Zheng and Wang, 2008). From the bidders' perspective, taking part in an auction allows them to analyze the competitors' cost structures and adjust their corporate strategies accordingly to achieve sustainable competitive advantages.

An auction can also be an aid to forecast new product success. Traditionally, pricing decisions on new products are made based on the evaluation of consumer preferences and willingness to pay, which could be overestimated in hypothetical settings. A wealth of studies have suggested that experimental auctions create an environment in which real products and real money are exchanged thus revealing customers' valuations (Rousu, Alexander and Lusk, 2007; Lusk, Feldkamp and Schroeder, 2004; Fox, Hayes and Shogren, 2002). Hence, an auction can be viewed as a marketing tool to test whether the price of a new product will be acceptable to future buyers.

In addition, an auction can be utilized as a medium for advertising (Rosenkrans, 2005) since it invites customers to participate in the decision-making process of price by placing bids according to their perceived valuations. To determine the true value of a product, consumers frequently search relevant information in various ways including the use of the Internet. As a result, auctioning as a transaction channel generates a large number of visits to the websites of the manufacturer as well as the competitors for their product/service offerings and other relevant information, which increases the company's exposure to would-be buyers. As such, auction may be viewed as an incentive to learn about an organization and build brand recognition.

Finally, many online auctions have become an efficient product distribution channel where multiple buying options are available to potential customers. Some good examples include auctions with such features as eBay's "Buy-it-now," Yahoo!'s "Buy-price," Amazon's "Take-it," and uBid's "uBuy-it." Under those circumstances, bidders may end the auction



immediately by purchasing the item being auctioned off at the posted price predetermined by the auctioneer. Such hybrid mechanisms have proven to increase competition, raise the auctioneer's revenue and reveal the bidders' true valuations (Engelbrecht-Wiggans and Katok, 2006; Engelbrecht-Wiggans, 1996).

#### 1.4 Taxonomy of Auctions

Numerous traditional and online auctions held everyday can be classified into four basic types: ascending-bid, descending-bid, sealed-bid first-price, and sealed-bid second-price (Vickrey, 1961; Klemperer, 1999). A brief description of each of them can be found in Table 1.1. They all can be run on site or via electronic systems such as fax, telephone and the Internet; they may also be applied to a single unit or multiple units of an object with or without a reserve price (Kuo et al., 2004).

Table 1.1

##### *Taxonomy of Auctions*

Auction format	Key features
1. Ascending-bid	<ul style="list-style-type: none"> <li>• The auctioneer starts with at a low price and continues to raise it until one bidder remains.</li> <li>• The winner pays at the final price for the object for sale.</li> <li>• Information about the bids submitted is available to bidders.</li> </ul>
2. Descending-bid	<ul style="list-style-type: none"> <li>• The auctioneer begins with a high price and progressively lowers it until one bidder indicates a willingness to buy.</li> <li>• The winner buys the object at the current price.</li> </ul>
3. Sealed-bid first-price	<ul style="list-style-type: none"> <li>• Information about the bids submitted is unavailable to bidders.</li> <li>• The highest bidder wins and pays his bidding price.</li> </ul>
4. Sealed-bid second-price	<ul style="list-style-type: none"> <li>• Information about the bids submitted is unavailable to bidders.</li> <li>• The highest bidder wins and pays a price equal to the second highest bid.</li> </ul>

Ascending-bid auctions: The price starts at a low level and continues to climb until one bidder remains. During the bidding process, information about the bids submitted is available to all participants in real time. This mechanism creates more intense competition among bidders than other types of auctions because the auctioneer has more opportunities to use manipulative tactics to affect the final price achieved (Cassady, 1967). As an example, the asking price could successively rise by a fixed or variable amount or stays the same for different lot sizes. An ascending-bid auction also has the advantage of price discovery, which is particularly important for selling objects with unknown market value such as antiques.

The English auction is the most commonly-used ascending-bid mechanism. It is so named because the format is traditionally adopted by leading English auction houses such as Sotheby's and Christie's. The scheme may be implemented either electronically or by human voice or by the clock, with the last one being termed the English clock auction. One of the main features of the English auction is that bidder exit information is not public. Even if a bidder does not show his willingness to bid at a low level, he can still reenter the bidding process later. As a consequence, the bidders cannot exactly identify their competitors at each bid level, and the unexpected participants can result in greater competition.

Another popular form of the ascending-bid mechanism is the Japanese auction, which is characterized by the requirement that every bidder submit a bid at each level in order to stay in the game; otherwise, he is considered a non-participant and cannot return to the competitive bidding process. Such a rule is designed to prevent a bidder from preempting the auction by making a jump bid (Klemperer, 1999). The Japanese auction is usually recommended when the number of bidders is too small to warrant an effective English auction (Buxton and Ashcroft, 2006).

Descending-bid mechanism: This is the converse of the ascending-bid mechanism in that the auctioneer starts with a high price and progressively lowers it. The current price is posted and known to all participants, and the first bidder to indicate a willingness to buy the object at the current price is the winner.

A descending-bid mechanism is also called a Dutch auction as it has long been used to sell flowers in the Netherlands albeit the system is also applied to sell fish in Israel and tobacco in Canada (McAfee and McMillan, 1987). It has been argued that bidding pressure is stronger in the descending-bid mechanism than in the ascending-bid mechanism because bidders may overestimate their competition (Cassady, 1967). A Yankee auction is a variant of the Dutch auction where multiple units of an object are being auctioned off and winning bidders pay the respective prices at which they bid instead of paying the same price as in the Dutch auction.

Sealed-bid first-price mechanism: Under this auction format, bidders submit sealed bids to the auctioneer without knowing anything about the bids from the competitors and the highest bidder wins by paying his bidding price. It requires bidders to take actions during a short period of time based on both the supposed market value of the object and their own willingness to pay rather than engaging in competition with others as in the ascending-bid and descending-bid mechanisms. Therefore, this mechanism is particularly useful when it is necessary to speed up the auction process.

Sealed-bid first-price auctions have been held for the sale of timber rights by the U.S. Forest Service as well as for the sale of dried fish in several prime markets in Tokyo, Japan (Cassady, 1967). The federal government also used this mechanism to sell offshore oil and gas leases (Milgrom, 2004). Another interesting application is in electricity trading in the United Kingdom at the turn of the new millennium (Klemperer, 2004).

Sealed-bid second-price mechanism: Sealed bids are relayed to the auctioneer and no information about the bids from other bidders is available. The highest bidder wins and pays a price equal to the second highest bid. Also dubbed a Vickrey auction (Vickrey, 1961), this mechanism has found widespread applications in multi-unit online auctions, where all winning bidders pay the highest non-winning bid price, such as the sale of treasury bills and other securities (Rothkopf, Thomas and Kahn, 1990).

Aside from the four basic auctioning systems discussed above, there are several other mechanisms that are in use, including silent auctions where there is no auctioneer and bids are written on a sheet of paper. Also, a Dutch-English auction is a hybrid mechanism in which the price is first progressively lowered until one bidder submits a bid to determine the starting price, and then other bidders outbid this price in an ascending-bid fashion until the final winner is determined. But an in-depth treatment of the Dutch auction is provided in the next section as it is the focal point of this dissertation study.

### 1.5 Dutch Auction

The Dutch auction is the most popular descending-bid mechanism, which derives its name from the century-long tradition of selling tulips and other flowers in the Netherlands. To understand its significance, I note that approximately 4 billion flowers and 400 million plants are auctioned off by about 8,000 different sellers annually in Aalsmeer, Holland (Katok and Roth, 2004). In particular, Flower Auction Aalsmeer (VBA) and Flower Auction Holland (BVH), the world's two largest flower auctions, trade over 30 million flowers in nearly 60,000 transactions daily and generate over \$2.4 billion in annual sales (Kambil and Heck, 1998).

The Dutch auction is also known as the clock auction or the open outcry descending-price auction. Before the event takes place, information about the object for sale such as the quantity

available and the minimum bid, if any, are displayed on a clock or a similar electronic device at the front of the auctioning location. The hand of the clock is initially set at an arbitrarily high price and it continues to fall according to the predetermined decrements until an individual stops the clock by pushing a button or calling out “mine.” He or she then acquires the object at the current price.

Unlike English auctions, which are more suitable for unique items such as antiques and art works, Dutch auction are more appropriate for the disposal of perishable goods or the sale of products whose value decreases with time. These include produce, fish, tobaccos, concert tickets, seats on an airplane, and container space on an ocean-going vessel among others. What is shown in Table 1.2 is just a small sample of the diverse applications of Dutch auctions in the real world.

Table 1.2

*Applications of Dutch Auctions*

Application area	Source
1. Cash management and financial management	Alderson, Brown and Lummer (1987)
2. Stock repurchase	Bagwell (1992); Gay, Kale and Noe (1996)
3. Sales of art works in Holland	Marchi (1995); Cassady (1967)
4. Fish sales in Taiwan, England and Israel	Crawford and Kuo (2003); Cassady (1967)
5. Tobacco sales in Canada	McAfee and McMillan (1987)
6. Cloth sale in Taipei and Tuscany	Crawford and Kuo (2003)
7. Flower and potted plant sales in Holland	Katok amd Roth (2004)
8. Vehicle slots in a container port in Australia	IPART (2007)
9. Initial public offering of Google	Robicheaux and Herrington (2007)

### 1.6 Key Concepts in Auction Theory

Familiarity with a few key terms is critical to a better understanding of the theory of auction in general and this dissertation research in particular. These include bidder’s valuation, bidder’s attitude towards risk and outcome evaluation. Each of them is delineated below.

Bidder's valuation is concerned with a bidder's estimation of the economic value of the object being auctioned off and there are two basic types of valuations: private and common. In a private-value auction, each participant knows how much he values the object to be sold but this information is private and unknown to his competitors. This is true when, for instance, each bidder assigns a different value to a painting and his utility is derived from possessing it as a consumption good instead of reselling it in the market in the future. In contrast, in a common-value auction, the actual value of the object is exactly the same for everybody but each bidder has his own information about how much it is worth (Krishna, 2002). A typical example is an oil lease auction where the market price of the oil underground is the same regardless who the winner is albeit the bidders might have different estimates of its value (Rose and Levin, 2008). Bidders' valuations can also be classified as independent or interdependent. In an auction with independent valuations, a bidder's assessment of the value of the object to be sold is not affected by other bidders' appraisals, whereas in an interdependent- or affiliated-valuation auction, the actions taken by the competitors have impact on a bidder's estimation of how much the object is worth (Krishna, 2002).

Bidder's attitude towards risk refers to an auction participant's behavior in response to uncertainties present in his decision-making process. Uncertainties faced in an action include, but are not limited to, the value of the object being auctioned off, the strategies to be followed by other bidders, and the private valuation information the competitors have. There are three different categories of attitudes towards risk: risk-averse, risk-neutral and risk-seeking. A bidder is said to be risk averse if he is conservative and willing to pay more money than his valuation to avoid losing in an auction. In contrast, a risk-seeking bidder is aggressive and willing to take the risk of not winning in an auction by submitting a relatively low bid. Somewhere in between lies

the attitude of risk neutrality, which means that a bidder is rational and will pay up to his valuation to acquire the object for sale (Maskin and Riley, 1984).

It has been found that the outcome of an auction depends heavily on every bidder's attitude towards risk. For instance, in a first-price auction, a risk-averse bidder may submit a higher bid than a risk-neutral or risk-seeking bidder with the same valuation to increase the likelihood of winning at the expense of a lower surplus (Klemperer, 1999). Similarly, in a dual-channel auction with a posted price, a risk-averse bidder is more likely to accept the posted price in order to end the auction earlier and ensure his winning status (Reynolds and Wooders, 2009).

Lastly, auction outcomes may be evaluated based on two criteria: revenue and efficiency. The former is frequently used in the private industry and refers to the final price at which the object is sold, while the latter measures the extent to which the object is sold to the bidder with the highest valuation and is more applicable to the sale of public assets by government agencies (David et al., 2007; Sakai, 2008). The issue of outcome evaluation is of particular importance when few bidders take part in the auctioning process. For example, in the spectrum auctions where licenses of certain radio frequency bands were auctioned off by the U.S. government in the 1990s and 2000s, the number of serious bidders was as low as two or three for a given license (Dasgupta and Maskin, 2000). Thus, the major concerns of auctioneers were to achieve the maximum revenue as well as allocate the resources efficiently although the two goals were conflicting. I am quick to point out that the first criterion is chosen to assess the outcome of a Dutch auction in this dissertation. As such, the sole objective of the treatment in Chapters 3, 4 and 5 is to seek the auctioneer's maximum expected revenue.

## 1.7 Purpose of Study

While auction theory has been studied for a long time in economics, marketing, finance, and other related fields, the growing popularity of auction in today's global market has prompted a call for a deeper understanding of bidder behaviors and bidding strategies, which constitute a major research area (Kauffman and Wood, 2006; Kim, 2007; Shen and Su, 2007). Another emerging stream of studies has focused on optimal mechanism design to improve the performance of auctions of various forms (Krishna, 2002; Bapna, Goes, Gupta, and Karuga, 2002; Segev, Beam, Shanthikumar, 2001).

This dissertation takes the second approach by considering a special class of auctions where the auctioneer strives to attain the highest possible revenue. Specifically, I set out to extensively review the current auction literature and identify a number of key parameters characterizing an auction model. Insightful observations are made to reveal several common principles underpinning sound auction schemes. Once I decide to zero in on Dutch mechanisms, several assumptions are made and mathematical programs are developed as well as solved. The findings not only help auctioneers in establishing the optimal bid levels for achieving the highest possible average price of the object for sale, but also equip them with some useful tools for adjusting the auctioning system under different conditions. It is believed that the results obtained from the investigation into the relatively uncharted waters fill a void in the current auction literature.

The remainder of the dissertation is organized as follows. In Chapter 2, I conduct a comprehensive review of existing works on critical dimensions of mechanism design and opt to focus on the decision analytic problem where a single unit of an object is sold in an open-bid, first-price Dutch auction with discrete bid levels and a fixed number of bidders. A constrained



nonlinear program is formulated accordingly and solved in Chapter 3 to reveal a few interesting properties possessed by the optimal bid levels as well as the maximum expected revenue.

Numerical examples are also given to illustrate and validate the theoretical propositions put forward. In Chapter 4, I extend the basic Dutch auction model by considering a realistic scenario in which the number of bidders is a Poisson distributed random variable rather than a constant.

The treatment in Chapter 5 mirrors that in Chapter 4 except that a secret reserve price is introduced into the competitive bidding process aside from uncertainty about how many bidders are present. Finally, in Chapter 6, I discuss the limitations of the present dissertation and outline several key items on the future research agenda.

## CHAPTER2

### LITERATURE REVIEW

#### 2.1 Overview

An auction is a mechanism through which an object of an unknown value is sold to one of the potential buyers with possibly different valuations. The final selling price is determined by competition among the bidders according to the rules set by the auctioneer on behalf of the seller. From a theoretical perspective, establishing those rules is equivalent to specifying the levels of the key parameters that characterize an auction model.

Optimal auction design (OAD) is concerned with setting the values of the parameter in an auction model so that its efficiency or the auctioneer's expected revenue is maximized. The problem of OAD has been studied extensively over the past two decades in the case of continuous bidding where the bidder can submit a bid of any size (Myerson, 1981; Riley and Samuelson, 1981; Krishna, 2002). In contrast, research on auctions with prescribed discrete bid levels is a relatively uncharted territory. Regardless of the nature of the bid levels, however, it is of paramount importance to start the design of an auction by identifying the factors that are most influential on the outcome. These factors are referred to as model parameters and, collectively, they uniquely define the format of the auction to be conducted.

Based on a comprehensive review of the relevant literature, Li and Kuo (2008) identify eight primary parameters of an auction model, including buy-out option, closing rule, price determination, direction of price change, transparency of information, lot size, reserve price, and bid increment/decrement. Among them, the first two tend to be applicable to online auctions only and, therefore, is not discussed below. In the rest of the chapter, each of the remaining six

parameters is examined in detail along with number of bidders, which is deemed by some to be another critical dimension of an auction. I also highlight their individual impacts on the auctioning process in addition to pointing out the differences between this dissertation and other existing works to manifest the contribution of the present study.

## 2.2 Price Determination

Price determination is concerned with setting the price at which the winning bidder should pay for the object to be sold in an auction. Based on this parameter, we can classify all auctions into one of the following two categories: uniform-price and discriminatory-price (Harris and Raviv, 1981; Goldreich, 2007). In a uniform-price auction with multiple units of an object, all winning bidders are charged the same unit price for the quantities they request, which is equal to the highest losing bid. In the case that only a single unit is to be auctioned off, a uniform-price auction is sometimes called a second-price auction because the highest bidder pays the second highest bid price (i.e., the highest losing bid). In contrast, in a discriminatory-price auction with multiple units of an object, each successful bidder pays the actual unit price he offers for the quantities he requests. If only a single unit is for sale, a discriminatory-price auction is also known as a first-price auction since the winner pays the highest bid.

The comparison between uniform-price auctions and discriminatory-price auctions is well documented in the current literature albeit the findings from some studies are conflicting with respect to the resulting expected revenue. As one of the pioneers in auction theory, Vickrey (1961) is in favor of uniform prices and suggests that all winners pay the highest losing bid. Along the same line, Sade, Schnitzlein and Zender (2006a, 2006b) find that the discriminatory-price auction is more susceptible to collusion and leads to a lower average revenue. In studying the Zambian foreign exchange market, Tenorio (1993) argues that the uniform-price auction is

revenue-superior to its discriminatory-price counterpart due to greater bidder participation. Katok and Roth's (2004) result shows that price discrimination is often viewed as being unfair. Furthermore, Goldreich (2007) analyzes data collected from 105 discriminatory-price and 178 uniform-price auctions to reach the conclusion that the expected revenue received from the latter is substantially higher than that from the former. This observation is consistent with the recent move of the U.S. Treasury from the discriminatory-price format to the uniform-price format in auctioning off securities to finance the public debt (Goldreich, 2007).

However, several researchers maintain that higher revenues are achievable through discriminatory mechanisms when the bidders are risk-averse in single-unit auctions (Holt, 1980; Harris and Raviv, 1981; Riley and Samuelson, 1981). Pekec and Tsetlin (2008) notes that uncertainty about the number of bidders is an important reason why a discriminatory-price system generates more revenues. The same result holds in auctions where multiple units are to be sold since the discriminatory-price mechanism tends to induce truthful bidding and yield a higher final price (List and Lucking-Reiley, 2000; Porter and Vragov, 2006; Engelmann and Grimm, 2003; Levin, 2005).

It appears that the differences between the two pricing rules are rooted in the uncertainty involved in the competitive bidding process as well as the bidder's attitude towards risk. Specifically, in a discriminatory-price auction, the participants face uncertainty about bid acceptance rather than the price of the object because the final price is determined by what they bid. In comparison, in a uniform-price auction, the bidders face uncertainty about both bid acceptance and the bidding price since the final selling price is determined by the competitor who places the highest losing bid (Harris and Raviv, 1981). Additionally, as observed by

Klemperer (1999), a risk-averse bidder may place a higher bid than a risk-neutral or risk-seeking bidder to avoid losing in a discriminatory-price auction.

### 2.3 Direction of Price Change

The direction of price change refers to the magnitude of the current bid relative to the immediately preceding bid during the course of the auction and it can take one of two forms: ascending-price and descending-price. In an ascending-price auction, the price of the object to be sold begins at a low level and it continues to go up until only one interested bidder remains. This auction form is consistent with the original meaning of the word “auction,” which is derived from the Latin *augere* and signifies “increasing” (McAfee and McMillan, 1987). The most common ascending-price selling mechanism is the English auction. In contrast, a descending-price auction works in the opposite way where the auctioneer starts the competitive process with an extremely high price and reduces it in steps until a bidder is willing to buy the object. The Dutch auction is the most familiar form of the descending-price mechanism.

A fundamental result in the auction literature on the direction of price change is the revenue equivalence theorem due to Vickrey (1961), who claims that both the English auction and the Dutch auction yield the same expected revenue. He proves it by demonstrating that, under the assumption of independent and private bidder valuations, the descending-price auction is equivalent to the first-price auction while the ascending-price auction is equivalent to the second-price auction. The proposition follows easily from a basic fact in auction theory that the revenue from a first-price auction is equal to that from a second-price auction on average (Krishna, 2002).

Some researchers question the revenue equivalence theorem and argue that a descending-price auction generates higher revenues than its ascending-price counterpart because of several

inherent advantages. One of them is that the bidding game speeds up quickly and every bidder knows his winning status right after submitting a bid (Katok and Roth, 2004). Also, information about bidders' true valuations is not revealed in the course of competition and the outcome is sudden with complete uncertainty. As a consequence, bidders may be anxious about losing enough to stop the auctioning process at a higher price (Cheema, 2003). Katok and Roth (2004) investigate multi-unit auctions for homogeneous goods where the exposure problem may occur (i.e., the bidders are exposed to the risk of buying only a small fraction of the desired quantity because the competition on the other subset of units is tougher than expected). They show that the average revenue received from descending-price auctions is higher than that from ascending-price auctions.

Other scholars maintain that the descending-price format is outperformed by its ascending-price counterpart since the former requires the knowledge about the upper bound on the eventual bidding price in order to avoid a loss (Katok and Roth, 2004). Thus, the descending-price auction is not appropriate for objects such as antiques whose value is unique. Katok and Roth (2004) discuss the free-riding problem in multi-unit auctions, where local bidders bid a small quantity at a high unit price so that the remainder is less than what the global bidder needs. Consequently, other local bidders can make a transaction at a very low unit price and this makes the descending-price auction less profitable.

#### 2.4 Transparency of Information

This parameter refers to the availability of bidding information to the bidders during the course of auction (Kuo et al., 2004) and it can take one of two forms: open-bid and sealed-bid. In an open-bid auction, information on bidding price (and quantity in the case of multi-unit auction) submitted by all participants is available to bidders in real time to stir up competition

among them. In contrast, in a sealed-bid auction, such information is unavailable and only the auctioneer has access to it to protect the privacy of the losing bidders. According to the extent to which the bidding information is released during the auctioning process, open-bid auctions can be further classified as completely open-bid auctions or partially open-bid auctions (e.g., reporting only the summary statistics or just the leading bid).

There has been considerable research on the differences between open- and sealed-bid auctions with respect to revenue generation although the traditional auction theory suggests the equivalence of revenue between the two formats. For instance, by assuming that each risk-neutral bidder has an independent and private valuation, Vickrey (1961) demonstrates that the expected revenue is the same for both types of single-unit auctions. But researchers have sought to understand how the revenue implications might change when some of the underlying assumptions (e.g., risk neutrality, private valuation independence, and information symmetry) do not hold.

Riley (1989), Ausubel (2004), as well as Milgrom and Weber (1982) find the open-bid auction to be more profitable than the sealed-bid auction if the bidders' valuations are positively correlated. One possible explanation is that when bidders are uncertain about how much the object for sale is worth, an open environment allows them to acquire useful information by observing the competitors' bidding behaviors. This may induce auction participants to bid more aggressively and eventually result in a higher selling price than in an environment where bidders' valuations are independent of each other.

Contrary to what is described above, a number of studies contend that an auctioneer will be financially better off in a sealed-bid auction when the assumptions of risk neutrality and single-unit object are dropped. If the bidder exhibits an attitude of risk aversion, there have been

reports that the open-bid auction generates lower revenues than its sealed-bid counterpart (Maskin and Riley, 1984; Riley and Samuelson, 1981; Riley, 1989; Li and Riley, 2007). In particular, Riley and Samuelson (1981) assert that the optimal revenue from the sealed-bid auction declines as the degree of risk aversion increases perhaps because the bidders have greater fear of loss and therefore are apt to place higher bids more quickly. In the case of multi-unit auctions, Alsemgeest, Noussair and Olson (1998) state that the sealed-bid auction yields higher revenues than the English auction when two units of an object are to be auctioned off. In the laboratory experiments conducted by Engelmann and Grimm (2003), it is found that uniform-price sealed-bid auctions with multiple units generate more revenues but are less efficient than their open-bid counterparts. This may be attributed to the fact that bidders are inclined to overbid their valuations in a less transparent setting.

When the presumption of information symmetry (i.e., bidder valuations are drawn from a common distribution) is relaxed, the findings reported in the literature are mixed (Maskin and Riley, 2000). On the one hand, some argue that asymmetries cause the expected revenues to drop in the open-bid auction. Klemperer (1998) and Bulow, Huang and Klemperer (1999) consider asymmetry across bidders in that one has an edge over the others (e.g., having more information or less budget constraints). It is concluded that such an advantage makes him more aggressive while his competitors face increased winner's curse (i.e., the danger for a winning bidder to overpay) so that they bid more conservatively. As a consequence, the bidder's probability of winning increases and the price he pays decreases. Li and Riley (2007) support this view by modeling the case where the auctioneer has much less information than the bidder to evaluate other potential competitors. In such a situation, he cannot set a high reserve price to squeeze more money out of the pocket of the bidder with the highest valuation. On the other



hand, several researchers maintain that the revenue comparison between the open-bid and the sealed-bid auctions is contingent on other factors than what is discussed previously. For example, Bulow and Klemperer (2002) use the concept of hazard rate to express the bidder's failure rate at any point in time. They show that the expected revenue is an increasing function of bidders' hazard rates in a sealed-bid single-unit auction.

Jap (2003) proposes to investigate whether revealing or hiding partial information during the course of auction would produce different outcomes. Towards that end, there appears to be a consensus among scholars that withholding bidders' identification increases the auctioneer's revenue in a first-price auction with risk-averse bidders (McAfee and McMillan, 1987; Mathews, 1987; Dyer, Kagel and Levin, 1989; Levin and Ozdenoren, 2004). Mithas and Jones (2007) consider the bid-ranking auction format in which a bidder only sees his rank rather than the levels of other bids. They report that greater competition is observed and bidders are more aggressive to stay ahead in the ranking list. Therefore, higher auction revenues are derived than those from the sealed-bid auction.

As a departure from the previous studies, Dufwenberg and Gneezy (2002) compare the effects of three types of information disclosure on revenues in first-price auctions: complete information disclosure in which the auctioneer announces all the bids submitted in the previous auctions, partial information disclosure in which only winning bids are made public, and no information disclosure. They demonstrate that the expected revenue is much lower in the first auction format than that in each of the other two formats. Thus, their suggestion to auctioneers is to keep information on losing bids secret.

Overall, the impact of information transparency on auction revenue is not all that clear. More research is called for to gain a deeper understanding of why some of the findings are

different or even conflicting. One possible reason is the assumptions upon which each model is based (Ausubel, 2004). Another possibility is the myriad of factors that affect the auctioneer's choice between the two auction formats, such as bidders' attitudes towards risk (Milgrom and Weber, 1982; Riley, 1989; Maskin and Riley, 1984) and their auction experiences (Ariely, Ockenfels and Roth, 2005).

## 2.5 Lot Size

Lot size refers to the quantity of an object to be auctioned off and it can be one single unit or multiple units. Based on the nature of goods to be sold, the latter can be further classified as multi-unit auctions of homogeneous goods or multi-unit auctions of heterogeneous goods. Multi-unit auctions are much more complicated to analyze than their single-unit counterparts since each bidder needs to indicate both the unit bidding price and the bid amount in his submission. The significance of multi-unit auctions lies in its increasing applications in business-to-business commerce in the private sector and government resource allocation procedures in the public sector of the economy. The two most popular online auctions of this type are the "Yankee auction" at Onsale.com and the "Dutch auction" at eBay.com.

Much of the early auction literature on lot sizing deals with single-unit auctions under various conditions, but the momentum has recently shifted to multi-unit auctions. A critical question to ask here is whether all of the units of an object should be sold via one multi-unit auction or a series of single-unit auctions to maximize the total revenue. If each bidder demands only one of the multiple units for sale, the basic results on single-unit auctions can be extended to this case and the revenue equivalence theorem will still hold under appropriate assumptions (Harris and Raviv, 1981; Ockenfels, Reiley and Sadrieh, 2007). If each bidder is allowed to win more than one unit, however, analyzing the strategic complexity and situational uncertainty

involved becomes a more challenging task (Tenorio, 1997; Engelbrecht-Wiggans and Kahn, 1998a, 1998b).

It has been suggested that the auctioneer will be financially worse off when bidders demand more than one unit in multi-unit auctions (Porter and Vragov, 2006). This may be explained by the concept of demand reduction, which refers to the bidders' incentive to bid below their valuations on the second or subsequent units. Likewise, Alsemgeest, Noussair and Olson (1998) note that the revenue from the English auction is lower in the two-unit requirement environment than in the single-unit requirement environment due to strategic bidder demand reduction. Another reason for the lower revenue is that multiple-unit auctions are more susceptible to collusions, where secret agreements are reached among bidders for their own gains. One example is a ring operation in which arrangements are made so that would-be buyers do not outbid one another in the auctioning process and one of them is chosen to be the winner. After the auction is closed, the winner sells what he gets to other bidders at the initial price. This is verified by Brusco and Lopomo (2002) in simultaneous ascending-price auctions of heterogeneous goods, where bidders signal detailed information about their valuations and divide the objects among themselves, thus buying their own share at a relatively low price. In addition, Grimm, Riedel and Wolfstetter (2003) investigate the GSM spectrum auctions in Germany and present evidence of low-price equilibria in simultaneous, ascending-bid multi-unit auctions.

Some progress has been made in the design of multi-unit auctions to improve the average selling prices of the objects over the past few years. For instance, Philippe, William and Michael (2005) study the buyer's option in multi-unit ascending-price auctions, where the first unit is auctioned off and the winner is given the right to purchase any number of additional units at the winning price. This process repeats until everything is sold. Their empirical results indicate no

significant difference in the auctioneer's expected revenues between the sequential single-unit auctions and the multi-unit auctions with buyer's option.

## 2.6 Number of Bidders

This is concerned with the size of the group of potential buyers taking part in an auction. Based on the auctioneer's knowledge about it, the parameter can take one of the following two forms: known and unknown. The former happens when, for example, only pre-qualified individuals are invited to bid in an auction. However, in the latter case, the auctioneer has no control over or information on exactly how many bidders are participating in the event. Arora et al. (2007) highlight the prevalence of uncertainty about the number of bidders in electronic marketplaces auctions conducted daily on eBay.

The significance of this parameter to auction design has been well recognized by a host of researchers (Pekec and Tsetlin, 2008; Geoffrion and Krishnan, 2003; Anandalingam et al., 2005). As an example, the auctioneer's expected revenue has been proven to be a monotone increasing function of the number of bidders provided that the latter is known and that the bidders have independent private valuations (Zheng, 2009). However, the result no longer holds if it is not known how many bidders will compete in the auction. Levin and Smith (1994) remark that if an uncertain number of risk-neutral bidders share a common distribution of valuations, the expected auction revenue decreases as the number of potential bidders increases when it grows beyond some cut-off point. A similar observation is made by Li and Zheng (2009) in auctions with an unknown number of bidders who have independent private valuations. Zheng (2009) echoes the theme by noting that the median winning bid is increased by \$45.46 by controlling the potential number of participants in timber auctions with an uncertain number of bidders. Clearly, an important implication of the findings from the last three studies is that it

might be economically beneficial to set an upper limit on the number of auction participants if the auctioneer does not know it.

There have been a few inquiries into the relative merits of known and unknown numbers of bidders in auctions albeit no consensus has been reached as to which of them maximizes the average revenue. One school of thought argues that an unknown number of bidders can increase the auctioneer's revenue in certain situations. For instance, under the assumption that each bidder has an affiliated private valuation, Pekec and Tsetlin (2008) as well as Matthews (1987) show that the auctioneer is better off with an uncertain number of bidders in a discriminatory-price auction. But Pekec and Tsetlin (2008) join Harstad et al. (2008) in contending that the opposite is true when the bidders' valuations are subject to a common distribution. Another school of thought suggests that the average revenues derived from auctions with a known and an unknown numbers of bidders are equivalent provided that the participants are risk averse (McAfee and McMillan, 1987). The same result is obtained by Levin and Ozdenoren (2004) in a study of uniform-price auctions with independent private bidders' valuations as well as by Harstad et al. (2008) in their research on uniform-price auctions with common bidders' valuations and a large number of bidders.

In sum, in the auction literature up to the late 1970s, it is typically assumed that the number of bidders is known with certainty. Starting in the 1980s, it has been treated as a random variable in optimal design of auctions to reflect the reality the oftentimes the auctioneer is not sure how many would-be buyers are going to show up when an auction begins (McAfee and McMillan, 1987; Levin and Ozdenoren, 2004). There is a relatively small body of work on the circumstances under which it is beneficial to mitigate or induce uncertainty about the number of

bidders. There is no clear-cut answer to the question at the present time, but it appears to be an interesting area to explore.

## 2.7 Reserve Price

A reserve price is the lowest price that an object can be sold for in an auction and it can take one of two forms: public and secret. A public reserve price is announced to the bidders before an auction starts, whereas a secret reserve price is never announced and information about it is unavailable to the bidders. In some auctions with a secret reserve price, bidders are notified whether it has been exceeded or not throughout the bidding process. To win in auctions with a reserve price, the bidder must place a bid that is higher than or equal to it. Setting a reserve price involves three sequential decisions. The first is whether a reserve price should be established. If so, then the next decision is whether it should be made public or kept secret. Finally, it is necessary to decide how high or low the reserve price should be.

The existing literature supports the notion that the use of a public reserve price increases the auctioneer's revenue. Riley and Samuelson (1981) claim that when each potential buyer's valuation of the object for sale is not influenced by how others value it, the public reserve price allows the auctioneer to extract additional revenue. The result is extended by Milgrom and Weber (1982) to the first-price sealed auction, the second-price auction and the ascending-bid English auction when the object to be sold is worth the same to every bidder but this value is not known with certainty. Their rationale is that announcing the reserve price provides more information to the bidders and hence affects their valuations, which has been tested with both real-world data and data from controlled experiments. As an example, Reiley (2006) examines first-price sealed-bid auctions of trading cards and confirms that the presence of a public reserve

price improves the final selling prices, but the magnitude of improvement depends on the goods to be auctioned off.

Unlike the case with a public reserve price, the results from studies on auctions involving a secret reserve price are mixed. On the one hand, some have shown that incorporating a secret reserve price into an auction leads to a higher auction revenue. For instance, Bajari and Hortaçsu (2003) observe that auctioneers who sell high-value books are more likely to use secret reserve prices in hopes of raising the winner's bid. It has also been suggested that although the inclusion of a secret reserve price reduces the probability of sale, it is associated with a higher ending price on average when there is a successful transaction (Dewan and Hsu, 2004; Bland, Black and Lawrimore, 2007). Lucking-Reiley et al. (2007) investigate auctions of one-cent coins on eBay and find that the presence of a secret reserve price increases the object's final selling price by about 15% overall. This may be accounted for by considering the secret reserve price as another competing bidder before the reserve price is reached and the auctioneer's expected revenue accrues as a result of the more fierce competition. On the other hand, however, Ockenfels, Reiley and Sadrieh (2007) question the existence of an economic equilibrium in which secret reserve prices improve online auction revenues. Their reasoning is that a secret reserve price deters serious bidders from entering the auction and hence lowers the expected revenue.

There has been much discussion about the respective advantages and disadvantages of public and secret reserve prices, but no definite conclusion has been reached as to which of them maximizes the average return. Proponents of the public reserve price such as Walley and Fortin (2005) feel that it will spark more bidding interest in the course of the auction and eventually lead to an increase in revenue. Katkar and Reiley (2006) concur, showing that auctions with a secret reserve price earn 10% less revenue and are 30% less likely to sell the goods than auctions

with a public reserve price. A similar finding is reported in Eklof and Lunander's (2003) simulation study, in which the auctioneer's revenue is approximately 10% higher if a public reserve price is in place. In contrast, those in support of the secret reserve price maintain that it outperforms its public counterpart in both the dependent-valuation setting (Vincent, 1995) and the independent-valuation setting (Rosenkranz and Schmitz, 2007). One of their main arguments is that a secret reserve price is frequently used with a minimum bid, which provides more information to bidders than a public reserve price. Moreover, a secret reserve price can encourage greater participation from the bidders and intensify the linkage between the value of the object for sale and the price to be paid. This is consistent with the findings of Brisset and Naegelen (2006) in auctions involving risk-averse bidders. Finally, Adams (2007) observes C5 Corvettes auctions on eBay and discovers that the secret reserve price is usually set at a much higher level than the public reserve price.

A handful of researchers have proposed that the reserve price should be an increasing function of the number of bidders to maximize the auctioneer's expected revenue (Cai et al., 2007; Rosenkranz and Schmitz, 2007; Reiley, 2006). In other words, the reserve price should be set low if there are not many bidders and it should be high when the number of auction participants is large. In the case where the number of bidders is determined endogenously, Engelbrecht-Wiggans (1987) proves that a lower reserve price may result in higher expected revenues if it entices more bidders to compete in an auction because the perceived financial risk is lower. In turn, the increasing number of bids submitted stirs up the competition and creates a more profitable auction. It has been noted that a high reserve price decreases both the number of bids and the probability of a successful transaction but increases the revenue that is actually received (Reiley, 2006).



## 2.8 Bid Increment/Decrement

This parameter refers to the amount by which the current bid is above (or below) the preceding bid in an ascending-price (or descending-price) auction. It can take one of three forms: a fixed bid increment (or decrement), multiple fixed bid increments (or decrements), and a minimum increment (or decrement). If an auction is designed with one or multiple fixed bid increments/decrements, then the bidding price goes up (or down) by them as the auction proceeds. Some good examples are Christie's LIVE™ online bidding, Sotheby's Internet bidding, and Phillips online bidding. If a minimal increment (or decrement) is required, however, each bid must be higher (or lower) than its immediate predecessor by at least that amount. The philosophy behind this rule is to encourage competition among bidders at the early stage of the auctioning process. In the absence of such a requirement, a bidder may submit a bid with an extremely small increment (or decrement) to outbid the current highest bidder in an ascending-price (descending-price) auction. All of eBay's "Minimum Bid Increment," Yahoo!'s "Bid Increment," Amazon's "Bid Increment," and uBid's "Bid Increment" fall under this category of bid changes in auctions.

Yamey (1972) is the first to discuss the impact of one fixed bid increment on oral auctions, where the price goes up steadily by the same amount and therefore the bids are evenly distributed. He observes the art auctions at Christie's and Sotheby's and finds that both auction houses increase the bid level by around 5% of the preceding bid. He inquires into the rationale of the practice and concludes that the rule is aimed at speeding up the auctioning process to reduce the costs as well as time of all parties involved. Yamey (1972) argues that this approach works well only if there is a big gap between the two highest bidders' valuations; but it is not applicable to agricultural products as they have continuous market demands and are supplied

without significant qualitative differentiation. Consequently, it is reasonable to expect the bidders' valuations for common-value goods to cluster. Thus, the aforementioned proportional-increment bidding rule is more appropriate for auctions of one-of-a-kind merchandises such as antiques.

As a special case of one fixed bid increment, Chwe (1989) investigates a first-price sealed-bid auction where bids submitted must be equal to multiples of a fixed increment. He proves that a unique Nash equilibrium bidding strategy exists and it converges to a steady state of the continuous bid auction as the bid increment approaches zero. Without regard to time or communication costs incurred during the bidding process, he also shows that the expected revenue from an auction with a fixed increment is consistently lower than that from a continuous bid auction. This is why the auctioneer always has an incentive to set a small bid increment.

As an extension of Chwe's (1989) work, Yu (1999) considers four widely-used auction formats in which bids can only be multiples of a fixed increment: the first-price sealed-bid auction, the second-price sealed-bid auction, the English auction and the Dutch auction. She demonstrates that the first-price and the Dutch auctions are strategically equivalent. Furthermore, the second-price auction tends to yield higher expected revenues than the first-price auction as the number of bidders increases. When the number of bid levels is extremely large so that the bid increment becomes very small, the auction with discrete bid levels converges to one with continuous bid increments.

As a departure from the two studies described above, Rothkopf and Harstad (1994) examine the effect of multiple fixed bid increments on a single-unit open-bid auction. In the case of two bidders with valuations taken from a common uniform distribution, it is proven that the bid levels should be evenly spaced in order to maximize the expected revenue and minimize

the expected loss of economic efficiency. When there are only two bid levels with uniformly distributed bidders' valuations, the bid increment should decrease as the auction progresses. Finally, if there are two bidders with the same exponentially distributed valuation, the bid increments should be increasing.

David et al. (2007) generalize the work of Rothkopf and Harstad (1994) by deriving a mathematical expression of the expected revenue with any bidder's valuation distribution, any number of bidders, and any number of discrete bid levels. Further, based on the same analytical model proposed by Rothkopf and Harstad (1994), David and her colleagues show how to determine the optimal number and values of discrete bid levels to maximize the expected revenue when bidders' valuations are independently drawn from a common uniform or exponential distribution. More specifically, under the presumption of uniformly distributed bidders' valuations, they prove that a fixed bid increment is optimal when only two bidders take part in the auction, whereas the bid increment should be decreasing as the auction proceeds when more than two bidders are present. They also point out that the optimal public reserve price is an increasing function of the number of bidders. However, under the assumption of exponentially distributed bidders' valuations, the bid increment should be increasing if two bidders are in the auction, while the bid increment initially decreases and then increases as the number of bidders increases.

Unlike David et al. (2007) or Rothkopf and Harstad (1994), who are mainly concerned with multiple fixed-bid increments in an (ascending-price) English auction, Yuen et al. (2002) focus on an online (descending-price) Dutch auction with discrete price decrements. They contend that the optimal bid decrements should be increasing if the function of the common bidder distribution is convex, but they should be decreasing when the function is concave.

Ahlberg (2009) studies the expected revenue from a sealed-bid auction where two units of an object are sold. A notable difference in his work is the assumption that the discrete bid levels are drawn from a known discrete joint distribution.

As for research on minimum bid increment, Bapna, Goes, Gupta and Karuga (2002) develop a theoretical model to analyze business-to-consumer online Yankee auctions. The empirical and simulation results they obtain reveal that an auctioneer should set a minimal bid increment to maximize the auctioneer's expected revenue. The crux of their argument is that such an increase creates a partition in the bidder's valuation space so that higher bid levels become feasible and identifiable to as many participants as possible. Bansal and Garg (2005) propose to impose an upper bound on the minimum bid increment to allow for competitive price discovery and truthful bidding in an online auction.

## 2.9 Focus and Uniqueness of Study

In the preceding sections, I provide a comprehensive survey of literature on seven key parameters of an auction model. A summary of the definitions is listed in Table 2.1 and their major findings can be found in Table 2.2 at the end of this chapter. Evidently, in theory, a large number of auction formats can be designed by combining those parameters at different levels. For instance, one can hold a multiple-unit, open-bid, second-price, ascending auction where the bid levels are continuous, the number of bidders is known with certainty, and a public reserve price is used. Alternatively, an auctioneer might choose to run a single-unit sealed-bid, first-price, descending auction with discrete bid levels, a secret reserve price and an unknown number of bidders. However, in practice, not all of the possible auction forms are useful in all occasions.

The main focus of this dissertation is on optimal design of a special class of auction models possessing the following characteristics: discriminatory price (Parameter 1), descending

price (Parameter 2), open bid (Parameter 3), single unit (Parameter 4), a known number of bidders (Parameter 5), no reserve price (Parameter 6), and multiple fixed decrements (Parameter 7). More precisely, I seek to develop a mathematical program to model the Dutch auction in which an object is to be sold, the bid levels are discrete, no reserve price is set, and the number of bidders is known with certainty. The eventual goal is to determine the optimal bid levels that will maximize the auctioneer's expected revenue (i.e., the final selling price). Subsequently, I examine two variations of the basic formulation by assuming that the number of bidders is unknown and a secret reserve price is in use, respectively, in the competitive bidding process.

The existing literature abounds in studies of single- or multi-unit English auctions where a public or secret reserve price may or may not be applied and the number of bidders may or may not be known for sure. In most of the auction models proposed, it is posited that the bid increment may be of any size; that is, the bid level is a continuous variable (Rothkopf and Harstad, 1994; Bapna, Goes and Gupta, 2003; David et al., 2007; Ahlberg, 2009). Note that such an assumption can be problematic for auctions with time and cost constraints since continuous bid levels will slow down the bidding process and hence increase administrative as well as personnel expenses. This is especially true for Dutch auctions of perishable goods (e.g., flowers and produce) and products whose value decreases with time (e.g., air flight seats and container ship space) (Cassady, 1967; Kambil and Heck, 1998; Carare and Rothkopf, 2005). The observation highlights the importance of establishing discrete bid levels in those situations to speed up the transactions.

However, as discussed in Section 2.8, there exists only a very small body of research on auctions with discrete bid levels despite their widespread practical applications. These include the works by Yamey (1972), Chwe (1989), Yu (1999), Rothkopf and Harstad (1994), David et al.

(2007), Mathews and Sengupta (2008) and Ahlberg (2009). It should be pointed out that all of them deal with setting optimal price increments in English or sealed-bid (instead of Dutch) auctions. In some cases (e.g., David et al., 2007), the provisional winner is awarded the object when there is a tie (instead of using random drawing as the tie-breaking rule). In others (e.g., Mathews and Sengupta, 2008), the winner pays the second highest (instead of the highest) price. Lastly, their underlying assumptions include a known (instead of unknown) number of bidders and no reserve price (instead of a secret reserve price). To the best of my knowledge, the only research work that resembles this dissertation is Yuen et al. (2002). They consider an online Dutch auction with discrete price decrements in which bids are submitted via a wireless network, but their study differs from mine in several respects as elaborated below:

- (1) Scope - The paper by Yuen et al. (2002) is centered around deriving the optimal bid decrements whereas I go beyond that to uncover the properties of both the optimal bid levels and the maximum expected revenue. A closed-form expression for computing the maximum expected revenue is also developed, which allows us to analyze the impact of each of the number of bidders and the number of bid levels on the auction outcome. In addition, I extend the basic model to explore two scenarios where the number of bidders is uncertain and a secret reserve price is used in the Dutch auction, respectively. These issues are not addressed by Yuen et al. (2002).
- (2) Communication costs - A salient feature of Yuen et al. (2002) is that bidders in the online Dutch auction submit their bids via a wireless Internet network. A time discounting factor is introduced into their model to reflect the communication cost of server resources, signal processing, and bandwidth incurred by the auctioneer. However, such a cost factor is dropped from the objective function in this analytical

- model since the present dissertation is primarily concerned with traditional Dutch auctions.
- (3) Methodology - Yuen et al. (2002) employ an iterative numerical method to obtain the optimal solution to a one-dimensional search problem transformed from a nonlinear program formulated for the Dutch auction. Instead, I take a more efficient approach by directly tackling the NLP and applying standard optimization techniques to determine the revenue-maximizing bid levels. Another difference is that I opt to provide rigorous mathematical proofs of the superiority of the optimal strategy over two heuristic strategies for establishing bid levels in Dutch auctions. In contrast, Yuen et al. (2002) demonstrate it by numerical illustrations based on a small set of problem instances randomly generated.
- (4) Certainty about number of bidders - It is assumed in Yuen et al. (2002) that the number of bidders in the Dutch auction is known and fixed, which is one of the assumptions made about the basic model in Chapter 3 as well. However, I consider more general settings in Chapters 4 and 5 where the number of bidders is subject to a Poisson distribution rather than being a constant. While inclusion of the factor of uncertainty in the treatment complicates the mathematical analysis of the problem, I believe that it is more reflective of reality since in many Dutch auctions the auctioneer has no information about or control over how many interested buyers will attend the event and bid.
- (5) Secret reserve price - In view of the important role played by reserve price in traditional or online bidding processes as delineated in Section 2.7, I incorporate a secret reserve price into this analysis involving a random number of bidders in

Chapter 5 and evaluate its effect on the optimal bid levels as well as the maximum expected revenue. The results have tremendous implications for the auctioneer but they are not available from Yuen et al. (2002), where no floor price of the object for sale is imposed in the Internet Dutch auctions they investigate.

In conclusion, this dissertation distinguishes itself from existing studies in the current literature by taking a novel approach to the problem of optimal auction design. More specifically, I draw up a mathematical program to model the Dutch auction in which the bid levels are discrete, the number of bidders may be known or unknown, a secret reserve price may be present or absent, and the overall goal is to determine the optimal bid levels for maximizing the auctioneer's expected revenue. I believe that this research is unique and its findings will contribute to a deeper understanding of the system dynamics in a competitive bidding process such as a Dutch auction that is most suitable for selling time-critical products or services.



Table 2.1

*Definitions of Auction Model Parameters*

<b>Model parameter</b>	<b>Definition of parameter</b>	<b>Parameter level</b>
Price determination	<ul style="list-style-type: none"> <li>• Price at which the winning bidder pays for the object to be sold in an auction</li> </ul>	<ul style="list-style-type: none"> <li>• Uniform price <ul style="list-style-type: none"> <li>○ In a uniform-price auction of multiple units of an object, all winning bidders are charged the same price, which is equal to the highest losing bid.</li> <li>○ In a uniform-price auction of a single unit of an object, the highest bidder pays the second highest bidding price (i.e., second-price auction).</li> </ul> </li> <li>• Discriminatory price <ul style="list-style-type: none"> <li>○ In a discriminatory-price auction of multiple units of an object, each winner pays his bidding price.</li> <li>○ In a discriminatory-price auction of a single unit of an object, the winner pays the highest bidding price (i.e., first-price auction).</li> </ul> </li> </ul>
Direction of price change	<ul style="list-style-type: none"> <li>• Direction in which bidding prices move during the course of auction</li> </ul>	<ul style="list-style-type: none"> <li>• Ascending-price: The auctioneer starts with a low price of the object to be sold and continues to raise it until only one interested bidder remains.</li> <li>• Descending-price: The auctioneer starts with an extremely high price of the object to be sold and continues to lower it until a bidder shows his willingness to buy.</li> </ul>
Transparency of information	<ul style="list-style-type: none"> <li>• Availability of bidding information to the bidders during the course of auction</li> </ul>	<ul style="list-style-type: none"> <li>• Open bid: Information on bidding price and quantity submitted by all bidders is available to them in real time.</li> <li>• Sealed bid: Information on bidding price or quantity submitted by all bidders is unavailable to them.</li> </ul>
Lot size	<ul style="list-style-type: none"> <li>• Number of units of an object to be sold</li> </ul>	<ul style="list-style-type: none"> <li>• Single unit: A single unit of an object is to be sold.</li> <li>• Multiple units: Multiple units of an object or different objects are to be sold.</li> </ul>
Number of bidders	<ul style="list-style-type: none"> <li>• Auctioneer's knowledge about how many bidders will take part in an auction</li> </ul>	<ul style="list-style-type: none"> <li>• Known: The number of bidders is known with certainty.</li> <li>• Unknown: The number of bidders is not known with certainty.</li> </ul>
Reserve price	<ul style="list-style-type: none"> <li>• Lowest acceptable price at which an object may be sold in an auction</li> </ul>	<ul style="list-style-type: none"> <li>• Public reserve price: A reserve price is imposed and announced to the bidders before an auction starts.</li> <li>• Secret reserve price: A reserve price is imposed but no information about it is available to the bidders.</li> </ul>
Bid increment/decrement	<ul style="list-style-type: none"> <li>• Amount by which the current bid is above or below the previous bid</li> </ul>	<ul style="list-style-type: none"> <li>• Single fixed bid increment/decrement: The bidding price goes up/down by a predetermined amount.</li> <li>• Multiple fixed bid increments/decrements: The bidding price goes up/down by one of the several predetermined amounts.</li> <li>• Minimum increment/decrement: The bidding price goes up/down by no less than a predetermined amount.</li> </ul>

Table 2.2

*Key Findings on Auction Model Parameters*

<b>Model parameter</b>	<b>Parameter levels</b>	<b>Major studies</b>	<b>Key results and comments</b>
Price determination	<ul style="list-style-type: none"> <li>• Uniform price vs. discriminatory price</li> </ul>	Vickrey (1961); Holt (1980); Harris and Raviv (1981); Riley and Samuelson (1981); Tenorio (1993); List and Lucking-Reiley (2000); Porter and Vragov (2006); Engelmann and Grimm (2003); Levin (2005); Sade, Schnitzlein and Zender (2006a, b); Goldreich (2007); Katok and Roth (2004); Pekec and Tsetlin (2008)	<ul style="list-style-type: none"> <li>• Mixed results on revenue comparison between uniform-price and discriminatory-price auctions</li> </ul>
Direction of price change	<ul style="list-style-type: none"> <li>• Ascending-price vs. descending-price</li> </ul>	Vickrey (1961); Cheema (2003); Katok and Roth (2004)	<ul style="list-style-type: none"> <li>• Equivalent revenue under standard assumptions</li> <li>• Descending-price auctions result in higher revenues in case of bidder exposure</li> <li>• Ascending-price auctions result in higher revenues in case of free riding</li> </ul>
Transparency of information	<ul style="list-style-type: none"> <li>• Open bid vs. sealed bid</li> </ul>	Vickrey (1961); Riley (1989); Ausubel (2004); Milgrom and Weber (1982); Maskin and Riley (1984, 2000); Riley and Samuelson (1981); Li and Riley (2007); Alsemgeest, Noussair and Olson (1998); Engelmann and Grimm (2003); Bulow and Klemperer (2002); Mithas and Jones (2007); McAfee and McMillan (1987); Mathews (1987); Dyer, Kagel and Levin (1989); Levin and Ozdenoren (2004); Dufwenberg and Gneezy (2002); Klemperer (1998); Bulow, Huang and Klemperer (1999)	<ul style="list-style-type: none"> <li>• Equivalent revenues under standard assumptions</li> <li>• Mixed results on revenue comparison between open-bid and sealed-bid auctions</li> <li>• Sharing less information with bidders increases auctioneer's expected revenue</li> </ul>

*(table continues)*

Table 2.2 (continued).

Model parameter	Parameter levels	Major studies	Key results and comments
Lot size	<ul style="list-style-type: none"> <li>• Single unit vs. multiple units</li> </ul>	<p>Harris and Raviv (1981); Tenorio (1997); Alsemgeest, Noussair and Olson (1998); Engelbrecht-Wiggans and Kahn (1998a, b); Brusco and Lopomo (2002); Grimm, Riedel and Wolfstetter (2003); Philippe, William and Michael (2005); Ockenfels, Reiley and Sadrieh (2007); Porter and Vragov (2006)</p>	<ul style="list-style-type: none"> <li>• Equivalent revenue in case each bidder bids one unit</li> <li>• Mixed results on revenue comparison between single-unit and multi-unit auctions when each bidder bids multiple units</li> </ul>
Number of bidders	<ul style="list-style-type: none"> <li>• Known number of bidders vs. unknown number of bidders</li> </ul>	<p>Pekec and Tsetlin (2008); Geoffrion and Krishnan (2003); Anandalingam et al. (2005); Levin and Smith (1994); Zheng (2009); Li and Zheng (2009); Pekec and Tsetlin (2008); Matthews (1987); Harstad et al. (2008); McAfee and McMillan (1987); Levin and Ozdenoren (2004)</p>	<ul style="list-style-type: none"> <li>• Revenue increases with a known number of bidder</li> <li>• Revenue decreases with an unknown but extremely large number of bidders</li> <li>• Mixed results on revenue comparison between auctions with known and unknown numbers of bidders</li> </ul>
Reserve price	<ul style="list-style-type: none"> <li>• Public reserve price vs. secrete reserve price</li> </ul>	<p>Riley and Samuelson (1981); Milgrom and Weber (1982); Vincent (1995); Bajari and Hortaçsu (2003); Dewan and Hsu (2004); Walley and Fortin (2005); Brisset and Naegelen (2006); Katkar and Reiley (2006); Reiley (2006); Rosenkranz and Schmitz (2007); Engelbrecht-Wiggans (1987); Cai, Riley and Ye (2007); Bland, Black and Lawrimore (2007); Lucking-Reiley, Bryan, Prasad and Reeves, (2007); Ockenfels, Reiley and Sadrieh (2007); Eklof and Lunander (2003); Adams (2007)</p>	<ul style="list-style-type: none"> <li>• Public reserve price increases revenue</li> <li>• Mixed results about secrete reserve price</li> <li>• Mixed results on revenue comparison between auctions with public and secrete reserve prices</li> <li>• Optimal reserve price increases with the number of bidders</li> </ul>
Bid increment/decrement	<ul style="list-style-type: none"> <li>• Single fixed increment vs. multiple fixed increments vs. minimum increment</li> </ul>	<p>Yamey (1972); Rothkopf and Harstad (1994); Chwe (1989); Bapna, Goes, Gupta and Karuga (2002); Bansal and Garg (2005); David et al. (2007); Yuen et al. (2002); Ahlberg (2009)</p>	<ul style="list-style-type: none"> <li>• New research area</li> <li>• Multiple bid increment levels lead to higher revenue as the number of bidders increases</li> </ul>

## CHAPTER 3

### A BASIC DUTCH AUCTION MODEL WITH DISCRETE BID LEVELS

#### 3.1 Introduction

In this chapter, I will focus on a special class of Dutch auctions where one unit of an object is to be sold in an open-bid, first-price, descending competitive process with discrete bid levels but without a reserve price. All participants are assumed to be rational and risk-neutral in that they will bid based on their valuations of the object. It is also assumed that all bidders' valuations are private and independently drawn from a common distribution. The implication is that a bidder's assessment of how much the object is worth is unknown to his competitors and will not be affected by their behaviors. A third assumption is that if there are two or more bidders willing to pay the highest asking price, then one of them will be randomly chosen as the winner. Finally, I posit that the main goal of the Dutch auction is to maximize the auctioneer's expected revenue by determining the optimal discrete bid levels.

#### 3.2 Auction Model

The problem under consideration is one in which one unit of an object is to be sold in a Dutch auction with  $n \geq 1$  potential bidders and  $m \geq 1$  distinct bid levels  $l_1, l_2, \dots, l_m$  with  $l_1 < l_2 < \dots < l_m$ . In this auction model, the auctioneer begins with an extraordinarily high price and continues to lower it in the sequence of  $l_m, l_{m-1}, \dots, l_2, l_1$ . The auction closes when one or more bidders are willing to accept the current asking price. Each bidder's valuation of the object,  $v_j$ , is a random variable subject to a common continuous distribution with probability density function (PDF)  $f(\cdot)$  and cumulative density function (CDF)  $F(\cdot)$ ,  $j = 1, 2, \dots, n$ .

Let  $Z$  be the expected revenue to be received by the auctioneer at the end of the auction. To derive a mathematical expression for  $Z$ , I need to multiply each bid level by its likelihood of occurrence and sum up the results. Since a Dutch auction requires only one bid to end, each participant will bid in accordance with his true valuation to avoid losing unexpectedly. Based on the assumptions made above, it is evident from Figure 3.1 that the final selling price of the object is  $l_i$  if and only if  $q \geq 1$  bidders' valuations are in the range  $[l_i, l_{i+1})$ , no one was willing to pay the higher price  $l_{i+1}$  announced previously, and the remaining  $n - q$  bidders' valuations are below  $l_i$ ,  $i = 1, 2, \dots, m$ . In case there is only one bid in that range (i.e.,  $q = 1$ ), the bidder is clearly the sole winner and is awarded the object by paying  $l_i$ ; otherwise (i.e.,  $q \geq 2$ ), a random drawing is necessary to select the winner.

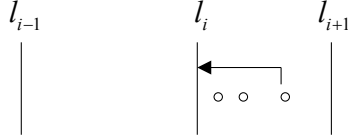


Figure 3.1. Dutch auction with a closing price of  $l_i$ .

Now, let  $P(l_i)$  denote the probability that the object is sold at the bid level  $l_i$ ,  $i = 1, 2, \dots, m$ . It follows that

$$P(l_i) = \sum_{q=1}^n \binom{n}{q} F(l_i)^{n-q} [F(l_{i+1}) - F(l_i)]^q \quad (3.1)$$

According to the binomial expansion  $\sum_{q=0}^n \binom{n}{q} a^{n-q} b^q = (a + b)^n$ , (3.1) can be simplified as

$$P(l_i) = F(l_{i+1})^n - F(l_i)^n \quad (3.2)$$

The expected revenue can then be written as

$$\begin{aligned}
 Z &= \sum_{i=1}^m l_i P(l_i) \\
 &= \sum_{i=1}^m l_i [F(l_{i+1})^n - F(l_i)^n]
 \end{aligned} \tag{3.3}$$

Thus, the Dutch auction model with discrete bid levels may be formulated as the following nonlinear program (NLP):

$$\begin{aligned}
 \text{Maximize} \quad & Z = \sum_{i=1}^m l_i [F(l_{i+1})^n - F(l_i)^n] \\
 \text{subject to:} \quad & l_{i+1} \geq l_i, \quad i = 1, 2, \dots, m-1 \\
 & l_i \geq 0, \quad i = 1, 2, \dots, m
 \end{aligned} \tag{3.4}$$

While the NLP in (3.4) above is very difficult to wrestle with in general, I will show that it becomes efficiently solvable provided that the bidders' valuations follow a uniform distribution on the support  $[0, \bar{v}]$ ; namely,  $U(0, \bar{v})^*$ . Without loss of generality, I can define

$l_1 = 0$ ,  $l_{m+1} = \bar{v}$ ,  $F(l_1) = 0$ , and  $F(l_{m+1}) = 1$ . Under these assumptions, one has  $F(l_i) = \frac{l_i}{\bar{v}}$  and

(3.4) reduces to

$$\begin{aligned}
 \text{Maximize} \quad & Z = \sum_{i=1}^m \frac{l_i}{\bar{v}^n} (l_{i+1}^n - l_i^n) \\
 \text{subject to:} \quad & l_{i+1} \geq l_i, \quad i = 1, 2, \dots, m \\
 & l_1 = 0 \\
 & l_{m+1} = \bar{v}
 \end{aligned} \tag{3.5}$$

In the theory of nonlinear programming, it is well known that an optimal solution to an NLP exists as long as the following two conditions are satisfied simultaneously (Winston, 1994):

---

\* Any general uniform distribution  $U(a, b)$  can be linearly transformed to  $U(0, \bar{v})$  with  $\bar{v} = b - a$ .

- (a) The objective function is concave; and
- (b) The constraint set is convex.

In the next section, it will be demonstrated that both requirements are met in (3.5) so that there is an optimal solution to the Dutch auction model under consideration.

### 3.3 Existence of Global Optimality

A closer look at the mathematical model in (3.5) reveals that each of the first  $m$  constraints is a half space, which obviously is a convex set. Moreover, each of the last two constraints is a hyperplane, which is also a convex set. Since the intersection of convex sets is still a convex set, it is clear that the constraint set of (3.5) is convex. It remains to show that the objective function is concave to ensure the existence of a global optimal solution. In the literature on NLP, there are three different approaches to proving the concavity of a continuous, twice differentiable function (Bazaraa et al., 2006). These include:

- (a) All eigenvalues of the Hessian of the function are negative
- (b) The principal minors of the Hessian of order  $i$  have the same sign as  $(-1)^i$
- (c) The leading principal minor of the Hessian of order  $i$  has the same sign as  $(-1)^i$

In what follows, I will decompose the Hessian of the objective function in (3.5) into two matrices and showing that individually each of them possesses the property described in (c) above. Since the same property is preserved when the two Hessians are added up to become the original Hessian, one can conclude that the objective function is indeed concave. Below is the first key finding in this chapter, and its proof is deferred to Appendix A.

**Proposition 3.1:**  $\sum_{i=1}^m \frac{l_i}{\bar{v}^n} (l_{i+1}^n - l_i^n)$  is concave in  $l_1, l_2, \dots, l_m$  if  $v_j \sim U(0, \bar{v})$ ,  $j = 1, 2, \dots, n$ .

The result below is presented without proof as it follows immediately from the concavity of the objective function as well as the convexity of the constraint set in (3.5), to which  $(l_1^*, l_2^*, \dots, l_m^*)$  is an optimal solution.

**Corollary 3.1:**  $(l_1^*, l_2^*, \dots, l_m^*)$  exists if  $v_j \sim U(0, \bar{v})$ ,  $j = 1, 2, \dots, n$ .

In light of the above result, the NLP in (3.5) may be solved by using one of the nonlinear optimization software packages currently available in the marketplace. Possible choices of the computer code include Solver in Microsoft Excel, MINOS, LINGO, and GRG2 among others (Nash, 1998).

In the sequel, I will develop a closed-form expression for calculating the revenue-maximizing bid levels. This is especially useful when I try to solve the Dutch auction problem but a computer program is not handy. In addition, it lays the foundation for proving some important properties of the optimal bid levels and the maximum expected revenue to be presented in the next two subsections.

Define

$$c_i = \begin{cases} 0, & i = 1 \\ 1, & i = 2 \\ \sqrt[n]{(n+1)c_{i-1}^n - nc_{i-2}c_{i-1}^{n-1}}, & i = 3, 4, \dots, m+1 \end{cases} \quad (3.6)$$

The optimal bid levels yielding the maximum expected revenue may be computed manually based on the following theorem, whose proof can be found in Appendix B.



**Proposition 3.2:**  $l_i^* = \frac{c_i}{c_{m+1}} \bar{v}$ ,  $i = 1, 2, \dots, m$

Proposition 3.2 implies that, given  $m$ ,  $n$  and  $\bar{v}$ , the optimal bid levels may be determined by  $c_1, c_2, \dots, c_{m+1}$ . For example, when  $m = 3$ ,  $n = 5$  and  $\bar{v} = 10$ , I have  $c_1 = 0$ ,  $c_2 = 1$ ,  $c_3 =$

$\sqrt[5]{(5+1) \times 1^5 - 5 \times 0 \times 1^4} \approx 1.4310$ , and  $c_4 = \sqrt[5]{(5+1) \times 1.4310^5 - 5 \times 1 \times 1.4310^4} \approx 1.7196$ . It follows

that  $l_1^* = \frac{0}{1.7196} \times 10 = 0$ ,  $l_2^* = \frac{1}{1.7196} \times 10 \approx 5.8153$ , and  $l_3^* = \frac{1.4310}{1.7196} \times 10 \approx 8.3217$ .

### 3.4 Properties of Optimal Bid Levels

In this section, I seek to uncover some interesting properties of the optimal bid levels in the Dutch auction of interest. Towards the end, what follows is an important result on their distribution pattern, whose proof is given in Appendix C.

**Proposition 3.3:**  $l_{i+1}^* - l_i^* < l_i^* - l_{i-1}^*$  if  $n > 1$  and  $l_{i+1}^* - l_i^* = l_i^* - l_{i-1}^*$  if  $n = 1$ ,  $i = 2, 3, \dots, m$ .

In light of Proposition 3.3, the bid levels should be evenly spaced (i.e.,  $l_{i+1}^* - l_i^* = l_i^* - l_{i-1}^*$ ) so as to maximize the expected revenue when only one bidder participates in the auction. The implication is that each bid decreases by the same amount as the asking price goes down from one level to another. But the bid decrements should be increasing (i.e.,  $l_{i+1}^* - l_i^* < l_i^* - l_{i-1}^*$ ) when two or more bidders compete for the object for sale, which means that the optimal bid levels become farther to each other as the asking price decreases. Interestingly, these are consistent with the findings of Yuen et al. (2002) when the communication cost is negligible.

The purpose of the next proposition is to characterize the behaviors of the revenue-maximization bid levels as a function of the number of auction participants. Specifically, it says that when the number of bidders in the Dutch auction increases, each of the optimal bid levels will go up albeit the bid decrements will still be increasing as described in Proposition 3.3. See Appendix D for the proof.

**Proposition 3.4:**  $l_i^*$  is an increasing function of  $n$ ,  $i = 2, 3, \dots, m$ .

Before proceeding further, it seems appropriate to refer to Figure 3.2 for a better understanding of Propositions 3.3 and 3.4 presented above. Here it is assumed that  $m = 6$  bid levels are to be set to maximize the auctioneer's expected revenue in a Dutch auction, and the number of bidders varies from  $n = 1$  to 2 to 15. A careful examination of each of the last two charts indicates that the vertical bars representing the optimal bid levels are getting farther away from each other as the asking price goes down from right to left when there are two or more bidders in the auction. Also, as shown in the first chart, the optimal bid levels are evenly spaced and the bid decrements are equal in the special case when there is only one bidder. It is also noted that the six bars associated with a larger  $n$  (e.g.,  $n = 15$ ) are to the right of their counterparts associated with a smaller  $n$  (e.g.,  $n = 2$ ).

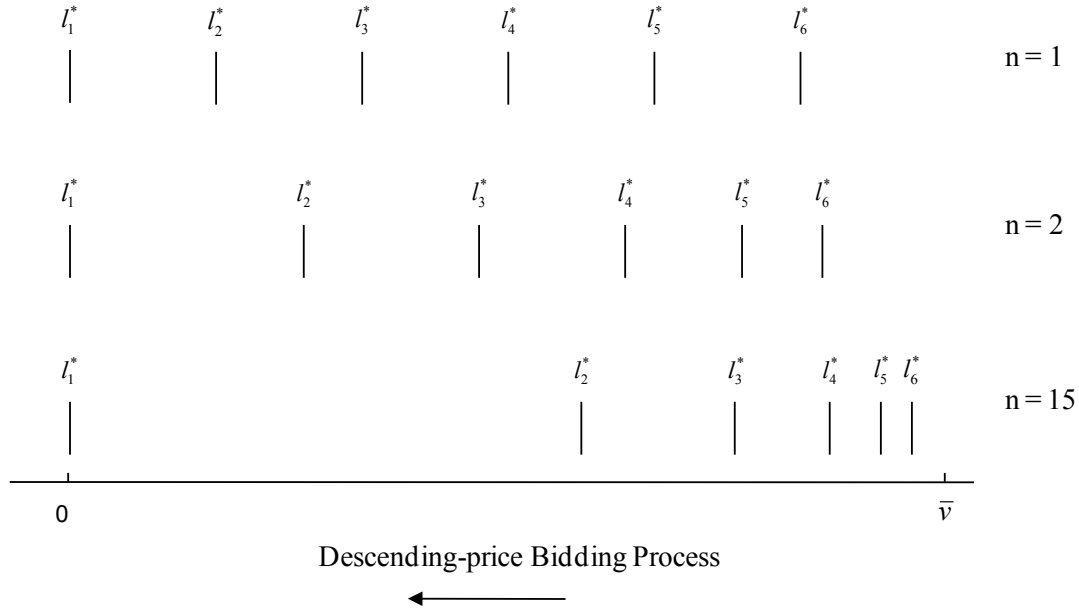


Figure 3.2. Optimal bid levels with  $m = 6$ .

The following observation is made to pave the way for presenting Proposition 3.5, whose proof can be found in Appendix E. Recall that  $c_i$  is defined in (3.6),  $i = 1, 2, \dots, m, m + 1$ .

**Lemma 3.1:** Given  $n$ ,  $c_1, c_2, \dots$  is an increasing series with decreasing increments.

**Proposition 3.5:** Given  $n$ ,  $l_m^*$  is an increasing function of  $m$ .

Proposition 3.5, whose proof is given in Appendix F, reveals the relationship between the highest optimal bid level and the number of bid levels. Assuming that  $n = 5$  bidders participate in a Dutch auction, I graphically illustrate the optimal solutions for  $m = 2, 3, 4, 5$  bid levels in Figure 3.3 with the right most vertical bar in each of the four charts representing the highest

optimal bid level. Notice that  $l_m^*$  with a larger  $m$  is closer to  $\bar{v}$ , which is the upper bound of any bidder's valuation, than its counterpart with a smaller  $m$ .

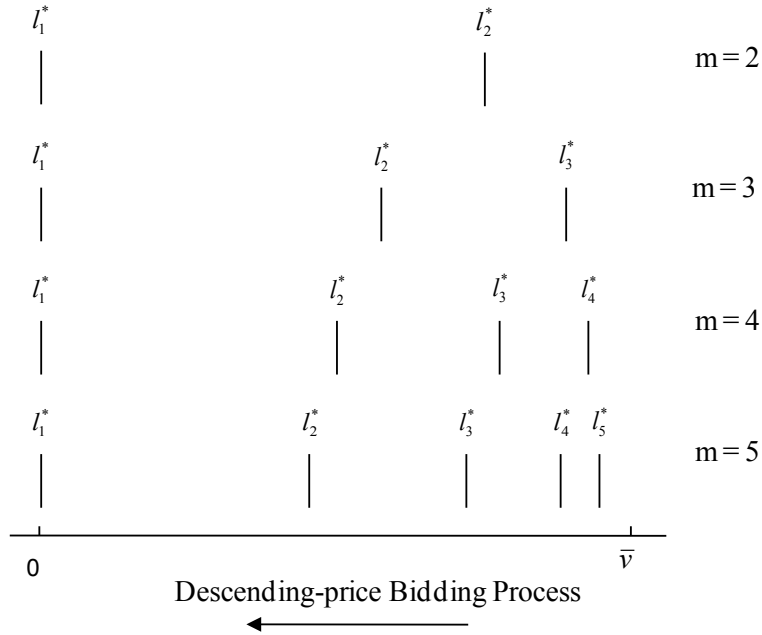


Figure 3.3. Optimal bid levels with  $n = 5$ .

### 3.5 Properties of Maximum Expected Revenue

Given the existence of a global optimal solution to the NLP in (3.5) and the ability to establish the revenue-maximization bid levels, I now direct the attention to two other critical issues about the Dutch auction model being studied: (1) up to how much can the auctioneer expect to make and (2) how does the maximum expected revenue change as the Dutch auction parameters vary? The answer to the first question can be found in Proposition 3.6 below, where a closed-form expression is provided for computing the maximum expected revenue

$$Z^* = \sum_{i=1}^m l_i^* P(l_i^*) \text{ and its proof is given in Appendix G.}$$

**Proposition 3.6:**  $Z^* = \frac{n}{n+1} l_m^* = \frac{n}{n+1} \frac{c_m}{c_{m+1}} \bar{v}$ .

Proposition 3.6 suggests that  $Z^*$  is no greater than  $l_m^* = \frac{c_m}{c_{m+1}} \bar{v}$  since  $\frac{n}{n+1} < 1$ , but the former will approach the latter as  $n$  increases. In particular, I have  $\lim_{n \rightarrow \infty} Z^* = \lim_{n \rightarrow \infty} \frac{n}{n+1} l_m^* \rightarrow l_m^*$ , which means that  $Z^*$  will be practically equal to  $l_m^*$  if  $n$  is sufficiently large. This appears to follow conventional wisdom since more bidders involved in the Dutch auction will create a more competitive environment, in which there is a better chance for someone to claim the object at an early stage of the bidding process when the asking price is high. Consequently, the closing price (hence the auctioneer's revenue) is closer to the highest optimal bid level.

In response to the second question posed above, I will seek to explore the relationships among  $Z^*$ ,  $m$ , and  $n$ . To begin, I see from Proposition 3.7 below that the maximum expected revenue increases with the number of auction participants. This is can be easily shown by

observing that  $l_m^* = \frac{c_m}{c_{m+1}} \bar{v}$  is an increasing function of  $n$  according to Propositions 3.2 and 3.4,

and  $\frac{n}{n+1}$  also increases with  $n$ .

**Proposition 3.7:** Given  $m$ ,  $Z^*$  is an increasing function of  $n$ .

Proposition 3.8 below says that the maximum expected revenue to be generated from the Dutch auction model increases with the number of bid levels. This follows immediately from Proposition 3.5 that  $l_m^*$  is an increasing function of  $m$  and from Proposition 3.6 that  $Z^* = \frac{n}{n+1} l_m^*$ .

**Proposition 3.8:** Given  $n$ ,  $Z^*$  is an increasing function of  $m$ .

This theorem makes intuitive sense because setting more bid levels within the common range of bidders' valuations  $[0, \bar{v}]$  can effectively squeeze more revenue out of them. However, the revenue is increasing at a diminishing rate. This is because, according to Lemma 3.1,

$c_1, c_2, \dots$  is an increasing series with decreasing increments, which implies that  $l_m^* = \frac{c_m}{c_{m+1}} \bar{v}$

increases with  $m$  at a decreasing rate for a fixed  $n$ . Consequently,  $Z^* = \frac{n}{n+1} \frac{c_m}{c_{m+1}} \bar{v}$  is an

increasing function of  $m$  but the increment slows down as the number of bid levels to be set increases.

### 3.6 A Comparative Study

In this section, two commonly-used heuristic strategies for setting bid levels are examined and compared with the optimal strategy  $(l_1^*, l_2^*, \dots, l_m^*)$  with increasing decrements to demonstrate the superiority of the latter over the former in the Dutch auction model. The first one, Strategy E, involves evenly-spaced bid levels with equal decrements and the other, Strategy D, uses unevenly-spaced bid levels with decreasing decrements as the asking price goes down.

To justify the comparative study, I note that the heuristic strategy with equal decrements has been considered by auction researchers (Schill, 1977; Yu, 1999). On the other hand, large auction houses such as Christie's and Sotheby's often increase the current asking price by around 5% as the next asking price in English auctions to speed up the bidding process, which is equivalent to setting bid levels with decreasing decrements in the Dutch auction (Yamey, 1972).

### 3.6.1 Strategy E

The first heuristic strategy calls for dividing the difference between the ceiling and the floor of the common range of bidders' valuations by the number of bid levels and the result is used as the fixed gap between two successive prices to be asked by the auctioneer. In terms of the Dutch auction previously discussed, such a rule translates into

$$l_{E,i} = \frac{(i-1)\bar{v}}{m}, i = 1, 2, \dots, m \quad (3.7)$$

where  $l_{E,i}$  is the  $i^{\text{th}}$  discrete bid level with  $l_{E,1} = 0$ . The bid levels resulting from this strategy with  $m = 5$  are graphically illustrated in Figure 3.4 below.

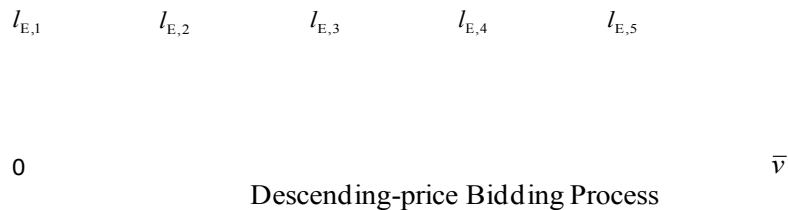


Figure 3.4. Evenly-spaced bid levels with fixed decrements and  $m = 5$ .

Let  $Z_E$  be the expected revenue the auctioneer will receive by following Strategy E. I then see from the objective function in (3.5) that

$$P(l_{E,i}) = \frac{l_{E,i+1}^n - l_{E,i}^n}{\bar{v}^n} \quad (3.8)$$

and

$$\begin{aligned} Z_E &= \sum_{i=1}^m \frac{l_{E,i}}{\bar{v}^n} (l_{E,i+1}^n - l_{E,i}^n) \\ &= \sum_{i=2}^m \frac{l_{E,i}}{\bar{v}^n} (l_{E,i+1}^n - l_{E,i}^n) \\ &= \sum_{i=2}^m \frac{1}{\bar{v}^n} \frac{(i-1)\bar{v}}{m} \left[ \left( \frac{i\bar{v}}{m} \right)^n - \left( \frac{(i-1)\bar{v}}{m} \right)^n \right] \\ &= \frac{\bar{v}}{m^{n+1}} \sum_{i=2}^m (i-1) \left[ i^n - (i-1)^n \right] \end{aligned} \quad (3.9)$$

### 3.6.2 Strategy D

The second heuristic strategy may be better understood by referring to Figure 3.5, where  $m = 5$  bid levels with decreasing decrements are to be considered in the Dutch auction. Let  $l_{D,i}$  be the  $i^{\text{th}}$  bid level,  $d > 0$  be the first (and largest) decrement from  $l_{D,m+1}$  to  $l_{D,m}$ , and the discounting rate (i.e.,  $\frac{l_{D,i} - l_{D,i-1}}{l_{D,i+1} - l_{D,i}}$ ) be  $\frac{1}{1+r}$  with the discounting factor  $r > 0$ ,  $i = 2, 3, \dots, m$ .

Thus, the decrement between any two successive bid prices may be written as

$$l_{D,i+1} - l_{D,i} = \frac{d}{(1+r)^{m-i}}.$$



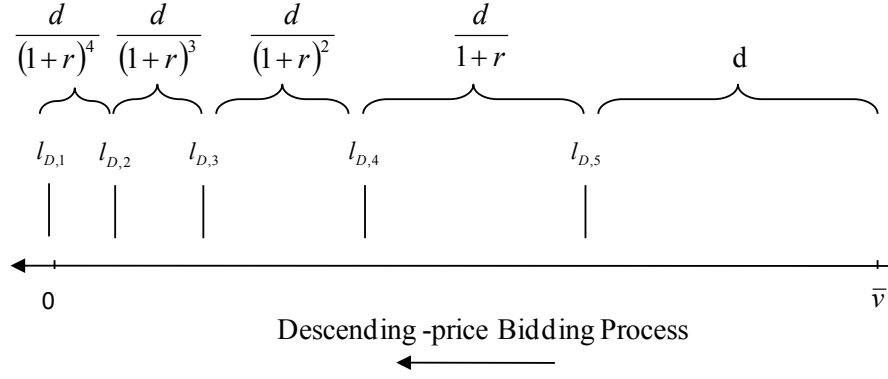


Figure 3.5. Unevenly-spaced bid levels with decreasing decrements and  $m = 5$ .

Since  $\sum_{i=1}^m (l_{D,i+1} - l_{D,i}) = \bar{v}$ , I have  $\sum_{i=0}^{m-1} \frac{d}{(1+r)^i} = \bar{v}$  and  $d = \frac{\bar{v}}{\sum_{i=0}^{m-1} (1+r)^{-i}}$ . Consequently,

$l_{D,1} = 0$  and  $l_{D,i}$ ,  $i = 2, \dots, m$ , can be written as

$$\begin{aligned}
 l_{D,i} &= \bar{v} - d \sum_{j=0}^{m-i} \frac{1}{(1+r)^j} \\
 &= \bar{v} - \frac{\sum_{j=0}^{m-i} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \bar{v} \\
 &= \frac{\sum_{j=m-i+1}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \bar{v}
 \end{aligned} \tag{3.10}$$

Let  $Z_D$  be the expected revenue the auctioneer will receive by following Strategy D. I can see from the objective function in (3.5) that

$$\begin{aligned}
Z_D &= \sum_{i=1}^m \frac{l_{D,i}}{\bar{v}^n} (l_{D,i+1}^n - l_{D,i}^n) \\
&= \sum_{i=2}^m \frac{l_{D,i}}{\bar{v}^n} (l_{D,i+1}^n - l_{D,i}^n) \\
&= \bar{v} \sum_{i=2}^m \frac{\sum_{j=m-i+1}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \left[ \left( \frac{\sum_{j=m-i}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \right)^n - \left( \frac{\sum_{j=m-i+1}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \right)^n \right]
\end{aligned} \tag{3.11}$$

### 3.6.3 Revenue Comparison

While it is trivial to see that the optimal strategy  $(l_1^*, l_2^*, \dots, l_m^*)$  outperforms both Strategy E and Strategy D with respect to the auctioneer's expected revenue, it is unclear which of the two heuristic rules leads to a higher final selling price on average in the Dutch auction model. To address this issue, I will make the following three observations and their proofs are presented in Appendices H, I and J, respectively.

$$\text{Let } t \geq 0, \tau_i = \frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}} \text{ and } \mu_i = \frac{\sum_{j=m-i+1}^{m-1} j(1+t)^{-j-1}}{\sum_{j=0}^{m-1} j(1+t)^{-j-1}}, i = 2, 3, \dots, m. \text{ It can be shown that}$$

**Lemma 3.2:**  $\tau_i < \mu_i, i = 2, 3, \dots, m.$

**Lemma 3.3:**  $\tau_i$  is a decreasing function of  $t, i = 2, 3, \dots, m.$

**Lemma 3.4:**  $\frac{\partial \tau_{i+1}}{\partial t} < \frac{\partial \tau_i}{\partial t} < 0, i = 2, 3, \dots, m.$

What follows is an important result on the revenue comparison between the two heuristic strategies being examined. I defer its proof to Appendix K.

**Proposition 3.9:**  $H_t = \bar{v} \sum_{i=2}^m \tau_i (\tau_{i+1}^n - \tau_i^n)$  is a decreasing function of  $t$ ,  $t \geq 0$ .

A close look at the above statement indicates that  $H_t = Z_D$  if  $t > 0$ . In case  $t = 0$ , however, one has  $\tau_i = \frac{i-1}{m}$ ,  $i = 2, 3, \dots, m$ , and  $H_t = \frac{\bar{v}}{m^{n+1}} \sum_{i=2}^m (i-1) [i^n - (i-1)^n] = Z_E$ . Since  $H_t$  decreases with  $t$ , I have  $Z_D < Z_E$ . This leads directly to the following theorem, where  $Z^*$  is based on  $(l_1^*, l_2^*, \dots, l_m^*)$  and the equality holds only if  $n = 1$ , in which case  $l_i^*$  are evenly spaced according to Proposition 3.3.

**Corollary 3.2:**  $Z^* \geq Z_E > Z_D$

It should be pointed out that although the highest possible expected revenue may be achieved by applying the optimal strategy developed in Section 3.3, establishing the optimal bid levels  $l_1^*, l_2^*, \dots, l_m^*$  may require a lot of computational efforts. In contrast, both Strategy E and Strategy D are much easier to implement, but the resulting final selling price of the object may be lower.

### 3.7 Numerical Illustrations

In this section, I design and solve a set of numerical examples of the Dutch auction problem under consideration to illustrate some of the major results that have been obtained thus far. The possible values to be taken on by the three key model parameters are specified as follows:  $\bar{v} = 10$ ,  $m \in \{3, 4, \dots, 17\}$  and  $n \in \{1, 2, 4, 10, 20\}$ . In light of (3.5), various nonlinear programs based on different combinations of  $m$  and  $n$  with  $\bar{v} = 10$  are set up. Solver, which is a Microsoft Excel add-in, is invoked to find the optimal solution to each of the  $1 \times 15 \times 5 = 75$  mathematical programming models. As an example, the NLP with  $\bar{v} = 10$ ,  $m = 3$  and  $n = 3$  is shown below with  $l_1$ ,  $l_2$ , and  $l_3$  being the decision variables and, by assumption,  $l_1 = 0$  and  $l_{m+1} = l_4 = \bar{v} = 10$ :

$$\begin{aligned} \text{Maximize} \quad & Z = \frac{1}{1,000} (l_1 l_2^3 - l_1^4 + l_2 l_3^3 - l_2^4 + 1,000 l_3 - l_3^4) \\ \text{subject to:} \quad & l_2 \geq l_1 \\ & l_3 \geq l_2 \\ & l_1 = 0 \end{aligned}$$

The results obtained are analyzed to help explain the properties of the optimal bid levels as well as the maximum expected revenue presented in the previous sections.

For instance, the optimal bid levels for NLPs with  $\bar{v} = 10$ ,  $m = 12$  and  $n = 1, 2, 4, 10, 20$  are displayed in Table 3.1 and the continuous versions of them are plotted in Figure 3.6. One can see from the two exhibits that  $l_i^*$  is moving farther away from  $l_{i+1}^*$  as the asking price goes down from right to left (in Table 3.1) or from top to bottom (in Figure 3.6) when  $n \geq 2$ ,  $i = 1, 2, \dots, 11$ . This observation concurs with the bid decrements shown in Table 3.2. As expected, the optimal bid levels are evenly spaced in the case of  $n = 1$ . All of these are consistent with Proposition 3.3.

It is also interesting to note from Table 3.1 and Figure 3.6 that  $l_i^*$  becomes larger as  $n$  increases from 1 to 2 to 4 to 10 to 20,  $i = 2, 3, \dots, 12$ , which is in line with Proposition 3.4. Additionally, a close look at Figure 3.6 reveals that each of the  $l_i^*$  curves tends to flatten out as  $n$  increases, which implies that  $l_i^*$  is an increasing function of  $n$  at a diminishing rate.

The highest optimal bid levels  $l_m^*$  for various combinations of  $m$  and  $n$  values with  $\bar{v} = 10$  are summarized in Tables 3.3 and the continuous versions of them are depicted in Figure 3.7. It is seen from the figure that  $l_m^*$  increases with  $m$  and  $n$ , which is in agreement with Propositions 3.4 and 3.5. Clearly,  $l_m^*$  will be closer to the highest bidder's valuation as both  $n$  and  $m$  increase.

Table 3.1

*Optimal Bid Levels with  $\bar{v} = 10$  and  $m = 12$*

n	$l_1^*$	$l_2^*$	$l_3^*$	$l_4^*$	$l_5^*$	$l_6^*$	$l_7^*$	$l_8^*$	$l_9^*$	$l_{10}^*$	$l_{11}^*$	$l_{12}^*$
1	0	0.8333	1.6666	2.4999	3.3333	4.1666	4.9999	5.8332	6.6666	7.4999	8.3332	9.1665
2	0	1.5769	2.7313	3.7103	4.5859	5.3909	6.1434	6.8547	7.5325	8.1823	8.8081	9.4132
4	0	2.8553	4.2697	5.2723	6.0732	6.7518	7.3469	7.8809	8.3679	8.8174	9.2361	9.6291
10	0	5.1110	6.4960	7.2816	7.8345	8.2642	8.6176	8.9189	9.1823	9.4168	9.6286	9.8219
20	0	6.7405	7.8488	8.3932	8.7496	9.0143	9.2249	9.4001	9.5501	9.6815	9.7985	9.9039

Table 3.2

*Optimal Bid Decrements with  $\bar{v} = 10$  and  $m = 12$*

n	$l_2^* - l_1^*$	$l_3^* - l_2^*$	$l_4^* - l_3^*$	$l_5^* - l_4^*$	$l_6^* - l_5^*$	$l_7^* - l_6^*$	$l_8^* - l_7^*$	$l_9^* - l_8^*$	$l_{10}^* - l_9^*$	$l_{11}^* - l_{10}^*$	$l_{12}^* - l_{11}^*$
1	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333	0.8333
2	1.5769	1.1544	0.9790	0.8756	0.8050	0.7525	0.7113	0.6778	0.6498	0.6259	0.6051
4	2.8553	1.4143	1.0027	0.8009	0.6785	0.5951	0.5340	0.4870	0.4495	0.4187	0.3929
10	5.1110	1.3850	0.7856	0.5529	0.4297	0.3534	0.3013	0.2634	0.2345	0.2118	0.1933
20	6.7405	1.1083	0.5444	0.3564	0.2646	0.2106	0.1751	0.1501	0.1314	0.1170	0.1055

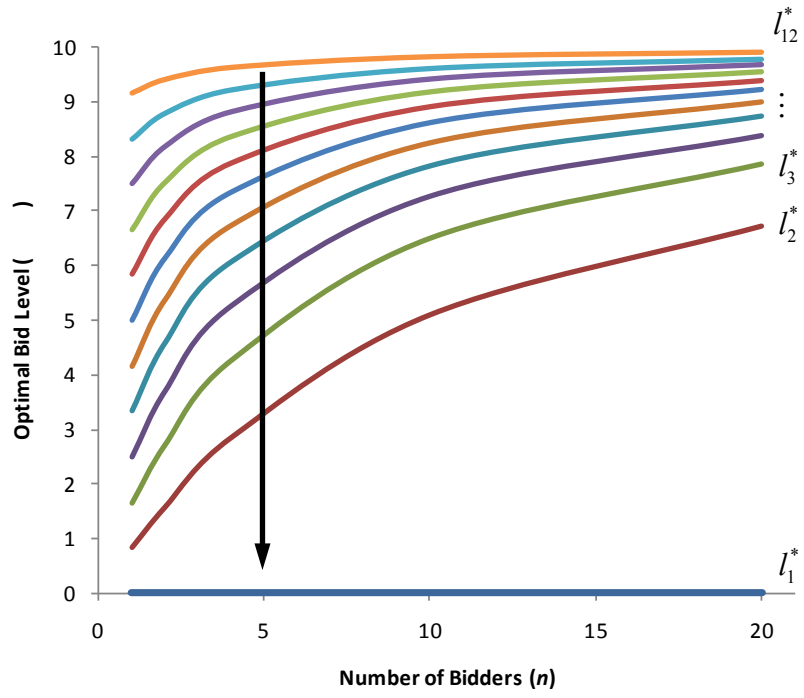


Figure 3.6. Optimal bid levels versus number of bidders with  $\bar{v} = 10$  and  $m = 12$ .

Table 3.3

Highest Optimal Bid Level versus Number of Bidders and Number of Bid Levels with  $\bar{v} = 10$

$n \backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	6.6667	7.5000	8.0000	8.3333	8.5714	8.7500	8.8889	9.0000	9.0909	9.1666	9.2307	9.2857	9.3333	9.3750	9.4117
2	7.3615	8.0906	8.5068	8.7751	8.9629	9.1002	9.2059	9.2943	9.3572	9.4132	9.4603	9.5013	9.5371	9.5766	9.5915
4	8.0983	8.6812	8.9950	9.1900	9.3224	9.4180	9.4902	9.5467	9.5919	9.6291	9.6600	9.6863	9.7088	9.7283	9.7453
10	8.9211	9.2943	9.4800	9.5899	9.6622	9.7132	9.7510	9.7801	9.8032	9.8219	9.8374	9.8505	9.8616	9.8712	9.8796
20	9.3513	9.5927	9.7064	9.7717	9.8137	9.8429	9.8643	9.8806	9.8935	9.9039	9.9125	9.9197	9.9258	9.9310	9.9356

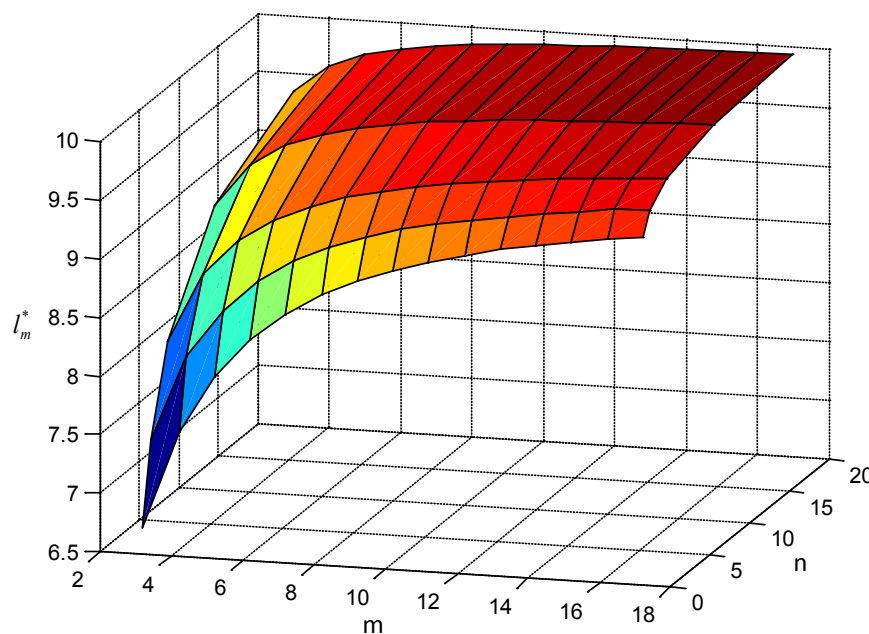


Figure 3.7. Highest optimal bid level versus number of bidders and number of bid levels with  $\bar{v} = 10$ .

The maximum expected revenues with various  $m$  and  $n$  values are shown in Table 3.4 and the continuous versions of them are plotted in Figure 3.8. One can tell from the two exhibits that  $Z^*$  increases with both the number of auction participants and the number of bid levels, which are consistent with Propositions 3.7 and 3.8. However, a careful examination of Figure 3.8 shows that the  $Z^*$  curves becomes flat as  $n$  and  $m$  approaches  $\infty$ . It is evident from Tables 3.5 and 3.6 that while  $Z^*$  increases with  $m$  and  $n$ , the increasing rate is decreasing.

In particular, Figure 3.8 shows that  $Z^*$  increases quickly with the number of bidders when  $n \leq 5$ , but the improvement slows down when  $n \geq 10$ . The implication is that there is a trade-off between making additional efforts to attract more bidders and achieving higher expected revenues. Thus, prudent decisions must be made by the auctioneer to strike a delicate balance between the marginal cost and the incremental benefit when designing Dutch auctions.



Table 3.4

Maximum Expected Revenue with  $\bar{v} = 10^*$

n	$Z_{m=3}^*$	$Z_{m=4}^*$	$Z_{m=5}^*$	$Z_{m=6}^*$	$Z_{m=7}^*$	$Z_{m=8}^*$	$Z_{m=9}^*$	$Z_{m=10}^*$	$Z_{m=11}^*$	$Z_{m=12}^*$	$Z_{m=13}^*$	$Z_{m=14}^*$	$Z_{m=15}^*$	$Z_{m=16}^*$	$Z_{m=17}^*$
1	3.3333	3.7500	4.0000	4.1667	4.2857	4.3750	4.4444	4.5000	4.5455	4.5833	4.6154	4.6429	4.6667	4.6875	4.7059
2	4.9077	5.3937	5.6712	5.8501	5.9749	6.0668	6.1373	6.1930	6.2382	6.2755	6.3069	6.3336	6.3566	6.3766	6.3944
4	6.4786	6.9450	7.1960	7.3520	7.4579	7.5344	7.5922	7.6373	7.6736	7.7033	7.7280	7.7490	7.7670	7.7826	7.7963
10	8.1101	8.4494	8.6182	8.7181	8.7838	8.8302	8.8645	8.8910	8.9120	8.9290	8.9432	8.9550	8.9651	8.9739	8.9815
20	8.9060	9.1359	9.2442	9.3064	9.3464	9.3742	9.3946	9.4102	9.4224	9.4323	9.4405	9.4474	9.4532	9.4582	9.4625

Table 3.5

Maximum Expected Revenue Increasing Rate between Bid Levels with  $\bar{v} = 10^{**}$

n	$\frac{Z_{m=4}^* - Z_{m=3}^*}{Z_{m=3}^*}$	$\frac{Z_{m=5}^* - Z_{m=4}^*}{Z_{m=4}^*}$	$\frac{Z_{m=6}^* - Z_{m=5}^*}{Z_{m=5}^*}$	$\frac{Z_{m=7}^* - Z_{m=6}^*}{Z_{m=6}^*}$	$\frac{Z_{m=8}^* - Z_{m=7}^*}{Z_{m=7}^*}$	$\frac{Z_{m=9}^* - Z_{m=8}^*}{Z_{m=8}^*}$	$\frac{Z_{m=10}^* - Z_{m=9}^*}{Z_{m=9}^*}$	$\frac{Z_{m=11}^* - Z_{m=10}^*}{Z_{m=10}^*}$	$\frac{Z_{m=12}^* - Z_{m=11}^*}{Z_{m=11}^*}$	$\frac{Z_{m=13}^* - Z_{m=12}^*}{Z_{m=12}^*}$	$\frac{Z_{m=14}^* - Z_{m=13}^*}{Z_{m=13}^*}$	$\frac{Z_{m=15}^* - Z_{m=14}^*}{Z_{m=14}^*}$	$\frac{Z_{m=16}^* - Z_{m=15}^*}{Z_{m=15}^*}$	$\frac{Z_{m=17}^* - Z_{m=16}^*}{Z_{m=16}^*}$
	1	12.50%	6.67%	4.17%	2.86%	2.08%	1.59%	1.25%	1.01%	0.83%	0.70%	0.60%	0.51%	0.45%
2	9.90%	5.14%	3.15%	2.13%	1.54%	1.16%	0.91%	0.73%	0.60%	0.50%	0.42%	0.36%	0.31%	0.28%
4	7.20%	3.62%	2.17%	1.44%	1.03%	0.77%	0.59%	0.47%	0.39%	0.32%	0.27%	0.23%	0.20%	0.18%
10	4.18%	2.00%	1.16%	0.75%	0.53%	0.39%	0.30%	0.24%	0.19%	0.16%	0.13%	0.11%	0.10%	0.08%
20	2.58%	1.19%	0.67%	0.43%	0.30%	0.22%	0.17%	0.13%	0.11%	0.09%	0.07%	0.06%	0.05%	0.05%

\*  $Z_{n=a}^*$  denotes the maximum expected revenue derived with  $a$  bidders in the Dutch auction.

\*\*  $Z_{m=b}^*$  denotes the maximum expected revenue with  $b$  bid levels in the Dutch auction.

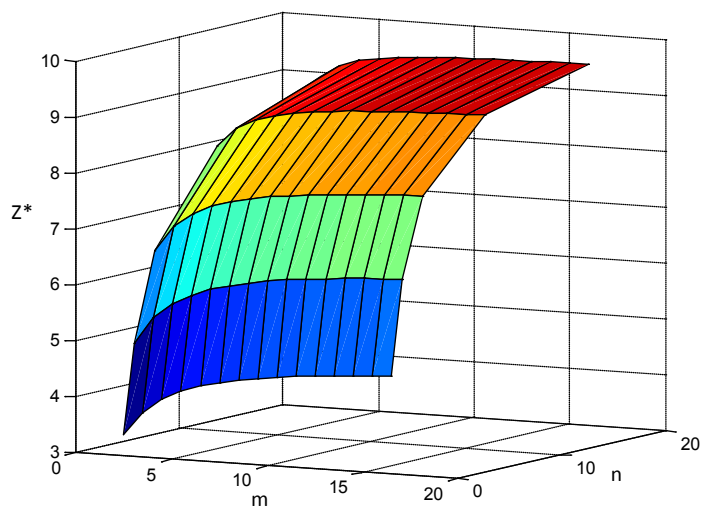


Figure 3.8. Maximum expected revenue versus number of bidders and number of bid levels with  $\bar{v} = 10$ .

Table 3.6

Maximum Expected Revenue Increasing Rate between Numbers of Bidders with  $\bar{v} = 10^*$

m	$\frac{Z_{n=2}^* - Z_{n=1}^*}{Z_{n=1}^*}$	$\frac{Z_{n=4}^* - Z_{n=2}^*}{Z_{n=2}^*}$	$\frac{Z_{n=10}^* - Z_{n=4}^*}{Z_{n=4}^*}$	$\frac{Z_{n=20}^* - Z_{n=10}^*}{Z_{n=10}^*}$
	3	47.23%	32.01%	25.18%
4	43.83%	28.76%	21.66%	8.13%
5	41.78%	26.89%	19.76%	7.26%
6	40.40%	25.67%	18.58%	6.75%
7	39.41%	24.82%	17.78%	6.40%
8	38.67%	24.19%	17.20%	6.16%
9	38.09%	23.71%	16.76%	5.98%
10	37.62%	23.32%	16.41%	5.84%
11	37.24%	23.01%	16.14%	5.73%
12	36.92%	22.75%	15.91%	5.64%
13	36.65%	22.53%	15.72%	5.56%
14	36.42%	22.35%	15.56%	5.50%
15	36.21%	22.19%	15.43%	5.44%
16	36.03%	22.05%	15.31%	5.40%
17	35.88%	21.92%	15.20%	5.36%

\*  $Z_{n=a}^*$  denotes the maximum expected revenue derived with  $a$  bidders in the Dutch auction.

What is depicted in Figure 3.9 is a graph of the maximum expected revenue as a percentage of the highest optimal level for  $m = 3, 4, \dots, 17$  with different values of  $n$ . As it turns out, the 15 nonlinear curves overlap and it can be shown easily that the best-fit function for them is  $y = \frac{n}{n+1}x$ . This does not come as a surprise since I know from Proposition 3.6 that the ratio of  $Z^*$  to  $l_m^*$  is  $\frac{n}{n+1}$  and it approaches 100% with an extremely large  $n$ . In other words, the maximum expected revenue is always less than the highest optimal bid level, but the gap between them narrows as more bidders participate in the Dutch auction. Notice also that the curves tend to become flat as  $n$  approaches  $\infty$ .

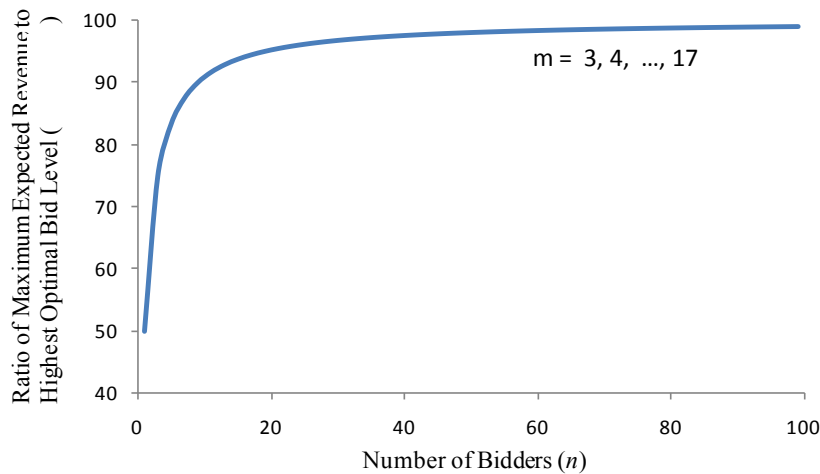


Figure 3.9. Maximum expected revenue as a percentage of highest optimal bid level with  $\bar{v} = 10$ .

Finally, I conclude this section with a numerical analysis of the three approaches to setting discrete bid levels in Dutch auctions discussed in Section 3.6: the optimal strategy,

Strategy E and Strategy D. The values to be taken on by each of the three key model parameters are:  $\bar{v} = 10$ ,  $m \in \{3, 5, 9, 13, 17\}$  and  $n \in \{1, 2, 4, 10, 20\}$ .

Suppose that  $\bar{v} = 10$ ,  $m = 5$  and  $n = 2$ . Based on Strategy E, the five bid levels obtained in (3.7) are  $l_{E,1} = \frac{(1-1) \times 10}{5} = 0$ ,  $l_{E,2} = \frac{(2-1) \times 10}{5} = 2$ ,  $l_{E,3} = \frac{(3-1) \times 10}{5} = 4$ ,  $l_{E,4} = \frac{(4-1) \times 10}{5} = 6$ , and  $l_{E,5} = \frac{(5-1) \times 10}{5} = 8$ . Additionally, I see from (3.8) that the respective probabilities of selling the object at those five prices are  $P(l_{E,1}) = (2^2 - 0^2)/100 = 0.04$ ,  $P(l_{E,2}) = (4^2 - 2^2)/100 = 0.12$ ,  $P(l_{E,3}) = (6^2 - 4^2)/100 = 0.2$ ,  $P(l_{E,4}) = (8^2 - 6^2)/100 = 0.28$ , and  $P(l_{E,5}) = (10^2 - 8^2)/100 = 0.36$ . The auctioneer is expected to receive an average revenue of  $Z_E = 0(0.04) + 2(0.12) + 4(0.2) + 6(0.28) + 8(0.36) = 5.6$ .

However, based on Strategy D with  $r = 25\%$ , it can be easily seen that the five bid levels obtained in (3.10) are

$$l_{D,1} = 0$$

$$l_{D,2} = \frac{\sum_{j=4}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 \approx 1.2185$$

$$l_{D,3} = \frac{\sum_{j=3}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 \approx 2.7416$$

$$l_{D,4} = \frac{\sum_{j=2}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 \approx 4.6454$$

$$l_{D,5} = \frac{\sum_{j=1}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 \approx 7.0252$$

The probabilities of selling the object at those five prices are  $P(l_{D,1}) = (1.2185^2 - 0^2)/100 \approx 0.0149$ ,  $P(l_{D,2}) = (2.7416^2 - 1.2185^2)/100 \approx 0.0603$ ,  $P(l_{D,3}) = (4.6454^2 - 2.7416^2)/100 \approx 0.1406$ ,  $P(l_{D,4}) = (7.0252^2 - 4.6454^2)/100 \approx 0.2777$ , and  $P(l_{D,5}) = (10^2 - 7.0252^2)/100 \approx 0.5065$ . Thus, the auction revenue is  $Z_D = 0(0.0148) + 1.22(0.0603) + 2.74(0.1406) + 4.65(0.2777) + 7.03(0.5065) \approx 5.3073$ .

These along with the results for each of the other  $1 \times 5 \times 5 - 1 = 24$  problem instances are summarized in Table 3.7 and the continuous versions of them are plotted in Figure 3.10. In terms of expected revenue, one sees from the two exhibits that the optimal strategy outperforms Strategy E except in the case of  $n = 1$ , when there is a tie between them. In turn, Strategy E outperforms Strategy D. All of these observations are consistent with the findings reported in Proposition 3.9.

It is also seen from Figure 3.10 that the discrepancies among the three sets of expected revenues are more significant when the number of bid levels is small while the number of auction participants is large. When  $n$  is small and  $m$  is large, however, the differences are almost negligible.

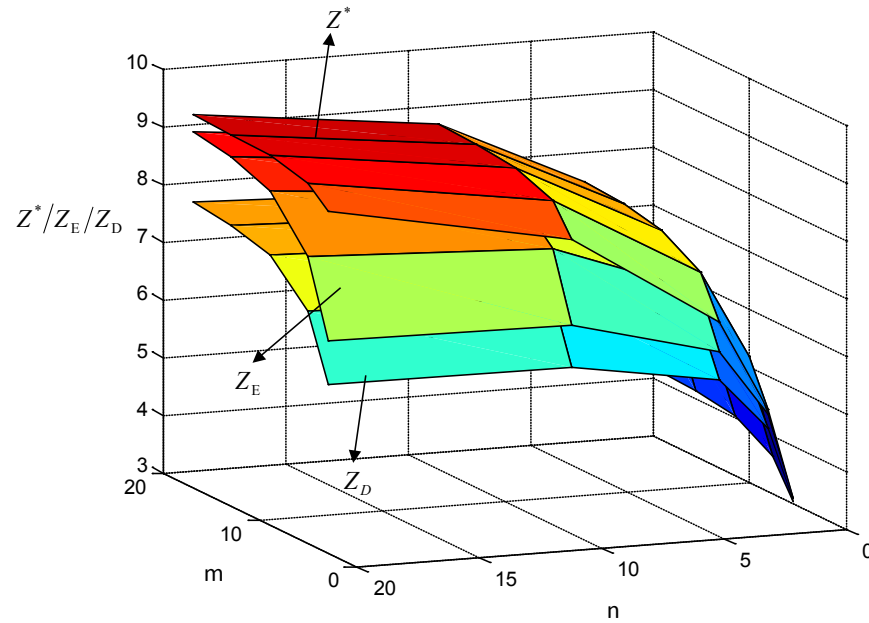


Figure 3.10. Expected revenues based on three different design strategies with  $\bar{v} = 10$ .

Table 3.7

Expected Revenues Based on Three Different Design Strategies with  $\bar{v} = 10$

n \ m	3			5			9			13			17		
	$Z^*$	$Z_E$	$Z_D$	$Z^*$	$Z_E$	$Z_D$	$Z^*$	$Z_E$	$Z_D$	$Z^*$	$Z_E$	$Z_D$	$Z^*$	$Z_E$	$Z_D$
1	3.3333	3.3333	3.2787	4.0000	4.0000	3.9029	4.4444	4.4444	4.2722	4.6154	4.6154	4.3798	4.7059	4.7059	4.4188
2	4.9077	4.8148	4.5792	5.6712	5.6000	5.3073	6.1373	6.0905	5.7163	6.3069	6.2722	5.8325	6.3944	6.3668	5.8743
4	6.4786	5.9671	5.4915	7.1960	6.8672	6.3480	7.5922	7.4033	6.8167	7.7280	7.5957	6.9478	7.7963	7.6944	6.9947
10	8.1101	6.6088	5.8848	8.6182	7.7729	6.9546	8.8645	8.4340	7.5492	8.9432	8.6573	7.7146	8.9815	8.7681	7.7737
20	8.9060	6.6657	5.9016	9.2442	7.9769	7.0232	9.3946	8.7759	7.6803	9.4405	9.0438	7.8690	9.4625	9.1731	7.9370

### 3.8 Simulation Analysis

Throughout Section 3.7, various NLPs are set up to model Dutch auctions with different  $m$  and  $n$  values while  $\bar{v}$  is fixed at 10. Solver is then run to determine the optimal bid levels  $(l_1^*, l_2^*, \dots, l_m^*)$  and the maximum expected revenue  $Z^*$ . In this section, the aim is to find  $Z^*$  by taking a simulation approach provided that  $(l_1^*, l_2^*, \dots, l_m^*)$  is given.

I begin by specifying the values to be taken on by each of the three key parameters as follows:  $\bar{v} = 10$ ,  $m \in \{6, 9, 12\}$ , and  $n \in \{1, 2, 4, 10, 20\}$ . For each combination of  $m$  and  $n$ , an NLP based on (3.5) is formulated to model the particular Dutch auction. It is subsequently solved to determine  $(l_1^*, l_2^*, \dots, l_m^*)$  and  $Z^*$  with the aid of Solver. As an alternative approach, MATLAB is employed to generate  $n$  values of the random variable subject to the uniform distribution  $U(0, 10)$ , to simulate the bids submitted by the auction participants based on their valuations. If  $k$  is the largest index such that the highest of the  $n$  bids is greater than or equal to  $l_k^*$ , then the final selling price of the object is set to be equal to  $l_k^*$ . Based on  $T = 100,000$  replications of the simulation experiment,  $Z^*$  is computed as the average selling price. Figure 3.11 shows a flow chart of the simulation process in which  $t$  is a counter. The two maximum expected revenues based on the optimization model and the simulation approach, respectively, are compared and the differences are calculated.

The process described above is repeated for each of the other  $3 \times 5 - 1 = 14$  pairs of  $m$  and  $n$ , and the findings are summarized in Table 3.8. As evidenced by the minor differences due to rounding errors, the simulation-based empirical results lend strong support to the optimization-based theoretical results. Consequently, the validity of the latter is well established.

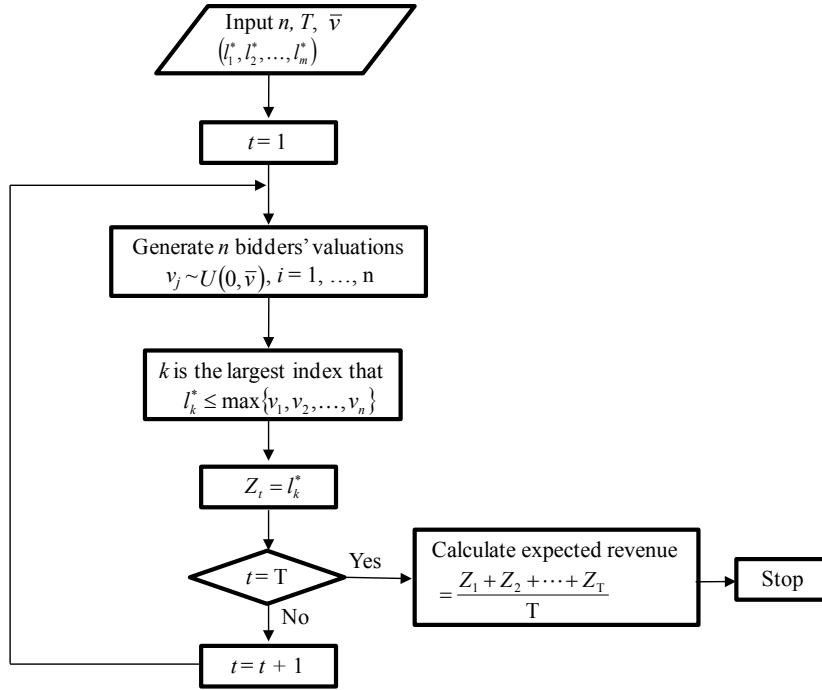


Figure 3.11. Flowchart of simulation process.

Table 3.8

*Maximum Expected Revenues Based on Two Different Approaches*

m	n	Theoretical	Simulation	Error
6	1	4.1667	4.1712	-0.0045
	2	5.8501	5.8503	-0.0002
	4	7.3520	7.3563	-0.0043
	10	8.7181	8.7198	-0.0017
	20	9.3064	9.3070	-0.0006
9	1	4.4444	4.4410	0.0034
	2	6.1373	6.1384	-0.0011
	4	7.5922	7.5976	-0.0054
	10	8.8645	8.8642	0.0003
	20	9.3946	9.3940	0.0006
12	1	4.5833	4.5802	0.0031
	2	6.2755	6.2752	0.0003
	4	7.7033	7.7000	0.0033
	10	8.9290	8.9321	-0.0031
	20	9.4323	9.4329	-0.0006



### 3.9 Conclusions

The main focus of this chapter is on the optimal design of a Dutch auction, where one unit of an object is to be sold in an open-bid, first-price, descending bidding process with discrete bid levels. Given the number of bidders as well as the distribution of their valuations, the objective is to determine the optimal bid levels that maximize the auctioneer's expected revenue. As a departure from the traditional methods used by applied economists and game-theorists, a nonlinear programming approach is taken to solve the problem. I strive to uncover some important properties of the optimal bid levels in addition to deriving closed-form expressions for calculating the optimal bid levels as well as the maximum expected revenue. Finally, a comparative study of the optimal strategy and two heuristics are carried out. The managerial and economic implications of the major findings from this chapter are discussed.

In the next chapter, I will direct the attention to a variant of the current Dutch auction model where the number of auction participants is a random variable rather than a fixed number. Similar results to those obtained in this chapter will be reported and analyzed to assess the impact of the uncertainty in the number of bidders on the Dutch auction outcome.

## CHAPTER 4

### A DUTCH AUCTION MODEL WITH AN UNCERTAIN NUMBER OF BIDDERS

#### 4.1 Introduction

In Chapter 3, a basic Dutch auction model is presented where the number of bidders is fixed and known to the auctioneer. In contrast, this chapter aims to consider a more general setting in which the number of auction participants is random rather than constant. Uncertainty about the size of bidder population is not uncommon in real-world auctioning processes. For instance, the prospect of hundreds of interested buyers arises in a large auto auction where people come and go constantly. The auctioneer has neither knowledge about nor control over the number of potential bidders. Theoretical works on auctions with an uncertain number of bidders include McAfee and McMillan (1987), Matthews (1987) and Harstad et al. (1990) among others. What is shown in Table 4.1 is a sample of studies involving practical applications of the same type of auctions in which the bidders do not know exactly how many competitors they are facing.

Table 4.1

*Auction Studies with an Uncertain Number of Bidders*

Focus of Study	Source
Wholesale used car auctions in Korea	Lee (2006)
Oil leases	Engelbrecht-Wiggans, Dougherty and Lohrenz (1986)
Spectrum auctions	Levin and Ozdenoren (2004)
Vehicle quota system in Singapore	Chou and Parlar (2005)
Governmental project	Paul and Carr (2005)
Online auctions on eBay	Ma, Wang and Lai (2004)

In the rest of this chapter, I make the same assumptions as those in Chapter 3 in setting the optimal bid levels as well as computing the auctioneer's maximum expected revenue. More specifically, it is postulated that rational and risk-neutral bidders have private and independent valuations about the object to be auctioned off. The final winner will be randomly selected if multiple bidders show their willingness to pay the current asking price in a Dutch auction. The only difference is that the number of bidders is stochastic rather than deterministic.

## 4.2 A New Model

As an extension of the basic auction model treated in the previous chapter, I begin by supposing that the number of bidders in the competitive bidding process is a Poisson random

variable; namely, the probability that there are  $n$  participants in the auction is  $P(n) = \frac{\lambda^n e^{-\lambda}}{n!}$ ,  $n =$

$0, 1, 2, \dots$ , where  $\lambda \geq 0$  is the average size of the bidder population during the course of auction.

The same assumption has been made in a number of studies in the existing auction literature (Ma, Wang and Lai, 2004; David et al., 2007; Caldentey and Vulcano, 2007; Gallien and Gupta, 2007) and has been empirically validated by real eBay auctions (Bajari and Hortacısu, 2003).

Recall from (3.5) that the expected revenue from a Dutch auction with  $m$  bid levels and  $n$

bidders is  $\sum_{i=1}^m \frac{l_i}{\bar{v}^n} (I_{i+1}^n - I_i^n)$  if  $v_j \sim U(0, \bar{v})$  with  $l_1 = 0$ ,  $l_{m+1} = \bar{v}$ ,  $F(l_1) = 0$  and  $F(l_{m+1}) = 1$ . Let  $Z_\lambda$

denote the expected revenue to be received from the new Dutch auction considered in this

chapter. It follows that

$$\begin{aligned}
Z_\lambda &= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \sum_{i=1}^m \frac{l_i}{\bar{v}^n} (l_{i+1}^n - l_i^n) \\
&= \sum_{i=1}^m \sum_{n=0}^{\infty} l_i e^{-\lambda} \left[ \frac{1}{n!} \left( \frac{\lambda l_{i+1}}{\bar{v}} \right)^n - \frac{1}{n!} \left( \frac{\lambda l_i}{\bar{v}} \right)^n \right] \\
&= \sum_{i=1}^m l_i e^{-\lambda} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\lambda l_{i+1}}{\bar{v}} \right)^n - \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\lambda l_i}{\bar{v}} \right)^n \right] \\
&= \sum_{i=1}^m l_i e^{-\lambda} \left( e^{\frac{\lambda l_{i+1}}{\bar{v}}} - e^{\frac{\lambda l_i}{\bar{v}}} \right) \tag{4.1}
\end{aligned}$$

Thus, the Dutch auction with an uncertain number of bidders may be formulated as the following NLP with  $l_1, l_2, \dots, l_m$  being the decision variables:

$$\begin{aligned}
\text{Maximize} \quad & Z_\lambda = \sum_{i=1}^m l_i e^{-\lambda} \left( e^{\frac{\lambda l_{i+1}}{\bar{v}}} - e^{\frac{\lambda l_i}{\bar{v}}} \right) \\
\text{subject to:} \quad & l_{i+1} \geq l_i, i = 1, \dots, m \\
& l_1 = 0 \\
& l_{m+1} = \bar{v} \tag{4.2}
\end{aligned}$$

In addition, the probability of selling the object can be computed as follows since no transaction occurs when  $n = 0$ :

$$\begin{aligned}
P_\lambda &= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \sum_{i=1}^m \frac{1}{\bar{v}^n} (l_{i+1}^n - l_i^n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \left( \frac{l_{m+1}^n}{\bar{v}^n} - \frac{l_1^n}{\bar{v}^n} \right) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \left( \frac{\bar{v}^n}{\bar{v}^n} - \frac{0^n}{\bar{v}^n} \right) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \\
&= 1 - e^{-\lambda} \tag{4.3}
\end{aligned}$$

### 4.3 Existence and Properties of Optimal Bid Levels

I start this section by showing the existence of an optimal solution to the new Dutch auction model under consideration. As discussed in Section 3.3, the combination of a convex constraint set and a concave objective function is a sufficient condition for global optimality in a nonlinear mathematical program. Note that all the constraints in (4.2) are identical to those in (3.5), which are convex. Moreover, the concavity of the objective function in (4.2) can be established by showing in Appendix L that the leading principal minor of its Hessian of order  $i$  has the same sign as  $(-1)^i$ . The first key result in this chapter is presented below:

**Proposition 4.1:**  $\sum_{i=1}^m l_i e^{-\lambda} \left( e^{\frac{\lambda_{i+1}}{\bar{v}}} - e^{\frac{\lambda_i}{\bar{v}}} \right)$  is concave in  $l_1, l_2, \dots, l_m$  if  $v_j \sim U(0, \bar{v})$ ,  $j = 1, 2, \dots, n$

In light of the following corollary, which is straightforward and its proof is omitted, one of the popular computer codes may be used to solve (4.2) efficiently and find the optimal bid levels as well as the maximum expected revenue to be received by the auctioneer.

**Corollary 4.1:**  $(l_1^*, l_2^*, \dots, l_m^*)$  exists if  $v_j \sim U(0, \bar{v})$ ,  $j = 1, 2, \dots, n$ .

Next, I shift the focus to uncovering some interesting properties possessed by the optimal bid levels. Toward that end, the theorem below characterizes their distribution pattern and its proof is placed in Appendix M.

**Proposition 4.2:**  $l_{i+1}^* - l_i^* < l_i^* - l_{i-1}^*$ ,  $i = 2, 3, \dots, m$ .

Proposition 4.2 states that when the number of participants in the Dutch auction is Poisson distributed, the bid decrements should be increasing as the asking price goes down step by step. Interestingly, the same conclusion is arrived at by David et al. (2007) in the context of English auctions. Unlike the case with the basic Dutch auction model involving a fixed number of bidders delineated in Chapter 3, however, evenly-spaced bid levels (i.e.,  $l_{i+1}^* - l_i^* = l_i^* - l_{i-1}^*$ ) are no longer optimal when only one bidder is present on average (i.e.,  $\lambda = 1$  instead of  $n = 1$ ). Obviously, randomness in the number of bidders has led to a different design scheme.

The following observation, which is parallel to Proposition 3.4 and whose proof can be found in Appendix N, describes the behaviors of  $l_1^*, l_2^*, \dots, l_m^*$  as a function of  $\lambda$ . More precisely, it says that in a Dutch auction with a larger average number of bidders each revenue-maximizing bid level should be set higher than its counterpart with a smaller average number of bidders.

**Proposition 4.3:**  $l_i^*$  is an increasing function of  $\lambda$ ,  $i = 2, 3, \dots, m$ .

To help explain Propositions 4.2 and 4.3 set forth above, I depict the optimal bid levels for  $m = 6$  and  $\lambda = 1, 2, 15$  in Figure 4.1. It is seen that  $l_6^*, l_5^*, \dots, l_2^*$  become farther away from each other as the asking price declines. In addition, as  $\lambda$  changes from 1 to 2 to 15, the optimal bid levels associated with a larger  $\lambda$  (e.g.,  $\lambda = 15$ ) are to the right of their counterparts associated with a smaller  $\lambda$  (e.g.,  $\lambda = 2$ ) with the exception of  $l_1^*$ .

However, there is a notable difference between the respective first charts in Figures 3.2 and 4.1. The optimal bid decrements are evenly spaced if exactly one bidder participates in the

auction in the former (i.e.,  $n = 1$ ), while the optimal bid decrements are increasing when the average number of bidders present is one in the latter (i.e.,  $\lambda = 1$ ).

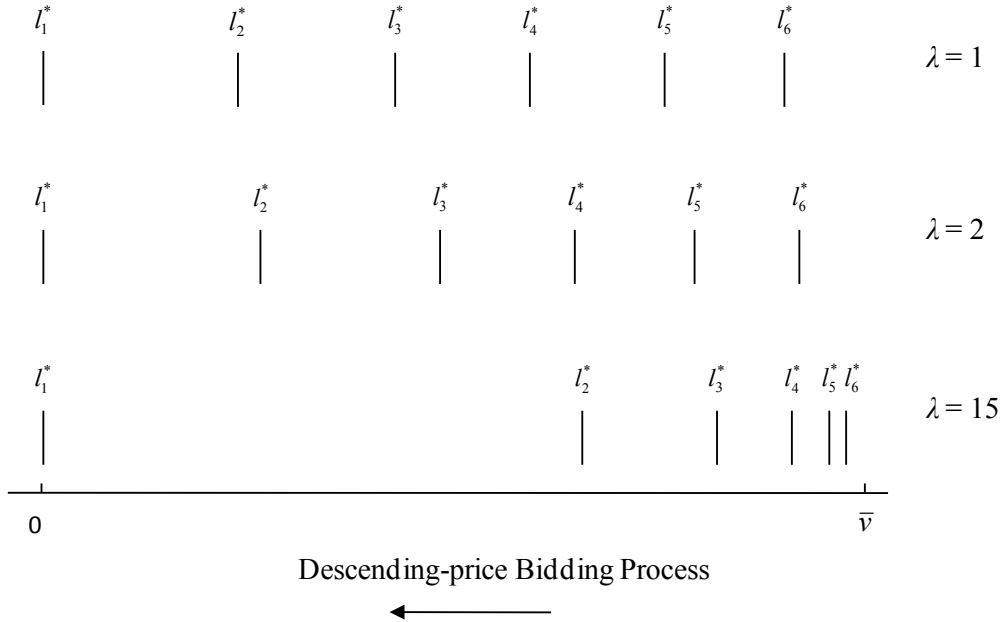


Figure 4.1. Optimal bid levels with  $m = 6$ .

The result presented below indicates that the highest optimal bid level will go up as more bid levels are set in the Dutch auction with an uncertain number of bidders and its proof is deferred to Appendix O. In view of Proposition 3.5, one sees that the general relationship between  $l_m^*$  and  $m$  does not change whether the number of auction participants is certain or not.

**Proposition 4.4:** Given  $\lambda$ ,  $l_m^*$  is an increasing function of  $m$ .

As an illustration, I plot the optimal bid levels based on  $\lambda = 5$  and  $m = 2, 3, 4, 5$  in Figure 4.2. Consistent with Proposition 4.4, the right most vertical bar representing the highest optimal

bid level associated with a larger  $m$  (e.g.,  $m = 5$ ) is to the right of its counterpart associated with a smaller  $m$  (e.g.,  $m = 2$ ).

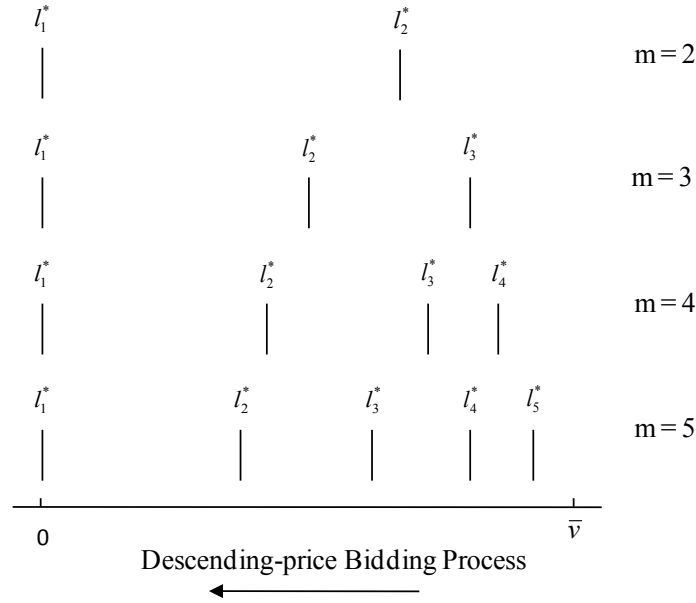


Figure 4.2. Optimal bid levels with  $\lambda = 5$ .

#### 4.4 Properties of Maximum Expected Revenue

Once the optimal bid levels are identified, one can compute the highest possible expected revenue to be received by the auctioneer by using (4.1) or the objective function in (4.2), that is,

$$Z_{\lambda}^* = \sum_{i=1}^m l_i^* e^{-\lambda} \left( e^{\frac{\lambda l_{i+1}^*}{\bar{v}}} - e^{\frac{\lambda l_i^*}{\bar{v}}} \right).$$

For the purpose of exploring the relationships between  $Z_{\lambda}^*$  and

other auction model parameters, however, I develop an alternative expression of  $Z_{\lambda}^*$  in

Proposition 4.5 below. The reader is referred to Appendix P for the proof.



**Proposition 4.5:**  $Z_\lambda^* = l_m^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_2^*}{\bar{v}} \right)} \right]$

The above theorem suggests that the maximum expected revenue is strictly less than the highest optimal bid level since  $0 < e^{-\lambda \left( 1 - \frac{l_2^*}{\bar{v}} \right)} < 1$  and  $\frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_2^*}{\bar{v}} \right)} \right] > 0$ . It should be pointed out, nevertheless, that although  $Z_\lambda^* < l_m^*$ , the gap between them gets smaller as  $\lambda$  becomes larger. This

is because, as will be shown in Appendix Q,  $\frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_2^*}{\bar{v}} \right)} \right]$  is a decreasing function of  $\lambda$  and

$$\lim_{\lambda \rightarrow \infty} \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_2^*}{\bar{v}} \right)} \right] = 0. \text{ Consequently, } Z_\lambda^* \text{ is practically equal to } l_m^* \text{ when } \lambda \text{ is sufficiently large.}$$

This seems to make intuitive sense since the competition among the bidders will grow if the number of auction participants increases. Therefore, there is a higher likelihood for the object to be sold at the outset of the bidding process so that the auctioneer's expected revenue is closer to the highest optimal bid level.

It is also observed that  $Z_\lambda^*$  may be calculated once  $l_2^*$  and  $l_m^*$  have been obtained. In comparison, it is proven in Proposition 3.6 of Chapter 3 that  $Z^*$  can be computed based solely on  $l_m^*$ . This is another testament of the effect that uncertainty about the number of bidders has on optimal design of Dutch auctions. To be more specific, more efforts are required in order for the auctioneer to figure out how much he is going to make in a Dutch auction when less information is available (i.e., from  $n$  to  $\lambda$ ) about how many people will attend the event.

In light of Proposition 4.5, one can show that the maximum expected revenue increases with the average number of auction participants. The proof of the following result is given in Appendix Q.

**Proposition 4.6:** Given  $m$ ,  $Z_\lambda^*$  is an increasing function of  $\lambda$ .

To conclude this section, I attempt to investigate how  $Z_\lambda^*$  varies in response to change in  $m$ . As stated in Proposition 4.7 below and as proven in Appendix R, the maximum expected revenue increases as more bid levels are established in the Dutch auction provided that the average (not exact) number of bidders is known and fixed.

**Proposition 4.7:** Given  $\lambda$ ,  $Z_\lambda^*$  is an increasing function of  $m$ .

Notice that the last two theorems concur with Propositions 3.7 and 3.8, respectively, for the basic Dutch auction model examined in Chapter 3. The agreement suggests that uncertainty about the number of bidders in the Dutch auction being studied has no bearing on the relationship between  $Z_\lambda^*$  and  $\lambda$  (or  $n$ ) or between  $Z_\lambda^*$  and  $m$ .

#### 4.5 A Comparative Study

Two frequently-used heuristic approaches to establishing bid levels, Strategy E and Strategy D, are introduced and analyzed compared with the optimal strategy in Section 3.6 of Chapter 3, where the number of bidders in the Dutch auction is assumed to be known with certainty. The main purpose of this section is to analyze both of them under a different scenario

where the certainty is replaced with uncertainty. Again, the following discussion is motivated by the fact that the two methods have long been employed by leading auction houses in the world as well as treated in the auction literature (Schill, 1977; Yamey, 1972; Yu, 1999).

Recall that Strategy E calls for dividing the difference between the ceiling and the floor of the common range of bidders' valuations by the number of bid levels to obtain the constant decrement between two successive bid levels. One then has  $l_{E,i} = \frac{(i-1)\bar{v}}{m}$ ,  $i = 1, 2, \dots, m$  and, based on (4.1), the probability of selling the object at  $l_{E,i}$  is

$$\begin{aligned} P(l_{E,i}) &= e^{-\lambda} \left( e^{\frac{\lambda l_{E,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{E,i}}{\bar{v}}} \right) \\ &= e^{-\lambda} \left[ e^{\frac{\lambda i}{m}} - e^{\frac{\lambda(i-1)}{m}} \right] \end{aligned} \quad (4.4)$$

Let  $Z_{\lambda,E}$  be the expected revenue to be generated from a Dutch auction with an uncertain number of bidders when Strategy E is applied to determine the bid levels. It follows that

$$\begin{aligned} Z_{\lambda,E} &= \sum_{i=1}^m l_{E,i} e^{-\lambda} \left( e^{\frac{\lambda l_{E,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{E,i}}{\bar{v}}} \right) \\ &= \sum_{i=1}^m \frac{(i-1)\bar{v}}{m} e^{-\lambda} \left[ e^{\frac{\lambda i}{m}} - e^{\frac{\lambda(i-1)}{m}} \right] \\ &= \frac{\bar{v} e^{-\lambda}}{m} \sum_{i=2}^m (i-1) \left[ e^{\frac{\lambda i}{m}} - e^{\frac{\lambda(i-1)}{m}} \right] \end{aligned} \quad (4.5)$$

On the other hand, Strategy D is designed based on the premise that the decrement between two consecutive bid levels changes at a fixed discounting rate of  $\frac{l_{D,i} - l_{D,i-1}}{l_{D,i+1} - l_{D,i}} = \frac{1}{1+r}$

with  $r > 0$  being the discounting factor. According to (3.10), I have  $l_{D,1} = 0$  and

$$l_{D,i} = \frac{\sum_{j=m-i+1}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \bar{v}, \quad i = 2, 3, \dots, m \text{ and } d > 0 \text{ being the first (and largest) decrement.}$$

Additionally, in light of (4.1), the probability of selling the object at  $l_{D,i}$  is

$$P(l_{D,i}) = e^{-\lambda} \left( e^{\frac{\lambda l_{D,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{D,i}}{\bar{v}}} \right) \quad (4.6)$$

Suppose that  $Z_{\lambda,D}$  is the expected revenue to be received by the auctioneer who applies Strategy D in the Dutch auction with an uncertain number of bidders. It is easy to see that

$$Z_{\lambda,D} = \sum_{i=1}^m l_{D,i} e^{-\lambda} \left( e^{\frac{\lambda l_{D,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{D,i}}{\bar{v}}} \right) \quad (4.7)$$

What follows is an important result on the comparison of action revenues based on

Strategies E and D, where  $\tau_i = \frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}}$ ,  $i = 2, 3, \dots, m$  and  $t \geq 0$ .

**Proposition 4.8:**  $F_t = e^{-\lambda \bar{v}} \sum_{i=2}^m \tau_i \left( e^{\lambda \tau_{i+1}} - e^{\lambda \tau_i} \right)$  is a decreasing function of  $t$ .

Note that  $F_t = Z_{\lambda,D}$  if  $t > 0$ . In case  $t = 0$ , however, one has  $\tau_i = \frac{i-1}{m} = \frac{l_{E,i}}{\bar{v}}$ ,

$i = 2, 3, \dots, m$ , and  $F_t = e^{-\lambda \bar{v}} \sum_{i=2}^m \frac{l_{E,i}}{\bar{v}} \left( e^{\frac{\lambda l_{E,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{E,i}}{\bar{v}}} \right) = \frac{\bar{v} e^{-\lambda}}{m} \sum_{i=2}^m (i-1) \left[ e^{\frac{\lambda i}{m}} - e^{\frac{\lambda(i-1)}{m}} \right] = Z_{\lambda,E}$ . Since

Proposition 4.8 indicates that  $F_t$  is a decreasing function of  $t$ , I have  $Z_{\lambda,E} > Z_{\lambda,D}$ .

I now report the last research finding in this chapter. Since it follows immediately from Proposition 4.8 and the fact that the optimal strategy  $(l_1^*, l_2^*, \dots, l_m^*)$  always outperforms both Strategy E and Strategy D even if the number of bidders in the Dutch auction is a random variable rather than a constant, no proof is given here.

**Corollary 4.2:**  $Z_\lambda^* > Z_{\lambda,E} > Z_{\lambda,D}$

One of the managerial implications of Corollary 4.2 is that when no optimization software package is handy to up the optimal bid levels, Strategy E is preferred to Strategy D as a heuristic substitute since the former not only leads to a greater expected revenue, but is also computationally more efficient.

#### 4.6 Numerical Illustrations

To develop a better understanding of the Dutch auction in which the number of bidders follows a Poisson distribution with a mean of  $\lambda$ , I will provide a collection of numerical examples for illustrative purposes in this section. The possible values to be assumed by the key model parameters are  $\bar{v} = 10$ ,  $m \in \{3, 4, \dots, 17\}$  and  $\lambda \in \{1, 2, 4, 10, 20\}$ .

In light of (4.2), various nonlinear programs based on the  $15 \times 5 = 75$  combinations of  $m$  and  $\lambda$  values with  $\bar{v} = 10$  are drawn up. For instance, if  $m = 3$  and  $\lambda = 3$ , the appropriate NLP is shown below with  $l_1$ ,  $l_2$ , and  $l_3$  being the decision variables:

$$\begin{aligned}
\text{Maximize} \quad & Z_\lambda = \frac{1}{e^3} \left[ l_1 \left( e^{\frac{3l_2}{10}} - e^{\frac{3l_1}{10}} \right) + l_2 \left( e^{\frac{3l_3}{10}} - e^{\frac{3l_2}{10}} \right) + l_3 \left( e^{\frac{3l_4}{10}} - e^{\frac{3l_3}{10}} \right) \right] \\
\text{subject to :} \quad & l_2 \geq l_1 \\
& l_3 \geq l_2 \\
& l_1 = 0 \\
& l_4 = 10
\end{aligned}$$

The above mathematical formulation may be easily solved by running Solver. The same procedure is repeated for each of the other  $75 - 1 = 74$  instances. All the results are analyzed to gain insight into the properties of the optimal bid levels as well as the maximum expected revenue presented in the previous sections of this chapter.

I set out to examine the behaviors of  $l_1^*, l_2^*, \dots, l_m^*$  and how they relate to  $m$  and  $\lambda$ . To begin, the optimal solutions to the Dutch auction models with  $\bar{v} = 10$ ,  $m = 12$  and  $\lambda = 1, 2, 4, 10, 20$  are summarized in Table 4.2 and Figure 4.3. One can see that  $l_i^*$  is moving farther away from  $l_{i+1}^*$  as the asking price moves from right to left (in Table 4.2) or from top to bottom (in Figure 4.3),  $i = 1, \dots, 11$ . This observation agrees with the bid decrements shown in Table 4.3 and is consistent with Proposition 4.2.

I also observe from the same exhibits that  $l_i^*$  becomes larger as  $\lambda$  increases from 1 to 2 to 4 to 10 to 20,  $i = 2, \dots, 12$ , which is in line with Proposition 4.3. Additionally, a close look at Figure 4.3 reveals that each of the  $l_i^*$  curves tends to flatten out as  $\lambda$  increases, which implies that  $l_i^*$  is an increasing function of  $\lambda$  at a diminishing rate.

Finally, Table 4.4 and Figure 4.4 summarize the highest optimal bid level  $l_m^*$  for various combinations of  $m$  and  $\lambda$  values with  $\bar{v} = 10$ . It is evident that  $l_m^*$  increases with  $m$  and  $\lambda$ , which is in agreement with Propositions 4.3 and 4.4. Thus one can expect  $l_m^*$  to get closer to the highest bidder's valuation as  $\lambda$  and  $m$  both become larger.

Table 4.2

Optimal Bid Levels with  $\bar{v} = 10$  and  $m = 12$

$\lambda$	$l_1^*$	$l_2^*$	$l_3^*$	$l_4^*$	$l_5^*$	$l_6^*$	$l_7^*$	$l_8^*$	$l_9^*$	$l_{10}^*$	$l_{11}^*$	$l_{12}^*$
1	0	1.0494	2.0465	2.9977	3.9066	4.7765	5.6104	6.4112	7.1814	7.9236	8.6398	9.3315
2	0	1.3180	2.4877	3.5389	4.4929	5.3661	6.1709	6.9171	7.6126	8.2639	8.8760	9.4535
4	0	2.0006	3.4703	4.6264	5.5766	6.3819	7.0801	7.6958	8.2462	8.7437	9.1974	9.6144
10	0	4.2128	5.8640	6.8390	7.5195	8.0387	8.4568	8.8062	9.1058	9.3679	9.6007	9.8100
20	0	6.2731	7.5761	8.2174	8.6301	8.9310	9.1666	9.3596	9.5228	9.6641	9.7885	9.8996

Table 4.3

Optimal Bid Decrements with  $\bar{v} = 10$  and  $m = 12$

$\lambda$	$l_2^* - l_1^*$	$l_3^* - l_2^*$	$l_4^* - l_3^*$	$l_5^* - l_4^*$	$l_6^* - l_5^*$	$l_7^* - l_6^*$	$l_8^* - l_7^*$	$l_9^* - l_8^*$	$l_{10}^* - l_9^*$	$l_{11}^* - l_{10}^*$	$l_{12}^* - l_{11}^*$
1	1.0494	0.9972	0.9512	0.9088	0.8699	0.8339	0.8008	0.7703	0.7422	0.7162	0.6917
2	1.3180	1.1698	1.0511	0.9540	0.8732	0.8048	0.7462	0.6955	0.6512	0.6122	0.5775
4	2.0006	1.4698	1.1560	0.9502	0.8054	0.6982	0.6157	0.5504	0.4975	0.4537	0.4169
10	4.2128	1.6511	0.9750	0.6806	0.5191	0.4181	0.3494	0.2996	0.2621	0.2328	0.2093
20	6.2731	1.3031	0.6413	0.4127	0.3009	0.2356	0.1930	0.1632	0.1413	0.1244	0.1111

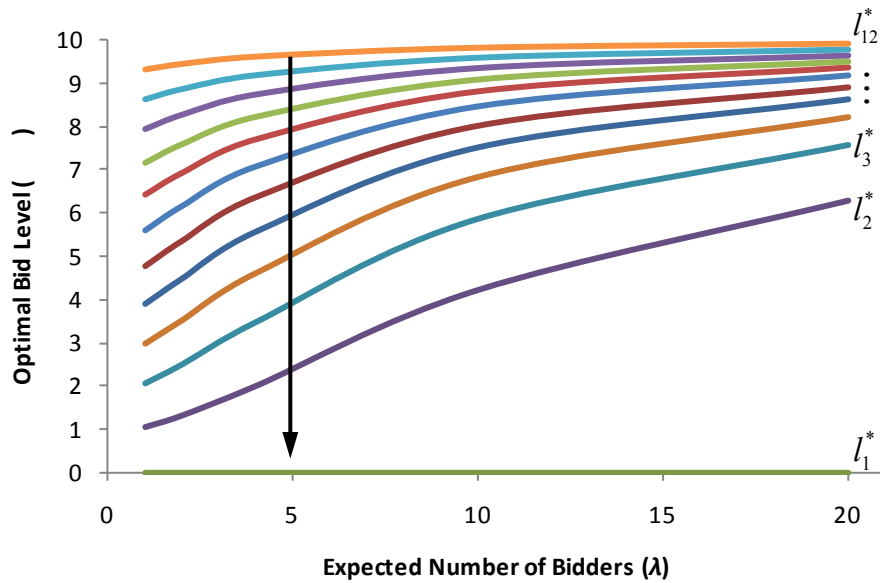
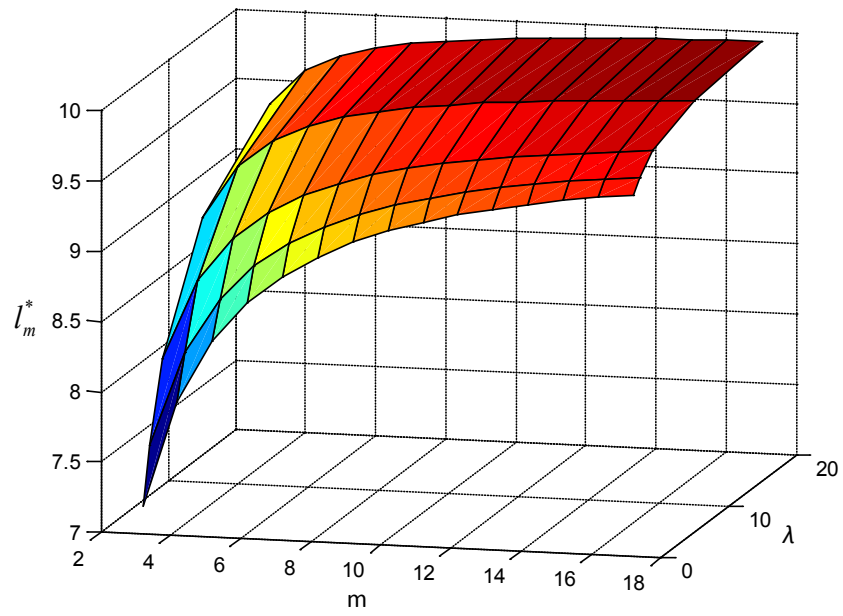


Figure 4.3. Optimal bid levels versus expected number of bidders with  $\bar{v} = 10$  and  $m = 12$ .



*Figure 4.4.* Highest optimal bid level versus expected number of bidders and number of bid levels with  $\bar{v} = 10$ .



Table 4.4

*Highest Optimal Bid Level versus Expected Number of Bidders and Number of Bid Levels with  $\bar{v} = 10$*

$\lambda \backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	7.1629	7.9139	8.3507	8.6363	8.8376	8.9872	9.1025	9.1944	9.2693	9.3314	9.3837	9.4286	9.4674	9.5011	9.5306
2	7.5538	8.2333	8.6182	8.8656	9.0381	9.1649	9.2623	9.3393	9.4018	9.4536	9.4970	9.5341	9.5660	9.5939	9.6184
4	8.1050	8.6735	8.9816	9.1743	9.3060	9.4016	9.4741	9.5310	9.5768	9.6144	9.6459	9.6727	9.6957	9.7157	9.7332
10	8.8825	9.2629	9.4539	9.5676	9.6427	9.6958	9.7354	9.7659	9.7902	9.8100	9.8263	9.8401	9.8519	9.8621	9.8709
20	9.3294	9.5774	9.6948	9.7623	9.8058	9.8361	9.8584	9.8754	9.8888	9.8996	9.9086	9.9161	9.9224	9.9279	9.9327

I now turn the attention to the behavior of  $Z_\lambda^*$  and the way it interacts with  $\lambda$  and  $m$ .

Towards that end, the maximum expected revenues generated from Dutch auctions for various combinations of  $m$  and  $\lambda$  values with  $\bar{v} = 10$  are summarized in Table 4.5 and their continuous versions are depicted in Figure 4.5. One can tell from the two exhibits that  $Z_\lambda^*$  increases with both the expected number of bidders and the number of bid levels, which are consistent with Propositions 4.6 and 4.7. However, a careful examination of Figure 4.5 shows that the revenue improvement is much slower for large values of  $\lambda$  and  $m$ . In fact, it is evident from Tables 4.6 and 4.7 that the increasing rate is diminishing.

One of the implications of the above observation is that there is an inherent trade-off between the additional efforts required to attract more potential buyers to bid and the resulting increase in auction revenue. As such, the auctioneer needs to determine what the most desirable number of participants is in the Dutch auction on average so that the expected revenue to be accrued from the competitive bidding process will be as high as possible.

Next, the continuous versions of the maximum expected revenues shown in in Tables 3.4 and 4.5 are plotted in Figure 4.5 for comparison purposes. As displayed in the figure,  $Z^*$  is invariably greater than  $Z_\lambda^*$  but the difference between them becomes negligible when  $m$  and  $\lambda$  (or  $n$ ) are large. It is worth noting that a similar finding has been reported on English auctions by David et al. (2007). The seeming convergence of  $Z_\lambda^*$  to  $Z^*$  in the presence of many bidders may be accounted for by the fact that when  $\lambda$  is large, the standard deviation of the Poisson distribution ( $\sqrt{\lambda}$ ) is small relative to its mean ( $\lambda$ ). Under such a circumstance, the random numbers of bidders center around  $\lambda$  and the dispersion is narrow. This is a close approximation to the stochastic case where there are  $\lambda$  bidders with probability 1, which in turn is essentially

equivalent to the deterministic case with exactly  $\lambda = n$  bidders. Consequently, the expected number of bidders ( $\lambda$ ) is almost identical to the actual number of bidders ( $n$ ) in the Dutch auction being considered.

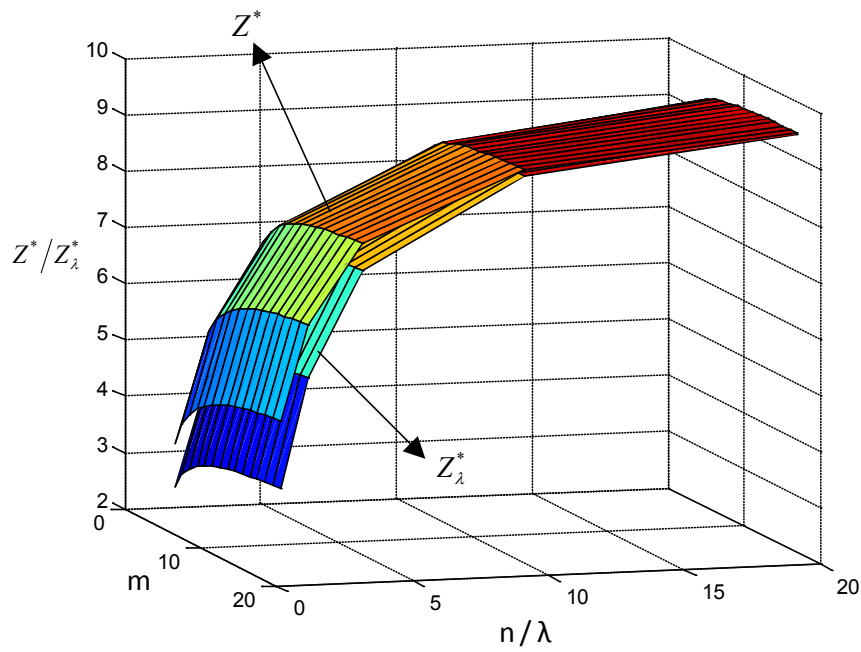


Figure 4.5. Maximum expected revenue versus number of bidders and number of bid levels with  $\bar{v} = 10$ .

Table 4.5

Maximum Expected Revenue with  $\bar{v} = 10^*$

$\lambda$	$Z_{\lambda,m=3}^*$	$Z_{\lambda,m=4}^*$	$Z_{\lambda,m=5}^*$	$Z_{\lambda,m=6}^*$	$Z_{\lambda,m=7}^*$	$Z_{\lambda,m=8}^*$	$Z_{\lambda,m=9}^*$	$Z_{\lambda,m=10}^*$	$Z_{\lambda,m=11}^*$	$Z_{\lambda,m=12}^*$	$Z_{\lambda,m=13}^*$	$Z_{\lambda,m=14}^*$	$Z_{\lambda,m=15}^*$	$Z_{\lambda,m=16}^*$	$Z_{\lambda,m=17}^*$
1	2.5869	2.8714	3.0384	3.1481	3.2257	3.2836	3.3283	3.3639	3.3930	3.4171	3.4375	3.4550	3.4701	3.4832	3.4949
2	4.1846	4.5894	4.8216	4.9722	5.0777	5.1558	5.2158	5.2635	5.3022	5.3343	5.3613	5.3844	5.4043	5.4217	5.4370
4	5.9820	6.4385	6.6898	6.8485	6.9578	7.0375	7.0982	7.1460	7.1845	7.2163	7.2430	7.2657	7.2852	7.3022	7.3171
10	7.9243	8.2852	8.4681	8.5777	8.6503	8.7019	8.7403	8.7701	8.7938	8.8131	8.8291	8.8426	8.8541	8.8641	8.8728
20	8.8372	9.0809	9.1968	9.2636	9.3068	9.3369	9.3590	9.3759	9.3892	9.4000	9.4089	9.4163	9.4226	9.4281	9.4328

Table 4.6

Maximum Expected Revenue Increasing Rate between Bid Levels with  $\bar{v} = 10$

$\lambda$	$\frac{Z_{\lambda,m=4}^* - Z_{\lambda,m=3}^*}{Z_{\lambda,m=3}^*}$	$\frac{Z_{\lambda,m=5}^* - Z_{\lambda,m=4}^*}{Z_{\lambda,m=4}^*}$	$\frac{Z_{\lambda,m=6}^* - Z_{\lambda,m=5}^*}{Z_{\lambda,m=5}^*}$	$\frac{Z_{\lambda,m=7}^* - Z_{\lambda,m=6}^*}{Z_{\lambda,m=6}^*}$	$\frac{Z_{\lambda,m=8}^* - Z_{\lambda,m=7}^*}{Z_{\lambda,m=7}^*}$	$\frac{Z_{\lambda,m=9}^* - Z_{\lambda,m=8}^*}{Z_{\lambda,m=8}^*}$	$\frac{Z_{\lambda,m=10}^* - Z_{\lambda,m=9}^*}{Z_{\lambda,m=9}^*}$	$\frac{Z_{\lambda,m=11}^* - Z_{\lambda,m=10}^*}{Z_{\lambda,m=10}^*}$	$\frac{Z_{\lambda,m=12}^* - Z_{\lambda,m=11}^*}{Z_{\lambda,m=11}^*}$	$\frac{Z_{\lambda,m=13}^* - Z_{\lambda,m=12}^*}{Z_{\lambda,m=12}^*}$	$\frac{Z_{\lambda,m=14}^* - Z_{\lambda,m=13}^*}{Z_{\lambda,m=13}^*}$	$\frac{Z_{\lambda,m=15}^* - Z_{\lambda,m=14}^*}{Z_{\lambda,m=14}^*}$	$\frac{Z_{\lambda,m=16}^* - Z_{\lambda,m=15}^*}{Z_{\lambda,m=15}^*}$	$\frac{Z_{\lambda,m=17}^* - Z_{\lambda,m=16}^*}{Z_{\lambda,m=16}^*}$
1	11.00%	5.81%	3.61%	2.47%	1.79%	1.36%	1.07%	0.86%	0.71%	0.60%	0.51%	0.44%	0.38%	0.33%
2	9.67%	5.06%	3.12%	2.12%	1.54%	1.16%	0.91%	0.74%	0.61%	0.51%	0.43%	0.37%	0.32%	0.28%
4	7.63%	3.90%	2.37%	1.59%	1.15%	0.86%	0.67%	0.54%	0.44%	0.37%	0.31%	0.27%	0.23%	0.20%
10	4.56%	2.21%	1.29%	0.85%	0.60%	0.44%	0.34%	0.27%	0.22%	0.18%	0.15%	0.13%	0.11%	0.10%
20	2.76%	1.28%	0.73%	0.47%	0.32%	0.24%	0.18%	0.14%	0.11%	0.09%	0.08%	0.07%	0.06%	0.05%

\*  $Z_{\lambda,m=a}^*$  denotes the maximum expected revenue derived from the Dutch auction with  $a$  bid levels.

Table 4.7

Maximum Expected Revenue Increasing Rate between Expected Numbers of Bidders with  $\bar{v} = 10^*$

m	$\frac{Z_{\lambda=2}^* - Z_{\lambda=1}^*}{Z_{\lambda=1}^*}$	$\frac{Z_{\lambda=4}^* - Z_{\lambda=2}^*}{Z_{\lambda=2}^*}$	$\frac{Z_{\lambda=10}^* - Z_{\lambda=4}^*}{Z_{\lambda=4}^*}$	$\frac{Z_{\lambda=20}^* - Z_{\lambda=10}^*}{Z_{\lambda=10}^*}$
3	61.76%	42.95%	32.47%	11.52%
4	59.83%	40.29%	28.68%	9.60%
5	58.69%	38.75%	26.58%	8.61%
6	57.94%	37.74%	25.25%	8.00%
7	57.41%	37.03%	24.33%	7.59%
8	57.02%	36.50%	23.65%	7.30%
9	56.71%	36.09%	23.13%	7.08%
10	56.47%	35.77%	22.73%	6.91%
11	56.27%	35.50%	22.40%	6.77%
12	56.10%	35.28%	22.13%	6.66%
13	55.96%	35.10%	21.90%	6.57%
14	55.84%	34.94%	21.70%	6.49%
15	55.74%	34.80%	21.54%	6.42%
16	55.65%	34.68%	21.39%	6.36%
17	55.57%	34.58%	21.26%	6.31%

I will close this section with a set of numerical examples to illustrate the optimal strategy and two heuristic strategies for establishing the bid levels described in Section 4.5 by focusing on the resulting expected revenues. The possible values of the model parameters are as follows:  $\bar{v} = 10$ ,  $m \in \{3, 5, 9, 13, 17\}$  and  $\lambda \in \{1, 2, 4, 10, 20\}$ .

Taking as an example the problem instance with  $\bar{v} = 10$ ,  $m = 5$  and  $\lambda = 2$ , one can easily formulate the appropriate NLP based on (4.2) and solve it by running Solver to obtain

$Z_{\lambda}^* = 4.8216$ . If Strategy E is used, then the five bid levels are  $l_{E,1} = \frac{(1-1) \times 10}{5} = 0$ ,

$l_{E,2} = \frac{(2-1) \times 10}{5} = 2$ ,  $l_{E,3} = \frac{(3-1) \times 10}{5} = 4$ ,  $l_{E,4} = \frac{(4-1) \times 10}{5} = 6$ , and  $l_{E,5} = \frac{(5-1) \times 10}{5} = 8$ .

\*  $Z_{\lambda=b}^*$  denotes the maximum expected revenue derived from the Dutch auction with  $b$  expected bidders.

Additionally, I see from (4.4) that the respective probabilities of selling the object at those five

$$\text{prices are } P(l_{E,1}) = e^{-2} \left( e^{\frac{2 \times 1}{5}} - e^{\frac{2 \times 0}{5}} \right) \approx 0.0666, \quad P(l_{E,2}) = e^{-2} \left( e^{\frac{2 \times 2}{5}} - e^{\frac{2 \times 1}{5}} \right) \approx 0.0993,$$

$$P(l_{E,3}) = e^{-2} \left( e^{\frac{2 \times 3}{5}} - e^{\frac{2 \times 2}{5}} \right) \approx 0.1481, \quad P(l_{E,4}) = e^{-2} \left( e^{\frac{2 \times 4}{5}} - e^{\frac{2 \times 3}{5}} \right) \approx 0.2210 \text{ and}$$

$$P(l_{E,5}) = e^{-2} \left( e^{\frac{2 \times 5}{5}} - e^{\frac{2 \times 4}{5}} \right) \approx 0.3297. \text{ Accordingly, the auctioneer is expected to receive an}$$

$$\text{average revenue of } Z_{\lambda,E} = 0(0.0666) + 2(0.0993) + 4(0.1481) + 6(0.2210) + 8(0.3297) \approx 4.7546$$

from the Dutch auction.

$$\text{However, based on Strategy D with } r = 25\%, \text{ I have } l_{D,1} = 0, \quad l_{D,2} = \frac{\sum_{j=4}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \approx 1.2185,$$

$$l_{D,3} = \frac{\sum_{j=3}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \approx 2.7416, \quad l_{D,4} = \frac{\sum_{j=2}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \approx 4.6454 \text{ and } l_{D,5} = \frac{\sum_{j=1}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \approx 7.0252.$$

According to (4.6), the probabilities of selling the object at the five prices are

$$P(l_{D,1}) = e^{-2} \left( e^{\frac{2 \times 1.2185}{10}} - e^{\frac{2 \times 0}{10}} \right) \approx 0.0374, \quad P(l_{D,2}) = e^{-2} \left( e^{\frac{2 \times 2.7416}{10}} - e^{\frac{2 \times 1.2185}{10}} \right) \approx 0.0615,$$

$$P(l_{D,3}) = e^{-2} \left( e^{\frac{2 \times 4.6454}{10}} - e^{\frac{2 \times 2.7416}{10}} \right) \approx 0.1085, \quad P(l_{D,4}) = e^{-2} \left( e^{\frac{2 \times 7.0252}{10}} - e^{\frac{2 \times 4.6454}{10}} \right) \approx 0.2089 \text{ and}$$

$$P(l_{D,5}) = e^{-2} \left( e^{\frac{2 \times 10}{10}} - e^{\frac{2 \times 7.0252}{10}} \right) \approx 0.4484. \text{ Thus, the expected auction revenue is } Z_{\lambda,D} = 0(0.0374)$$

$$+ 1.2185(0.0615) + 2.7416(0.1085) + 4.6454(0.2089) + 7.0252(0.4484) \approx 4.4930.$$

The computational process described above is repeated for each of the other  $5 \times 5 - 1 = 24$  pairs of  $m$  and  $\lambda$ . All of the results are summarized in Table 4.8 and the continuous versions of them are plotted in Figure 4.6. One can see from the two exhibits that, as suggested by Corollary 4.2, the optimal strategy completely outperforms the two heuristic strategies; further, Strategy E consistently yields greater expected revenues than Strategy D does. I also observe that the differences among the three revenue surfaces in Figure 4.6 are more pronounced if the number of bid levels is small while the expected number of auction participants is large. When  $\lambda$  is small and  $m$  is large, however, the differences are almost negligible.

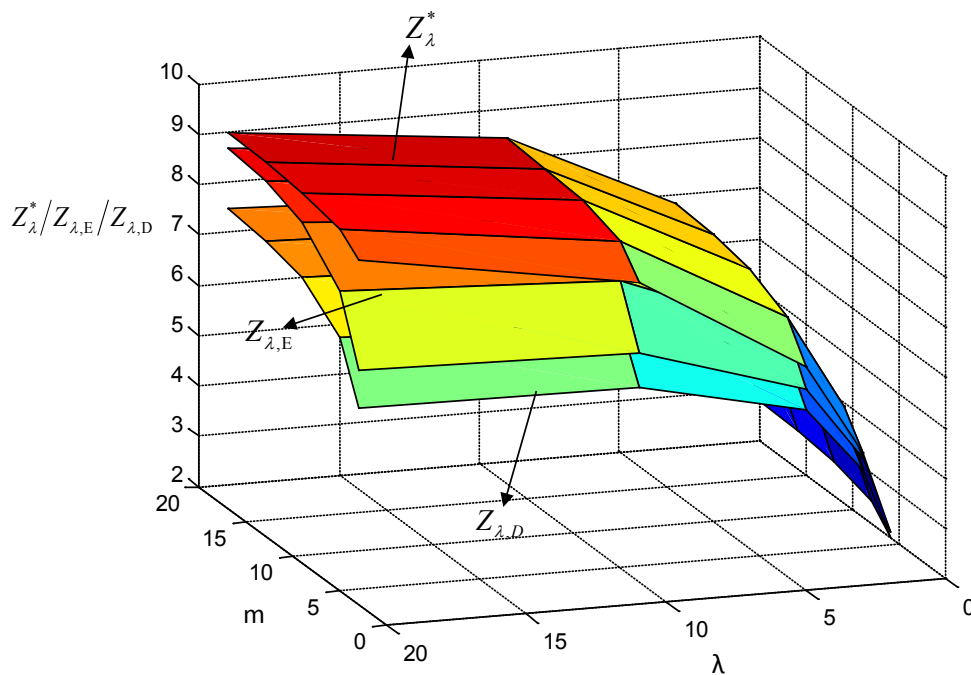


Figure 4.6. Expected revenues based on three different design strategies with  $\bar{v} = 10$ .

Table 4.8

*Expected Revenues Based on Three Different Design Strategies with  $\bar{v} = 10$*

$\lambda$	$Z$	3			5			9			13			17		
		$Z_{\lambda}^*$	$Z_{\lambda,E}$	$Z_{\lambda,D}$	$Z_{\lambda}^*$	$Z_{\lambda,E}$	$Z_{\lambda,D}$	$Z_{\lambda}^*$	$Z_{\lambda,E}$	$Z_{\lambda,D}$	$Z_{\lambda}^*$	$Z_{\lambda,E}$	$Z_{\lambda,D}$	$Z_{\lambda}^*$	$Z_{\lambda,E}$	$Z_{\lambda,D}$
1		2.5869	2.5668	2.4711	3.0384	3.0256	2.8999	3.3283	3.3211	3.1469	3.4375	3.4326	3.2180	3.4949	3.4911	3.2437
2		4.1846	4.0766	3.8573	4.8216	4.7545	4.4930	5.2158	5.1785	4.8520	5.3613	5.3356	4.9541	5.4370	5.4174	4.9909
4		5.9820	5.5564	5.1281	6.6898	6.4346	5.9577	7.0982	6.9601	6.4155	7.2430	7.1489	6.5439	7.3171	7.2457	6.5899
10		7.9243	6.5435	5.8456	8.4681	7.6871	6.8935	8.7403	8.3437	7.4776	8.8291	8.5666	7.6408	8.8728	8.6773	7.6991
20		8.8372	6.6624	5.9007	9.1968	7.9627	7.0190	9.3590	8.7538	7.6714	9.4089	9.0204	7.8584	9.4328	9.1495	7.9257



## 4.7 Simulation Analysis

The present section aims to validate the theoretical value of  $Z_\lambda^*$  by computing the maximum expected revenue via a simulation approach and then comparing it with the one obtained by solving the mathematical program in (4.2). I start by specifying the values to be taken on by each of the three model parameters as follows:  $\bar{v} = 10$ ,  $m \in \{6, 9, 12\}$  and  $\lambda \in \{1, 2, 4, 10, 20\}$ . For each combination of  $m$  and  $\lambda$ , an NLP based on (4.2) is formulated to model the particular Dutch auction and is solved to determine  $(l_1^*, l_2^*, \dots, l_m^*)$  as well as  $Z_\lambda^*$  with the aid of Solver.

A flow chart for guiding the simulation process is given in Figure 4.7, where  $t$  is a counter and  $T = 100,000$  is the maximum number of runs allowed. To begin, MATLAB is employed to generate a value of the random variable following the Poisson distribution  $\text{Poisson}(\lambda)$  to represent the actual number of bidders in the Dutch auction. Then,  $n$  values of the random variable subject to the uniform distribution over the range  $[0, 10]$  are generated to represent the bids submitted by the auction participants. Let  $k$  be the largest index such that the highest of the  $n$  bids is greater than or equal to  $l_k^*$ . It follows that the final selling price of the object is  $l_k^*$ . Finally,  $Z_\lambda^*$  is computed as the average selling price over the 100,000 simulation runs.

The procedure described above is repeated for each of the other  $3 \times 5 - 1 = 14$  pairs of  $m$  and  $\lambda$ , and the two streams of  $Z_\lambda^*$  are summarized in Table 4.9. In view of the minor differences between them due most likely to rounding errors, the simulation-based empirical results lend strong support to the optimization-based theoretical results on the maximum expected revenue derived from the Dutch auction with an uncertain number of bidders. As a consequence, the validity of the latter is well established.

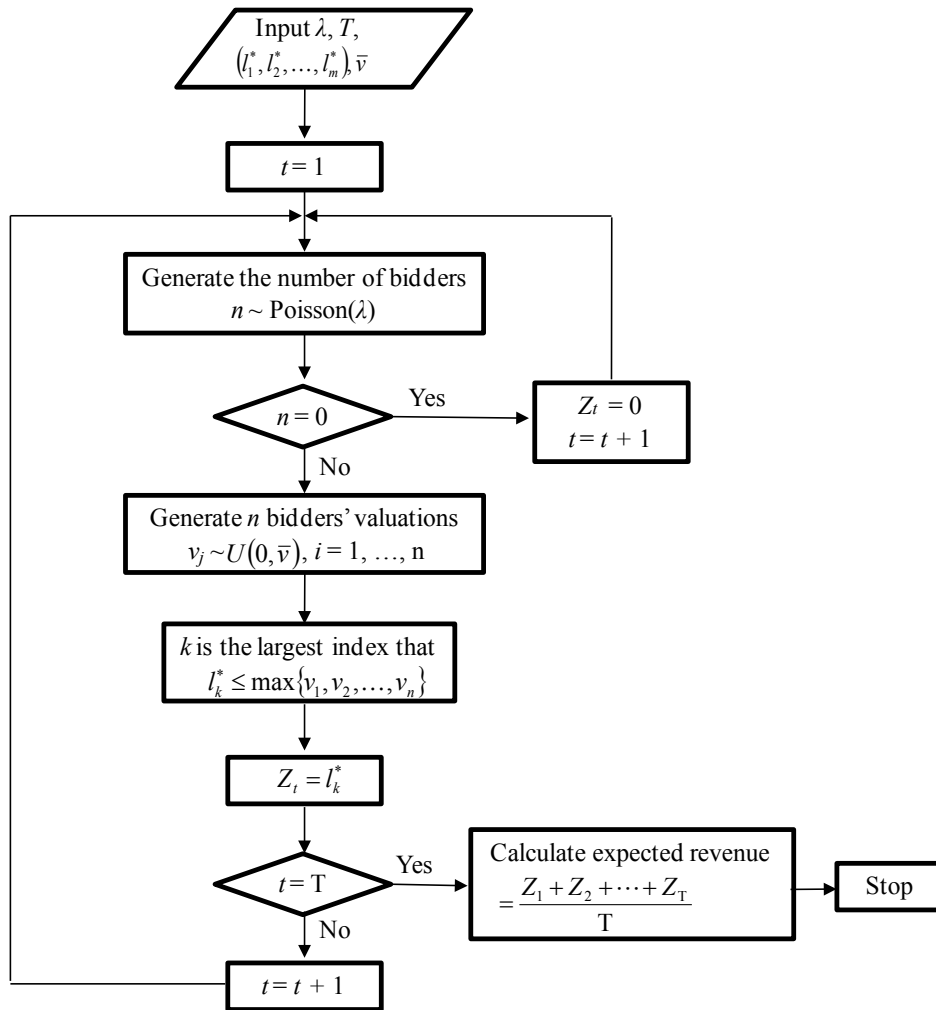


Figure 4.7. Flowchart of simulation process.

Table 4.9

*Maximum Expected Revenues Based on Two Different Approaches*

m	$\lambda$	Theoretical	Simulation	Error
6	1	3.1481	3.1544	-0.0063
	2	4.9722	4.9768	-0.0046
	4	6.8485	6.8438	0.0047
	10	8.5777	8.5713	0.0064
	20	9.2636	9.2541	0.0095
9	1	3.3283	3.3210	0.0073
	2	5.2158	5.2127	0.0031
	4	7.0982	7.1027	-0.0045
	10	8.7403	8.7332	0.0071
	20	9.3590	9.3522	0.0068
12	1	3.4171	3.4081	0.0090
	2	5.3343	5.3418	-0.0075
	4	7.2163	7.2219	-0.0056
	10	8.8131	8.8082	0.0049
	20	9.4000	9.4082	-0.0082

## 4.8 Conclusions

The main focus of this chapter is on the optimal design of Dutch auction where the number of bidders is a random number rather than a constant. I set out to reveal some important properties of the optimal bid levels and the maximum expected revenue in this extension of the basic models discussed previously. Consistent with the treatment in Chapter 3, a comparative study of three design strategies is conducted and numerical examples along with graphical illustrations are used to help explain the key propositions developed.

Throughout the presentation, much emphasis has been placed on how uncertainty about the number of bidders affects the optimal design of Dutch auctions. As it turns out, the only impact is on the distribution pattern of optimal bid levels when  $\lambda = 1$  and the maximum expected revenue to be received by the auctioneer. Most of the properties possessed by  $l_1^*, l_2^*, \dots, l_m^*$  and  $Z^*$  in Chapter 3 are preserved in Chapter 4.

It should be pointed out that our empirical studies show the maximum expected revenue in the presence of randomness of bidder participation to be strictly lower than its counterpart when the number of bidder is fixed when both  $\lambda$  and  $n$  assumes the same value. Nonetheless, Harstad (1990) and Klemperer (1999) have argued that if  $\lambda = n$  then  $Z_\lambda^* = Z^*$  so long as the bidders are risk-neutral and their private valuations are independent. Apparently, more research is called for to account for the discrepancy between them in the future.

## CHAPTER 5

# A DUTCH AUCTION MODEL WITH A SECRET RESERVE PRICE AND AN UNCERTAIN NUMBER OF BIDDERS

### 5.1 Introduction

This chapter is concerned with a Dutch auction where the number of bidders is a random variable and the auctioneer reserves the right not to sell the object if the highest bid is lower than some threshold amount called the reserve (or reservation) price. The purpose of setting such a lowest acceptable bid, which may be secret or public, is to prevent the object from “being given away” in a temporary soft market (Cassady, 1967). A secret reserve price is never made available to the bidders whereas a public one is announced before the auctioning process begins. The major thrust of the present chapter is to tackle a variant of the auctioning problem discussed in Chapter 4 by introducing a secret reserve price into the model. In short, I am considering a Dutch auction where an object is to be sold in an open-bid, first-price, descending competitive process with discrete bid levels and a secret reserve price when the number of bidders is uncertain.

Auctions with a secret reserve price have been studied and practiced extensively for many years. For instance, Cassady (1967) discusses Dutch auctions where neither the existence of a reserve price nor its value is announced prior to the start of the event. Sotheby’s and Christie’s often use a reserve price in traditional open-cry English auctions and never reveal what the minimum figure is (Ashenfelter, 1989). In online auctions conducted on eBay, potential buyers are usually told at the beginning that there is a reserve price but no knowledge about its

value is available. During the course of auction, however, they will be informed if the reserve price has been met. It has been reported that 14% of the e-Bay auctions in 2003 were held with a secret reserve price (Bajari and Hortaçsu, 2003). What is summarized in Table 5.1 are some research works on auctions involving a secret reserve price.

Table 5.1

*Auction Studies with a Secret Reserve Price*

Focus of Study	Sources
Federal sales of offshore oil and gas leases	Hendricks, Porter and Wilson (1994), Hendricks and Porter (1988), Porter (1995)
Wine and art work auctions	Ashenfelter (1989)
Timber auctions	Li and Perrigne (2003)
Coin auctions on eBay	Bajari and Hortacsu (2003)
Cattle auctions	Zulehner (2009)
Forest contracts in Canada	Brisset and Naegelen (2006)
Golf drivers auctions on eBay	Hossain (2008)

In the following discussions, the assumptions made in Chapters 3 and 4 remain applicable for determining the optimal bid levels, the maximum expected revenue, and the probability of selling the object of interest. More specifically, it is posited that rational and risk-neutral bidders arrive at the auction site according to a Poisson distribution with a mean of  $\lambda \geq 0$ , whose PDF is

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}, n = 0, 1, 2, \dots$$

All bidders have private and independent valuations about the

object to be auctioned off. Lastly, the winner will be randomly selected if multiple bidders show their willingness to pay at the highest acceptable bid level that is greater than or equal to the secret reserve price.

## 5.2 Another New Model

Let  $R$  denote the secret reserve price. Since this lowest acceptable bid normally increases with the value of the object for sale, it seems reasonable to express  $R$  as a fraction of the highest possible bidder's valuation, i.e.,  $R = \beta\bar{v}$  with  $0 \leq \beta \leq 1$ . It is required that

$\beta\bar{v} = l_{S,1} < l_{S,2} < \dots < l_{S,m}$ , where  $l_{S,i}$  is the  $i^{\text{th}}$  bid level in the new Dutch auction model,  $i = 1, 2, \dots, m$ ; namely, the  $m$  bid levels all have to be greater than or equal to the secret reserve price.

Recall from (3.5) that the expected revenue from a Dutch auction with  $m$  bid levels and  $n$  bidders is  $Z = \sum_{i=1}^m \frac{l_i}{\bar{v}^n} (l_{i+1}^n - l_i^n)$  if  $v_j \sim U(0, \bar{v})$  with  $l_1 = 0$ ,  $F(l_1) = 0$ ,  $l_{m+1} = \bar{v}$ , and  $F(l_{m+1}) = 1$ .

If  $Z_{S,\lambda}$  is the expected revenue from the Dutch auction with a secret reserve price and an uncertain number of bidders, it can be found by multiplying the expected revenue from a Dutch auction with a particular number of bidders by its likelihood of occurrence and summing up the results. Thus,  $Z_{S,\lambda}$  has the same expression as  $Z_\lambda$  derived in Chapter 4 where a secret reserve price is absent. The only difference is that the discrete bid levels are all bounded below by  $R$ .

Based on the above discussions, an NLP for the new Dutch auction under consideration is presented below:

$$\begin{aligned}
 \text{Maximize} \quad & Z_{S,\lambda} = \sum_{i=1}^m l_{S,i} e^{-\lambda} \left( e^{\frac{\lambda l_{S,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{S,i}}{\bar{v}}} \right) \\
 \text{subject to:} \quad & l_{S,i+1} \geq l_{S,i}, i = 1, 2, \dots, m \\
 & l_{S,1} = \beta\bar{v} \\
 & l_{S,m+1} = \bar{v}
 \end{aligned} \tag{5.1}$$

In addition, the probability of sale,  $P_{S,\lambda}$ , can be expressed as follows in light of (4.3):

$$\begin{aligned}
P_{S,\lambda} &= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \sum_{i=1}^m \frac{1}{\bar{v}^n} (l_{S,i+1}^n - l_{S,i}^n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \left( \frac{l_{S,m+1}^n}{\bar{v}^n} - \frac{l_{S,1}^n}{\bar{v}^n} \right) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \left[ \frac{\bar{v}^n}{\bar{v}^n} - \frac{(\beta \bar{v})^n}{\bar{v}^n} \right] \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} (1 - \beta^n) \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} - \sum_{n=1}^{\infty} \frac{(\beta \lambda)^n e^{-\lambda}}{n!} \\
&= 1 - e^{-\lambda} - e^{-\lambda(1-\beta)} (1 - e^{-\lambda\beta}) \\
&= 1 - e^{-\lambda(1-\beta)}
\end{aligned} \tag{5.2}$$

### 5.3 Existence and Properties of Optimal Bid Levels

Recall that the convexity of the constraint set and the concavity of the objective function are two sufficient conditions for the existence of global optimality of a nonlinear mathematical program. Note that the constraint set of (5.1) is convex since each of the first  $m$  constraints is a half space and each of the last two constraints is a hyperplane. Moreover, the objective function in (5.1) is the same as that in (4.2), whose concavity has been proven in Appendix L. The following proposition is set forth without proof, where  $(l_{S,1}^*, l_{S,2}^*, \dots, l_{S,m}^*)$  is an optimal solution to (5.1).

**Proposition 5.1:**  $(l_{S,1}^*, l_{S,2}^*, \dots, l_{S,m}^*)$  exists if  $v_j \sim U(0, \bar{v})$ ,  $j = 1, 2, \dots, n$ .



Given the existence of the optimal bid levels in the Dutch auction of interest, I now seek

to characterize their behaviors. To begin, define  $l'_{S,i} = \frac{l_{S,i}}{\bar{v}} - \beta$  with

$l'_{S,1} = \frac{l_{S,1}}{\bar{v}} - \beta = \frac{\beta\bar{v}}{\bar{v}} - \beta = 0$  and  $l'_{S,m+1} = \frac{l_{S,m+1}}{\bar{v}} - \beta = \frac{\bar{v}}{\bar{v}} - \beta = 1 - \beta$ . The objective function in

(5.1) can be rewritten as

$$\begin{aligned}
Z_{S,\lambda} &= \sum_{i=1}^m \bar{v}(l'_{S,i} + \beta) e^{-\lambda} \left[ e^{\lambda(l'_{S,i+1} + \beta)} - e^{\lambda(l'_{S,i} + \beta)} \right] \\
&= \bar{v} e^{-\lambda} \sum_{i=1}^m (l'_{S,i} + \beta) \left[ e^{\lambda(l'_{S,i+1} + \beta)} - e^{\lambda(l'_{S,i} + \beta)} \right] \\
&= \bar{v} e^{-\lambda} e^{\lambda\beta} \sum_{i=1}^m (l'_{S,i} + \beta) (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) \\
&= \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m l'_{S,i} (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) + \beta \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) \\
&= \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m l'_{S,i} (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) + \beta \bar{v} e^{-\lambda(1-\beta)} (e^{\lambda l'_{S,m+1}} - e^{\lambda l'_{S,1}}) \\
&= \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m l'_{S,i} (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) + \beta \bar{v} e^{-\lambda(1-\beta)} [e^{\lambda(1-\beta)} - 1] \\
&= \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m l'_{S,i} (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) + \beta \bar{v} [1 - e^{-\lambda(1-\beta)}]
\end{aligned} \tag{5.3}$$

Thus, (5.1) reduces to

$$\begin{aligned}
\text{Maximize} \quad & Z_{S,\lambda} = \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m l'_{S,i} (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) + \beta \bar{v} [1 - e^{-\lambda(1-\beta)}] \\
\text{subject to:} \quad & l'_{S,i+1} \geq l'_{S,i}, i = 1, 2, \dots, m \\
& l'_{S,1} = 0 \\
& l'_{S,m+1} = 1 - \beta
\end{aligned} \tag{5.4}$$

Since the coefficient  $\bar{v} e^{-\lambda(1-\beta)} > 0$  and the constant term  $\beta \bar{v} [1 - e^{-\lambda(1-\beta)}]$  plays no role in the optimization process, solving (5.4) is equivalent to solving (5.5) below with

$$Z'_{S,\lambda} = \frac{Z_{S,\lambda} - \beta \bar{v} [1 - e^{-\lambda(1-\beta)}]}{\bar{v} e^{-\lambda(1-\beta)}}.$$

$$\begin{aligned}
\text{Maximize} \quad & Z'_{S,\lambda} = \sum_{i=1}^m l'_{S,i} (e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}}) \\
\text{subject to:} \quad & l'_{S,i+1} > l'_{S,i}, i = 1, 2, \dots, m \\
& l'_{S,1} = 0 \\
& l'_{S,m+1} = 1 - \beta
\end{aligned} \tag{5.5}$$

Due to the fact that the objective function and the constrain set in (5.5) have the same structure as that in (4.2)<sup>1</sup>, their respective optimal solutions will share the same properties as discussed in Section 4.3. As a result, the following propositions are straightforward and their proofs are omitted.

**Proposition 5.2:**  $l^*_{S,i+1} - l^*_{S,i} < l^*_{S,i} - l^*_{S,i-1}, i = 2, 3, \dots, m.$

**Proposition 5.3:** Given  $m$  and  $\beta$ ,  $l^*_{S,i}$  is an increasing function of  $\lambda$ ,  $i = 2, \dots, m.$

**Proposition 5.4:** Given  $\lambda$  and  $\beta$ ,  $l^*_{S,m}$  is an increasing function of  $m.$

Proposition 5.2 implies that when a secret reserve price is used in the Dutch auction with an uncertain number of bidders, the bid decrements should be increasing as the asking price

---

<sup>1</sup> If we let  $l'_{\lambda,i} = \frac{l_{\lambda,i}}{\bar{v}}$  and  $Z'_\lambda = \frac{Z_\lambda}{\bar{v}e^{-\lambda}}$ , then (4.2) reduces to

$$\begin{aligned}
\text{Maximize} \quad & Z'_\lambda = \sum_{i=1}^m l'_{\lambda,i} (e^{\lambda l'_{\lambda,i+1}} - e^{\lambda l'_{\lambda,i}}) \\
\text{subject to:} \quad & l'_{i+1} \geq l'_i, i = 1, 2, \dots, m \\
& l'_1 = 0 \\
& l'_{m+1} = 1
\end{aligned} \tag{5.6}$$

decreases step by step in order to maximize the auctioneer's expected revenue. This is aligned with Proposition 4.2 where no secret reserve price is in place.

Proposition 5.3 suggests that each optimal bid level should be set higher as more bidders are present in the Dutch auction on average. In view of Propositions 4.3 and 3.4, one sees that the relationship between the optimal bid levels and the size of bidder population holds no matter whether a secret reserve price is used or whether the number of auction participants is certain.

I plot the optimal bid levels for  $m = 6$ ,  $\beta = 0.2$  and  $\lambda = 1, 2, 15$  in Figure 5.1 to illustrate Propositions 5.2 and 5.3 presented above. The lowest optimal bid level,  $l_{s,1}^*$ , is the secret reserve price imposed by the auctioneer. As  $\lambda$  changes from 1 to 2 to 15,  $l_{s,i}^*$ ,  $i = 2, 3, \dots, 6$ , associated with a larger  $\lambda$  (e.g.,  $\lambda = 15$ ) are to the right of their counterparts associated with a smaller  $\lambda$  (e.g.,  $\lambda = 2$ ). In addition,  $l_6^*, l_5^*, \dots, l_2^*$  become farther away from each other as the asking price goes down from right to left. Compared with Figure 4.1 where no secret reserve price is introduced in the Dutch auction, the same distribution pattern of optimal bid levels is observed here except that each bid level must be higher than or equal to the secret reserve price  $R = \beta\bar{v}$ .

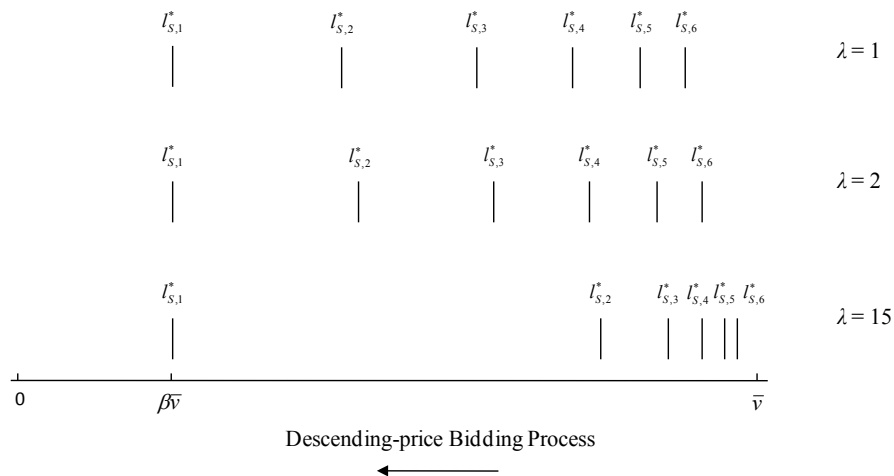


Figure 5.1. Optimal bid levels with  $m = 6$  and  $\beta = 0.2$ .

Proposition 5.4 indicates that the highest optimal bid level,  $l_{S,m}^*$ , will go up as more bid levels are set in the Dutch auction with a secret reserve price and a random number of bidders. This is consistent with the findings reported in Propositions 4.4 and 3.5. It is concluded that neither uncertainty about the number of auction participants nor the existence of secret reserve price has any influence on the relationship between the highest optimal bid level and the number of bid levels.

The optimal bid levels based on  $\lambda = 5$ ,  $\beta = 0.2$  and  $m = 2, 3, 4, 5$  are depicted in Figure 5.2 to help explain Proposition 5.4. The right most vertical bar representing the highest optimal bid level associated with a larger  $m$  (e.g.,  $m = 5$ ) is to the right of its counterpart associated with a smaller  $m$  (e.g.,  $m = 2$ ). This is the same pattern observed in Figure 4.2 except that the lowest optimal bid level is now equal to the secret reserve price.

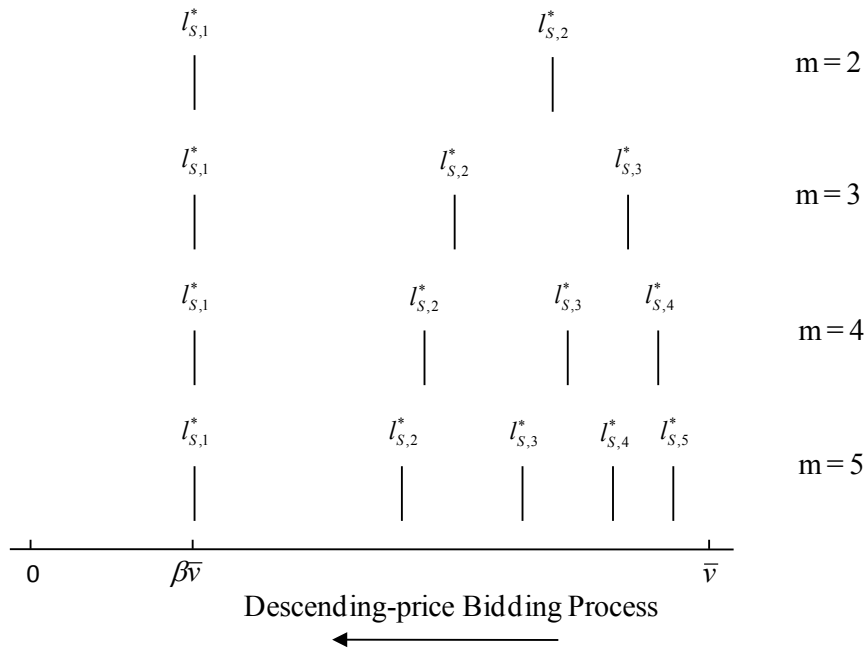


Figure 5.2. Optimal bid levels with  $\lambda = 5$  and  $\beta = 0.2$ .

The purpose of the next two propositions is to characterize the behaviors of the revenue-maximization bid levels as a function of the secret reserve price. Specifically, it is argued in Proposition 5.5 that each of the optimal bid levels will go up if the secret reserve price increases. The proof can be found in Appendix T. Obviously, proving Proposition 5.5 is equivalent to proving  $\frac{\partial l_{S,i}^*}{\partial \beta} > 0, \forall \beta \in (0,1], i = 2, 3, \dots, m$ . In Proposition 5.6, whose proof is given in Appendix U, it is claimed that the increasing rate of  $l_{S,i}^*$  is greater than that of  $l_{S,i+1}^*$  as the secret reserve price increases,  $i = 2, 3, \dots, m-1$ .

**Proposition 5.5:** Given  $\lambda$  and  $m$ ,  $l_{S,i}^*$  is an increasing function of  $\beta$ ,  $i = 2, 3, \dots, m$ .

**Proposition 5.6:** Given  $\lambda$  and  $m$ ,  $\frac{\partial l_{S,i}^*}{\partial \beta} > \frac{\partial l_{S,i+1}^*}{\partial \beta}, \forall \beta \in (0,1], i = 2, 3, \dots, m-1$ .

Assuming that an average of  $\lambda = 5$  bidders participate in the Dutch auction with  $m = 6$  bid levels, I plot the optimal bid levels for  $\beta = 0.1, 0.2, 0.3$  in Figure 5.3. It is interesting to note that the six vertical bars associated with a larger  $\beta$  (e.g.,  $\beta = 0.3$ ) are to the right of their counterparts associated with a smaller  $\beta$  (e.g.,  $\beta = 0.1$ ). Besides, a careful comparison between the first and the third charts in Figure 5.3 reveals that the increasing rate of left vertical bars (e.g.,  $l_{S,2}^*$ ) from  $\beta = 0.1$  to  $\beta = 0.3$  is greater than that of the right vertical bars (e.g.,  $l_{S,6}^*$ ) from  $\beta = 0.1$  to  $\beta = 0.3$ .

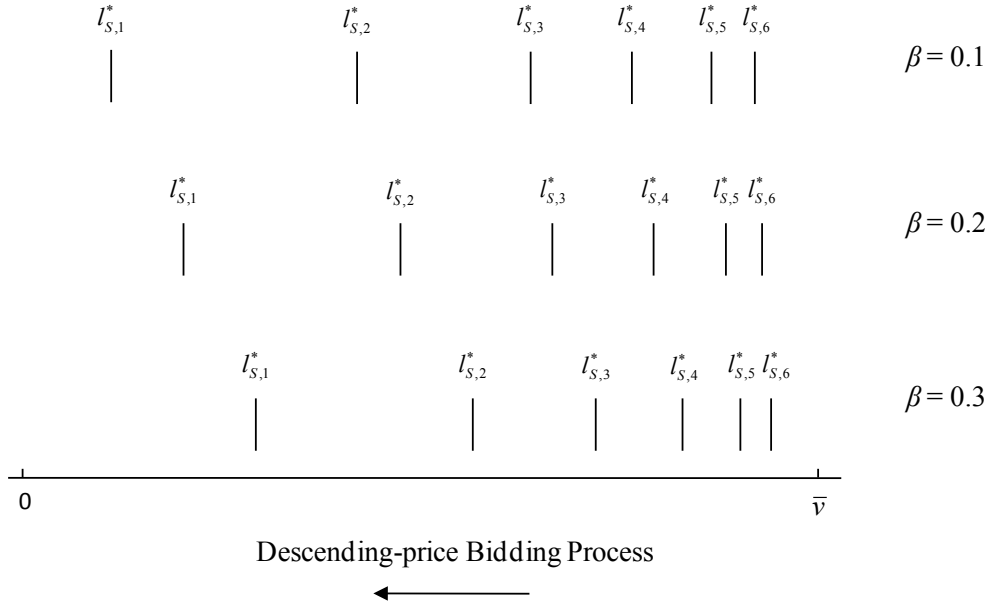


Figure 5.3. Optimal bid levels with  $\lambda = 5$  and  $m = 6$ .

#### 5.4 Properties of Maximum Expected Revenue

Once the global optimal solution to the NLP in (5.1) has been found, one can evaluate the corresponding expected revenue to be received from the Dutch auction by computing  $Z_{S,\lambda}^*$ . For the sake of uncovering the relationships between the maximum expected revenue and other model parameters, however, I develop a mathematical expression for calculating  $Z_{S,\lambda}^*$  and present it in Proposition 5.7 below, whose proof is provided in Appendix V.

**Proposition 5.7:** 
$$Z_{S,\lambda}^* = l_{S,m}^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right] - \beta \bar{v} e^{-\lambda(1-\beta)}$$

Under the condition that  $\beta = 0$ , which means that no secret reserve price is involved in the

Dutch auction,  $Z_{S,\lambda}^*$  reduces to  $Z_{S,\lambda}^* = l_{S,m}^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right]$ . This is essentially the same as

$Z_{\lambda}^* = l_m^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_2^*}{\bar{v}} \right)} \right]$  in Proposition 4.5. The result is not surprising since the auctioning

model examined in Chapter 4 is a special case of that discussed in this chapter with a secret reserve price of zero.

One also sees from the above theorem that  $Z_{S,\lambda}^* < l_{S,m}^*$  since  $0 < e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} < 1$ ,

$\frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right] > 0$  and  $\beta \bar{v} e^{-\lambda(1-\beta)} > 0$ . In particular, I have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} (l_{S,m}^* - Z_{S,\lambda}^*) &= \lim_{\lambda \rightarrow \infty} \left\{ \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right] + \beta \bar{v} e^{-\lambda(1-\beta)} \right\} \\ &= \lim_{\lambda \rightarrow \infty} \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right] + \beta \bar{v} \lim_{\lambda \rightarrow \infty} e^{-\lambda(1-\beta)} \\ &= 0 \end{aligned}$$

This suggests that  $Z_{S,\lambda}^*$  will be practically equal to  $l_{S,m}^*$  if  $\lambda$  is sufficiently large. Consistent with Proposition 4.5 of Chapter 4, I note that  $Z_{S,\lambda}^*$  can be determined based solely on  $l_{S,2}^*$  and  $l_{S,m}^*$  but nothing else.

In the remainder of this section, I seek to understand the links between  $Z_{S,\lambda}^*$ ,  $m$  and  $\lambda$ .

First of all, it is demonstrated in Proposition 5.8 below that the maximum expected revenue to be

generated from the Dutch auction of interest increases with the expected number of bidders. The proof can be found in Appendix W.

**Proposition 5.8:** Given  $m$  and  $\beta$ ,  $Z_{S,\lambda}^*$  is an increasing function of  $\lambda$ .

This theorem apparently follows conventional wisdom since more bidders in the Dutch auction will create a more competitive environment in which the object for sale will be claimed at an early stage of the bidding process while the asking price is high. Consequently, the closing price (hence the auctioneer's revenue) is closer to the highest optimal bid level.

As for the relationship between  $Z_{S,\lambda}^*$  and  $m$ , I see from Proposition 5.9 below that the maximum expected revenue increases with the number of bid levels to be established in the Dutch auction provided that a secret reserve price is used and the number of bidders is subject to a Poisson distribution with a known mean. This can be easily shown by observing that

$Z_\lambda^* = \frac{Z_\lambda^*}{\bar{v}e^{-\lambda}}$  in (5.6) is an increasing function of  $m$  because, according to Proposition 4.7,  $Z_\lambda^*$

increases with  $m$ . It follows immediately that  $Z_{S,\lambda}^*$  in (5.5) is also an increasing function of  $m$

due to the equivalence between (5.5) and (5.6). As a result,  $Z_{S,\lambda}^* = \bar{v}e^{-\lambda(1-\beta)}Z_{S,\lambda}^* + \beta\bar{v}[1 - e^{-\lambda(1-\beta)}]$

also increases with  $m$  since both  $\bar{v}e^{-\lambda(1-\beta)}$  and  $\beta\bar{v}[1 - e^{-\lambda(1-\beta)}]$  are independent of  $m$ .

**Proposition 5.9:** Given  $\lambda$  and  $\beta$ ,  $Z_{S,\lambda}^*$  is an increasing function of  $m$ .



It should be mentioned that Propositions 5.8 and 5.9 presented above are, respectively, consistent with Propositions 3.7 and 3.8 of Chapter 3 as well as Propositions 4.6 and 4.7 of Chapter 4. One then arrives at the conclusion that the impact of each of  $\lambda$  and  $m$  on  $Z_{S,\lambda}^*$  remains the same regardless of the existence or nonexistence of a secret reserve price or the certainty or uncertainty about the number of participants in the Dutch auction.

## 5.5 A Comparative Study

This section aims to derive the discrete bid levels in the Dutch auction based on two widely-used heuristic approaches discussed in Chapters 3 and 4: Strategy E (i.e., evenly-spaced bid levels with fixed decrements) and Strategy D (i.e., unevenly-spaced bid levels with decreasing decrements). Apart from highlighting the superiority of the optimal strategy delineated in the previous sections over the two heuristic strategies, I also compare the expected auction revenues that those two suboptimal design schemes will yield.

### 5.5.1 Strategy E

Strategy E calls for dividing the difference between the ceiling of the common range of bidders' valuations and the secret reserve price by the number of bid levels to be established, and the result is set as the fixed gap between two successive bid levels. Mathematically, I have

$$l_{SE,i} = \frac{(i-1)(1-\beta)\bar{v}}{m} + \beta\bar{v}, \quad i = 1, 2, \dots, m \quad (5.7)$$

where  $l_{SE,i}$  is the  $i^{\text{th}}$  discrete bid level with  $l_{SE,1} = \beta\bar{v}$  and, as before, it is assumed that  $l_{SE,m+1} = \bar{v}$ .

In light of (4.4), the probability of selling the object at  $l_{SE,i}$  is

$$\begin{aligned}
P(l_{SE,i}) &= e^{-\lambda} \left( e^{\frac{\lambda l_{SE,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{SE,i}}{\bar{v}}} \right) \\
&= e^{-\lambda} \left[ e^{\frac{\lambda i(1-\beta)}{m} + \lambda\beta} - e^{\frac{\lambda(i-1)(1-\beta)}{m} + \lambda\beta} \right] \\
&= e^{-\lambda(1-\beta)} \left[ e^{\frac{\lambda i(1-\beta)}{m}} - e^{\frac{\lambda(i-1)(1-\beta)}{m}} \right] \tag{5.8}
\end{aligned}$$

The  $m = 5$  bid levels resulting from this strategy with a secret reserve price in the Dutch auction are graphically displayed in Figure 5.4 below.

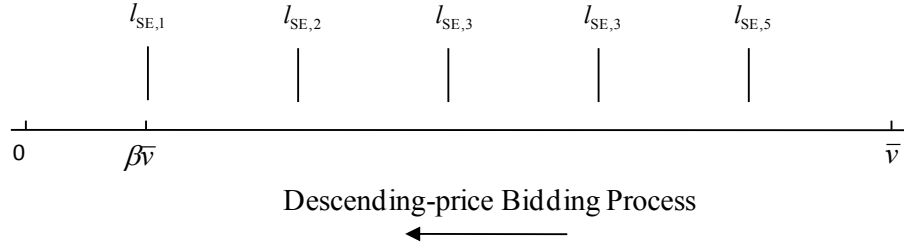


Figure 5.4. Evenly-spaced bid levels with fixed decrements with  $m = 5$ .

Let  $Z_{S,E}$  be the expected revenue to be generated by following Strategy E from a Dutch auction with a secret reserve price and an uncertain number of bidders. I see from (5.7) and (5.8) that

$$\begin{aligned}
Z_{S,E} &= \sum_{i=1}^m l_{SE,i} P(l_{SE,i}) \\
&= \sum_{i=1}^m \left[ \frac{(i-1)(1-\beta)\bar{v}}{m} + \beta\bar{v} \right] e^{-\lambda(1-\beta)} \left[ e^{\frac{\lambda i(1-\beta)}{m}} - e^{\frac{\lambda(i-1)(1-\beta)}{m}} \right] \\
&= \frac{(1-\beta)\bar{v}}{m} e^{-\lambda(1-\beta)} \sum_{i=1}^m (i-1) \left[ e^{\frac{\lambda i(1-\beta)}{m}} - e^{\frac{\lambda(i-1)(1-\beta)}{m}} \right] + \beta\bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m \left[ e^{\frac{\lambda i(1-\beta)}{m}} - e^{\frac{\lambda(i-1)(1-\beta)}{m}} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{(1-\beta)\bar{v}}{m} e^{-\lambda(1-\beta)} \sum_{i=1}^m (i-1) \left[ e^{\frac{\lambda i(1-\beta)}{m}} - e^{\frac{\lambda(i-1)(1-\beta)}{m}} \right] + \beta\bar{v} e^{-\lambda(1-\beta)} [e^{\lambda(1-\beta)} - 1] \\
&= \frac{(1-\beta)\bar{v}}{m} e^{-\lambda(1-\beta)} \sum_{i=1}^m (i-1) \left[ e^{\frac{\lambda i(1-\beta)}{m}} - e^{\frac{\lambda(i-1)(1-\beta)}{m}} \right] + \beta\bar{v} [1 - e^{-\lambda(1-\beta)}] \tag{5.9}
\end{aligned}$$

### 5.5.2 Strategy D

Strategy D may be better grasped by referring to Figure 5.5, where  $m = 5$  bid levels with decreasing decrements are to be established in the Dutch auction with a secret reserve price and an uncertain number of bidders. This strategy is designed based on the premise that the decrement between any two successive bid levels changes at a fixed discounting rate of

$$\frac{l_{SD,i} - l_{SD,i-1}}{l_{SD,i+1} - l_{SD,i}} = \frac{1}{1+r} \text{ with } r > 0 \text{ being the discounting factor and } d > 0 \text{ being the first (and largest)}$$

decrement. Let  $l_{SD,i}$  be the  $i^{\text{th}}$  bid level and assume that  $l_{SD,m+1} = \bar{v}$ . I then have

$$l_{SD,i+1} - l_{SD,i} = \frac{d}{(1+r)^{m-i}}, \quad i = 1, 2, \dots, m.$$

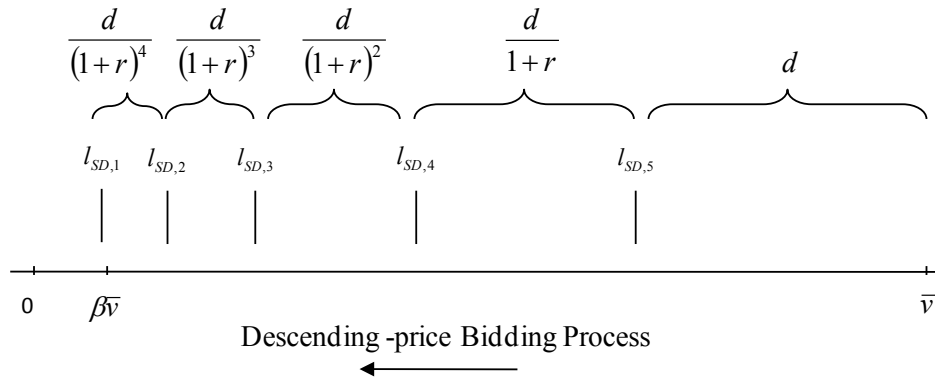


Figure 5.5. Unevenly-spaced bid levels with decreasing decrements with  $m = 5$ .

Since  $\sum_{i=1}^m (l_{SD,i+1} - l_{SD,i}) = l_{SD,m+1} - l_{SD,1} = \bar{v} - \beta\bar{v} = (1 - \beta)\bar{v}$ , I have  $\sum_{i=0}^{m-1} \frac{d}{(1+r)^i} = (1 - \beta)\bar{v}$  and

$d = \frac{(1 - \beta)\bar{v}}{\sum_{i=0}^{m-1} (1+r)^{-i}}$ . Consequently,  $l_{SD,i}$ ,  $i = 2, \dots, m$ , can be written as

$$\begin{aligned}
l_{SD,i} &= \bar{v} - d \sum_{j=0}^{m-i} \frac{1}{(1+r)^j} \\
&= \bar{v} - (1 - \beta)\bar{v} \frac{\sum_{j=0}^{m-i} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \\
&= \frac{\sum_{j=m-i+1}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \bar{v} + \frac{\sum_{j=0}^{m-i} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \beta\bar{v} \\
&= \frac{\sum_{j=m-i+1}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \bar{v} + \left[ 1 - \frac{\sum_{j=m-i+1}^{m-1} (1+r)^{-j}}{\sum_{j=0}^{m-1} (1+r)^{-j}} \right] \beta\bar{v}
\end{aligned} \tag{5.10}$$

Thus, the probability of selling the object at  $l_{SD,i}$  is

$$P(l_{SD,i}) = e^{-\lambda} \left( e^{\frac{\lambda l_{SD,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{SD,i}}{\bar{v}}} \right) \tag{5.11}$$

Let  $Z_{S,D}$  be the expected revenue to be generated by following Strategy D in a Dutch auction

with a secret reserve price and an uncertain number of bidders. It follows from (5.11) that

$$\begin{aligned}
Z_{S,D} &= \sum_{i=1}^m l_{SD,i} P(l_{SD,i}) \\
&= \sum_{i=1}^m l_{SD,i} e^{-\lambda} \left( e^{\frac{\lambda l_{SD,i+1}}{\bar{v}}} - e^{\frac{\lambda l_{SD,i}}{\bar{v}}} \right)
\end{aligned} \tag{5.12}$$

### 5.5.3 Revenue comparison

What follows is an important result that enables us to compare the respective auction revenues from the use of Strategy E and Strategy D, and its proof is given in Appendix X. As

defined in Section 3.6.3 and Section 4.5,  $\tau_i = \frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}}$ ,  $t \geq 0$ ,  $i = 2, 3, \dots, m$ .

**Proposition 5.10:**  $G_t = \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m [(1-\beta)\tau_i + \beta] [e^{\lambda(1-\beta)\tau_{i+1}} - e^{\lambda(1-\beta)\tau_i}]$  is a decreasing function of  $t$ ,  $t \geq 0$ .

It is interesting to note that  $G_t = Z_{S,D}$  if  $t > 0$ . In case  $t = 0$ , however, one has  $\tau_i = \frac{i-1}{m}$ ,

$i = 2, 3, \dots, m$ , and

$$\begin{aligned}
G_t &= \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m [(1-\beta)\tau_i + \beta] [e^{\lambda(1-\beta)\tau_{i+1}} - e^{\lambda(1-\beta)\tau_i}] \\
&= \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m \left[ (1-\beta) \frac{i-1}{m} + \beta \right] \left[ e^{\lambda(1-\beta) \frac{i}{m}} - e^{\lambda(1-\beta) \frac{i-1}{m}} \right] \\
&= \frac{1-\beta}{m} \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m (i-1) \left[ e^{\frac{\lambda(1-\beta)i}{m}} - e^{\frac{\lambda(1-\beta)(i-1)}{m}} \right] + \beta \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m \left[ e^{\frac{\lambda(1-\beta)i}{m}} - e^{\frac{\lambda(1-\beta)(i-1)}{m}} \right] \\
&= \frac{1-\beta}{m} \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m (i-1) \left[ e^{\frac{\lambda(1-\beta)i}{m}} - e^{\frac{\lambda(1-\beta)(i-1)}{m}} \right] + \beta \bar{v} e^{-\lambda(1-\beta)} e^{\lambda(1-\beta)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1-\beta}{m} \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m (i-1) \left[ e^{\frac{\lambda(1-\beta)i}{m}} - e^{\frac{\lambda(1-\beta)(i-1)}{m}} \right] + \beta \bar{v} \\
&= Z_{S,E}
\end{aligned}$$

Given that  $G_t$  is a decreasing function of  $t$ ,  $G_t = Z_{S,D}$  if  $t > 0$  and  $G_t = Z_{S,E}$  if  $t = 0$ , I conclude that  $Z_{S,E} > Z_{S,D}$ . This leads immediately to the corollary below without proof since the optimal strategy represented by  $(l_{S,1}^*, l_{S,2}^*, \dots, l_{S,m}^*)$  outperforms any nonoptimal methods including Strategies E and D for setting the bid levels in the Dutch auction being studied. In turn, Strategy E is more attractive than Strategy D from both the economic and the computational perspectives.

**Corollary 5.1:**  $Z_{S,\lambda}^* > Z_{S,E} > Z_{S,D}$

## 5.6 Numerical Illustrations

To gain a better understanding of the impact of a secret reserve price on the optimal bid levels and the maximum expected revenue in a Dutch auction with a secret reserve price and an uncertain number of bidders, I present a collection of numerical examples for illustrative purposes in this section. The possible values to be assumed by the key model parameters are  $\bar{v} = 10$ ,  $m \in \{3, 4, \dots, 17\}$ ,  $\lambda \in \{1, 2, 4, 10, 20\}$  and  $\beta \in \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$ .

In light of (5.1), various nonlinear programs based on different combinations of  $m$ ,  $\lambda$  and  $\beta$  values with  $\bar{v} = 10$  are drawn up and the optimal solutions are found. For instance, the appropriate NLP with  $\bar{v} = 10$ ,  $m = 3$ ,  $\lambda = 2$  and  $\beta = 0.2$  is shown below with  $l_{S,1}$ ,  $l_{S,2}$ , and  $l_{S,3}$  being the decision variables:

$$\begin{aligned}
\text{Maximize} \quad & Z_{S,\lambda} = \frac{1}{e^2} \left[ l_{S,1} \left( e^{\frac{l_{S,2}}{5}} - e^{\frac{l_{S,1}}{5}} \right) + l_{S,2} \left( e^{\frac{l_{S,3}}{5}} - e^{\frac{l_{S,2}}{5}} \right) + l_{S,3} \left( e^{\frac{l_{S,4}}{5}} - e^{\frac{l_{S,3}}{5}} \right) \right] \\
\text{subject to :} \quad & l_{S,2} \geq l_{S,1} \\
& l_{S,3} \geq l_{S,2} \\
& l_{S,1} = 2 \\
& l_{S,4} = 10
\end{aligned}$$

The above optimization model may be solved by running Solver to determine the three revenue-maximization bid levels as well as the highest possible expected revenue. The same procedure is repeated for each of the other  $15 \times 5 \times 9 - 1 = 674$  instances and all of the results are summarized for subsequent analysis.

### 5.6.1 Optimal bid levels

I start by examining the behaviors of  $l_{S,1}^*, l_{S,2}^*, \dots, l_{S,m}^*$  and how they relate to  $m$ ,  $\lambda$  and  $\beta$ . A few observations are in order. First of all, Table 5.2 and Figure 5.6 show the optimal solutions to the Dutch auction models with a secret reserve price when  $\bar{v} = 10$ ,  $m = 12$ ,  $\beta = 0.2$  and  $\lambda = 1, 2, 4, 10, 20$ . One can see that  $l_{S,i}^*$  is moving farther away from  $l_{S,i+1}^*$  as the asking price goes down from right to left in Table 5.2 or from top to bottom in Figure 5.6,  $i = 1, \dots, 11$ . This concurs with the bid decrements shown in Table 5.3 and is consistent with Proposition 5.2.

As for the connection between  $l_{S,i}^*$  and  $\lambda$ , I observe from the same exhibits that  $l_{S,i}^*$  becomes larger as  $\lambda$  increases from 1 to 2 to 4 to 10 to 20,  $i = 2, \dots, 12$ , which is in line with Proposition 5.3. Additionally, a close look at Figure 5.6 reveals that each of the  $l_{S,i}^*$  curves tends to become flat as  $\lambda$  increases. This implies that  $l_{S,i}^*$  increases with  $\lambda$  at a decreasing rate.

Table 5.2

Optimal Bid Levels with  $\bar{v} = 10$ ,  $m = 12$  and  $\beta = 0.2$

$\lambda$	$l_{S,1}^*$	$l_{S,2}^*$	$l_{S,3}^*$	$l_{S,4}^*$	$l_{S,5}^*$	$l_{S,6}^*$	$l_{S,7}^*$	$l_{S,8}^*$	$l_{S,9}^*$	$l_{S,10}^*$	$l_{S,11}^*$	$l_{S,12}^*$
1	2	2.8016	3.5725	4.3153	5.0318	5.7239	6.3930	7.0406	7.6682	8.2768	8.8677	9.4418
2	2	2.9638	3.8450	4.6565	5.4085	6.1089	6.7646	7.3806	7.9615	8.5109	9.0320	9.5276
4	2	3.3662	4.4561	5.3607	6.1328	6.8057	7.4016	7.9360	8.4203	8.8631	9.2707	9.6483
10	2	4.8360	6.1805	7.0325	7.6488	8.1289	8.5211	8.8519	9.1377	9.3891	9.6134	9.8157
20	2	6.5186	7.6717	8.2696	8.6629	8.9531	9.1819	9.3703	9.5302	9.6690	9.7914	9.9009

Table 5.3

Optimal Bid Decrements with  $\bar{v} = 10$ ,  $m = 12$  and  $\beta = 0.2$

$\lambda$	$l_{S,2}^* - l_{S,1}^*$	$l_{S,3}^* - l_{S,2}^*$	$l_{S,4}^* - l_{S,3}^*$	$l_{S,5}^* - l_{S,4}^*$	$l_{S,6}^* - l_{S,5}^*$	$l_{S,7}^* - l_{S,6}^*$	$l_{S,8}^* - l_{S,7}^*$	$l_{S,9}^* - l_{S,8}^*$	$l_{S,10}^* - l_{S,9}^*$	$l_{S,11}^* - l_{S,10}^*$	$l_{S,12}^* - l_{S,11}^*$
1	0.8016	0.7710	0.7428	0.7165	0.6920	0.6691	0.6476	0.6275	0.6087	0.5909	0.5741
2	0.9638	0.8812	0.8115	0.7519	0.7005	0.6556	0.6161	0.5809	0.5494	0.5211	0.4956
4	1.3662	1.0899	0.9046	0.7721	0.6729	0.5959	0.5344	0.4843	0.4427	0.4076	0.3776
10	2.8360	1.3445	0.8520	0.6163	0.4801	0.3921	0.3308	0.2858	0.2514	0.2243	0.2023
20	4.5186	1.1531	0.5979	0.3933	0.2902	0.2288	0.1884	0.1599	0.1387	0.1225	0.1095

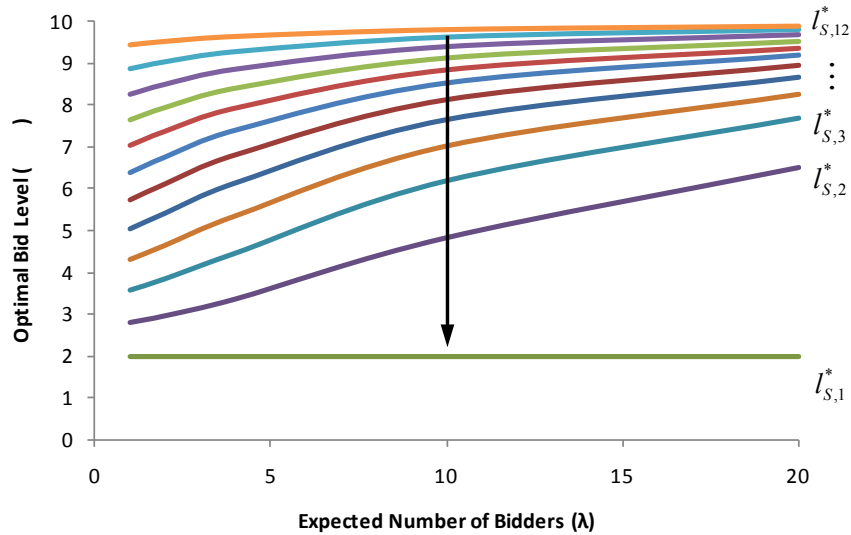


Figure 5.6. Optimal bid levels versus expected number of bidders with  $\bar{v} = 10$ ,  $m = 12$  and  $\beta = 0.2$ .



In regard to the issue of how  $l_{S,m}^*$  varies as  $m$  and  $\lambda$  change, Table 5.4 and Figure 5.7 summarize the highest optimal bid levels for various combinations of  $m$  and  $\lambda$  values with  $\bar{v} = 10$  and  $\beta = 0.2$ . It is obvious from a review of those two exhibits that  $l_{S,m}^*$  increases with  $m$  and  $\lambda$ , which is in agreement with Propositions 5.3 and 5.4. Thus, one can expect  $l_{S,m}^*$  to get closer to the highest possible bidder's valuation,  $\bar{v}$ , as  $\lambda$  and  $m$  become larger. It is worth pointing out that similar findings on the behaviors of  $l_{S,1}^*, l_{S,2}^*, \dots, l_{S,m}^*$  and their relationships with  $m$  and  $\lambda$  have been reported in Chapter 4, where Dutch auctions without a secret reserve price are treated under the assumption that the number of participations is uncertain.

Table 5.4

Highest Optimal Bid Level versus Expected Number of Bidders and Number of Bid Levels with  $\bar{v}=10$  and  $\beta = 0.2$

$\lambda \backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	7.6582	8.2716	8.6303	8.8658	9.0323	9.1562	9.2519	9.3281	9.3902	9.4417	9.4852	9.5225	9.5548	9.5829	9.6077
2	7.9269	8.4923	8.8158	9.0251	9.1716	9.2799	9.3631	9.4291	9.4827	9.5276	9.5649	9.5967	9.6241	9.6481	9.6692
4	8.3315	8.8183	9.0864	9.2558	9.3723	9.4574	9.5222	9.5732	9.6144	9.6483	9.6768	9.7010	9.7218	9.7399	9.7558
10	8.9646	9.3065	9.4815	9.5870	9.6573	9.7073	9.7447	9.7737	9.7968	9.8157	9.8314	9.8446	9.8559	9.8657	9.8743
20	9.3628	9.5928	9.7036	9.7681	9.8099	9.8392	9.8608	9.8773	9.8904	9.9009	9.9097	9.9170	9.9233	9.9286	9.9333

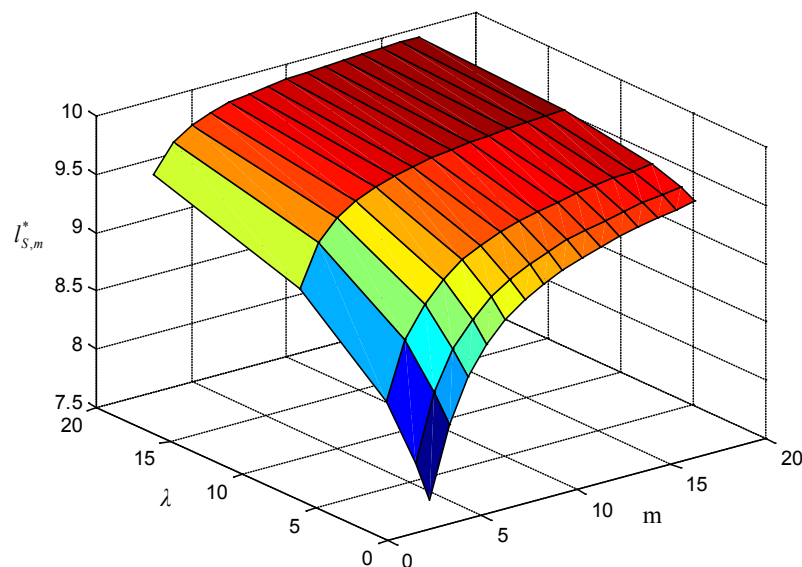


Figure 5.7. Highest optimal bid level versus expected number of bidders and number of bid levels with  $\bar{v}=10$  and  $\beta = 0.2$ .

Finally, the optimal bid levels for the NLPs with  $\bar{v} = 10$ ,  $m = 12$ ,  $\lambda = 4$ , and  $\beta = 0.0$ , 0.1, ..., 0.8 are displayed in Table 5.5 and the continuous versions of them are plotted in Figure 5.8. One can see from the two exhibits that  $l_i^*$  becomes larger as  $\beta$  increases from 0.0 to 0.8,  $i = 2, 3, \dots, 12$ , which is aligned with Proposition 5.5. In particular, note that  $\beta = 0.0$  implies the absence of a secret reserve price and (5.1) reduces to (4.2) in Chapter 4. In general, Table 5.5 and Figure 5.8 disclose that the use of a higher secret reserve price leads to higher optimal bid levels in the Dutch auction being considered. Also, a careful examination of each curve in Figure 5.8 indicates that the increasing rate of  $l_{S,i}^*$  increases from top to bottom in Figure 5.8,  $i = 1, \dots, 11$ . This observation agrees with Proposition 5.6.

Table 5.5

*Optimal Bid Levels with  $\bar{v} = 10$ ,  $m = 12$  and  $\lambda = 4$*

$\beta$	$\beta\bar{v}$	$l_{S,1}^*$	$l_{S,2}^*$	$l_{S,3}^*$	$l_{S,4}^*$	$l_{S,5}^*$	$l_{S,6}^*$	$l_{S,7}^*$	$l_{S,8}^*$	$l_{S,9}^*$	$l_{S,10}^*$	$l_{S,11}^*$	$l_{S,12}^*$
0	0	0	2.0005	3.4703	4.6263	5.5766	6.3819	7.0801	7.6958	8.2462	8.7437	9.1975	9.6144
0.1	1	1	2.6663	3.9432	4.9747	5.8383	6.5801	7.2297	7.8071	8.3267	8.7987	9.2311	9.6299
0.2	2	2	3.3662	4.4561	5.3608	6.1329	6.8057	7.4015	7.9360	8.4203	8.8631	9.2707	9.6483
0.3	3	3	4.0993	5.0104	5.7872	6.4637	7.0623	7.5991	8.0853	8.5296	8.9386	9.3174	9.6701
0.4	4	4	4.8641	5.6063	6.2562	6.8339	7.3536	7.8258	8.2583	8.6573	9.0274	9.3726	9.6959
0.5	5	5	5.6592	6.2444	6.7701	7.2473	7.6840	8.0856	8.4580	8.8058	9.1316	9.4378	9.7267
0.6	6	6	6.4816	6.9221	7.3277	7.7037	8.0539	8.3816	8.6896	8.9801	9.2548	9.5156	9.7635
0.7	7	7	7.3296	7.6392	7.9311	8.2072	8.4691	8.7181	8.9555	9.1823	9.3994	9.6076	9.8076
0.8	8	8	8.1995	8.3918	8.5783	8.7586	8.9317	9.0978	9.2607	9.4196	9.5728	9.7200	9.8621

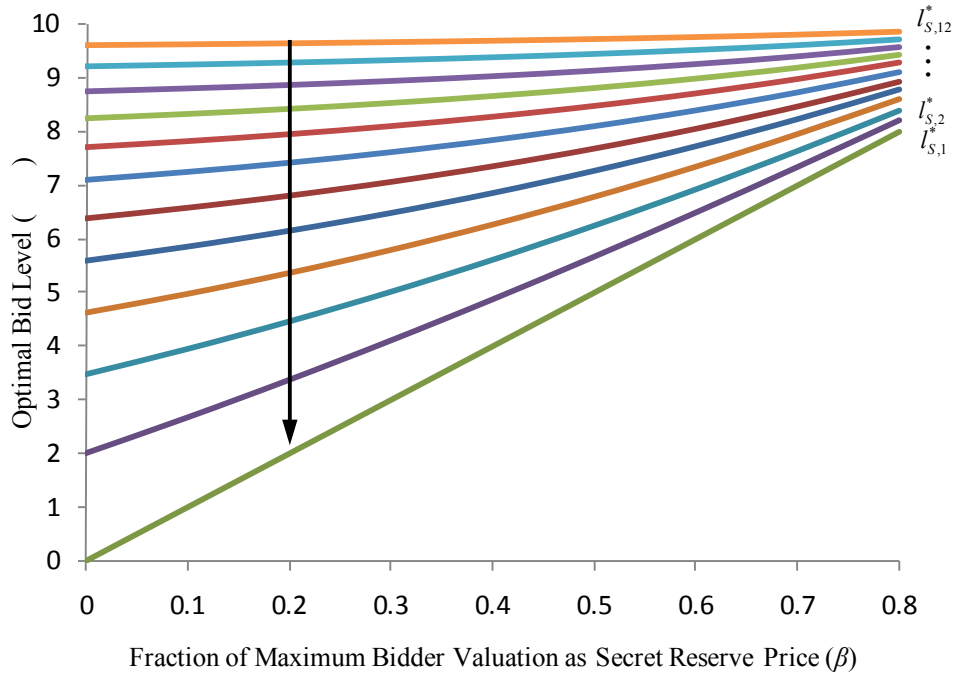


Figure 5.8. Optimal bid levels versus fraction of maximum bidder’s valuation as secret reserve price with  $\bar{v} = 10$ ,  $m = 12$  and  $\lambda = 4$ .

### 5.6.2 Maximum expected revenue

I now turn our attention to the behaviors of  $Z_{S,\lambda}^*$  and how it interacts with  $\lambda$ ,  $m$  and  $\beta$ .

Towards that end, maximum expected revenues generated from the present Dutch auction for various combinations of  $m$  and  $\lambda$  values with  $\bar{v} = 10$  and  $\beta = 0.2$  are summarized in Table 5.6, and their continuous versions are graphed in Figures 5.9. It is seen from the two exhibits that  $Z_{S,\lambda}^*$  increases with both the expected number of bidders and the number of bid levels, which are in line with Propositions 5.8 and 5.9. However, the  $Z_{S,\lambda}^*$  curves tend to flatten out as both  $\lambda$  and  $m$  increase. This suggests that the improvement in auction revenue is more limited for large

values of  $\lambda$  and  $m$ . In fact, the diminishing rate of improvement is evident from a close look at both Tables 5.7 and 5.8.

Table 5.6

Maximum Expected Revenue with  $\bar{v} = 10$  and  $\beta = 0.2^*$

$\lambda$	$Z_{S,\lambda,m=3}^*$	$Z_{S,\lambda,m=4}^*$	$Z_{S,\lambda,m=5}^*$	$Z_{S,\lambda,m=6}^*$	$Z_{S,\lambda,m=7}^*$	$Z_{S,\lambda,m=8}^*$	$Z_{S,\lambda,m=9}^*$	$Z_{S,\lambda,m=10}^*$	$Z_{S,\lambda,m=11}^*$	$Z_{S,\lambda,m=12}^*$	$Z_{S,\lambda,m=13}^*$	$Z_{S,\lambda,m=14}^*$	$Z_{S,\lambda,m=15}^*$	$Z_{S,\lambda,m=16}^*$	$Z_{S,\lambda,m=17}^*$
1	2.8368	3.0326	3.1481	3.2242	3.2781	3.3183	3.3495	3.3743	3.3946	3.4115	3.4257	3.4379	3.4484	3.4576	3.4658
2	4.4991	4.7945	4.9656	5.0771	5.1555	5.2137	5.2586	5.2942	5.3232	5.3473	5.3676	5.3849	5.3999	5.4130	5.4245
4	6.2465	6.6082	6.8104	6.9395	7.0289	7.0945	7.1447	7.1843	7.2163	7.2428	7.2651	7.2840	7.3003	7.3145	7.3270
10	8.0217	8.3387	8.5028	8.6025	8.6692	8.7169	8.7527	8.7804	8.8026	8.8208	8.8359	8.8486	8.8596	8.8690	8.8773
20	8.8735	9.0978	9.2066	9.2700	9.3113	9.3403	9.3616	9.3780	9.3909	9.4014	9.4101	9.4174	9.4236	9.4289	9.4336

Table 5.7

Maximum Expected Revenue Increasing Rate between Bid Levels with  $\bar{v} = 10$  and  $\beta = 0.2$

$\lambda$	$\frac{Z_{S,\lambda,m=4}^* - Z_{S,\lambda,m=3}^*}{Z_{S,\lambda,m=3}^*}$	$\frac{Z_{S,\lambda,m=5}^* - Z_{S,\lambda,m=4}^*}{Z_{S,\lambda,m=4}^*}$	$\frac{Z_{S,\lambda,m=6}^* - Z_{S,\lambda,m=5}^*}{Z_{S,\lambda,m=5}^*}$	$\frac{Z_{S,\lambda,m=7}^* - Z_{S,\lambda,m=6}^*}{Z_{S,\lambda,m=6}^*}$	$\frac{Z_{S,\lambda,m=8}^* - Z_{S,\lambda,m=7}^*}{Z_{S,\lambda,m=7}^*}$	$\frac{Z_{S,\lambda,m=9}^* - Z_{S,\lambda,m=8}^*}{Z_{S,\lambda,m=8}^*}$	$\frac{Z_{S,\lambda,m=10}^* - Z_{S,\lambda,m=9}^*}{Z_{S,\lambda,m=9}^*}$	$\frac{Z_{S,\lambda,m=11}^* - Z_{S,\lambda,m=10}^*}{Z_{S,\lambda,m=10}^*}$	$\frac{Z_{S,\lambda,m=12}^* - Z_{S,\lambda,m=11}^*}{Z_{S,\lambda,m=11}^*}$	$\frac{Z_{S,\lambda,m=13}^* - Z_{S,\lambda,m=12}^*}{Z_{S,\lambda,m=12}^*}$	$\frac{Z_{S,\lambda,m=14}^* - Z_{S,\lambda,m=13}^*}{Z_{S,\lambda,m=13}^*}$	$\frac{Z_{S,\lambda,m=15}^* - Z_{S,\lambda,m=14}^*}{Z_{S,\lambda,m=14}^*}$	$\frac{Z_{S,\lambda,m=16}^* - Z_{S,\lambda,m=15}^*}{Z_{S,\lambda,m=15}^*}$	$\frac{Z_{S,\lambda,m=17}^* - Z_{S,\lambda,m=16}^*}{Z_{S,\lambda,m=16}^*}$
1	6.90%	3.81%	2.42%	1.67%	1.23%	0.94%	0.74%	0.60%	0.50%	0.42%	0.36%	0.31%	0.27%	0.24%
2	6.57%	3.57%	2.25%	1.54%	1.13%	0.86%	0.68%	0.55%	0.45%	0.38%	0.32%	0.28%	0.24%	0.21%
4	5.79%	3.06%	1.89%	1.29%	0.93%	0.71%	0.55%	0.45%	0.37%	0.31%	0.26%	0.22%	0.19%	0.17%
10	3.95%	1.97%	1.17%	0.78%	0.55%	0.41%	0.32%	0.25%	0.21%	0.17%	0.14%	0.12%	0.11%	0.09%
20	2.53%	1.20%	0.69%	0.45%	0.31%	0.23%	0.17%	0.14%	0.11%	0.09%	0.08%	0.07%	0.06%	0.05%

\*  $Z_{S,\lambda,m=a}^*$  denotes the maximum expected revenue derived from the Dutch auction with a secret reserve price, an average of  $\lambda$  bidders, and  $m = a$  bid levels.

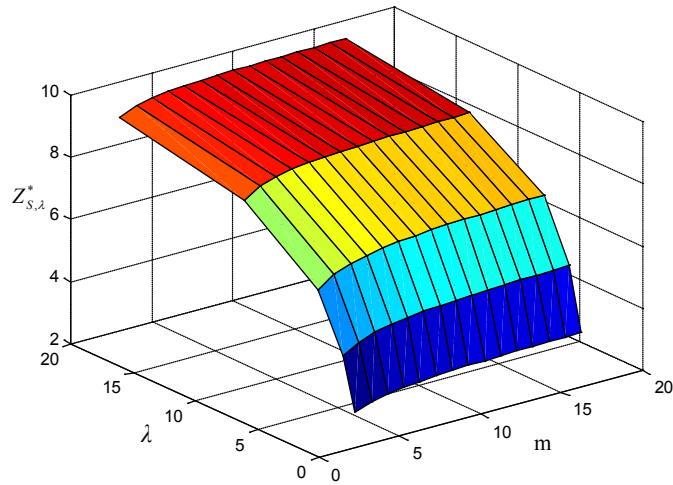


Figure 5.9. Maximum expected revenue versus expected number of bidders and number of bid levels with  $\bar{v} = 10$  and  $\beta = 0.2$ .

Table 5.8

Maximum Expected Revenue Increasing Rate between Expected Numbers of Bidders with  $\bar{v} = 10$  and  $\beta = 0.2$ \*

m	$\frac{Z_{S,\lambda=2}^* - Z_{S,\lambda=1}^*}{Z_{S,\lambda=1}^*}$	$\frac{Z_{S,\lambda=4}^* - Z_{S,\lambda=2}^*}{Z_{S,\lambda=2}^*}$	$\frac{Z_{S,\lambda=10}^* - Z_{S,\lambda=4}^*}{Z_{S,\lambda=4}^*}$	$\frac{Z_{S,\lambda=20}^* - Z_{S,\lambda=10}^*}{Z_{S,\lambda=10}^*}$
3	58.60%	38.84%	28.42%	10.62%
4	58.10%	37.83%	26.19%	9.10%
5	57.74%	37.15%	24.85%	8.28%
6	57.47%	36.68%	23.96%	7.76%
7	57.27%	36.34%	23.34%	7.41%
8	57.12%	36.07%	22.87%	7.15%
9	57.00%	35.87%	22.51%	6.96%
10	56.90%	35.70%	22.22%	6.81%
11	56.81%	35.56%	21.98%	6.68%
12	56.74%	35.45%	21.79%	6.58%
13	56.69%	35.35%	21.62%	6.50%
14	56.63%	35.27%	21.48%	6.43%
15	56.59%	35.19%	21.36%	6.37%
16	56.55%	35.13%	21.25%	6.31%
17	56.52%	35.07%	21.16%	6.27%

\*  $Z_{S,\lambda=b}^*$  denotes the maximum expected revenue derived from the Dutch auction with a secret reserve price and an average of  $b$  bidders.

Apparently, there is a clear trade-off between investing additional resources to entice more would-be buyers as well as to go through more bid levels and achieving greater revenues in a Dutch auction with a secret reserve price and an uncertain number of bidders. Prudent decisions must be made by the auctioneer to strike a delicate balance between the associated costs and benefits.

Table 5.9 and Figure 5.10 display the expected revenues for various combinations of  $m$ ,  $\lambda$  and  $\beta$  values with  $\bar{v} = 10$ . It appears that  $Z_{s,\lambda}^*$  is generally a concave function of  $\beta$  for all the  $5 \times 15 = 75$  pairs of  $\lambda$  and  $m$ , which suggests the existence of an optimal secret reserve price  $\beta^*$ . In particular, given  $\lambda$ , the revenue-maximization  $\beta^*$  becomes larger as  $m$  decreases from 17 to 3. Conversely, given  $m$ , the revenue-maximization  $\beta^*$  becomes larger as  $\lambda$  increases from 1 to 2 to 4 to 10 to 20. All of these point to the fact that the concavity of  $Z_{s,\lambda}^*$  in  $\beta$  is a direct consequence of the interplay between  $m$  and  $\lambda$ .

Finally, recall that (4.2) is a special case of (5.1) with  $\beta = 0$ . A careful review of Table 5.9 reveals that the expected revenue from a Dutch auction model with a positive  $\beta$  (i.e., with a secret reserve price) may be greater than, equal to, or smaller than its counterpart with  $\beta = 0$  (i.e., without a secret reserve price) depending on the magnitude of  $\beta$ . This observation serves to explain the mixed views expressed by academicians and practitioners alike on the impact of secret reserve price on auction revenue in the existing literature. For example, Vishwanath and Barnett (2005) argue in their empirical study on eBay that an auction with a secret reserve price gives rise to a more competitive bidding environment and, thus, results in a higher final selling price. Hossain (2008) finds that the initial bids submitted by bidders will be higher if the reserve price is kept secret in auctions on eBay. In contrast, Ockenfels et al. (2007) contend that a secret



reserve price deters serious bidders from entering the auction and hence lowers the expected revenue. The insight I gain above may be deemed as a unified theory on the influence of  $\beta$  on  $Z_{S,\lambda}^*$  in the context of Dutch auctions with a random number of bidders.

Table 5.9

*Maximum Expected Revenue versus Secret Reserve Price, Expected Number of Bidders and Number of Bid Levels with  $\bar{v} = 10$*

$\lambda$	$\beta$	$\beta\bar{v}$	m								
			3	4	6	8	10	12	14	16	17
1	0.0	0	2.5869	2.8714	3.1481	3.2836	3.3639	3.4171	3.4550	3.4832	3.4949
	0.1	1	2.7383	2.9772	3.2103	3.3246	3.3925	3.4375	3.4695	3.4934	3.5033
	0.2	2	2.8368	3.0326	3.2242	3.3183	3.3743	3.4115	3.4379	3.4576	3.4658
	0.3	3	2.8714	3.0271	3.1797	3.2549	3.2996	3.3293	3.3505	3.3663	3.3728
	0.4	4	2.8297	2.9484	3.0652	3.1228	3.1572	3.1800	3.1962	3.2084	3.2134
	0.5	5	2.6972	2.7828	2.8673	2.9090	2.9340	2.9505	2.9623	2.9712	2.9748
	0.6	6	2.4573	2.5142	2.5705	2.5984	2.6151	2.6262	2.6341	2.6400	2.6425
	0.7	7	2.0909	2.1242	2.1572	2.1736	2.1835	2.1900	2.1946	2.1981	2.1995
	0.8	8	1.5764	1.5918	1.6071	1.6147	1.6193	1.6222	1.6245	1.6260	1.6268
2	0.0	0	4.1846	4.5894	4.9722	5.1558	5.2635	5.3343	5.3844	5.4217	5.4370
	0.1	1	4.3612	4.7111	5.0438	5.2040	5.2982	5.3603	5.4042	5.4369	5.4503
	0.2	2	4.4991	4.7945	5.0771	5.2137	5.2942	5.3473	5.3849	5.4130	5.4245
	0.3	3	4.5789	4.8211	5.0539	5.1670	5.2338	5.2778	5.3091	5.3325	5.3421
	0.4	4	4.5751	4.7659	4.9503	5.0403	5.0935	5.1287	5.1537	5.1724	5.1800
	0.5	5	4.4541	4.5963	4.7347	4.8024	4.8426	4.8692	4.8881	4.9022	4.9080
	0.6	6	4.1717	4.2697	4.3654	4.4125	4.4405	4.4591	4.4723	4.4822	4.4862
	0.7	7	3.6708	3.7302	3.7885	3.8173	3.8345	3.8459	3.8540	3.8601	3.8626
	0.8	8	2.8770	2.9055	2.9337	2.9476	2.9559	2.9614	2.9654	2.9683	2.9696
4	0.0	0	5.9820	6.4385	6.8485	7.0375	7.1460	7.2163	7.2657	7.3022	7.3171
	0.1	1	6.1158	6.5262	6.8984	7.0712	7.1709	7.2357	7.2812	7.3149	7.3287
	0.2	2	6.2465	6.6082	6.9395	7.0945	7.1843	7.2428	7.2840	7.3145	7.3270
	0.3	3	6.3600	6.6704	6.9578	7.0933	7.1721	7.2236	7.2600	7.2869	7.2980
	0.4	4	6.4310	6.6880	6.9285	7.0429	7.1097	7.1535	7.1844	7.2074	7.2169
	0.5	5	6.4156	6.6180	6.8094	6.9012	6.9551	6.9905	7.0155	7.0342	7.0418
	0.6	6	6.2401	6.3878	6.5291	6.5974	6.6376	6.6642	6.6830	6.6970	6.7027
	0.7	7	5.7816	5.8770	5.9692	6.0142	6.0408	6.0584	6.0708	6.0802	6.0840
	0.8	8	4.8392	4.8882	4.9361	4.9596	4.9736	4.9829	4.9895	4.9944	4.9965
10	0.0	0	7.9243	8.2852	8.5777	8.7019	8.7701	8.8131	8.8426	8.8641	8.8728
	0.1	1	7.9692	8.3096	8.5888	8.7086	8.7747	8.8165	8.8453	8.8663	8.8748
	0.2	2	8.0217	8.3387	8.6025	8.7169	8.7804	8.8208	8.8486	8.8690	8.8773
	0.3	3	8.0832	8.3736	8.6191	8.7269	8.7872	8.8257	8.8523	8.8718	8.8798
	0.4	4	8.1540	8.4138	8.6374	8.7369	8.7931	8.8290	8.8541	8.8724	8.8799
	0.5	5	8.2291	8.4532	8.6501	8.7392	8.7898	8.8225	8.8453	8.8621	8.8689
	0.6	6	8.2829	8.4655	8.6295	8.7051	8.7485	8.7765	8.7964	8.8110	8.8169
	0.7	7	8.2239	8.3585	8.4824	8.5406	8.5743	8.5964	8.6119	8.6234	8.6281
	0.8	8	7.7542	7.8352	7.9118	7.9485	7.9700	7.9842	7.9942	8.0016	8.0047
20	0.0	0	8.8372	9.0809	9.2636	9.3369	9.3759	9.4000	9.4163	9.4281	9.4328
	0.1	1	8.8539	9.0886	9.2665	9.3384	9.3768	9.4006	9.4168	9.4285	9.4332
	0.2	2	8.8735	9.0978	9.2700	9.3403	9.3780	9.4014	9.4174	9.4289	9.4336
	0.3	3	8.8968	9.1091	9.2745	9.3427	9.3796	9.4026	9.4182	9.4296	9.4342
	0.4	4	8.9255	9.1235	9.2805	9.3460	9.3817	9.4041	9.4194	9.4305	9.4350
	0.5	5	8.9619	9.1424	9.2886	9.3507	9.3848	9.4063	9.4211	9.4318	9.4362
	0.6	6	9.0092	9.1677	9.2996	9.3568	9.3885	9.4087	9.4226	9.4328	9.4370
	0.7	7	9.0646	9.1945	9.3063	9.3561	9.3841	9.4021	9.4146	9.4238	9.4276
	0.8	8	9.0499	9.1412	9.2232	9.2610	9.2827	9.2967	9.3066	9.3139	9.3169

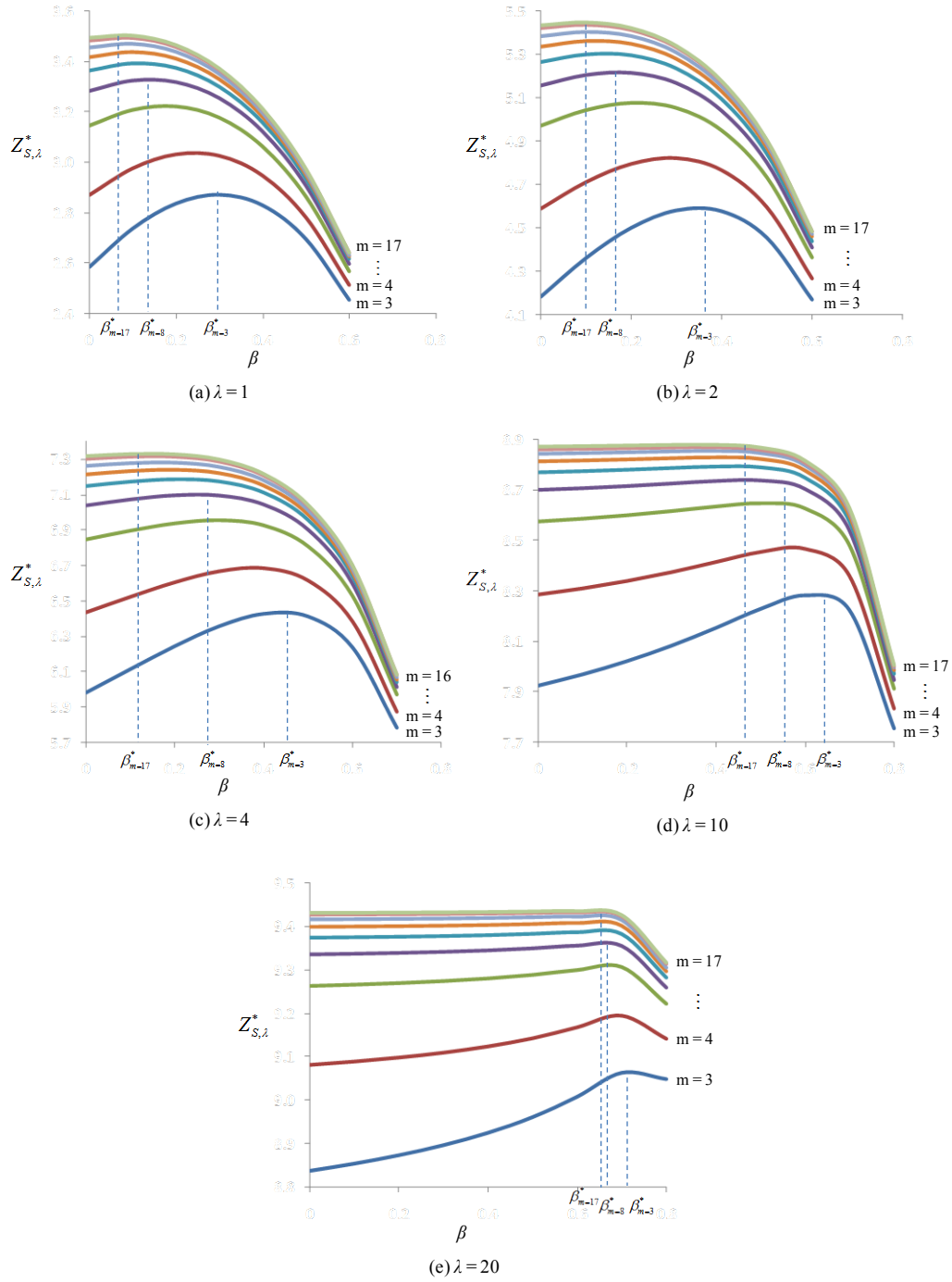


Figure 5.10. Maximum expected revenue versus secret reserve price, expected number of bidders and number of bid levels with  $\bar{v} = 10$ .<sup>†</sup>

<sup>†</sup>  $\beta_{m=c}^*$  denotes the optimal secret reserve price in the Dutch auction with an uncertain number of bidders and  $m = c$  bid levels.

### 5.6.3 Revenue comparison

I will end this section with a set of numerical examples to validate our theoretical analysis of the three different strategies discussed in Section 5.5 for establishing bid levels in a Dutch auction with a secret reserve price, where the number of bidders is a Poisson random variable with a mean of  $\lambda$ . The possible values to be assumed by the model parameters are:

$$\bar{v} = 10, \beta = 0.2, m \in \{3, 5, 9, 13, 17\} \text{ and } \lambda \in \{1, 2, 4, 10, 20\}.$$

Taking as an example the problem instance with  $\bar{v} = 10, \beta = 0.2, m = 5$  and  $\lambda = 2$ , one can easily set up the appropriate NLP based on (5.1) and solve it to obtain  $Z_{S,\lambda}^* = 4.9656$ . If Strategy E is used, the five bid levels based on (5.7) are

$$l_{SE,1} = \frac{(1-1) \times (1-0.2) \times 10}{5} + 0.2 \times 10 = 2.0$$

$$l_{SE,2} = \frac{(2-1) \times (1-0.2) \times 10}{5} + 0.2 \times 10 = 3.6$$

$$l_{SE,3} = \frac{(3-1) \times (1-0.2) \times 10}{5} + 0.2 \times 10 = 5.2$$

$$l_{SE,4} = \frac{(4-1) \times (1-0.2) \times 10}{5} + 0.2 \times 10 = 6.8$$

$$l_{SE,5} = \frac{(5-1) \times (1-0.2) \times 10}{5} + 0.2 \times 10 = 8.4$$

Additionally, I see from (5.8) that the respective probabilities of selling the object at those five prices are

$$P(l_{SE,1}) = e^{-2(1-0.2)} \left[ e^{\frac{2 \times 1 \times (1-0.2)}{5}} - e^{\frac{2 \times 0 \times (1-0.2)}{5}} \right] \approx 0.0761$$

$$P(l_{SE,2}) = e^{-2(1-0.2)} \left[ e^{\frac{2 \times 2 \times (1-0.2)}{5}} - e^{\frac{2 \times 1 \times (1-0.2)}{5}} \right] \approx 0.1049$$

$$P(l_{SE,3}) = e^{-2(1-0.2)} \left[ e^{\frac{2 \times 3 \times (1-0.2)}{5}} - e^{\frac{2 \times 2 \times (1-0.2)}{5}} \right] \approx 0.1444$$

$$P(l_{SE,4}) = e^{-2(1-0.2)} \left[ e^{\frac{2 \times 4 \times (1-0.2)}{5}} - e^{\frac{2 \times 3 \times (1-0.2)}{5}} \right] \approx 0.1989$$

$$P(l_{SE,5}) = e^{-2(1-0.2)} \left[ e^{\frac{2 \times 5 \times (1-0.2)}{5}} - e^{\frac{2 \times 4 \times (1-0.2)}{5}} \right] \approx 0.2739$$

It follows that the auctioneer is expected to receive an average revenue of  $Z_{S,E} \approx 2.0(0.0761) + 3.6(0.1049) + 5.2(0.1444) + 6.8(0.1989) + 8.4(0.2739) \approx 4.9340$  from the Dutch auction.

However, based on Strategy D with  $r = 25\%$ , one sees from (5.10) that the five bid levels are

$$l_{SD,1} = 2.0000$$

$$l_{SD,2} = \frac{\sum_{j=4}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 + \left[ 1 - \frac{\sum_{j=4}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \right] \times 0.2 \times 10 \approx 2.9748$$

$$l_{SD,3} = \frac{\sum_{j=3}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 + \left[ 1 - \frac{\sum_{j=3}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \right] \times 0.2 \times 10 \approx 4.1932$$

$$l_{SD,4} = \frac{\sum_{j=2}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 + \left[ 1 - \frac{\sum_{j=2}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \right] \times 0.2 \times 10 \approx 5.7163$$

$$l_{SD,5} = \frac{\sum_{j=1}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \times 10 + \left[ 1 - \frac{\sum_{j=1}^4 (1.25)^{-j}}{\sum_{j=0}^4 (1.25)^{-j}} \right] \times 0.2 \times 10 \approx 7.6202$$

According to (5.11), the probabilities of selling the object at those five prices are

$$P(l_{SD,1}) = e^{-2} \left( e^{\frac{2 \times 2.9748}{10}} - e^{\frac{2 \times 2}{10}} \right) \approx 0.0435$$

$$P(l_{SD,2}) = e^{-2} \left( e^{\frac{2 \times 4.1932}{10}} - e^{\frac{2 \times 2.9748}{10}} \right) \approx 0.0677$$

$$P(l_{SD,3}) = e^{-2} \left( e^{\frac{2 \times 5.7163}{10}} - e^{\frac{2 \times 4.1932}{10}} \right) \approx 0.1115$$

$$P(l_{SD,4}) = e^{-2} \left( e^{\frac{2 \times 7.6202}{10}} - e^{\frac{2 \times 5.7163}{10}} \right) \approx 0.1967$$

$$P(l_{SD,5}) = e^{-2} \left( e^{\frac{2 \times 10}{10}} - e^{\frac{2 \times 7.6202}{10}} \right) \approx 0.3787$$

Thus, the auctioneer's expected auction revenue is  $Z_{S,D} = 2.000(0.0435) + 2.9748(0.0677) + 4.1932(0.1115) + 5.7163(0.1967) + 7.6202(0.3787) \approx 4.7661$ .

The procedure described above is repeated for each of the other  $5 \times 5 - 1 = 24$  pairs of  $m$  and  $\lambda$  values. All of the results are summarized in Table 5.10 and the continuous versions of them are plotted in Figure 5.11. One can see from the two exhibits that, as suggested in Corollary 5.1, the optimal strategy outperforms the two heuristic strategies of E and D with respect to expected auction revenue in all cases. Moreover, Strategy E consistently leads to a higher expected revenue than Strategy D does. I also observe that the differences among the three revenue surfaces in Figure 5.11 are more pronounced if the number of bid levels is small while the expected number of auction participants is large. When  $\lambda$  is small and  $m$  is large,

however, the differences are insignificant. It is interesting to notice that the same observations are made in Section 4.6 of Chapter 4, where no secret reserve price is introduced in the Dutch auction model.

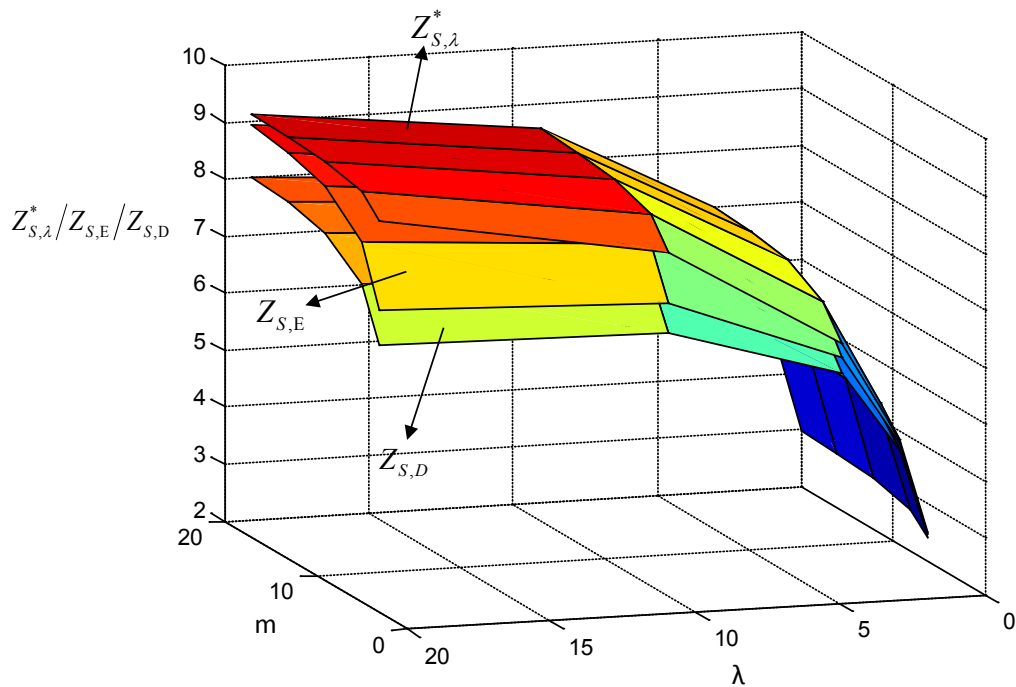


Figure 5.11. Expected revenues based on three different design strategies with  $\bar{v} = 10$  and  $\beta = 0.2$ .

Table 5.10

*Expected Revenues Based on Three Different Design Strategies with  $\bar{v} = 10$  and  $\beta = 0.2$*

$\lambda$	$m$ $Z$	3			5			9			13			17		
		$Z_{S,\lambda}^*$	$Z_{S,E}$	$Z_{S,D}$	$Z_{S,\lambda}^*$	$Z_{S,E}$	$Z_{S,D}$	$Z_{S,\lambda}^*$	$Z_{S,E}$	$Z_{S,D}$	$Z_{S,\lambda}^*$	$Z_{S,E}$	$Z_{S,D}$	$Z_{S,\lambda}^*$	$Z_{S,E}$	$Z_{S,D}$
1		2.8368	2.8278	2.7699	3.1481	3.1424	3.0639	3.3495	3.3463	3.2341	3.4257	3.4235	3.2831	3.4658	3.4640	3.3009
2		4.4991	4.4474	4.3115	4.9656	4.9332	4.7663	5.2586	5.2405	5.0250	5.3676	5.3551	5.0989	5.4245	5.4150	5.1256
4		6.2465	6.0182	5.7351	6.8104	6.6717	6.3512	7.1447	7.0688	6.6933	7.2651	7.2131	6.7897	7.3270	7.2876	6.8243
10		8.0217	7.1345	6.6161	8.5028	7.9952	7.4177	8.7527	8.4904	7.8601	8.8359	8.6607	7.9833	8.8773	8.7461	8.0273
20		8.8735	7.3204	6.7176	9.2066	8.3320	7.6036	9.3616	8.9303	8.1136	9.4101	9.1307	8.2584	9.4336	9.2283	8.3103

## 5.7 Simulation Analysis

This section is centered on validating the theoretical value of  $Z_{S,\lambda}^*$  in (5.1) by computing the maximum expected revenue via a simulation approach. The procedure to be followed is similar to those described in Section 3.8 and Section 4.7, and its flow chart is presented in Figure 5.12.

To begin, I specify the values to be taken on by each of the four model parameters as follows:  $\bar{v} = 10$ ,  $\beta = 0.2$ ,  $m \in \{6, 9, 12\}$ , and  $\lambda \in \{1, 2, 4, 10, 20\}$ . For each combination of  $m$  and  $\lambda$  values, an NLP based on (5.1) is formulated to model the particular Dutch auction and is solved to determine  $(l_{S,1}^*, l_{S,2}^*, \dots, l_{S,m}^*)$  as well as  $Z_{S,\lambda}^*$  with the aid of Solver. Let  $T$  be the maximum number of simulation runs allowed (e.g.,  $T = 100,000$ ) and  $t$  is a counter. MATLAB is employed to generate a value of the random variable subject to the Poisson distribution  $\text{Poisson}(\lambda)$  to simulate the actual number of bidders in the Dutch auction. If  $n = 0$ , then the final selling price is equal to zero; otherwise,  $n \geq 1$  values of the random variable subject to the uniform distribution  $U(0, 10)$  are generated to simulate the bids submitted by the auction participants. The final selling price of the object is set as zero in case the highest bid is smaller than  $l_{S,1}^*$ ; otherwise, let  $k$  be the largest index such that the highest bid is greater than or equal to  $l_{S,k}^*$  and the final selling price is equal to  $l_{S,k}^*$ . Finally, the expected revenue  $Z_{S,\lambda}^*$  is computed as the average selling price over the  $T$  simulation runs.

The process described above is repeated for each of the other  $3 \times 5 - 1 = 14$  pairs of  $m$  and  $\lambda$ , and the two sets of  $Z_{S,\lambda}^*$  based on different approaches are summarized in Table 5.11. As evidenced by the minor differences due most likely to rounding errors, the simulation-based empirical results lend strong support to the optimization-based theoretical results on the maximum expected revenue. Consequently, the validity of the latter is established.



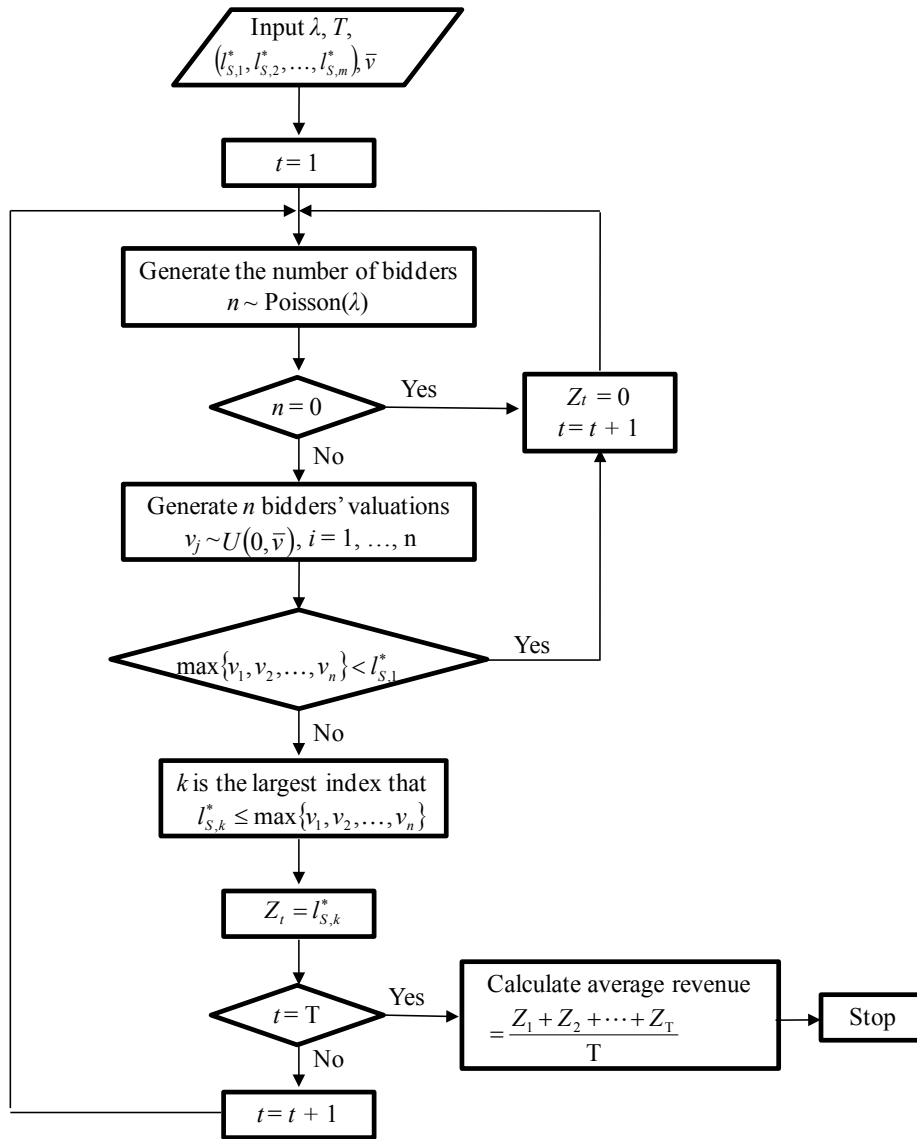


Figure 5.12. Flowchart of simulation process.

Table 5.11

*Maximum Expected Revenues Based on Two Different Approaches*

m	$\lambda$	Optimization	Simulation	Error
6	1	3.2242	3.2144	0.0098
	2	5.0771	5.0669	0.0102
	4	6.9395	6.9350	0.0045
	10	8.6025	8.6078	-0.0053
	20	9.2700	9.2732	-0.0032
9	1	3.3495	3.3440	0.0055
	2	5.2586	5.2627	-0.0041
	4	7.1447	7.1464	-0.0017
	10	8.7527	8.7481	0.0046
	20	9.3616	9.3713	-0.0097
12	1	3.4115	3.4081	-0.0084
	2	5.3473	5.3534	-0.0061
	4	7.2428	7.2337	0.0091
	10	8.8208	8.8180	0.0028
	20	9.4014	9.3999	0.0015

## 5.8 Conclusions

The main emphasis of this chapter is on investigating the effect of secret reserve price on the optimal design of Dutch auctions where the number of bidders is uncertain. As it turns out, most of the properties possessed by the optimal bid levels and the maximum expected revenue for the models examined in Chapters 3 and 4 are preserved in the new model studied in the present chapter.

The impact of  $\beta$  on each of  $l_{S,i}^*$ ,  $Z_{S,\lambda}^*$  is also analyzed both theoretically and numerically. In particular, I make an insightful observation about  $\beta$  that helps explain the conflicting findings on the role of secret reserve price in the auction literature. Unfortunately, our attempt to develop a closed-form expression for computing the revenue-maximization  $\beta^*$  proved unsuccessful.

This “unfinished business” will be on our future research agenda.

## CHAPTER 6

### CONCLUSIONS

#### 6.1 Summary of Dissertation

Although traditional auctions have existed in human society from time immortal, they have been an uncommon way of trading commodities until the emergence of the Internet in the mid 1990s when online auctions started to thrive. Today, exceedingly large volumes of goods, services and financial instruments worth trillions of dollars are auctioned off annually while, at the same time, new auctions are being designed and implemented to sell a wide spectrum of tangible as well as intangible products in the global economy. The rapidly growing popularity of auctions of all formats has spawned considerable interest in studying the renewed selling mechanisms over the past decade. This dissertation is prepared in response to a call from the academic community for a deeper understanding of how to set the rules of different types of auctions to achieve desirable goals.

In undertaking this project, I chose to make an inquiry into an important but relatively unexplored area of research by focusing on the optimal design of single-unit Dutch auctions where the bidding prices are restricted to a finite set of values and the main aim is to ensure that the average selling price of the object (i.e., the auctioneer's expected revenue) is as high as possible. As a significant departure from the traditional methods employed by applied economists and game-theorists, a novel approach to the problem is taken by formulating the descending-bid auctioning process as a constrained mathematical program and using standard nonlinear optimization techniques to solve it. For each of the basic auction model and its two extensions considered in this study (see Table 6.1), I seek to characterize the properties of the

optimal bid levels in addition to developing closed-form expressions for computing the auctioneer's maximum expected revenue and the probability of reaching a deal, respectively. Furthermore, the superiority of the recommended optimal strategy over two heuristic procedures is demonstrated both theoretically and empirically. Major findings are presented in the form of propositions, which are summarized in Table 6.2, and their economic implications are discussed.

Table 6.1

*Dutch Auction Models Examined*

Model	Chapter	Characteristics
1	3	A basic model with a fixed number of bidders but without a secret reserve price
2	4	An extended model with an uncertain number of bidders but without a secret reserve price
3	5	An extended model with an uncertain number of bidders and a secret reserve price

Table 6.2

*Key Research Findings*

Model	Properties of optimal bid levels	Properties of maximum expected revenue
1	<ul style="list-style-type: none"> <li>• Increasing if <math>n &gt; 1</math> and equal if <math>n = 1</math></li> <li>• Increasing with <math>n</math></li> <li>• Increasing with <math>m</math></li> </ul>	<ul style="list-style-type: none"> <li>• Closed-form expression a function of <math>n</math>, <math>m</math> and <math>\bar{v}</math></li> <li>• Increasing with <math>n</math></li> <li>• Increasing with <math>m</math></li> </ul>
2	<ul style="list-style-type: none"> <li>• Increasing for <math>n \geq 1</math></li> <li>• Increasing with <math>\lambda</math></li> <li>• Increasing with <math>m</math></li> </ul>	<ul style="list-style-type: none"> <li>• A function of <math>l_m^*</math>, <math>l_2^*</math>, <math>\lambda</math> and <math>\bar{v}</math></li> <li>• Increasing with <math>\lambda</math></li> <li>• Increasing with <math>m</math></li> </ul>
3	<ul style="list-style-type: none"> <li>• Increasing for <math>n \geq 1</math></li> <li>• Increasing with <math>\lambda</math></li> <li>• Increasing with <math>m</math></li> <li>• Increasing with <math>\beta</math></li> </ul>	<ul style="list-style-type: none"> <li>• A function of <math>l_{S,m}^*</math>, <math>l_{S,2}^*</math>, <math>\lambda</math> and <math>\bar{v}</math></li> <li>• Increasing with <math>\lambda</math></li> <li>• Increasing with <math>m</math></li> <li>• Likely a concave function of <math>\beta</math></li> </ul>

## 6.2 Limitations of Study

Whilst the results reported in this dissertation appear interesting and make intuitive sense, I am mindful that some of the fundamental assumptions underlying the three auction models investigated in Chapters 3, 4 and 5, respectively, may be too restrictive from the practical standpoint. It should also be pointed out that I encountered difficulties in mathematically proving several seemingly obvious propositions. These along with other related issues must be properly addressed in order for our research findings to be applied to a broader range of settings. In what follows, each of the model limitations is delineated.

- (1) Independent valuation - Throughout this study, it is assumed that a bidder's assessment of how much the object to be auctioned off is worth will not be influenced by other bidders' appraisals. While this assumption has been validated and used in many auctions (Vickrey, 1961; Myerson, 1981; Riley and Samuelson, 1981), it may not hold in some situations where the value of the object being sold is highly uncertain. For instance, the participants in a Dutch auction of the oil drilling rights in a wildcat venture might change their valuations when the asking price continues to go down but no one expresses any interest to buy them. Competitive selling mechanisms involving bidders with interdependent (or affiliated) value like this have been extensively researched in the auction, economics, and game theory literatures (Krishna, 2002).
- (2) Risk neutrality - Another assumption made about our models is that all bidders in the Dutch auction are risk-neutral in that they will pay up to their valuations for the object for sale. In reality, however, it is quite likely that some of them are risk-averse

(i.e., conservative) and others are risk seeking (i.e., aggressive). Since the impact of the bidder's attitude towards risk under other auction formats has been well documented (Klemperer, 1999; Reynolds and Wooders, 2009), it should be of interest to investigate the role it plays in the outcome of the Dutch auctions examined in this dissertation.

- (3) Single unit - The current project has been centered on Dutch auctions in which only one unit of an object is to be sold (i.e., the lot size is one) partly because they are easier to wrestle with than their multi-unit counterparts. In case two or more units are to be auctioned off, the analysis becomes rather involved and sometimes intractable depending on the values at which the other model parameters are set. An important question to ask here is whether it is more profitable or efficient to hold multiple one-unit auctions or to conduct one auction where all of the units are treated as a bundle. In that regard, preliminary results have been obtained by Palfrey (1983) as well as Avery and Hendershott (2000) for English bidding processes under a number of simplifying assumptions. Given the practical significance of multi-unit Dutch auctions and the limited attention that they have received, the topic appears to be a fruitful avenue to pursue in the near future.
- (4) Secret reserve price - As described in Section 2.7, public reserve prices and secret reserve prices are often used in auctions in hopes of raising the final selling price of the object for sale. In Chapter 5, I incorporate a secret reserve price into the Dutch auction model with a random number of bidders and explore its interactions with the optimal bid levels as well as the auctioneer's maximum expected revenue. It remains to make an inquiry into its public reserve price counterpart as well as those with a

fixed number of bidders to fully understand the effect of such minimum acceptable bids on the auction outcome. A careful review of studies along this line, which include Riley and Samuelson (1981), Milgrom and Weber (1982), and Riley (2006), serves as a good starting point for further research.

- (5) Distribution of bidders' valuations - Unlike the uniform distribution considered in this dissertation and other scholarly publications (Mathews, 2006; Sasa Pekec and Tsetlin, 2008; Chou and Parlar, 2005), bidders' valuations are assumed to follow a common exponential (Rothkopf and Harstad, 1994; David et al., 2007) or triangular distribution (Bapna et al., 2008) by others. Interestingly, Chen et al. (2007) conduct a laboratory experiment of sealed-bid auctions in which bidders' valuations are taken from an unknown distribution. Analysis of Dutch auctions with these continuous probability distributions of bidders' valuations will be part of our work sometime down the road.

### 6.3 Concluding Remarks

This dissertation is motivated primarily by our observation that there is a clear lack of theoretical studies on optimal design of Dutch auctions with discrete bid levels in the existing literature despite their increasing importance in the global marketplace. Research in this area is still in its infancy but I argue that its significance has loomed large enough to warrant more dedicated consideration. The findings reported here represent part of the efforts in that direction, and they should only be viewed as a first step toward a complete understanding of how the critical dimensions of a Dutch auction model interplay with each other and how each of the key parameters can be controlled by the auctioneer to reach the desired goals.

A great deal remains to be done. In particular, the limitations discussed in the preceding section collectively pose a challenge to us to develop an expanded repertoire of formulations as well as solution approaches to deal with different variants of the Dutch auction models examined in this study. It is hoped that the present work will shed some light on these unfinished business to be completed in the future.



APPENDIX A

PROOF OF PROPOSITION 3.1

According to the objective function in (3.5), I have the following, for  $l_i$ ,  $i = 2, \dots, m$ ,

$$\begin{aligned} \frac{\partial Z}{\partial l_i} &= \frac{1}{\bar{v}^n} \frac{\partial}{\partial l_i} (l_i l_{i+1}^n - l_i^{n+1} + l_{i-1} l_i^n) \\ &= \frac{1}{\bar{v}^n} [l_{i+1}^n - (n+1)l_i^n + n l_{i-1} l_i^{n-1}] \end{aligned} \quad (\text{A.1})$$

$$\frac{\partial^2 Z}{\partial l_i^2} = \frac{n}{\bar{v}^n} [-(n+1)l_i^{n-1} + (n-1)l_{i-1} l_i^{n-2}] \quad (\text{A.2})$$

$$\frac{\partial^2 Z}{\partial l_i \partial l_{i+1}} = \frac{n}{\bar{v}^n} l_{i+1}^{n-1} \quad (\text{A.3})$$

and

$$\frac{\partial^2 Z}{\partial l_i \partial l_j} = 0, j > i + 1 \quad (\text{A.4})$$

In particular, since  $l_1 = 0$ , one has

$$\frac{\partial Z}{\partial l_2} = \frac{1}{\bar{v}^n} [l_3^n - (n+1)l_2^n] \quad (\text{A.5})$$

$$\frac{\partial^2 Z}{\partial l_2^2} = -\frac{n(n+1)}{\bar{v}^n} l_2^{n-1} \quad (\text{A.6})$$

$$\frac{\partial^2 Z}{\partial l_2 \partial l_3} = \frac{n}{\bar{v}^n} l_3^{n-1} \quad (\text{A.7})$$

and

$$\frac{\partial^2 Z}{\partial l_2 \partial l_j} = 0, j > 3 \quad (\text{A.8})$$

Let  $H(Z)$  be the Hessian of the objective function in (3.5). It follows that

$$H(Z) = \begin{bmatrix} \frac{\partial^2 Z}{\partial l_2^2} & \frac{\partial^2 Z}{\partial l_2 \partial l_3} & 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial^2 Z}{\partial l_3 \partial l_2} & \frac{\partial^2 Z}{\partial l_3^2} & \frac{\partial^2 Z}{\partial l_3 \partial l_4} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\partial^2 Z}{\partial l_4 \partial l_3} & \frac{\partial^2 Z}{\partial l_4^2} & \frac{\partial^2 Z}{\partial l_4 \partial l_5} & \cdots & 0 & 0 \\ 0 & 0 & \frac{\partial^2 Z}{\partial l_5 \partial l_4} & \frac{\partial^2 Z}{\partial l_5^2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{\partial^2 Z}{\partial l_{m-1}^2} & \frac{\partial^2 Z}{\partial l_{m-1} \partial l_m} \\ 0 & 0 & 0 & 0 & \cdots & \frac{\partial^2 Z}{\partial l_m \partial l_{m-1}} & \frac{\partial^2 Z}{\partial l_m^2} \end{bmatrix}$$

Based on (A1.2), (A1.3), (A1.4), (A1.6), (A1.7) and (A1.8), I have

$$H(Z) = \frac{n}{\bar{v}^n} \begin{bmatrix} -(n+1)l_2^{n-1} & l_3^{n-1} & 0 & \cdots & 0 \\ l_3^{n-1} & -l_3^{n-2}[(n+1)l_3 - (n-1)l_2] & l_4^{n-1} & \cdots & 0 \\ 0 & l_4^{n-1} & -l_4^{n-2}[(n+1)l_4 - (n-1)l_3] & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -l_m^{n-2}[(n+1)l_m - (n-1)l_{m-1}] \end{bmatrix}$$

Let  $l_{i+1}^{n-1} = l_i^{n-1} + \Delta(l_i, n)$ ,  $i = 2, \dots, m-1$ . The Hessian can be rewritten as

$$H(Z) = \frac{n}{\bar{v}^n} \begin{bmatrix} -(n+1)l_2^{n-1} & l_3^{n-1} & 0 & \cdots & 0 \\ l_2^{n-1} + \Delta(l_2, n) & -l_3^{n-2}[(n+1)l_3 - (n-1)l_2] & l_4^{n-1} & \cdots & 0 \\ 0 & l_3^{n-1} + \Delta(l_3, n) & -l_4^{n-2}[(n+1)l_4 - (n-1)l_3] & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -l_m^{n-2}[(n+1)l_m - (n-1)l_{m-1}] \end{bmatrix}$$

Now,  $H(Z)$  is decomposed into  $H_1(Z)$  and  $H_2(Z)$  below so that  $H(Z) = H_1(Z) + H_2(Z)$ :

$$H_1(Z) = \frac{n}{\bar{v}^n} \begin{bmatrix} -2l_2^{n-1} & l_3^{n-1} & 0 & \cdots & 0 \\ l_2^{n-1} & -2l_3^{n-1} & l_4^{n-1} & \cdots & 0 \\ 0 & l_3^{n-1} & -2l_4^{n-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -2l_m^{n-1} \end{bmatrix}$$

and

$$H_2(Z) = \frac{n}{\bar{v}^n} \begin{bmatrix} -(n-1)l_2^{n-1} & 0 & 0 & \cdots & 0 \\ \Delta(l_2, n) & -(n-1)l_3^{n-2}(l_3 - l_2) & 0 & \cdots & 0 \\ 0 & \Delta(l_3, n) & -(n-1)l_4^{n-2}(l_4 - l_3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -(n-1)l_m^{n-2}(l_m - l_{m-1}) \end{bmatrix}$$

I am going to prove that the leading principal minor of each of  $H_1(Z)$  and  $H_2(Z)$  of order  $i$  has the same sign as  $(-1)^i$ . To begin, let  $M_i$  be the leading principal minor of  $H_1(Z)$  of order  $i$ . One then has

$$\begin{aligned} M_i &= \frac{n}{\bar{v}^n} \left( -2l_i^{n-1}M_{i-1} - \frac{n}{\bar{v}^n}l_i^{n-1}l_{i-1}^{n-1}M_{i-2} \right) \\ &= \left( \frac{n}{\bar{v}^n} \right)^2 \left[ -2l_i^{n-1} \left( -2l_{i-1}^{n-1}M_{i-2} - \frac{n}{\bar{v}^n}l_{i-1}^{n-1}l_{i-2}^{n-1}M_{i-3} \right) - l_i^{n-1}l_{i-1}^{n-1}M_{i-2} \right] \\ &= \left( \frac{n}{\bar{v}^n} \right)^2 \left( 3l_i^{n-1}l_{i-1}^{n-1}M_{i-2} + 2\frac{n}{\bar{v}^n}l_i^{n-1}l_{i-1}^{n-1}l_{i-2}^{n-1}M_{i-3} \right) \\ &= \left( \frac{n}{\bar{v}^n} \right)^3 \left( -4l_i^{n-1}l_{i-1}^{n-1}l_{i-2}^{n-1}M_{i-3} - 3\frac{n}{\bar{v}^n}l_i^{n-1}l_{i-1}^{n-1}l_{i-2}^{n-1}l_{i-3}^{n-1}M_{i-4} \right) \\ &\quad \dots \\ &= \left( \frac{n}{\bar{v}^n} \right)^{i-2} \left[ (-1)^{i-2}(i-1) \left( \prod_{k=4}^i l_k^{n-1} \right) M_2 + (-1)^{i-2}(i-2) \frac{n}{\bar{v}^n} \left( \prod_{k=3}^i l_k^{n-1} \right) M_1 \right] \end{aligned}$$

Because  $M_1 = -2\frac{n}{\bar{v}^n}l_2^{n-1}$  and  $M_2 = 3\left(\frac{n}{\bar{v}^n}\right)^2 l_2^{n-1}l_3^{n-1}$ , I have

$$\begin{aligned} M_i &= \left( \frac{n}{\bar{v}^n} \right)^{i-2} \left[ (-1)^{i-2}(i-1) \left( \prod_{k=4}^i l_k^{n-1} \right) M_2 + (-1)^{i-2}(i-2) \frac{n}{\bar{v}^n} \left( \prod_{k=3}^i l_k^{n-1} \right) M_1 \right] \\ &= (-1)^i \left( \frac{n}{\bar{v}^n} \right)^i \left[ 3(i-1) \prod_{k=2}^i l_k^{n-1} - 2(i-2) \prod_{k=2}^i l_k^{n-1} \right] \\ &= (-1)^i \left( \frac{n}{\bar{v}^n} \right)^i [3(i-1) - 2(i-2)] \prod_{k=2}^i l_k^{n-1} \\ &= (-1)^i \left( \frac{n}{\bar{v}^n} \right)^i (i+1) \prod_{k=2}^i l_k^{n-1} \end{aligned}$$

It is clear that  $M_i$  has the same sign as  $(-1)^i$ . Likewise, let  $N_i$  be the leading principal minor of  $H_2(Z)$  of order  $i$ . The same approach may be followed to show that

$$N_i = (-1)^i \left[ \frac{n(n-1)}{\bar{v}^n} \right]^i l_2^{n-1} \prod_{k=2}^{i-1} l_{k+1}^{n-2} (l_{k+1} - l_k)$$

Again, it is easy to see that  $N_i$  has the same sign as  $(-1)^i$ .

Since both of the leading principal minors of  $H_1(Z)$  and  $H_2(Z)$  of order  $i$  have the same sign as  $(-1)^i$ , the leading principal minor of  $H(Z) = H_1(Z) + H_2(Z)$  of order  $i$  has the same sign as  $(-1)^i$ . This implies that the objective function in (3.5) is concave in  $(l_1, l_2, \dots, l_m)$ , which completes the proof.  $\square$

APPENDIX B

PROOF OF PROPOSITION 3.2

A necessary condition for the objective function in (3.5) to achieve its maximum is

$$\frac{\partial Z}{\partial l_i} = 0, \quad i = 2, 3, \dots, m. \quad \text{For } i = 2, \text{ I see from (A.5) that}$$

$$\frac{1}{v^n} \left[ (l_3^*)^n - (n+1)(l_2^*)^n \right] = 0$$

This leads to the expression below:

$$l_3^* = (n+1)^{\frac{1}{n}} l_2^*$$

If  $c_3 = (n+1)^{\frac{1}{n}}$  then  $l_3^* = c_3 l_2^*$ . Next, for  $i \geq 3$ , one sees from (A.1) that

$$\frac{1}{v^n} \left[ (l_{i+1}^*)^n - (n+1)(l_i^*)^n + n l_{i-1}^* (l_i^*)^{n-1} \right] = 0$$

or

$$l_{i+1}^* = \left[ (n+1)(l_i^*)^n - n l_{i-1}^* (l_i^*)^{n-1} \right]^{\frac{1}{n}}$$

When  $i = 3$ ,

$$\begin{aligned} l_4^* &= \left[ (n+1)(l_3^*)^n - n l_2^* (l_3^*)^{n-1} \right]^{\frac{1}{n}} \\ &= \left[ (n+1)(c_3 l_2^*)^n - n l_2^* (c_3 l_2^*)^{n-1} \right]^{\frac{1}{n}} \\ &= \left[ (n+1)(c_3)^n - n(c_3)^{n-1} \right]^{\frac{1}{n}} l_2^* \end{aligned}$$

If  $c_4 = \left[ (n+1)(c_3)^n - n(c_3)^{n-1} \right]^{\frac{1}{n}}$  then  $l_4^* = c_4 l_2^*$ .

When  $i = 4$ ,

$$\begin{aligned} l_5^* &= \left[ (n+1)(l_4^*)^n - n l_3^* (l_4^*)^{n-1} \right]^{\frac{1}{n}} \\ &= \left[ (n+1)(c_4 l_2^*)^n - n c_3 l_2^* (c_4 l_2^*)^{n-1} \right]^{\frac{1}{n}} \\ &= \left[ (n+1)(c_4)^n - n c_3 (c_4)^{n-1} \right]^{\frac{1}{n}} l_2^* \end{aligned}$$

If  $c_5 = \left[ (n+1)(c_4)^n - nc_3(c_4)^{n-1} \right]^{\frac{1}{n}}$  then  $l_5^* = c_5 l_2^*$ .

⋮

Finally, when  $i = m$ ,

$$\begin{aligned} l_{m+1}^* &= \left[ (n+1)(l_m^*)^n - nl_{m-1}^*(l_m^*)^{n-1} \right]^{\frac{1}{n}} \\ &= \left[ (n+1)(c_m l_2^*)^n - n(c_{m-1} l_2^*)(c_m l_2^*)^{n-1} \right]^{\frac{1}{n}} \\ &= \left[ (n+1)c_m^n - nc_{m-1}c_m^{n-1} \right]^{\frac{1}{n}} l_2^* \end{aligned}$$

If  $c_{m+1} = \left[ (n+1)c_m^n - nc_{m-1}c_m^{n-1} \right]^{\frac{1}{n}}$  then  $l_{m+1}^* = c_{m+1} l_2^*$ . Because of the assumption that  $l_{m+1}^* = \bar{v}$ , I

have  $\bar{v} = c_{m+1} l_2^*$  or  $l_2^* = \frac{1}{c_{m+1}} \bar{v}$ . It follows that, for  $i = 3, 4, \dots, m$ ,

$$l_i^* = c_i l_2^* = \frac{c_i}{c_{m+1}} \bar{v}$$

where  $c_i = \left[ (n+1)c_{i-1}^n - nc_{i-2}c_{i-1}^{n-1} \right]^{\frac{1}{n}}$ .

This along with the fact that  $l_1^* = 0$  leads to

$$l_i^* = \frac{c_i}{c_{m+1}} \bar{v}, \quad i = 1, 2, \dots, m$$

with

$$c_i = \begin{cases} 0, & i = 1 \\ 1, & i = 2 \\ \sqrt[n]{(n+1)c_{i-1}^n - nc_{i-2}c_{i-1}^{n-1}}, & i = 3, 4, \dots, m \end{cases}$$

This completes the proof.  $\square$



APPENDIX C

PROOF OF PROPOSITION 3.3

A necessary condition for the objective function in (3.5) to achieve its maximum is

$\frac{\partial z}{\partial l_i} = 0$ ,  $i = 2, 3, \dots, m$ . Thus, I have

$$\frac{1}{v^n} \left[ (l_{i+1}^*)^n - (n+1)(l_i^*)^n + n l_{i-1}^* (l_i^*)^{n-1} \right] = 0 \quad (\text{C.1})$$

Equation (C.1) can be rewritten as

$$\left( \frac{l_{i+1}^*}{l_i^*} \right)^n = (n+1) - n \frac{l_{i-1}^*}{l_i^*}$$

or

$$\left( \frac{l_{i+1}^*}{l_i^*} \right)^n = 1 + \frac{n(l_i^* - l_{i-1}^*)}{l_i^*}$$

or

$$\frac{l_{i+1}^* - l_i^*}{l_i^*} = \left[ 1 + \frac{n(l_i^* - l_{i-1}^*)}{l_i^*} \right]^{\frac{1}{n}} - 1 \quad (\text{C.2})$$

Note that  $(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots \geq 1 + nx$  with the strict equality holding when

$n=1$  or  $x=0$ . Thus,  $1+x \geq (1+nx)^{\frac{1}{n}}$  or  $x \geq (1+nx)^{\frac{1}{n}} - 1$ . Substituting  $x = \frac{l_i^* - l_{i-1}^*}{l_i^*} \neq 0$  in the

last inequality gives

$$\frac{l_i^* - l_{i-1}^*}{l_i^*} \geq \left[ 1 + \frac{n(l_i^* - l_{i-1}^*)}{l_i^*} \right]^{\frac{1}{n}} - 1 \quad (\text{C.3})$$

It follows immediately from (C.2) that

$$\frac{l_{i+1}^* - l_i^*}{l_i^*} \leq \frac{l_i^* - l_{i-1}^*}{l_i^*}$$

or

$$l_{i+1}^* - l_i^* \leq l_i^* - l_{i-1}^*,$$

where the strict equality holds when  $n = 1$ . This completes the proof.  $\square$

APPENDIX D

PROOF OF PROPOSITION 3.4

According to Proposition 3.2,  $l_i^* = \frac{c_i}{c_{m+1}} \bar{v}$  with  $c_1 = 0$ ,  $c_2 = 1$ , and

$c_i = \left[ (n+1)c_{i-1}^n - nc_{i-2}c_{i-1}^{n-1} \right]^{\frac{1}{n}}$ ,  $i = 3, 4, \dots, m$ . I now show by induction that  $c_i$  decreases with  $n$

and has the same decreasing rate as the function  $n^{\frac{i-2}{n}}$ ,  $i \geq 3$ .

To begin, let  $i = 3$  and one has Note that  $c_3 = \left[ (n+1)c_2^n - nc_1c_2^{n-1} \right]^{\frac{1}{n}} = (n+1)^{\frac{1}{n}}$ . Clearly,  $c_3$

decreases with  $n$  and has the same decreasing rate as the function  $n^{\frac{3-2}{n}} = n^{\frac{1}{n}}$ . Next, assume that

the statement is also true for  $i = k$ ,  $k \geq 4$ ; namely,  $c_k$  decreases with  $n$  and has the decreasing

rate as  $n^{\frac{k-2}{n}}$ . Now, consider the case where  $i = k+1$ . Note that

$$\begin{aligned} c_{k+1} &= \left[ (n+1)c_k^n - nc_{k-1}c_k^{n-1} \right]^{\frac{1}{n}} \\ &= c_k \left[ (n+1) - n \frac{c_{k-1}}{c_k} \right]^{\frac{1}{n}} \end{aligned}$$

It is seen that both  $c_k$  and  $\left[ (n+1) - n \frac{c_{k-1}}{c_k} \right]^{\frac{1}{n}}$  decreases with  $n$  and have the same decreasing rates

as  $n^{\frac{k-2}{n}}$  and  $n^{\frac{1}{n}}$ , respectively. Thus,  $c_{k+1}$  decreases with  $n$  and its decreasing rate is the same as

$$n^{\frac{k-2}{n}} \cdot n^{\frac{1}{n}} = n^{\frac{k-1}{n}} = n^{\frac{(k+1)-2}{n}}.$$

In sum,  $c_i$  decreases with  $n$  and has the same decreasing rate as  $n^{\frac{i-2}{n}}$ ,  $i \geq 3$ . It follows

that  $c_{m+1}$  decreases with  $n$  and has the same decreasing rate as  $n^{\frac{(m+1)-2}{n}} = n^{\frac{m-1}{n}}$ . Since

$\frac{m-1}{n} > \frac{i-2}{n}$  for  $i = 3, 4, \dots, m$ ,  $c_{m+1}$  decreases at a higher rate than  $c_i$  does, I see that

$l_i^* = \frac{c_i}{c_{m+1}} \bar{v}$  is an increasing function of  $n$ ,  $i = 2, 3, \dots, m$ .  $\square$

APPENDIX E

PROOF OF LEMMA 3.1

Recall from (3.6) that  $c_1 = 0$  and  $c_2 = 1$ . For  $i \geq 2$ , I have

$$\begin{aligned}
c_{i+1} &= \left[ (n+1)c_i^n - nc_{i-1}c_i^{n-1} \right]^{\frac{1}{n}} \\
&= c_i \left[ (n+1) - n \frac{c_{i-1}}{c_i} \right]^{\frac{1}{n}} \\
&= c_i \left[ 1 + n \left( 1 - \frac{c_{i-1}}{c_i} \right) \right]^{\frac{1}{n}}
\end{aligned} \tag{E.1}$$

Clearly,  $c_{i+1} > c_i$ , and  $c_1, c_2, \dots$  is an increasing series. Moreover, one sees from (E.1) that

$$\frac{c_{i+1}}{c_i} = \left[ 1 + n \left( \frac{c_i - c_{i-1}}{c_i} \right) \right]^{\frac{1}{n}}$$

Thus, for  $i \geq 2$ ,

$$\frac{c_{i+1}}{c_i} - 1 = \left[ 1 + n \left( \frac{c_i - c_{i-1}}{c_i} \right) \right]^{\frac{1}{n}} - 1$$

or

$$\begin{aligned}
\frac{c_{i+1} - c_i}{c_i} &= \left[ 1 + n \left( \frac{c_i - c_{i-1}}{c_i} \right) \right]^{\frac{1}{n}} - 1 \\
&\leq 1 + \frac{c_i - c_{i-1}}{c_i} - 1 \\
&= \frac{c_i - c_{i-1}}{c_i}
\end{aligned}$$

Note that the above inequality follows from our observation in Appendix C that  $1 + x \geq (1 + nx)^{\frac{1}{n}}$ .

This leads to the conclusion that  $c_{i+1} - c_i \leq c_i - c_{i-1}$  with the strict equality holding when  $n = 1$ .

In summary,  $c_1, c_2, \dots$  is an increasing series whose increments are decreasing if  $n > 1$  and constant if  $n = 1$ .  $\square$



APPENDIX F

PROOF OF PROPOSITION 3.5

Consider the series  $c_1, c_2, \dots$ , and let  $m, m' \in N$  with  $m' = m + k$ ,  $k \geq 1$ . If  $m$  bid levels are to be set up, then  $l_m^* = \frac{c_m}{c_{m+1}} \bar{v}$ . If  $m'$  bid levels are to be established, however,

$$l_{m'}^* = \frac{c_{m'}}{c_{m'+1}} \bar{v} = \frac{c_{m+k}}{c_{m+k+1}} \bar{v}.$$

I know from Lemma 3.1 that  $c_{i+1} - c_i \leq c_i - c_{i-1}$ , or,  $c_i \geq \frac{c_{i-1} + c_{i+1}}{2}$ ,  $i = 2, 3, \dots, m$ . On the other hand,  $\frac{c_{i-1} + c_{i+1}}{2} \geq \sqrt{c_{i-1}c_{i+1}}$  with the strict equality holding when  $c_{i-1} = c_{i+1}$ . It follows

that  $c_i \geq \frac{c_{i-1} + c_{i+1}}{2} > \sqrt{c_{i-1}c_{i+1}}$  or  $c_i > \sqrt{c_{i-1}c_{i+1}}$  or  $c_i^2 > c_{i-1}c_{i+1}$  or  $\frac{c_i}{c_{i+1}} > \frac{c_{i-1}}{c_i}$  since  $c_{i-1} \neq c_i$ ,

$i = 2, 3, \dots, m$ . Thus I have  $\frac{c_{m'}}{c_{m'+1}} = \frac{c_{m+k}}{c_{m+k+1}} > \frac{c_{m+k-1}}{c_{m+k}} > \dots > \frac{c_{m+1}}{c_{m+2}} > \frac{c_m}{c_{m+1}}$  or  $\frac{c_{m'}}{c_{m'+1}} > \frac{c_m}{c_{m+1}}$ .

Consequently,  $\frac{c_{m'}}{c_{m'+1}} \bar{v} > \frac{c_m}{c_{m+1}} \bar{v}$  or  $l_{m'}^* > l_m^*$ . This completes the proof that, given  $n$ ,  $l_m^*$  is an

increasing function of  $m$ .  $\square$

APPENDIX G

PROOF OF PROPOSITION 3.6

As before, a necessary condition for  $(l_1^*, l_2^*, \dots, l_m^*)$  to be the optimal bid levels in (3.5) is

that  $\frac{\partial Z}{\partial l_i^*} = 0$ ,  $i = 2, 3, \dots, m$ ; that is,

$$\frac{1}{\bar{v}^n} \left[ (l_{i+1}^*)^n - (n+1)(l_i^*)^n + n l_{i-1}^* (l_i^*)^{n-1} \right] = 0$$

or

$$(l_{i+1}^*)^n - (n+1)(l_i^*)^n + n l_{i-1}^* (l_i^*)^{n-1} = 0$$

Multiplying both sides of the equality by  $l_i^*$ , one has

$$l_i^* (l_{i+1}^*)^n - (n+1)(l_i^*)^{n+1} + n l_{i-1}^* (l_i^*)^n = 0$$

or

$$\frac{1}{n+1} l_i^* (l_{i+1}^*)^n - (l_i^*)^{n+1} + \frac{n}{n+1} l_{i-1}^* (l_i^*)^n = 0$$

Adding  $\frac{n}{n+1} l_i^* (l_{i+1}^*)^n$  to both sides of the equality, I obtain

$$l_i^* (l_{i+1}^*)^n - (l_i^*)^{n+1} + \frac{n}{n+1} l_{i-1}^* (l_i^*)^n = \frac{n}{n+1} l_i^* (l_{i+1}^*)^n \quad (\text{G.1})$$

Because  $l_1^* = 0$  and  $l_{m+1}^* = \bar{v}$ , the maximum expected auction revenue may be expressed as

$$\begin{aligned} Z^* &= \frac{1}{\bar{v}^n} \sum_{i=1}^m \left[ l_i^* (l_{i+1}^*)^n - (l_i^*)^{n+1} \right] \\ &= \frac{1}{\bar{v}^n} \left[ l_1^* (l_2^*)^n - (l_1^*)^{n+1} + l_2^* (l_3^*)^n - (l_2^*)^{n+1} + \dots + l_i^* (l_{i+1}^*)^n - (l_i^*)^{n+1} + \dots + l_{m-1}^* (l_m^*)^n - (l_{m-1}^*)^{n+1} + l_m^* (l_{m+1}^*)^n - (l_m^*)^{n+1} \right] \\ &= \frac{1}{\bar{v}^n} \left[ 0 + l_2^* (l_3^*)^n - (l_2^*)^{n+1} + \dots + l_i^* (l_{i+1}^*)^n - (l_i^*)^{n+1} + \dots + l_{m-1}^* (l_m^*)^n - (l_{m-1}^*)^{n+1} + l_m^* (l_{m+1}^*)^n - (l_m^*)^{n+1} \right] \\ &= \frac{1}{\bar{v}^n} \left\{ \left[ \frac{n}{n+1} l_1^* (l_2^*)^n + l_2^* (l_3^*)^n - (l_2^*)^{n+1} \right] + \dots + l_i^* (l_{i+1}^*)^n - (l_i^*)^{n+1} + \dots + l_{m-1}^* (l_m^*)^n - (l_{m-1}^*)^{n+1} + l_m^* (l_{m+1}^*)^n - (l_m^*)^{n+1} \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\bar{v}^n} \left\{ \left[ \frac{n}{n+1} l_2^* (l_3^*)^n + l_3^* (l_4^*)^n - (l_3^*)^{n+1} \right] + \cdots + l_i^* (l_{i+1}^*)^n - (l_i^*)^{n+1} + \cdots + l_{m-1}^* (l_m^*)^n - (l_{m-1}^*)^{n+1} + l_m^* (l_{m+1}^*)^n - (l_m^*)^{n+1} \right\} \\
&= \dots \\
&= \frac{1}{\bar{v}^n} \left\{ \left[ \frac{n}{n+1} l_{m-2}^* (l_{m-1}^*)^n + l_{m-1}^* (l_m^*)^n - (l_{m-1}^*)^{n+1} \right] + l_m^* (l_{m+1}^*)^n - (l_m^*)^{n+1} \right\} \\
&= \frac{1}{\bar{v}^n} \left[ \frac{n}{n+1} l_{m-1}^* (l_m^*)^n + l_m^* (l_{m+1}^*)^n - (l_m^*)^{n+1} \right] \\
&= \frac{1}{\bar{v}^n} \frac{n}{n+1} l_m^* (l_{m+1}^*)^n \\
&= \frac{1}{\bar{v}^n} \frac{n}{n+1} l_m^* \bar{v}^n \\
&= \frac{n}{n+1} l_m^*
\end{aligned}$$

Since  $l_m^* = \frac{c_m}{c_{m+1}}$ , I have  $Z^* = \frac{n}{n+1} l_m^* = \frac{n}{n+1} \frac{c_m}{c_{m+1}}$ . The proof is complete.  $\square$

## APPENDIX H

### PROOF OF LEMMA 3.2

Case I: when  $t > 0$  and  $i = 2, \dots, m$ , I have

$$\sum_{j=0}^{m-1} \frac{1}{(1+t)^j} = \frac{1 - \frac{1}{(1+t)^m}}{1 - \frac{1}{1+t}} = \frac{(1+t)^m - 1}{t(1+t)^{m-1}}$$

and

$$\sum_{j=m+1-i}^{m-1} \frac{1}{(1+t)^j} = \frac{\frac{1}{(1+t)^{m-i+1}} - \frac{1}{(1+t)^m}}{1 - \frac{1}{1+t}} = \frac{(1+t)^{i-1} - 1}{t(1+t)^{m-1}}$$

Thus,

$$\tau_i = \frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}} = \frac{(1+t)^{i-1} - 1}{(1+t)^m - 1}$$

In addition, I have

$$\begin{aligned} \sum_{j=0}^{m-1} j(1+t)^{-j-1} &= \sum_{j=1}^{m-1} (1+t)^{-j-1} + \sum_{j=2}^{m-1} (1+t)^{-j-1} + \dots + \sum_{j=m-1}^{m-1} (1+t)^{-j-1} \\ &= \frac{\frac{1}{(1+t)^2} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} + \frac{\frac{1}{(1+t)^3} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} + \dots + \frac{\frac{1}{(1+t)^{m-1}} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} + \frac{\frac{1}{(1+t)^m} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} \\ &= \frac{\frac{1}{1+t} - \frac{1}{(1+t)^m}}{t} + \frac{\frac{1}{(1+t)^2} - \frac{1}{(1+t)^m}}{t} + \dots + \frac{\frac{1}{(1+t)^{m-2}} - \frac{1}{(1+t)^m}}{t} + \frac{\frac{1}{(1+t)^{m-1}} - \frac{1}{(1+t)^m}}{t} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{t(1+t)} \left[ 1 - \frac{1}{(1+t)^{m-1}} \right] + \frac{1}{t(1+t)^2} \left[ 1 - \frac{1}{(1+t)^{m-2}} \right] + \cdots + \frac{1}{t(1+t)^{m-1}} \left( 1 - \frac{1}{1+t} \right) \\
&= \frac{1}{t(1+t)} + \frac{1}{t(1+t)^2} + \cdots + \frac{1}{t(1+t)^{m-1}} - \frac{1}{t} \left[ \frac{1}{(1+t)^m} + \cdots + \frac{1}{(1+t)^m} \right] \\
&= \frac{1}{t(1+t)} - \frac{1}{t(1+t)^m} - \frac{m-1}{t(1+t)^m} \\
&= \frac{1}{t^2} - \frac{1}{t^2(1+t)^{m-1}} - \frac{m-1}{t(1+t)^m}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=m+1-i}^{m-1} j(1+t)^{-j-1} &= (m+1-i) \sum_{j=m+1-i}^{m-1} (1+t)^{-j-1} + \sum_{j=m+2-i}^{m-1} (1+t)^{-j-1} + \cdots + \sum_{j=m-1}^{m-1} (1+t)^{-j-1} \\
&= (m+1-i) \frac{\frac{1}{(1+t)^{m+2-i}} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} + \frac{\frac{1}{(1+t)^{m+3-i}} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} + \cdots + \frac{\frac{1}{(1+t)^m} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} \\
&= (m+1-i) \frac{\frac{1}{(1+t)^{m+1-i}} - \frac{1}{(1+t)^m}}{t} + \frac{\frac{1}{(1+t)^{m+2-i}} - \frac{1}{(1+t)^m}}{t} + \cdots + \frac{\frac{1}{(1+t)^m} - \frac{1}{(1+t)^{m+1}}}{1 - \frac{1}{1+t}} \\
&= (m+1-i) \frac{1}{t(1+t)^{m+1-i}} \left[ 1 - \frac{1}{(1+t)^{i-1}} \right] + \frac{1}{t(1+t)^{m+2-i}} \left[ 1 - \frac{1}{(1+t)^{i-2}} \right] \\
&\quad + \cdots + \frac{1}{t(1+t)^{m-1}} \left( 1 - \frac{1}{1+t} \right) \\
&= (m+1-i) \frac{1}{t(1+t)^{m+1-i}} + \frac{1}{t(1+t)^{m+2-i}} + \cdots + \frac{1}{t(1+t)^{m-1}} \\
&\quad - (m+1-i) \frac{1}{t(1+t)^m} - \frac{1}{t(1+t)^m} - \cdots - \frac{1}{t(1+t)^m} \\
&= (m-i) \frac{1}{t(1+t)^{m+1-i}} + \frac{\frac{1}{t(1+t)^{m+1-i}} - \frac{1}{t(1+t)^m}}{1 - \frac{1}{1+t}} - \frac{m+1-i}{t(1+t)^m} - \frac{i-2}{t(1+t)^m} \\
&= \frac{m-i}{t(1+t)^{m+1-i}} + \frac{1}{t^2(1+t)^{m-i}} - \frac{1}{t^2(1+t)^{m-1}} - \frac{m-1}{t(1+t)^m}
\end{aligned}$$



Thus,

$$\begin{aligned}
\mu_i &= \frac{\sum_{j=m-i+1}^{m-1} j(1+t)^{-j-1}}{\sum_{j=0}^{m-1} j(1+t)^{-j-1}} \\
&= \frac{\frac{m-i}{t(1+t)^{m+1-i}} + \frac{1}{t^2(1+t)^{m-i}} - \frac{1}{t^2(1+t)^{m-1}} - \frac{m-1}{t(1+t)^m}}{\frac{1}{t^2} - \frac{1}{t^2(1+t)^{m-1}} - \frac{m-1}{t(1+t)^m}} \\
&= \frac{(m-i)t(1+t)^{i-1} + (1+t)^i - (1+t) - t(m-1)}{(1+t)^m - (1+t) - t(m-1)} \\
&= \frac{(1+t)^i - 1 + t[(m-i)(1+t)^{i-1} - m]}{(1+t)^m - 1 - tm} \\
&> \frac{(1+t)^i - 1}{(1+t)^m - 1} \\
&> \frac{(1+t)^{i-1} - 1}{(1+t)^m - 1} \\
&= \frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}} \\
&= \tau_i
\end{aligned}$$

Case II: when  $t = 0$  and  $i = 2, \dots, m$ , I have

$$\tau_i = \frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}} = \frac{i-1}{m}$$

and

$$\mu_i = \frac{\sum_{j=m-i+1}^{m-1} j(1+t)^{-j-1}}{\sum_{j=0}^{m-1} j(1+t)^{-j-1}} = \frac{\sum_{j=m-i+1}^{m-1} j}{\sum_{j=0}^{m-1} j} = \frac{(2m-i)(i-1)}{m(m-1)}$$

Thus,

$$\begin{aligned} \mu_i - \tau_i &= \frac{(2m-i)(i-1)}{m(m-1)} - \frac{i-1}{m} \\ &= \frac{(2m-i)(i-1) - (m-1)(i-1)}{m(m-1)} \\ &= \frac{(i-1)(m-i+1)}{m(m-1)} \\ &> 0 \end{aligned}$$

In sum, I have  $\mu_i > \tau_i$ . This completes the proof.  $\square$

APPENDIX I

PROOF OF LEMMA 3.3

The first derivative of  $\tau_i$  with respect to  $s$  is

$$\begin{aligned} \frac{\partial \tau_i}{\partial t} &= -\frac{\sum_{j=m-i+1}^{m-1} j(1+t)^{-j-1}}{\sum_{j=0}^{m-1} (1+t)^{-j}} + \frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j} \sum_{j=0}^{m-1} j(1+t)^{-j-1}}{\left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2} \\ &= \frac{-\sum_{j=0}^{m-1} (1+t)^{-j} \sum_{j=m-i+1}^{m-1} j(1+t)^{-j-1} + \sum_{j=m-i+1}^{m-1} (1+t)^{-j} \sum_{j=0}^{m-1} j(1+t)^{-j-1}}{\left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2} \end{aligned}$$

Based on Lemma 3.2,

$$\frac{\sum_{j=m-i+1}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}} < \frac{\sum_{j=m-i+1}^{m-1} j(1+t)^{-j-1}}{\sum_{j=0}^{m-1} j(1+t)^{-j-1}}$$

I have

$$\sum_{j=0}^{m-1} j(1+t)^{-j-1} \sum_{j=m-i+1}^{m-1} (1+t)^{-j} - \sum_{j=0}^{m-1} (1+t)^{-j} \sum_{j=m-i+1}^{m-1} j(1+t)^{-j-1} < 0$$

Thus,

$$\frac{\partial \tau_i}{\partial t} < 0$$

This completes the proof that  $\tau_i$  is a decreasing function of  $t$ ,  $i = 2, \dots, m$ .  $\square$

APPENDIX J

PROOF OF LEMMA 3.4

Note that

$$\begin{aligned}\tau_{i+1} &= \frac{\sum_{j=m-i}^{m-1} (1+t)^{-j}}{\sum_{j=0}^{m-1} (1+t)^{-j}} \\ &= \tau_i + \frac{(1+t)^{-(m-i)}}{\sum_{j=0}^{m-1} (1+t)^{-j}}\end{aligned}$$

The first derivative of  $\tau_{i+1}$  with respect to  $t$  can be rewritten as

$$\begin{aligned}\frac{\partial \tau_{i+1}}{\partial t} &= \frac{\partial \tau_i}{\partial t} + \frac{-(m-i)(1+t)^{-(m-i)-1}}{\sum_{j=0}^{m-1} (1+t)^{-j}} + \frac{(1+t)^{-(m-i)} \sum_{j=0}^{m-1} j(1+t)^{-j-1}}{\left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2} \\ &= \frac{\partial \tau_i}{\partial t} - \frac{(m-i)(1+t)^{-(m-i)-1} \sum_{j=0}^{m-1} (1+t)^{-j} - (1+t)^{-(m-i)} \sum_{j=0}^{m-1} j(1+t)^{-j-1}}{\left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2}\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial \tau_{i+1}}{\partial t} - \frac{\partial \tau_i}{\partial t} &= - \frac{(m-i)(1+t)^{-(m-i)-1} \sum_{j=0}^{m-1} (1+t)^{-j} - (1+t)^{-(m-i)} \sum_{j=0}^{m-1} j(1+t)^{-j-1}}{\left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2} \\ &= - \frac{(1+t)^{-(m-i)} \left[ (m-i) \sum_{j=0}^{m-1} (1+t)^{-j-1} - \sum_{j=0}^{m-1} j(1+t)^{-j-1} \right]}{\left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2}\end{aligned}$$

$$\begin{aligned}
&= -\frac{(1+t)^{-(m-i)} \left[ (m-i) \frac{(1+t)^m - 1}{t(1+t)^m} - \frac{1}{t^2} + \frac{1}{t^2(1+t)^{m-1}} + \frac{m-1}{t(1+t)^m} \right]}{\left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2} \\
&= -\frac{(1+t)^{-(m-i)} \left[ (1+t)^m (t(m-i) - 1) + t(i-1) + t + 1 \right]}{t^2(1+t)^m \left[ \sum_{j=0}^{m-1} (1+t)^{-j} \right]^2} \\
&< 0
\end{aligned}$$

This completes the proof.  $\square$

APPENDIX K

PROOF OF PROPOSITION 3.9



According to  $H_t = \bar{v} \sum_{i=1}^m \tau_i (\tau_{i+1}^n - \tau_i^n)$ , I have

$$\frac{\partial Z_s}{\partial t} = \bar{v} \sum_{i=1}^m \frac{\partial \tau_i}{\partial t} (\tau_{i+1}^n - \tau_i^n) + \bar{v} n \sum_{i=1}^m \tau_i \left( \tau_{i+1}^{n-1} \frac{\partial \tau_{i+1}}{\partial t} - \tau_i^{n-1} \frac{\partial \tau_i}{\partial t} \right) \quad (\text{K.1})$$

Since  $\tau_{i+1} = \tau_i + \frac{(1+t)^{-(m-i)}}{\sum_{j=0}^{m-1} (1+t)^{-j}} > \tau_i > 0$ , I have  $\tau_{i+1}^{n-1} > \tau_i^{n-1} > 0$ . In addition, based on Lemma 3.4,

$$\frac{\partial \tau_{i+1}}{\partial t} < \frac{\partial \tau_i}{\partial t}, \text{ it follows that } \tau_{i+1}^{n-1} \frac{\partial \tau_{i+1}}{\partial t} < \tau_i^{n-1} \frac{\partial \tau_i}{\partial t}; \text{ that is, } \tau_{i+1}^{n-1} \frac{\partial \tau_{i+1}}{\partial t} - \tau_i^{n-1} \frac{\partial \tau_i}{\partial t} < 0.$$

Note that Lemma 3.3 says  $\tau_i$  is a decreasing function of  $t$ , I can get  $\frac{\partial \tau_s}{\partial t} < 0$ . Thus, I see

from (K.1),  $\frac{\partial Z_s}{\partial t} = \bar{v} \sum_{i=1}^m \frac{\partial \tau_i}{\partial t} (\tau_{i+1}^n - \tau_i^n) + \bar{v} n \sum_{i=1}^m \tau_i \left( \tau_{i+1}^{n-1} \frac{\partial \tau_{i+1}}{\partial t} - \tau_i^{n-1} \frac{\partial \tau_i}{\partial t} \right) < 0$ . This completes the

proof that  $H_t$  is a decreasing function of  $t$ .  $\square$

APPENDIX L

PROOF OF PROPOSITION 4.1

Let  $L_i = \frac{l_i}{v}$ ,  $i = 1, 2, \dots, m$ , and the NLP in (4.3) can be rewritten as

$$\begin{aligned}
\text{Maximize} \quad & Z_\lambda = \bar{v} e^{-\lambda} \sum_{i=1}^m L_i [e^{\lambda L_{i+1}} - e^{\lambda L_i}] \\
\text{subject to:} \quad & L_{i+1} \geq L_i, i = 1, \dots, m \\
& L_1 = 0 \\
& L_{m+1} = 1
\end{aligned} \tag{L.1}$$

For each  $L_i$ ,  $i = 2, 3, \dots, m$ , I have

$$\frac{\partial Z_\lambda}{\partial L_i} = \bar{v} e^{-\lambda} [e^{\lambda L_{i+1}} - e^{\lambda L_i} - \lambda L_i e^{\lambda L_i} + \lambda L_{i-1} e^{\lambda L_i}] \tag{L.2}$$

$$\begin{aligned}
\frac{\partial^2 Z_\lambda}{\partial L_i^2} &= \bar{v} e^{-\lambda} [-\lambda e^{\lambda L_i} - \lambda e^{\lambda L_i} - \lambda^2 L_i e^{\lambda L_i} + \lambda^2 L_{i-1} e^{\lambda L_i}] \\
&= \bar{v} e^{-\lambda} [-2\lambda e^{\lambda L_i} - \lambda^2 L_i e^{\lambda L_i} + \lambda^2 L_{i-1} e^{\lambda L_i}] \\
&= \bar{v} \lambda e^{-\lambda} e^{\lambda L_i} [-2 - \lambda(L_i - L_{i-1})]
\end{aligned} \tag{L.3}$$

$$\frac{\partial^2 Z_\lambda}{\partial L_i \partial L_{i+1}} = \bar{v} \lambda e^{-\lambda} e^{\lambda L_{i+1}} \tag{L.4}$$

and

$$\frac{\partial^2 Z_\lambda}{\partial L_i \partial L_j} = 0, \quad j > i+1 \tag{L.5}$$

Let  $H(Z_\lambda)$  be the Hessian of the objective function in (L.1), it follows that

$$H(Z_\lambda) = \begin{bmatrix} \frac{\partial^2 Z_\lambda}{\partial L_2^2} & \frac{\partial^2 Z_\lambda}{\partial L_2 \partial L_3} & 0 & 0 & \dots & 0 & 0 \\ \frac{\partial^2 Z_\lambda}{\partial L_3 \partial L_2} & \frac{\partial^2 Z_\lambda}{\partial L_3^2} & \frac{\partial^2 Z_\lambda}{\partial L_3 \partial L_4} & 0 & \dots & 0 & 0 \\ 0 & \frac{\partial^2 Z_\lambda}{\partial L_4 \partial L_3} & \frac{\partial^2 Z_\lambda}{\partial L_4^2} & \frac{\partial^2 Z_\lambda}{\partial L_4 \partial L_5} & \dots & 0 & 0 \\ 0 & 0 & \frac{\partial^2 Z_\lambda}{\partial L_5 \partial L_4} & \frac{\partial^2 Z_\lambda}{\partial L_5^2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{\partial^2 Z_\lambda}{\partial L_{m-1}^2} & \frac{\partial^2 Z_\lambda}{\partial L_{m-1} \partial L_m} \\ 0 & 0 & 0 & 0 & \dots & \frac{\partial^2 Z_\lambda}{\partial L_m \partial L_{m-1}} & \frac{\partial^2 Z_\lambda}{\partial L_m^2} \end{bmatrix}$$

Based on (L.2), (L.3), (L.4) and (L.5), I have

$$H(Z_\lambda) = \bar{v} \lambda e^{-\lambda} \begin{bmatrix} e^{\lambda L_2} [-2 - \lambda(L_2 - L_1)] & e^{\lambda L_3} & 0 & \dots & 0 \\ e^{\lambda L_3} & e^{\lambda L_3} [-2 - \lambda(L_3 - L_2)] & e^{\lambda L_4} & \dots & 0 \\ 0 & e^{\lambda L_4} & e^{\lambda L_4} [-2 - \lambda(L_4 - L_3)] & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & e^{\lambda L_m} [-2 - \lambda(L_m - L_{m-1})] \end{bmatrix}$$

Let  $e^{\lambda L_{i+1}} = e^{\lambda L_i} + \Delta(L_i, \lambda)$ ,  $i = 2, \dots, m-1$ . The Hessian can be rewritten as

$$H(Z_\lambda) = \bar{v} \lambda e^{-\lambda} \begin{bmatrix} e^{\lambda L_2} [-2 - \lambda(L_2 - L_1)] & e^{\lambda L_2} + \Delta(L_2, \lambda) & 0 & \dots & 0 \\ e^{\lambda L_2} + \Delta(L_2, \lambda) & e^{\lambda L_3} [-2 - \lambda(L_3 - L_2)] & e^{\lambda L_3} + \Delta(L_3, \lambda) & \dots & 0 \\ 0 & e^{\lambda L_3} + \Delta(L_3, \lambda) & e^{\lambda L_4} [-2 - \lambda(L_4 - L_3)] & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & e^{\lambda L_m} [-2 - \lambda(L_m - L_{m-1})] \end{bmatrix}$$

Now,  $H(Z_\lambda)$  is decomposed into  $H_1(Z_\lambda)$  and  $H_2(Z_\lambda)$  below so that  $H(Z_\lambda) = H_1(Z_\lambda) + H_2(Z_\lambda)$ :

$$H_1(Z_\lambda) = \bar{v}\lambda e^{-\lambda} \begin{bmatrix} -2e^{\lambda L_2} & e^{\lambda L_3} & 0 & \cdots & 0 \\ e^{\lambda L_2} & -2e^{\lambda L_3} & e^{\lambda L_4} & \cdots & 0 \\ 0 & e^{\lambda L_3} & -2e^{\lambda L_4} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -2e^{\lambda L_m} \end{bmatrix}$$

and

$$H_2(Z_\lambda) = \bar{v}\lambda e^{-\lambda} \begin{bmatrix} -\lambda e^{\lambda L_2}(L_2 - L_1) & 0 & 0 & \cdots & 0 \\ \Delta(L_2, \lambda) & -\lambda e^{\lambda L_3}(L_3 - L_2) & 0 & \cdots & 0 \\ 0 & \Delta(L_3, n) & -\lambda e^{\lambda L_4}(L_4 - L_3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\lambda e^{\lambda L_m}(L_m - L_{m-1}) \end{bmatrix}$$

I am going to prove that each of the respective leading principal minors of  $H_1(Z_\lambda)$  and  $H_2(Z_\lambda)$  of order  $i$  has the same sign as  $(-1)^i$ . To begin, let  $M_i$  be the leading principal minor of  $H_1(Z_\lambda)$  of order  $i$ . One then has

$$M_1 = -2\bar{v}\lambda e^{-\lambda} e^{\lambda L_2} < 0,$$

$$M_2 = 3(\bar{v}\lambda e^{-\lambda})^2 e^{\lambda L_2} e^{\lambda L_3} > 0,$$

$$M_3 = -4(\bar{v}\lambda e^{-\lambda})^3 e^{\lambda L_2} e^{\lambda L_3} e^{\lambda L_4} < 0,$$

$$M_4 = 15(\bar{v}\lambda e^{-\lambda})^4 e^{\lambda L_2} e^{\lambda L_3} e^{\lambda L_4} e^{\lambda L_5} > 0,$$

and

$$M_i = (-1)^i 2^i (\bar{v}\lambda e^{-\lambda})^i \prod_{k=1}^i e^{\lambda L_k}, \quad i \geq 5$$

It is clear that  $M_i$  has the same sign as  $(-1)^i$ ,  $i = 1, 2, \dots$

Likewise, let  $N_i$  be the leading principal minor of  $H_2(Z_\lambda)$  of order  $i$ . A similar approach may be followed to demonstrate that

$$N_i = (-1)^i (\bar{v} \lambda^2 e^{-\lambda})^i \prod_{k=1}^i e^{\lambda L_k} (L_k - L_{k-1})$$

It is seen that  $N_i$  has the same sign as  $(-1)^i$ ,  $i = 1, 2, \dots$

Since the respective leading principal minors of  $H_1(Z_\lambda)$  and  $H_2(Z_\lambda)$  of order  $i$  have the same sign as  $(-1)^i$ , the leading principal minor of  $H(Z_\lambda) = H_1(Z_\lambda) + H_2(Z_\lambda)$  of order  $i$  has the same sign as  $(-1)^i$ . This implies that the objective function in (A12.1) is concave in

$L_1, L_2, \dots, L_m$  and, hence, in  $l_1, l_2, \dots, l_m$ , which complete our proof.  $\square$

APPENDIX M

PROOF OF PROPOSITION 4.2

A necessary condition for  $(L_1^*, L_2^*, \dots, L_m^*)$  to be an optimal solution to the NLP in (L.1) is

$\frac{\partial Z_\lambda}{\partial L_i^*} = 0$ ,  $i = 2, 3, \dots, m$ . Thus, I have

$$e^{\lambda L_{i+1}^*} - e^{\lambda L_i^*} - \lambda e^{\lambda L_i^*} (L_i^* - L_{i-1}^*) = 0 \quad (\text{M.1})$$

or

$$\frac{e^{\lambda L_{i+1}^*}}{e^{\lambda L_i^*}} - 1 - \lambda (L_i^* - L_{i-1}^*) = 0$$

or

$$e^{\lambda(L_{i+1}^* - L_i^*)} = 1 + \lambda (L_i^* - L_{i-1}^*) \quad (\text{M.2})$$

Note that, according to Taylor expansion, I have

$$\begin{aligned} e^{\lambda(L_{i+1}^* - L_i^*)} &= 1 + \lambda (L_i^* - L_{i-1}^*) + \frac{\lambda^2 (L_i^* - L_{i-1}^*)^2}{2!} + \frac{\lambda^3 (L_i^* - L_{i-1}^*)^3}{3!} + \dots \\ &> 1 + \lambda (L_i^* - L_{i-1}^*) \end{aligned} \quad (\text{M.3})$$

It follows from (M.2) and (M.3) that

$$e^{\lambda(L_{i+1}^* - L_i^*)} < e^{\lambda(L_i^* - L_{i-1}^*)}$$

or

$$L_{i+1}^* - L_i^* < L_i^* - L_{i-1}^*$$

This completes the proof that  $l_{i+1}^* - l_i^* < l_i^* - l_{i-1}^*$ ,  $i = 2, 3, \dots, m$ .  $\square$



APPENDIX N

PROOF OF PROPOSITION 4.3

Let  $k \in \{0\} \cup N$ ,  $x > 0$ , and

$$LN(k, x) = \underbrace{\ln(1 + \ln(1 + \cdots + \ln(1 + x)))}_{k \text{ ln}(\cdot)\text{'s involved}} \quad (\text{N.1})$$

Especially,  $LN(0, x) = x$ .

A necessary condition for  $(L_1^*, L_2^*, \dots, L_m^*)$  to be an optimal solution to the NLP in (L.1) is

$$\frac{\partial Z_\lambda}{\partial L_i^*} = 0, \quad i = 2, 3, \dots, m, \text{ or, } e^{\lambda L_{i+1}^*} - e^{\lambda L_i^*} - \lambda e^{\lambda L_i^*} (L_i^* - L_{i-1}^*) = 0$$

For  $i = 2$ , the following holds since  $L_1^* = \frac{L_1^*}{v} = \frac{0}{v} = 0$ :

$$e^{\lambda L_3^*} - e^{\lambda L_2^*} - \lambda e^{\lambda L_2^*} L_2^* = 0$$

or

$$e^{\lambda L_3^*} = e^{\lambda L_2^*} (1 + \lambda L_2^*)$$

Thus,

$$\lambda L_3^* = \lambda L_2^* + \ln(1 + \lambda L_2^*) \quad (\text{N.1})$$

$$L_3^* = \frac{\lambda L_2^* + \ln(1 + \lambda L_2^*)}{\lambda}$$

$$= \frac{1}{\lambda} \sum_{k=0}^1 LN(k, \lambda L_2^*)$$

For  $i = 3$ ,

$$e^{\lambda L_4^*} - e^{\lambda L_3^*} - \lambda e^{\lambda L_3^*} (L_3^* - L_2^*) = 0$$

or

$$e^{\lambda L_4^*} = e^{\lambda L_3^*} (\lambda L_3^* - \lambda L_2^* + 1)$$

Thus, I have

$$\begin{aligned}
\lambda L_4^* &= \lambda L_3^* + \ln(1 + \ln(1 + \lambda L_2^*)) \\
&= \lambda L_2^* + \ln(1 + \lambda L_2^*) + \ln(1 + \ln(1 + \lambda L_2^*)) \\
&= \sum_{k=0}^2 LN(k, \lambda L_2^*)
\end{aligned}$$

or

$$L_4^* = \frac{1}{\lambda} \sum_{k=0}^2 LN(k, \lambda L_2^*)$$

Continuing in the same fashion, one can obtain the general expression below and see that  $L_i^*$  is a function of  $L_2^*$  as well as  $\lambda$ ,  $i = 3, 4, \dots, m$ .

$$L_i^* = \frac{1}{\lambda} \sum_{k=0}^{i-2} LN(k, \lambda L_2^*) \quad (\text{N.2})$$

For the Dutch auction with  $m = 2$ , one has  $L_{m+1}^* = L_3^* = 1$ . One then sees from (N.1) that

$$\lambda L_2^* + \ln(1 + \lambda L_2^*) = \lambda$$

or

$$\ln(1 + \lambda L_2^*) = \lambda(1 - L_2^*)$$

or

$$1 + \lambda L_2^* = e^{\lambda(1-L_2^*)}$$

or

$$\lambda L_2^* = e^{\lambda(1-L_2^*)} - 1$$

When  $m = 3$ ,  $L_{m+1}^* = L_4^* = 1$  and I see from (N.2) that

$$\lambda L_2^* + \ln(1 + \lambda L_2^*) + \ln(1 + \ln(1 + \lambda L_2^*)) = \lambda$$

or

$$\ln(1 + \lambda L_2^*) + \ln(1 + \ln(1 + \lambda L_2^*)) = \lambda(1 - L_2^*)$$

or

$$(1 + \lambda L_2^*) (1 + \ln(1 + \lambda L_2^*)) = e^{\lambda(1-L_2^*)}$$

or

$$\ln(1 + \lambda L_2^*) = \frac{e^{\lambda(1-L_2^*)}}{1 + \lambda L_2^*} - 1$$

or

$$1 + \lambda L_2^* = e^{\frac{e^{\lambda(1-L_2^*)}}{1 + \lambda L_2^*} - 1}$$

or

$$\lambda L_2^* = e^{\frac{e^{\lambda(1-L_2^*)}}{1 + \lambda L_2^*} - 1} - 1$$

When  $m = 4$ ,  $L_{m+1}^* = L_5^* = 1$  and I see from (N.2) again that

$$\lambda L_2^* + \ln(1 + \lambda L_2^*) + \ln(1 + \ln(1 + \lambda L_2^*)) + \ln(1 + \ln(1 + \ln(1 + \lambda L_2^*))) = \lambda$$

or

$$\ln(1 + \lambda L_2^*) + \ln(1 + \ln(1 + \lambda L_2^*)) + \ln(1 + \ln(1 + \ln(1 + \lambda L_2^*))) = \lambda(1 - L_2^*)$$

or

$$(1 + \lambda L_2^*) (1 + \ln(1 + \lambda L_2^*)) (1 + \ln(1 + \ln(1 + \lambda L_2^*))) = e^{\lambda(1-L_2^*)}$$

or

$$1 + \ln(1 + \ln(1 + \lambda L_2^*)) = \frac{e^{\lambda(1-L_2^*)}}{(1 + \lambda L_2^*) (1 + \ln(1 + \lambda L_2^*))}$$

or

$$1 + \ln(1 + \lambda L_2^*) = e^{\frac{e^{\lambda(1-L_2^*)}}{(1 + \lambda L_2^*) (1 + \ln(1 + \lambda L_2^*))} - 1}$$

or

$$\lambda L_2^* = e^{\frac{e^{\lambda(1-L_2^*)}}{(1+\lambda L_2^*)(1+\ln(1+\lambda L_2^*))} - 1} - 1$$

In general, for the Dutch auction with  $m$  bid levels, one has

$$\lambda L_2^* = e^{\frac{e^{\lambda(1-L_2^*)}}{(1+\lambda L_2^*)(1+\ln(1+\lambda L_2^*)) \cdots (1+\ln(1+\cdots+\ln(1+\lambda L_2^*)))} - 1} - 1 \quad (\text{N.3})$$

It follows that, for  $i = 2, 3, \dots, m$ ,

$$LN(k, \lambda L_2^*) = \underbrace{\ln(\ln \cdots \ln)}_{k-1 \ln(\cdot)\text{'s involved}} \left( e^{\frac{e^{\lambda(1-L_2^*)}}{(1+\lambda L_2^*)(1+\ln(1+\lambda L_2^*)) \cdots (1+\ln(1+\cdots+\ln(1+\lambda L_2^*)))} - 1} + 1 \right) + 1 \quad (\text{N.4})$$

Since the increasing rate of the exponential function is greater than the decreasing rate of  $\frac{1}{\lambda}$ ,

$\frac{1}{\lambda} LN(k, \lambda L_2^*)$  is an increasing function of  $\lambda$ . As a result,  $L_i^* = \frac{1}{\lambda} \sum_{k=0}^{i-2} LN(k, \lambda L_2^*)$  is an increasing

function of  $\lambda$ ; so is  $l_i^*$ ,  $i = 2, 3, \dots, m$ . This completes the proof.  $\square$

APPENDIX O

PROOF OF PROPOSITION 4.4

To maximize the objective function in (L.1), the following must hold for

$i = 3, 4, \dots, m + 1$  according to (N.2):

$$L_i^* = \frac{1}{\lambda} \sum_{k=0}^{i-2} LN(k, \lambda L_2^*),$$

where  $LN(k, \lambda L_2^*) = \underbrace{\ln(1 + \ln(1 + \dots + \ln(1 + \lambda L_2^*)))}_{k \ln(\cdot)\text{'s involved}}$ . Obviously,

$$L_i^* = L_{i-1}^* + \frac{1}{\lambda} LN(i-2, \lambda L_2^*) \quad (\text{O.1})$$

If  $m$  bid levels are to be set up, then  $L_{m+1}^* = 1$  and (N.2) becomes

$$\frac{1}{\lambda} \sum_{k=0}^{m-1} LN(k, \lambda L_2^*) = 1 \quad (\text{O.2})$$

In addition, according to (O.1),  $L_{m+1}^* = L_m^* + \frac{1}{\lambda} LN(m-1, \lambda L_2^*) = 1$ . Thus,  $L_m^*$  can be written as

$$L_m^* = 1 - \frac{1}{\lambda} LN(m-1, \lambda L_2^*) \quad (\text{O.3})$$

However, if  $m' = m + p$ ,  $p = 1, 2, \dots$ , bid levels are to be established, then  $L_{m'+1}^* = 1$ . (N.2)

becomes

$$\frac{1}{\lambda} \sum_{k=0}^{m'-1} LN(k, \lambda L_2^*) = 1$$

That is,

$$\frac{1}{\lambda} \sum_{k=0}^{m+p-1} LN(k, \lambda L_2^*) = 1 \quad (\text{O.4})$$

Again, according to (O.1), one gets  $L_{m'+1}^* = L_{m'}^* + \frac{1}{\lambda} LN(m'-1, \lambda L_2^*) = 1$ . Thus,  $L_{m'}^*$  can be

expressed as

$$\begin{aligned}
L_{m'}^* &= 1 - \frac{1}{\lambda} LN(m' - 1, \lambda L_2^*) \\
&= 1 - \frac{1}{\lambda} LN(m + p - 1, \lambda L_2^*)
\end{aligned} \tag{O.5}$$

Combing (O.2) and (O.4), one can get

$$\sum_{k=0}^{m-1} LN(k, \lambda L_2^*) = \sum_{k=0}^{m+p-1} LN(k, \lambda L_2^*) \tag{O.6}$$

Since  $LN(k, \lambda L_2^*) > 0$ ,  $k = 0, 1, \dots$  and  $p = 1, 2, \dots$ , the right-hand side of (O.6) includes more positive items than its left-hand side. This leads to  $L_2^* < L_2^*$ , which indicates that  $l_2^* = \bar{\nu} L_2^*$  is a decreasing function of  $m$ . Thus,

$$\begin{aligned}
LN(m + p - 1, \lambda L_2^*) &= \underbrace{\ln(1 + \ln(1 + \dots + \ln(1 + \lambda L_2^*)))}_{m+p-1 \text{ ln}(\cdot)\text{'s involved}} \\
&< \underbrace{\ln(1 + \ln(1 + \dots + \ln(1 + \lambda L_2^*)))}_{m-1 \text{ ln}(\cdot)\text{'s involved}} \\
&< \underbrace{\ln(1 + \ln(1 + \dots + \ln(1 + \lambda L_2^*)))}_{m-1 \text{ ln}(\cdot)\text{'s involved}} \\
&= LN(m - 1, \lambda L_2^*)
\end{aligned} \tag{O.7}$$

Combing (O.3), (O.5) and (O.7), I have

$$\begin{aligned}
L_{m'}^* &= 1 - \frac{1}{\lambda} LN(m + p - 1, \lambda L_2^*) \\
&> 1 - \frac{1}{\lambda} LN(m - 1, \lambda L_2^*) \\
&= L_m^*
\end{aligned}$$

This completes the proof that  $l_m^* = \bar{\nu} L_m^*$  is an increasing function of  $m$ .  $\square$



APPENDIX P

PROOF OF PROPOSITION 4.5

Since the optimal bid levels must satisfy the equality (A13.1) for  $i = 2, 3, \dots, m$ , I have

$$\sum_{i=2}^m \left[ e^{\lambda L_{i+1}^*} - e^{\lambda L_i^*} - \lambda e^{\lambda L_i^*} (L_i^* - L_{i-1}^*) \right] = 0$$

This leads to

$$\begin{aligned} & \sum_{i=2}^m \lambda e^{\lambda L_i^*} (L_i^* - L_{i-1}^*) \\ &= \sum_{i=2}^m (e^{\lambda L_{i+1}^*} - e^{\lambda L_i^*}) \\ &= e^{\lambda L_3^*} - e^{\lambda L_2^*} + e^{\lambda L_4^*} - e^{\lambda L_3^*} + \dots + e^{\lambda L_{m+1}^*} - e^{\lambda L_m^*} \\ &= e^{\lambda} - e^{\lambda L_2^*} \end{aligned}$$

or

$$\sum_{i=2}^m e^{\lambda L_i^*} (L_i^* - L_{i-1}^*) = \frac{1}{\lambda} (e^{\lambda} - e^{\lambda L_2^*}) \quad (\text{P.1})$$

In addition, given  $L_1^* = 0$ , one has

$$\begin{aligned} & \sum_{i=2}^m e^{\lambda L_i^*} (L_i^* - L_{i-1}^*) \\ &= \sum_{i=2}^m L_i^* e^{\lambda L_i^*} - \sum_{i=2}^m L_{i-1}^* e^{\lambda L_i^*} \\ &= \sum_{i=2}^m L_i^* e^{\lambda L_i^*} - \sum_{i=2}^{m-1} L_i^* e^{\lambda L_{i+1}^*} \end{aligned} \quad (\text{P.2})$$

Combining (P.1) and (P.2), I have

$$\sum_{i=2}^m L_i^* e^{\lambda L_i^*} - \sum_{i=2}^{m-1} L_i^* e^{\lambda L_{i+1}^*} = \frac{1}{\lambda} (e^{\lambda} - e^{\lambda L_2^*}) \quad (\text{P.3})$$

Based on (4.1) and (P.3), one can express the maximum expected auction revenue as

$$\begin{aligned}
Z_\lambda^* &= \sum_{i=1}^m l_i^* e^{-\lambda} \left( e^{\frac{\lambda_{i+1}^*}{\bar{v}}} - e^{\frac{\lambda_i^*}{\bar{v}}} \right) \\
&= \sum_{i=2}^m l_i^* e^{-\lambda} \left( e^{\frac{\lambda_{i+1}^*}{\bar{v}}} - e^{\frac{\lambda_i^*}{\bar{v}}} \right) \\
&= \bar{v} e^{-\lambda} \sum_{i=2}^m L_i^* \left( e^{\lambda_{i+1}^*} - e^{\lambda_{i-1}^*} \right) \\
&= \bar{v} e^{-\lambda} \left( \sum_{i=2}^m L_i^* e^{\lambda_{i+1}^*} - \sum_{i=2}^m L_i^* e^{\lambda_{i-1}^*} \right) \\
&= \bar{v} e^{-\lambda} \left[ L_m^* e^\lambda - \frac{1}{\lambda} \left( e^\lambda - e^{\lambda_{i_2}^*} \right) \right] \\
&= \bar{v} \left\{ L_m^* - \frac{1}{\lambda} \left[ 1 - e^{-\lambda(1-L_2^*)} \right] \right\} \\
&= L_m^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{L_2^*}{\bar{v}} \right)} \right]
\end{aligned}$$

(P.4)

This completes the proof.  $\square$

APPENDIX Q

PROOF OF PROPOSITION 4.6

Observe from (P.4) that

$$\begin{aligned}
Z_{\lambda}^* &= l_m^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_m^*}{\bar{v}} \right)} \right] \\
&= \bar{v} L_m^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda (1 - L_m^*)} \right] \\
&= \bar{v} L_m^* - \bar{v} \frac{e^{\lambda (1 - L_m^*)} - 1}{\lambda e^{\lambda (1 - L_m^*)}}
\end{aligned} \tag{Q.1}$$

According to (N.2) with  $L_{m+1}^* = 1$ , I have

$$L_{m+1}^* = \frac{1}{\lambda} \sum_{k=0}^{m-1} LN(k, \lambda L_2^*)$$

or

$$\lambda = \sum_{k=0}^{m-1} LN(k, \lambda L_2^*)$$

or

$$\lambda - \lambda L_2^* = \sum_{k=1}^{m-1} LN(k, \lambda L_2^*)$$

or

$$\begin{aligned}
e^{\lambda (1 - L_2^*)} &= e^{\sum_{k=1}^{m-1} LN(k, \lambda L_2^*)} \\
&= \prod_{k=0}^{m-2} [1 + LN(k, \lambda L_2^*)]
\end{aligned} \tag{Q.2}$$

Thus, (Q.1) can be rewritten as

$$Z_{\lambda}^* = \bar{v} L_m^* - \bar{v} \frac{\prod_{k=0}^{m-2} [1 + LN(k, \lambda L_2^*)] - 1}{\lambda \prod_{k=0}^{m-2} [1 + LN(k, \lambda L_2^*)]} \tag{Q.3}$$

Note that  $\frac{\prod_{k=0}^{m-2} [1 + LN(k, \lambda L_2^*)] - 1}{\lambda \prod_{k=0}^{m-2} [1 + LN(k, \lambda L_2^*)]}$  is a decreasing function of  $\lambda$ . On the other hand, as shown in

Proposition 4.3,  $l_m^*$  is an increasing function of  $\lambda$ ; so is  $L_m^*$ . Consequently,  $Z_\lambda^*$  is an increasing function of  $\lambda$ . The proof is complete.  $\square$

APPENDIX R

PROOF OF PROPOSITION 4.7

According to (P.4), (O.3) and (Q.2), I have

$$\begin{aligned}
Z_\lambda^* &= l_m^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left(1 - \frac{l_2^*}{\bar{v}}\right)} \right] \\
&= \bar{v} l_m^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda(1-L_2^*)} \right] \\
&= \bar{v} \left\{ 1 - \frac{1}{\lambda} LN(m-1, \lambda L_2^*) - \frac{1}{\lambda} \left[ 1 - e^{-\lambda(1-L_2^*)} \right] \right\} \\
&= \bar{v} \left( 1 - \frac{1}{\lambda} \right) - \frac{\bar{v}}{\lambda} \left[ LN(m-1, \lambda L_2^*) - e^{-\lambda(1-L_2^*)} \right] \\
&= \bar{v} \left( 1 - \frac{1}{\lambda} \right) - \frac{\bar{v}}{\lambda} \left\{ LN(m-1, \lambda L_2^*) - \frac{1}{\prod_{k=0}^{m-2} [1 + LN(k, \lambda L_2^*)]} \right\}
\end{aligned}$$

According to (O.2), that is,  $\frac{1}{\lambda} \sum_{k=0}^{m-1} LN(k, \lambda L_2^*) = 1$ , one has

$$\sum_{k=0}^{m-1} LN(k, \lambda L_2^*) = \lambda \tag{R.1}$$

Since  $LN(k, \lambda L_2^*) > 0$ ,  $k = 0, 1, \dots$ , the right-hand side of (R.1) includes more positive items as  $m$  increases. It follows that  $LN(k, \lambda L_2^*)$  is a decreasing function of  $m$ . This leads to the result that  $L_2^*$  decreases with  $m$ ; so is  $l_2^* = \bar{v} L_2^*$ . It implies that each of  $LN(m-1, \lambda L_2^*)$  and

$\prod_{k=0}^{m-2} [1 + LN(k, \lambda L_2^*)]$  is also decreasing function of  $m$ . Thus,  $Z_\lambda^*$  is an increasing function of  $m$ .

This completes the proof.  $\square$



APPENDIX S

PROOF OF PROPOSITION 4.8

According to Lemma 3.3, which says  $\tau_i$  is a decreasing function of  $t \geq 0$ ,  $i = 2, 3, \dots, m$ .

This implies that  $\frac{\partial \tau_i}{\partial t} < 0$ . One then has

$$\frac{\partial \tau_{i+1}}{\partial t} e^{\lambda \tau_{i+1}} < 0 \quad (\text{S.1})$$

and

$$\frac{\partial \tau_i}{\partial t} e^{\lambda \tau_i} < 0 \quad (\text{S.2})$$

Combining (S.1), (S.2) and Lemma 3.4, one gets

$$\frac{\partial \tau_{i+1}}{\partial t} e^{\lambda \tau_{i+1}} < \frac{\partial \tau_i}{\partial t} e^{\lambda \tau_i} < 0$$

or

$$\frac{\partial \tau_{i+1}}{\partial t} e^{\lambda \tau_{i+1}} - \frac{\partial \tau_i}{\partial t} e^{\lambda \tau_i} < 0 \quad (\text{S.3})$$

Based on (S.3) and  $\frac{\partial \tau_i}{\partial t} < 0$ , I have

$$\begin{aligned} & \frac{\partial Z_{\lambda, s}}{\partial t} \\ &= e^{-\lambda \bar{v}} \sum_{i=1}^m \frac{\partial \tau_i}{\partial t} (e^{\lambda \tau_{i+1}} - e^{\lambda \tau_i}) + e^{-\lambda \bar{v}} \sum_{i=1}^m \tau_i \left( \lambda \frac{\partial \tau_{i+1}}{\partial t} e^{\lambda \tau_{i+1}} - \lambda \frac{\partial \tau_i}{\partial t} e^{\lambda \tau_i} \right) \\ &< 0 \end{aligned}$$

This completes the proof that  $F_t$  is a decreasing function of  $t$ .  $\square$

APPENDIX T

PROOF OF PROPOSITION 5.5

Recall that  $LN(k, x) = \underbrace{\ln(1 + \ln(1 + \dots + \ln(1 + x)))}_{k \text{ ln}(\cdot)\text{'s involved}}$ ,  $k \in \{0\} \cup N$ ,  $x > 0$ , with  $LN(0, x) = x$ .

A necessary condition for  $(l'_{S,1}, l'_{S,2}, \dots, l'_{S,m})$  to be an optimal solution to the NLP in (5.4) is

$$\frac{\partial Z_{S,\lambda}}{\partial l'_{S,i}} = 0 \text{ or } e^{\lambda l'_{i+1}} - e^{\lambda l'_i} - \lambda e^{\lambda l'_i} (l'_i - l'_{i-1}) = 0, \quad i = 2, 3, \dots, m. \text{ For } i = 3, 4, \dots, m, \text{ a general}$$

expression of  $l'_{S,i}$  as a function of  $l'_{S,2}$  in light of (N.2) is

$$l'_{S,i} = \frac{1}{\lambda} \sum_{k=0}^{i-2} LN(k, \lambda l'_{S,2}) \quad (\text{T.1})$$

Since  $l'_{S,m+1} = 1 - \beta$ , one has

$$1 - \beta = \frac{1}{\lambda} \sum_{k=0}^{m-1} LN(k, \lambda l'_{S,2})$$

or

$$\lambda(1 - \beta) = \sum_{k=0}^{m-1} LN(k, \lambda l'_{S,2})$$

or

$$\lambda(1 - \beta) = \sum_{k=1}^{m-1} LN(k, \lambda l'_{S,2}) + \lambda l'_{S,2}$$

Since  $l'_{S,2} = \frac{l'_{S,2}}{\bar{v}} - \beta$ , one gets

$$\lambda(1 - \beta) = \sum_{k=1}^{m-1} LN \left[ k, \lambda \left( \frac{l'_{S,2}}{\bar{v}} - \beta \right) \right] + \lambda \left( \frac{l'_{S,2}}{\bar{v}} - \beta \right)$$

or

$$\sum_{k=1}^{m-1} LN \left[ k, \lambda \left( \frac{l'_{S,2}}{\bar{v}} - \beta \right) \right] + \lambda \left( \frac{l'_{S,2}}{\bar{v}} - \beta \right) - \lambda(1 - \beta) = 0$$

or

$$\sum_{k=1}^{m-1} LN \left[ k, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] + \lambda \frac{l_{S,2}^*}{\bar{v}} = \lambda \quad (T.2)$$

Since  $LN \left[ k, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] = \underbrace{\ln \left( 1 + \ln \left( 1 + \dots + \ln \left( 1 + \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right) \right) \right)}_{k \ln(\cdot)'s \text{ involved}}$ , it follows that

$$\begin{aligned} & \frac{\partial LN \left[ k, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right]}{\partial \beta} \\ &= \left\{ 1 + LN \left[ k-1, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \left\{ 1 + LN \left[ k-2, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \dots \left\{ 1 + LN \left[ 0, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \left( \frac{\lambda}{\bar{v}} \frac{\partial l_{S,2}^*}{\partial \beta} - \lambda \right) \\ &= \lambda \left( \frac{1}{\bar{v}} \frac{\partial l_{S,2}^*}{\partial \beta} - 1 \right) \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \end{aligned} \quad (T.3)$$

Taking the partial derivative with respect to  $\beta$  on both sides of (T.2) gives

$$\sum_{k=1}^{m-1} \frac{\partial LN \left[ k, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right]}{\partial \beta} + \frac{\lambda}{\bar{v}} \frac{\partial l_{S,2}^*}{\partial \beta} = 0$$

In light of (T.3), the above equation can be written as

$$\sum_{k=1}^{m-1} \lambda \left( \frac{1}{\bar{v}} \frac{\partial l_{S,2}^*}{\partial \beta} - 1 \right) \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} + \frac{\lambda}{\bar{v}} \frac{\partial l_{S,2}^*}{\partial \beta} = 0$$

or

$$\left( \frac{\partial l_{S,2}^*}{\partial \beta} - \bar{v} \right) \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} + \frac{\partial l_{S,2}^*}{\partial \beta} = 0$$

or

$$\frac{\partial l_{S,2}^*}{\partial \beta} \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} + \frac{\partial l_{S,2}^*}{\partial \beta} = \bar{v} \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}$$

or

$$\frac{\partial l_{S,2}^*}{\partial \beta} = \frac{\bar{v} \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} \quad (\text{T.4})$$

Since  $l_{S,i}^* = \frac{l_{S,i}^*}{\bar{v}} - \beta$ , (T.1) can be rewritten as

$$\frac{l_{S,i}^*}{\bar{v}} - \beta = \frac{1}{\lambda} \sum_{k=0}^{i-2} LN \left[ k, \lambda \left( \frac{l_{S,i}^*}{\bar{v}} - \beta \right) \right]$$

or

$$l_{S,i}^* = \beta \bar{v} + \frac{\bar{v}}{\lambda} \sum_{k=0}^{i-2} LN \left[ k, \lambda \left( \frac{l_{S,i}^*}{\bar{v}} - \beta \right) \right]$$

It follows from (T.3) and (T.4) that

$$\begin{aligned}
\frac{\partial l_{S,i}^*}{\partial \beta} &= \frac{\partial}{\partial \beta} \left\{ \beta \bar{v} + \frac{\bar{v}}{\lambda} \sum_{k=0}^{i-2} LN \left[ k, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\} \\
&= \frac{\partial}{\partial \beta} \left\{ \beta \bar{v} + \bar{v} \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) + \frac{\bar{v}}{\lambda} \sum_{k=1}^{i-2} LN \left[ k, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\} \\
&= \bar{v} + \frac{\partial l_{S,2}^*}{\partial \beta} - \bar{v} + \frac{\bar{v}}{\lambda} \sum_{k=1}^{i-2} \lambda \left( \frac{1}{\bar{v}} \frac{\partial l_{S,2}^*}{\partial \beta} - 1 \right) \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \\
&= \frac{\partial l_{S,2}^*}{\partial \beta} + \left( \frac{\partial l_{S,2}^*}{\partial \beta} - \bar{v} \right) \sum_{k=1}^{i-2} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \\
&= \frac{\bar{v} \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} + \bar{v} \frac{\left\{ \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \right\}}{\left\{ 1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \right\}} - 1 \left\{ \sum_{k=1}^{i-2} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \right\} \\
&= \frac{\bar{v} \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} - \frac{\bar{v}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} \sum_{k=1}^{i-2} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \\
&= \frac{\bar{v}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} \left\{ \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} - \sum_{k=1}^{i-2} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1} \right\} \\
&= \frac{\bar{v} \sum_{k=i-1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} \tag{T.5}
\end{aligned}$$

Obviously,  $\frac{\partial l_{S,i}^*}{\partial \beta} > 0$  and it indicates that  $l_{S,i}^*$  increases with  $\beta$ ,  $i = 2, 3, \dots, m$ . This completes the proof.  $\square$

APPENDIX U

PROOF OF PROPOSITION 5.6



As shown in (T.4) and (T.5),

$$\begin{aligned}
\frac{\partial l_{S,i}^*}{\partial \beta} &= \frac{\bar{v} \sum_{k=i-1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} \\
&> \frac{\bar{v} \sum_{k=i}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}}{1 + \sum_{k=1}^{m-1} \prod_{j=0}^{k-1} \left\{ 1 + LN \left[ j, \lambda \left( \frac{l_{S,2}^*}{\bar{v}} - \beta \right) \right] \right\}^{-1}} \\
&= \frac{\partial l_{S,i+1}^*}{\partial \beta}
\end{aligned}$$

This completes the proof that  $\frac{\partial l_{S,i}^*}{\partial \beta} > \frac{\partial l_{S,i+1}^*}{\partial \beta} > 0$ ,  $i = 2, 3, \dots, m-1$ .  $\square$

APPENDIX V

PROOF OF PROPOSITION 5.7

An optimal solution  $(l'_{S,1}^*, l'_{S,2}^*, \dots, l'_{S,m}^*)$  to the NLP in (5.4) must satisfy the condition that

$\frac{\partial Z_{S,\lambda}}{\partial l'_{S,i}^*} = 0$ ,  $i = 2, 3, \dots, m$ . Thus, I have

$$\begin{aligned} \frac{\partial Z_{S,\lambda}^*}{\partial l'_{S,i}^*} &= \bar{v} e^{-\lambda(1-\beta)} \frac{\partial}{\partial l'_{S,i}^*} \left[ l'_{S,i}^* \left( e^{\lambda l'_{S,i+1}^*} - e^{\lambda l'_{S,i}^*} \right) + l'_{S,i-1}^* e^{\lambda l'_{S,i}^*} \right] \\ &= \bar{v} e^{-\lambda(1-\beta)} \left( e^{\lambda l'_{S,i+1}^*} - e^{\lambda l'_{S,i}^*} - \lambda l'_{S,i}^* e^{\lambda l'_{S,i}^*} + \lambda l'_{S,i-1}^* e^{\lambda l'_{S,i}^*} \right) \\ &= 0 \end{aligned}$$

This leads to the expression below:

$$e^{\lambda l'_{S,i+1}^*} - e^{\lambda l'_{S,i}^*} - \lambda e^{\lambda l'_{S,i}^*} (l'_{S,i}^* - l'_{S,i-1}^*) = 0 \quad (\text{V.1})$$

Clearly, I have

$$\sum_{i=2}^m \left[ e^{\lambda l'_{S,i+1}^*} - e^{\lambda l'_{S,i}^*} - \lambda e^{\lambda l'_{S,i}^*} (l'_{S,i}^* - l'_{S,i-1}^*) \right] = 0$$

or

$$\begin{aligned} \sum_{i=2}^m \lambda e^{\lambda l'_{S,i}^*} (l'_{S,i}^* - l'_{S,i-1}^*) &= \sum_{i=2}^m \left( e^{\lambda l'_{S,i+1}^*} - e^{\lambda l'_{S,i}^*} \right) \\ &= e^{\lambda l'_{S,m+1}^*} - e^{\lambda l'_{S,2}^*} \\ &= e^{\lambda(1-\beta)} - e^{\lambda l'_{S,2}^*} \end{aligned}$$

or

$$\sum_{i=2}^m e^{\lambda l'_{S,i}^*} (l'_{S,i}^* - l'_{S,i-1}^*) = \frac{1}{\lambda} \left[ e^{\lambda(1-\beta)} - e^{\lambda l'_{S,2}^*} \right] \quad (\text{V.2})$$

Note that an alternative expression of (V.2) is

$$\begin{aligned} \sum_{i=2}^m e^{\lambda l'_{S,i}^*} (l'_{S,i}^* - l'_{S,i-1}^*) &= \sum_{i=2}^m l'_{S,i}^* e^{\lambda l'_{S,i}^*} - \sum_{i=2}^m l'_{S,i-1}^* e^{\lambda l'_{S,i}^*} \\ &= \sum_{i=1}^m l'_{S,i}^* e^{\lambda l'_{S,i}^*} - \sum_{i=1}^{m-1} l'_{S,i}^* e^{\lambda l'_{S,i+1}^*} \end{aligned} \quad (\text{V.3})$$

Combining (V.2) and (V.3), one can get

$$\sum_{i=1}^m l_{S,i}^* e^{\lambda l_{S,i}^*} - \sum_{i=1}^{m-1} l_{S,i}^* e^{\lambda l_{S,i+1}^*} = \frac{1}{\lambda} \left[ e^{\lambda(1-\beta)} - e^{\lambda l_{S,2}^*} \right] \quad (\text{V.4})$$

Thus, the maximum expected revenue from the Dutch auction model formulated as (5.4) is

$$\begin{aligned} Z_{S,\lambda}^* &= \bar{v} e^{-\lambda(1-\beta)} \sum_{i=1}^m l_{S,i}^* \left( e^{\lambda l_{S,i+1}^*} - e^{\lambda l_{S,i}^*} \right) + \beta \bar{v} \left[ 1 - e^{-\lambda(1-\beta)} \right] \\ &= \bar{v} e^{-\lambda(1-\beta)} \left( \sum_{i=1}^m l_{S,i}^* e^{\lambda l_{S,i+1}^*} - \sum_{i=1}^m l_{S,i}^* e^{\lambda l_{S,i}^*} \right) + \beta \bar{v} \left[ 1 - e^{-\lambda(1-\beta)} \right] \\ &= \bar{v} e^{-\lambda(1-\beta)} \left( l_{S,m}^* e^{\lambda(1-\beta)} + \frac{1}{\lambda} e^{\lambda l_{S,2}^*} - \frac{1}{\lambda} e^{\lambda(1-\beta)} \right) + \beta \bar{v} \left[ 1 - e^{-\lambda(1-\beta)} \right] \\ &= \bar{v} l_{S,m}^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda(1-\beta-l_{S,2}^*)} \right] + \beta \bar{v} \left[ 1 - e^{-\lambda(1-\beta)} \right] \\ &= \bar{v} \left( \frac{l_{S,m}^*}{\bar{v}} - \beta \right) - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \beta - \frac{l_{S,2}^*}{\bar{v}} + \beta \right)} \right] + \beta \bar{v} \left[ 1 - e^{-\lambda(1-\beta)} \right] \\ &= l_{S,m}^* - \beta \bar{v} - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right] + \beta \bar{v} \left[ 1 - e^{-\lambda(1-\beta)} \right] \\ &= l_{S,m}^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right] - \beta \bar{v} e^{-\lambda(1-\beta)} \end{aligned}$$

This completes the proof that  $Z_{S,\lambda}^* = l_{S,m}^* - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda \left( 1 - \frac{l_{S,2}^*}{\bar{v}} \right)} \right] - \beta \bar{v} e^{-\lambda(1-\beta)}$ .  $\square$

APPENDIX W

PROOF OF PROPOSITION 5.8

Recall that  $l'_{S,i} = \frac{l_{S,i}}{\bar{v}} - \beta$ ,  $i = 1, 2, \dots, m$ , with  $l'_{S,1} = 0$  and  $l'_{S,m+1} = 1 - \beta$ . According to

Proposition 5.7,  $Z_{S,\lambda}^*$  can be rewritten as

$$\begin{aligned} Z_{S,\lambda}^* &= \bar{v}l'_{S,m} + \beta\bar{v} - \frac{\bar{v}}{\lambda} \left[ 1 - e^{-\lambda(1-\beta-l'_{S,2}^*)} \right] - \beta\bar{v}e^{-\lambda(1-\beta)} \\ &= \bar{v}l'_{S,m} + \beta\bar{v} - \bar{v} \frac{e^{\lambda(1-\beta-l'_{S,2}^*)} - 1}{\lambda e^{\lambda(1-\beta-l'_{S,2}^*)}} - \beta\bar{v}e^{-\lambda(1-\beta)} \end{aligned} \quad (\text{W.1})$$

Since  $l'_{S,i}$  satisfies the condition that the partial derivative of  $Z_{S,\lambda}$  with respect to  $l'_{S,i}$  be set to be equal to zero as described in (V.1), namely,  $e^{\lambda l'_{S,i+1}} - e^{\lambda l'_{S,i}} + \lambda e^{\lambda l'_{S,i}} (l'_{S,i} - l'_{S,i-1}) = 0$ .

Using the notation in (N.1),  $Ln(i, x) = \underbrace{\ln(1 + \ln(1 + \dots + \ln(1 + x)))}_{i \text{ number of } \ln(\cdot)}$ , especially,  $Ln(0, x) = x$ ,

one can see,

for  $i = 2$ ,

$$e^{\lambda l'_{S,3}} - e^{\lambda l'_{S,2}} - \lambda l'_{S,2} e^{\lambda l'_{S,2}} = 0$$

or

$$e^{\lambda l'_{S,3}} = e^{\lambda l'_{S,2}} (1 + \lambda l'_{S,2})$$

Thus, it can be seen that

$$\begin{aligned} l'_{S,3} &= \frac{\lambda l'_{S,2} + \ln(1 + \lambda l'_{S,2})}{\lambda} \\ &= \frac{1}{\lambda} \sum_{k=0}^1 Ln(k, \lambda l'_{S,2}) \end{aligned}$$

For  $i = 3$ ,

$$e^{\lambda l'_{S,4}} - e^{\lambda l'_{S,3}} - \lambda e^{\lambda l'_{S,3}} (l'_{S,3} - l'_{S,2}) = 0$$

or

$$e^{\lambda'_{S,4}} = e^{\lambda'_{S,3}} (\lambda'_{S,3} - \lambda'_{S,2} + 1)$$

Thus, I have

$$\begin{aligned} \lambda'_{S,4} &= \lambda'_{S,3} + \ln(1 + \ln(1 + \lambda'_{S,2})) \\ &= \lambda'_{S,2} + \ln(1 + \lambda'_{S,2}) + \ln(1 + \ln(1 + \lambda'_{S,2})) \\ &= \sum_{k=0}^2 Ln(k, \lambda'_{S,2}) \end{aligned}$$

That is,

$$l'_{S,4} = \frac{1}{\lambda} \sum_{k=0}^2 Ln(k, \lambda'_{S,2})$$

Similarly, this leads to the general expression below:

$$l'_{S,i} = \frac{1}{\lambda} \sum_{k=0}^{i-2} Ln(k, \lambda'_{S,2}), \quad i = 3, \dots, m \quad (\text{W.2})$$

According to (W.2) and  $l'_{S,m+1} = 1 - \beta$ , one can get

$$l'_{S,m+1} = \frac{1}{\lambda} \sum_{k=0}^{m-1} Ln(k, \lambda'_{S,2}) = 1 - \beta \quad (\text{W.3})$$

or

$$\sum_{k=0}^{m-1} Ln(k, \lambda'_{S,2}) = \lambda(1 - \beta)$$

or

$$\sum_{k=1}^{m-1} Ln(k, \lambda'_{S,2}) = \lambda(1 - \beta) - \lambda'_{S,2}$$

or

$$\begin{aligned} e^{\lambda(1-\beta-l'_{S,2})} &= e^{\sum_{k=1}^{m-1} Ln(k, \lambda'_{S,2})} \\ &= \prod_{k=0}^{m-2} (1 + Ln(k, \lambda'_{S,2})) \end{aligned} \quad (\text{W.4})$$

Thus, (W.1) can be rewritten as

$$\begin{aligned}
Z_{S,\lambda}^* &= \bar{v}l_{S,m}^* + \beta\bar{v} - \bar{v} \frac{e^{\lambda(1-\beta-l_{S,2}^*)} - 1}{\lambda e^{\lambda(1-\beta-l_{S,2}^*)}} - \beta\bar{v}e^{-\lambda(1-\beta)} \\
&= \bar{v}l_{S,m}^* + \beta\bar{v} - \bar{v} \frac{\prod_{k=0}^{m-2} (1 + Ln(k, \lambda l_{S,2}^*)) - 1}{\lambda \prod_{k=0}^{m-2} (1 + Ln(k, \lambda l_{S,2}^*))} - \beta\bar{v}e^{-\lambda(1-\beta)}
\end{aligned} \tag{W.5}$$

Note that  $\frac{\prod_{k=0}^{m-2} (1 + Ln(k, \lambda l_{S,2}^*)) - 1}{\lambda \prod_{k=0}^{m-2} (1 + Ln(k, \lambda l_{S,2}^*))}$  decreases with  $\lambda$ . In addition,  $l_{S,m}^*$  is an increasing function of  $\lambda$

as shown in Proposition 5.3, and  $\beta\bar{v}e^{-\lambda(1-\beta)}$  is a decreasing function of  $\lambda$ . This completes the

proof that  $Z_{S,\lambda}^*$  is an increasing function of  $\lambda$ .  $\square$



APPENDIX X

PROOF OF PROPOSITION 5.10

Taking the partial derivative of  $G_t$  with respect to  $t$ , one gets

$$\begin{aligned}
\frac{\partial G_t}{\partial t} &= \bar{v}e^{-\lambda(1-\beta)} \sum_{i=1}^m (1-\beta) \frac{\partial \tau_i}{\partial t} \left[ e^{\lambda(1-\beta)\tau_{i+1}} - e^{\lambda(1-\beta)\tau_i} \right] \\
&\quad + \bar{v}e^{-\lambda(1-\beta)} \sum_{i=1}^m [(1-\beta)\tau_i + \beta] \left[ \lambda(1-\beta) \frac{\partial \tau_{i+1}}{\partial t} e^{\lambda(1-\beta)\tau_{i+1}} - \lambda(1-\beta) \frac{\partial \tau_i}{\partial t} e^{\lambda(1-\beta)\tau_i} \right] \\
&= \bar{v}e^{-\lambda(1-\beta)} (1-\beta) \sum_{i=1}^m \frac{\partial \tau_i}{\partial t} \left[ e^{\lambda(1-\beta)\tau_{i+1}} - e^{\lambda(1-\beta)\tau_i} \right] \\
&\quad + \lambda \bar{v}e^{-\lambda(1-\beta)} (1-\beta) \sum_{i=1}^m [(1-\beta)\tau_i + \beta] \left[ \frac{\partial \tau_{i+1}}{\partial t} e^{\lambda(1-\beta)\tau_{i+1}} - \frac{\partial \tau_i}{\partial t} e^{\lambda(1-\beta)\tau_i} \right] \\
&= \bar{v}e^{-\lambda(1-\beta)} (1-\beta) \left\{ \sum_{i=1}^m \frac{\partial \tau_i}{\partial t} \left[ e^{\lambda(1-\beta)\tau_{i+1}} - e^{\lambda(1-\beta)\tau_i} \right] + \lambda \sum_{i=1}^m [(1-\beta)\tau_i + \beta] \left[ \frac{\partial \tau_{i+1}}{\partial t} e^{\lambda(1-\beta)\tau_{i+1}} - \frac{\partial \tau_i}{\partial t} e^{\lambda(1-\beta)\tau_i} \right] \right\}
\end{aligned}$$

According to Lemma 3.3,  $\frac{\partial \tau_i}{\partial t} < 0$ . This along with the fact that  $\tau_{i+1} > \tau_i$  leads to

$$\frac{\partial \tau_i}{\partial t} \left[ e^{\lambda(1-\beta)\tau_{i+1}} - e^{\lambda(1-\beta)\tau_i} \right] < 0 \tag{X.1}$$

In addition, Lemma 3.4 says that  $\frac{\partial \tau_{i+1}}{\partial t} < \frac{\partial \tau_i}{\partial t}$ . It follows that

$$\frac{\partial \tau_{i+1}}{\partial t} e^{\lambda(1-\beta)\tau_{i+1}} < \frac{\partial \tau_{i+1}}{\partial t} e^{\lambda(1-\beta)\tau_i} < \frac{\partial \tau_i}{\partial t} e^{\lambda(1-\beta)\tau_i}$$

or

$$\frac{\partial \tau_{i+1}}{\partial t} e^{\lambda(1-\beta)\tau_{i+1}} - \frac{\partial \tau_i}{\partial t} e^{\lambda(1-\beta)\tau_i} < 0 \tag{X.2}$$

Combining (X.1) and (X.2), I have

$$\frac{\partial Z_S}{\partial t} = \bar{v}e^{-\lambda(1-\beta)} (1-\beta) \left\{ \sum_{i=1}^m \frac{\partial \tau_i}{\partial t} \left[ e^{\lambda(1-\beta)\tau_{i+1}} - e^{\lambda(1-\beta)\tau_i} \right] + \lambda \sum_{i=1}^m [(1-\beta)\tau_i + \beta] \left[ \frac{\partial \tau_{i+1}}{\partial t} e^{\lambda(1-\beta)\tau_{i+1}} - \frac{\partial \tau_i}{\partial t} e^{\lambda(1-\beta)\tau_i} \right] \right\}$$

$< 0$

This completes the proof.  $\square$

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