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Inner matrices and Darlington synthesis

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INNER MATRICES AND DARLINGTON SYNTHESIS

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Abstract. We describe and parameterize the solutions of the scalar valued Darlington synthesis problem. In the case of rational data we derive a simple procedure for producing all possible solutions.

1. Introduction

In this note we outline a function theoretic approach to the Darlington synthesis problem from electrical network theory. We consider the following (scalar valued) version of this problem: Given a function $a(z)$ belonging to the Hardy space H^2 of the unit disk $\mathbb{D}$, do there exist functions b, c, and d also belonging to H^2 such that the matrix

$$U = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \tag{1}$$

is unitary a.e. on the unit circle $\partial\mathbb{D}$? Such a matrix U is called a (2×2) inner matrix. It is known [1, 5, 6] that this problem is solvable if and only if $\| a \|_\infty \leq 1$ and a is pseudocontinuable of bounded type (see [7, 14]). The second condition is equivalent (via [7, Th. 2.2.1]) to asserting that a is noncyclic for the backward shift operator on H^2.

We require a basic working knowledge of the classical Hardy space H^2 and of the inner-outer factorization theory for H^2 functions (see [8]). Our main tool is the following theorem, proved in Section 3:

Theorem 1. *If ϕ is a nonconstant inner function, then U is unitary a.e on $\partial\mathbb{D}$ and $\det U = \phi$ if and only if*

(1) *a, b, c, d belong to $(z\phi H^2)^\perp$.*
(2) *$\widehat{a} = d$ and $\widehat{b} = c$.*
(3) *$|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$.*

Here $\widehat{\cdot}$ denotes a certain involution of the backward shift invariant subspace $(z\phi H^2)^\perp$ of H^2. We discuss this involution in Section 2 and briefly remark that it can be used to provide an explicit analog of Beurling's theorem for the backward shift operator on H^2.

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The remainder of this note concerns the application of the preceding theorem to the description and parameterization of the solutions to the scalar valued Darlington synthesis problem. Using our method we derive a simple procedure for parameterizing all solutions given rational data $a(z)$.

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2. The Backward Shift

Before proceeding we require a few facts about the backward shift operator $B : H^2 \to H^2$. This is the bounded linear operator defined by

$$(Bf)(z) := \frac{f(z) - f(0)}{z}$$

for f in H^2. In terms of Taylor coefficients at the origin, the backward shift operator simply sends the sequence $(a_0, a_1, \dots)$ to $(a_1, a_2, \dots)$. Our interest lies in the study of B-invariant subspaces of H^2 and their application to the theory of inner matrices.

A subspace $\mathcal{M}$ of H^2 is called B-invariant (henceforth simply "invariant") if $B\mathcal{M} \subseteq \mathcal{M}$. It is well-known (see [4]) that the proper, nontrivial invariant subspaces for the backward shift operator are precisely the subspaces $(\phi H^2)^\perp$ where ϕ is a nonconstant inner function. In terms of boundary functions on $\partial\mathbb{D}$ we have

$$(\phi H^2)^\perp = H^2 \cap \phi \overline{z H^2}. \tag{2}$$

Fix a nonconstant inner function ϕ and consider the antilinear involution $\widehat{\cdot}$ on $(\phi H^2)^\perp$ defined by

$$\widehat{f} := \overline{fz}\phi. \tag{3}$$

Although it is evident from (2) that (3) defines an involution of $(\phi H^2)^\perp$, we can check this directly. For any h in H^2 we have

$$\langle \widehat{f}, \overline{zh} \rangle = \langle \overline{fz}\phi, \overline{zh} \rangle = \langle \phi h, f \rangle = 0$$

and hence $\overline{fz}\phi$, despite its appearance, belongs to H^2. The computation

$$\langle \widehat{f}, \phi h \rangle = \langle \overline{fz}\phi, \phi h \rangle = \langle \overline{fz}, h \rangle = 0$$

shows that $\widehat{f}$ also belongs to $(\phi H^2)^\perp$.

We will call this operator the conjugation operator on $(\phi H^2)^\perp$ and refer to $\widehat{f}$ as the conjugate of f. Each function f belonging to $(\phi H^2)^\perp$ can be written uniquely in the form

$$f = f_1 + i f_2$$

where f_1 and f_2 belong to $(\phi H^2)^\perp$ and are self-conjugate. Indeed, this decomposition is given by the equation

$$f = \tfrac{1}{2}(f + \widehat{f}) + i\tfrac{1}{2i}(f - \widehat{f}).$$

Moreover, writing $f = f_1 + if_2$ we have the formula $\widehat{f} = f_1 - if_2$ for the conjugate function.

This decomposition immediately yields an explicit function-theoretic characterization of $(\phi H^2)^\perp$ [10], and hence functions which are pseudocontinuable of bounded type [7, 14]. Suppose that ζ is a point on $\partial\mathbb{D}$ such that ϕ has a nontangential limiting value at ζ of unit modulus and c is a unimodular constant satisfying $c^2 = \overline{\zeta}\phi(\zeta)$. By (3), a self-conjugate function f satisfies $f = \overline{fz}\phi$ a.e on $\partial\mathbb{D}$ and hence $f(z) = cr(z)k_\zeta(z)$ where

$$k_\zeta(z) = \frac{1 - \overline{\phi(\zeta)}\phi(z)}{1 - \overline{\zeta}z}$$

and $r(z)$ is a function in the Smirnov class N^+ whose boundary values are real a.e. on $\partial\mathbb{D}$. Such functions are described explicitly in [11, 12].

From (2) it follows that a function f in H^2 belongs to $(\phi H^2)^\perp$ if and only if there exists a function g, having the same outer factor as f, such that $g = \overline{fz}\phi$ a.e. on $\partial\mathbb{D}$. In this case we have $g = \widehat{f}$.

Given a pair of conjugate functions $f, \widehat{f}$ belonging to $(\phi H^2)^\perp$, we write

$$f = I_f F, \quad \widehat{f} = I_{\widehat{f}} F$$

where I_f and $I_{\widehat{f}}$ are inner functions and F denotes the common outer factor of f and $\widehat{f}$. From (3) we deduce that the equation

$$I_{\widehat{f}} F = \overline{I_f F z}\phi$$

holds a.e. on $\partial\mathbb{D}$. Since I_f is inner this is equivalent to

$$I_f I_{\widehat{f}} = \frac{\overline{Fz}\phi}{F}.$$

and hence the inner function $I_f I_{\widehat{f}}$ depends only upon F and ϕ, not on the particular pair of conjugate functions in $(\phi H^2)^\perp$ with common outer factor F. We denote this inner function $\mathcal{I}_F$ and call it the *associated inner function for* F (with respect to ϕ). It satisfies the equation

$$\mathcal{I}_F F = \overline{Fz}\phi \tag{4}$$

a.e. on $\partial\mathbb{D}$.

Pulling our observations together we conclude that for any outer function F in $(\phi H^2)^\perp$ there exists a unique inner function $\mathcal{I}_F$ such that if I is an inner function, then the function IF belongs to $(\phi H^2)^\perp$ if and only if I is a divisor of $\mathcal{I}_F$.

3. Matrix Inner Functions

If a matrix U of the form (1) is unitary a.e. on $\partial\mathbb{D}$, then its determinant $\det U$ must be an inner function, say ϕ. It turns out that the entries of U (including a itself) belong to $(z\phi H^2)^\perp$, the backward shift invariant subspace of H^2 generated by ϕ. The precise relationship between the inner function $\det U$ and the entries of U is given in the following proposition (from [10], the thesis of the author):

Theorem 1. *If ϕ is a nonconstant inner function, then U is unitary a.e on $\partial\mathbb{D}$ and $\det U = \phi$ if and only if*

(1) *a, b, c, d belong to $(z\phi H^2)^\perp$.*
(2) *$\widehat{a} = d$ and $\widehat{b} = c$.*
(3) *$|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$.*

Proof. ($\Rightarrow$) If the matrix U is unitary a.e. on $\partial\mathbb{D}$, then the determinant

$$\phi = ad + bc$$

is inner. Comparing entries in the identity $U = (U^*)^{-1}$ yields the equations

$$\begin{aligned} a &= \overline{d}\phi, \\ b &= \overline{c}\phi \end{aligned}$$

a.e. on $\partial\mathbb{D}$ which establish conditions (1) and (2). The identity $UU^* = I$ yields condition (3).

($\Leftarrow$) Suppose now that conditions (1), (2), and (3) hold. Write $a = I_a F$, $b = I_b G$, $c = I_c G$, and $d = I_d F$ where I_a, I_b, I_c, I_d are inner and F, G are outer. Consider the entries in the matrix product

$$UU^* = \begin{pmatrix} I_a F & -I_b G \\ I_c G & I_d F \end{pmatrix} \begin{pmatrix} \overline{I_a F} & \overline{I_c G} \\ \overline{-I_b G} & \overline{I_d F} \end{pmatrix}.$$

Condition (3) ensures that the entries on the main diagonal of the product are both identically 1. The upper right corner of the product is the function

$$X = I_a F \overline{I_c G} - I_b G \overline{I_d F}$$

which we must show vanishes identically. A few manipulations yield

$$I_a I_d \frac{F}{\overline{F}} - I_b I_c \frac{G}{\overline{G}} = X \frac{I_c I_d}{\overline{FG}}.$$

Since a, d and b, c are conjugates, we see that $I_a I_d = \mathcal{I}_F$ and $I_b I_c = \mathcal{I}_G$ and hence (by (4) with $z\phi$ in place of ϕ)

$$X \frac{I_c I_d}{\overline{FG}} = \phi - \phi = 0$$

and hence X vanishes identically. A similar argument shows that the bottom left corner of the matrix product vanishes and thus U is unitary a.e. on $\partial\mathbb{D}$. We now compute the determinant:

$$\det U = ad + bc = I_a I_d F^2 + I_b I_c G^2 = \mathcal{I}_F F^2 + \mathcal{I}_G G^2 = |F|^2\phi + |G|^2\phi = \phi$$

which completes the proof. $\square$

The preceding theorem tells us that 2×2 inner functions resemble quaternions of unit modulus, for we have

$$U = \begin{pmatrix} a & -b \\ \widehat{b} & \widehat{a} \end{pmatrix} \tag{5}$$

where $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$. Moreover, the formula for the determinant of U assumes the form

$$\phi = a\widehat{a} + b\widehat{b} \tag{6}$$

$$= \mathcal{I}_F F^2 + \mathcal{I}_G G^2. \tag{7}$$

4. Darlington Synthesis

Results of Arov [1] and Douglas and Helton [6] tell us that the scalar valued Darlington synthesis problem with data $a(z)$ has a solution if and only if

(1) $\| a \|_\infty \leq 1$,
(2) a is pseudocontinuable of bounded type (see [7, 14] for details).

By a well-known theorem of Douglas, Shapiro, and Shields [7], condition (2) above is equivalent to asserting that a belongs to $(\phi H^2)^\perp$ for some nonconstant inner function ϕ. We can soon say much more.

Theorem 1 clearly implies that the two conditions above are necessary for the problem to have a solution. Moreover, any solution U must be of the form (5) described by the theorem and the initial data $a(z)$ belongs to the backward shift invariant subspace generated the determinant of U.

Now suppose that $a(z)$ is a function that satisfies $\| a \|_\infty \leq 1$ and belongs to $(z\phi H^2)^\perp$ for some nonconstant inner function ϕ. It is not immediately obvious that a function $b(z)$ can be found such that $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$ and that b also belongs to $(z\phi H^2)^\perp$ (unless, of course, a is an inner function). Although the existence of such a b can be inferred from Theorem 1 and the results of the aforementioned authors, we provide a direct proof (from [10]) based on our methods. The fact that b can be chosen to lie in $(z\phi H^2)^\perp$, it turns out, is the key to our entire theory.

Theorem 2. *If the function $a(z)$ belongs to $(z\phi H^2)^\perp$ for some nonconstant inner function ϕ and $\| a \|_\infty \leq 1$, then there exists a function $b(z)$ in $(z\phi H^2)^\perp$ such that $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$.*

Proof. If $a(z)$ is an inner function, then the result is trivial. Therefore assume that a is not an inner function. As before, let $a = I_a F$ where I_a and F denote the inner and outer factors of a, respectively. Since $\mathcal{I}_F F = \overline{F}\phi$ a.e. on $\partial\mathbb{D}$ we find that

$$\phi - \mathcal{I}_F F^2 = \phi(1 - |F|^2)$$

there. Since F is not an inner function and $\phi - \mathcal{I}_F F^2$ belongs to H^∞, it follows that

$$\int_{\partial\mathbb{D}} \log(1 - |F|^2) = \int_{\partial\mathbb{D}} \log|\phi - \mathcal{I}_F F^2| > -\infty$$

and hence there exists an outer function G in H^∞ such that $|G|^2 = 1 - |F|^2$ a.e. on $\partial\mathbb{D}$. Therefore

$$IG^2 = \phi - \mathcal{I}_F F^2 = \phi|G|^2$$

for some inner function I. Since $IG = \overline{G}\phi$ a.e. on $\partial\mathbb{D}$, we see that G belongs to $(z\phi H^2)^\perp$ and hence $I = \mathcal{I}_G$. We may let $b = I_b G$ where I_b is an inner function dividing $\mathcal{I}_G$. □

Using Theorems 1 and 2 we can now describe the solution sets for the scalar valued Darlington synthesis problem. Moreover, in the case of rational data $a(z)$ we can use our theory to produce a simple algorithm for producing all possible solutions.

5. Primitive Solution Sets

Suppose that we are given data $a(z)$ such that the scalar valued Darlington synthesis problem is solvable and that the matrix U is one particular solution. By Theorem 1, $\det U = \phi$ is an inner function and

$$U = \begin{pmatrix} a & -b \\ \widehat{b} & \widehat{a} \end{pmatrix}$$

where $\widehat{a}$ and $\widehat{b}$ denote the conjugates of a and b in $(z\phi H^2)^\perp$ as defined in Section 2. From the solution U we can construct infinitely many other solutions via a simple process.

If I_1 and I_2 are inner functions (possibly constant), then the matrix

$$U' = \begin{pmatrix} a & -I_1 b \\ I_2\widehat{b} & I_1 I_2 \widehat{a} \end{pmatrix} \tag{8}$$

is another solution. As such, it must be of the form dictated by Theorem 1. Indeed, we may write

$$U' = \begin{pmatrix} a & -(I_1 b) \\ \widetilde{(I_1 b)} & \widetilde{a} \end{pmatrix}$$

where $\det U' = I_1 I_2 \phi$ and

$$\widetilde{f} := \overline{f}(I_1 I_2 \phi)$$

denotes the conjugation operator on $(zI_1 I_2 \phi H^2)^\perp$. Noting that $\det U$ divides $\det U'$, we now consider solutions with minimal determinant.

We say that a solution U is primitive if the inner function $\phi = \det U$ is the minimal inner function such that $\det U$ divides $\det U'$ for any other solution U'. This is equivalent to requiring that ϕ is the minimal inner function such that a belongs to $(z\phi H^2)^\perp$. Note also that every primitive solution shares the same determinant, up to a unimodular constant factor. We call the inner function ϕ the minimal determinant for the problem (with data $a(z)$). Recall that Arov [2, 3] considered a similar concept ("minimal denominators") in the more general operator valued setting.

Any solution U' can be written in terms of a primitive solution via (8). Indeed, suppose that U' is a solution with determinant $U' = \phi\theta$ where θ is an inner function and ϕ is the minimal determinant for the problem. Let the outer functions F and G be defined as in the preceding sections and let $\mathcal{I}_F$ and $\mathcal{I}_G$ denote the associated inner functions for F and G with respect to ϕ. In terms of boundary functions, we may write the solution U' as

$$U' = \begin{pmatrix} a & -c \\ \overline{c}\phi\theta & \overline{a}\phi\theta \end{pmatrix}$$

where c belongs to $(z\phi\theta H^2)^\perp$ and has outer factor G. Since the conjugate $\widehat{a}$ of a in $(z\phi H^2)^\perp$ is $\overline{a}\phi$ we conclude that $\det U'$ equals

$$\phi\theta = \mathcal{I}_F F^2 \theta + I G^2 \theta$$

where I is some inner function. Comparing this with (7) we conclude that $I = \mathcal{I}_G$. In particular, the product of c and $\overline{c}\phi\theta$ (the boundary function for the conjugate of c in $(z\phi\theta H^2)^\perp$) equals $\mathcal{I}_G\theta G^2$. Therefore U' can be written in the form (8) for some inner functions I_1 and I_2 such that $I_1I_2 = \theta$.

We can therefore completely describe all possible solutions to our problem by describing all primitive solutions. We call a complete collection of primitive solutions sharing the same minimal determinant a primitive solution set. Since the minimal determinant is determined only up to a unimodular constant factor, there will be infinitely many primitive solution sets. These can be easily related to one another via (8) where the inner functions I_1 and I_2 are unimodular constants.

Fix a minimal determinant ϕ to our problem. Our present task, therefore, is to describe all solutions U with determinant ϕ. By Theorem 1, we may identify each such solution with its upper right entry, $b(z)$. Since the outer factor G of b is completely determined by condition (3) of Theorem 1, we may actually identify each solution with the inner factor of b.

The inner factor of b must be a divisor of $\mathcal{I}_G$ (which is determined by (7)) and hence there is a bijective correspondence between matrices in our primitive solution set and the inner divisors of $\mathcal{I}_G$. In particular, a primitive solution set has a natural partial ordering which arises from this correspondence.

Example 1. If $\mathcal{I}_G$ is constant, then each primitive solution set consists of precisely one solution. In other words, the solution to the general problem is essentially unique since all solutions can be constructed via (8) from a single primitive solution. In this situation b must be a self-conjugate outer function.

As an illustration, consider the data

$$a(z) = \frac{1+\phi(z)}{2}$$

where ϕ is any inner function. This function generates $(z\phi H^2)^\perp$ by [7, Th. 3.1.5] and hence any solution with determinant ϕ is primitive. Note also that a is self-conjugate and that the outer function

$$b(z) = \frac{1-\phi(z)}{2i}$$

belongs to $(z\phi H^2)^\perp$ and is also self-conjugate (hence $\mathcal{I}_G = 1$). The matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is therefore the unique solution with minimal determinant ϕ.

Example 2. If $\mathcal{I}_G$ is a finite Blaschke product of order n, then a primitive solution set contains at most 2^n solutions, the exact number depending upon the multiplicity of the zeroes of $\mathcal{I}_G$. This situation occurs frequently in the case where the data $a(z)$ is rational. When $\mathcal{I}_G$ is a finite Blaschke product, a primitive solution set is linearly ordered if and only if $\mathcal{I}_G$ is a power of a single Blaschke factor.

Example 3. If $\mathcal{I}_G$ is an infinite Blaschke product, then a primitive solution set is uncountably infinite and cannot be linearly ordered. Indeed, the primitive solutions are in a bijective, order-preserving correspondence with the Blaschke subproducts of $\mathcal{I}_G$ and hence in correspondence with the subsets of the natural numbers.

Example 4. If $\mathcal{I}_G$ is a singular inner function, then the primitive solution set is uncountably infinite. It is linearly ordered only when $\mathcal{I}_G$ is an atomic inner function corresponding to a point mass. As a consequence of this example and the preceding several, a primitive solution set can never be countably infinite.

Example 5. If $\mathcal{I}_G$ is the square of an inner function, then symmetric primitive solutions exist. By a symmetric solution, we mean here a solution U such that $U = U^t$ where U^t denotes the transpose of U. Observe that if $\mathcal{I}_G = I^2$ where I is an inner function, then the function $b = IG$ belongs to $(z\phi H^2)^\perp$ and is self-conjugate. This yields the primitive solution

$$\begin{pmatrix} a & -b \\ b & \widehat{a} \end{pmatrix}.$$

Using (8) with $I_1 = -i$ and $I_2 = i$ we obtain the symmetric solution

$$\begin{pmatrix} a & ib \\ ib & \widehat{a} \end{pmatrix}.$$

6. Rational Data

We now obtain primitive solution sets for the Darlington synthesis problem for rational data $a(z)$. The involution technique developed here, we believe, can shed new light on an old problem. We first go through the solution step-by-step.

Suppose that we are given a rational function $a(z)$ belonging to H^∞ and satisfying $\| a \|_\infty \leq 1$. The function a, being rational, is noncyclic for the backward shift operator and hence the scalar valued Darlington synthesis problem with data $a(z)$ is solvable. Since the problem is trivial if a is an inner function, we assume that a is not a finite Blaschke product.

We may write

$$a(z) = \frac{P(z)}{R(z)}$$

where $P(z)$ is a polynomial relatively prime to

$$R(z) = (1 - \overline{\lambda_1}z) \cdots (1 - \overline{\lambda_n}z).$$

Since a belongs to H^∞ we must have $|\lambda_k| < 1$ for each $k = 1, 2, \ldots, n$. There are two cases, depending on the degree m of $P(z)$.

Case I: If $m \leq n$, then $a(z)$ belongs to $(z\phi H^2)^\perp$ where ϕ denotes the finite Blaschke product

$$\phi(z) = \prod_{k=1}^{n} \frac{z - \lambda_k}{1 - \overline{\lambda_k} z}.$$

Any function belonging to $(z\phi H^2)^\perp$ is of the form $Q(z)/R(z)$ where $Q(z)$ is a polynomial of degree $\leq n$. A short calculation shows that

$$\widehat{Q/R}(z) = \frac{Q^\#(z)}{R(z)} \tag{9}$$

where the polynomial $Q^{\#}(z)$ is defined by

$$Q^{\#}(z) = z^n \overline{Q(1/\overline{z})}.$$

In particular, we need the following special cases of (9):

$$\widehat{a}(z) = \frac{P^{\#}(z)}{R(z)}, \quad \phi(z) = \frac{R^{\#}(z)}{R(z)}. \tag{10}$$

The second formula follows immediately from the fact that the functions ϕ and 1 both belong to $(z\phi H^2)^{\perp}$ and are conjugates.

The finite Blaschke product ϕ is the minimal determinant corresponding to the data $a(z)$. To see this, observe that (via a partial fraction decomposition) ϕ is the minimal inner function such that a belongs to $(z\phi H^2)^{\perp}$. Alternatively, apply Theorem 3.1.5 of [7] after noting that the inner factor of $\widehat{a}$ (namely the finite Blaschke product corresponding to the zeros of $P^{\#}(z)$) is relatively prime to ϕ since otherwise $P(z)$ would not be relatively prime to $R(z)$. Thus if we produce all solutions U with determinant ϕ we will have succeeded in describing a primitive solution set for the data $a(z)$.

By Theorem 1 and (6) we seek solutions U of the form

$$U = \begin{pmatrix} a & -b \\ \widehat{b} & \widehat{a} \end{pmatrix}$$

where

$$\phi = a\widehat{a} + b\widehat{b}.$$

Write

$$b(z) = \frac{Q(z)}{R(z)}$$

where $Q(z)$ is an unknown polynomial of degree $\leq n$. By (9) and (10) we must solve the equation

$$\frac{R^{\#}}{R} = \frac{P^{\#}P}{R^2} + \frac{Q^{\#}Q}{R^2} \tag{11}$$

for the polynomial $Q(z)$. This reduces to the simple equation

$$Q^{\#}Q = R^{\#}R - P^{\#}P \tag{12}$$

for $Q(z)$. Several aspects of (12) are worth mentioning.

First, observe that (12) can be obtained directly from $a(z)$ without factoring $R(z)$ into linear terms. Second, we have not yet made use of the assumption that $\| a \|_{\infty} \leq 1$. It turns out that the solvability of (12) actually implies that $\| a \|_{\infty} \leq 1$. If we can find a polynomial $Q(z)$ satisfying (12), then we can solve (11). However (11) is merely another way of saying that

$$\phi = a\widehat{a} + \frac{Q^{\#}Q}{R^2}$$

which implies that (since $\widehat{a} = \overline{a}\phi$ a.e. on $\partial\mathbb{D}$)

$$(1 - |a|^2) = \frac{Q^\# Q}{R^\# R} = \left|\frac{Q}{R}\right|^2 \geq 0$$

holds a.e. on $\partial\mathbb{D}$. Since a is bounded, we see that $\| a \|_\infty \leq 1$ as claimed.

To solve (12) for the unknown polynomial $Q(z)$ (and hence find the function $b = Q/R$) we consider inner-outer factorizations. Let us write $b = I_b G$ and $\widehat{b} = I_{\widehat{b}} G$ where I_b and $I_{\widehat{b}}$ are inner functions and G denotes the common outer factor of b and $\widehat{b}$. Since

$$b\widehat{b} = \frac{R^\# R - P^\# P}{R^2}$$

we see that

$$\mathcal{I}_G G^2 = I_b I_{\widehat{b}} G^2 = \frac{R^\# R - P^\# P}{R^2} \tag{13}$$

where $\mathcal{I}_G$ denotes the associated inner function for G. As noted in Section 5, we need only find the outer function G and the inner function $\mathcal{I}_G$ to completely parameterize all solutions U with determinant ϕ. To find these functions, we must merely produce the inner-outer factorization of

$$\frac{R^\# R - P^\# P}{R^2},$$

a rational function easily obtained from the data $a(z)$. We can simplify this even further.

Since the outer factor of any function in $(z\phi H^2)^\perp$ also lies in $(z\phi H^2)^\perp$, it follows that G belongs to $(z\phi H^2)^\perp$. Hence G is of the form

$$G(z) = \frac{S(z)}{R(z)}$$

where $S(z)$ is a polynomial of degree $\leq n$. Moreover, since $G(z)$ and $R(z)$ are outer functions, it follows that the polynomial $S(z)$ is also an outer function. Thus (13) reduces to

$$\mathcal{I}_G S^2 = R^\# R - P^\# P$$

where $\mathcal{I}_G$ is a finite Blaschke product (possibly constant) whose zeroes are precisely the zeros of $R^\# R - P^\# P$ (a polynomial of degree at most $2n$) which lie inside the unit disk.

We can factor $R^\# R - P^\# P$ into inner and outer factors without necessarily knowing its zeroes, obtaining S^2 and hence S. This immediately yields the (possibly identical) solutions

$$\begin{pmatrix} P/R & -S/R \\ S^\#/R & P^\#/R \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} P/R & -S^\#/R \\ S/R & P^\#/R \end{pmatrix}$$

to our problem. Finding the other (if any) primitive solutions with determinant ϕ is trickier.

Since $G = S/R$ is an outer function in $(z\phi H^2)^{\perp}$, we have

$$\widehat{G} = \mathcal{I}_G G = \frac{S^{\#}}{R}.$$

Therefore the desired inner function $\mathcal{I}_G$ is given by the formula

$$\mathcal{I}_G = \frac{S^{\#}}{S}.$$

Since S is an outer function, the zeroes of $\mathcal{I}_G$ must be precisely the zeros of $S^{\#}$ which lie inside the open unit disk. However, S and $S^{\#}$ may have common zeros which lie on $\partial\mathbb{D}$. We can discard these without actually finding them by simply calculating the greatest common divisor of the polynomials S and $S^{\#}$. Without loss of generality, we assume that this has been done and hence that the zeroes of $S^{\#}$ all lie inside the open unit disk.

Once the zeroes of $S^{\#}$ have been found, we can easily complete our primitive solution set since these solutions can be identified with the functions

$$b(z) = I_b G = I_b \frac{S}{R}$$

where I_b is an inner divisor of $\mathcal{I}_G$. The polynomials $Q(z)$ are simply the functions $I_b S$.

Case II: If $m > n$, then we use $z^{m-n}\phi$ in place of ϕ and define

$$Q^{\#}(z) = z^m \overline{Q(1/\overline{z})}$$

for polynomials $Q(z)$ of degree $\leq m$. The only substantial difference is that the polynomial $R^{\#}R - P^{\#}P$ is now of degree $\leq 2m$. The details are left to the reader.

Why is the polynomial $R^{\#}R - P^{\#}P$ so important? Since (if $m \leq n$)

$$\frac{R^{\#}R - P^{\#}P}{R^2} = \phi - a\widehat{a},$$

we see that the roots of $R^{\#}R - P^{\#}P$ correspond to the roots of the function $\phi - a\widehat{a}$ in the complex plane. On $\partial\mathbb{D}$ this equation simplifies to

$$\frac{R^{\#}R - P^{\#}P}{R^2} = \phi(1 - |a|^2)$$

and hence the roots of $R^{\#}R - P^{\#}P$ which lie on $\partial\mathbb{D}$ are exactly the points at which $|a| = 1$. Since the zeroes of $R^{\#}R - P^{\#}P$ occur in pairs symmetric with respect to $\partial\mathbb{D}$, we find that the number of zeros inside the unit disk (counted according to multiplicity) depends on the degree of $R^{\#}R - P^{\#}P$ and the number of times (according to multiplicity) that the data function $a(z)$ assumes its maximum possible modulus of one on $\partial\mathbb{D}$. The number of solutions in a primitive solution set therefore depends qualitatively on how many times the data $a(z)$ assumes extreme values. See Example 1, for instance.

We remark that the Schur-Cohn algorithm [13] can detect the number of zeroes of a polynomial inside the disk, on its boundary, and outside. Therefore in many situations, we can produce information on the number of solutions in a primitive solution set without explicitly finding the roots of polynomials.

We conclude now with a procedure which produces a complete primitive solution set to the scalar valued Darlington synthesis problem.

Algorithm:

Suppose that we are given a rational function $a(z)$ satisfying $\| a \|_\infty \leq 1$.

(1) *Write $a(z) = P(z)/R(z)$ where $R(z)$ has constant term 1 and $P(z)$ is relatively prime to $R(z)$. Let the degrees of P and R be denoted m and n, respectively.*

(2) *If $m \leq n$, then form the polynomial $R^\# R - P^\# P$ (of degree at most $2n$) using the definition $Q^\#(z) = z^n \overline{Q(1/\overline{z})}$ for polynomials $Q(z)$ of degree $\leq n$.*

(a) *The outer factor of $R^\# R - P^\# P$ is a polynomial S^2 of degree $\leq 2n$. The matrices*

$$\begin{pmatrix} P/R & -S/R \\ S^\#/R & P^\#/R \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} P/R & -S^\#/R \\ S/R & P^\#/R \end{pmatrix}$$

are primitive solutions with determinant $\phi = R^\#/R$.

(b) *Find the roots of the polynomial*

$$S' := \frac{S^\#}{\gcd(S, S^\#)}$$

(of degree $N \leq n$). These zeroes all lie inside the unit disk.

(c) *For each subset $\{\omega_1, \ldots, \omega_k\}$ of the roots of S' such that $k \leq \lfloor \frac{N}{2} \rfloor$,*

$$T(z) = S(z) \prod_{j=1}^{k} \frac{z - \omega_j}{1 - \overline{\omega_j} z}$$

is a polynomial of degree $N - k$ yielding the primitive solutions

$$\begin{pmatrix} P/R & -T/R \\ T^\#/R & P^\#/R \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} P/R & -T^\#/R \\ T/R & P^\#/R \end{pmatrix}.$$

This yields a complete set of primitive solutions with determinant ϕ.

(3) *If $m > n$, then form the polynomial $R^\# R - P^\# P$ (of degree at most $2m$) using the definition $Q^\#(z) = z^m \overline{Q(1/\overline{z})}$ for polynomials $Q(z)$ of degree $\leq m$. Proceed as in the previous case.*

References

1. Arov, D. Z., *Darlington's method in the study of dissipative systems*, Dokl. Akad. Nauk SSSR **201** (1971), no. 3, 559–562.
2. Arov, D. Z., *Stable dissipative linear stationary dynamical scattering systems*, Operator Theory: Advances and Applications **134** (2002), 99–136.
3. Arov, D. Z., *Realization of matrix-valued functions according to Darlington*, Izv. Akad. Nauk SSSR, Ser. Mat. **7** (1973), no. 6, 1295–1326.
4. Cima, J. A., Ross, W. T., *The Backward Shift on the Hardy Space*, American Mathematical Society, Providence R.I., 2000.

5. deWilde, P., *Roomy scattering matrix synthesis*, Technical Report (Berkeley, 1971).
6. Douglas, R. G., Helton, J. William, *Inner dilations of analytic m/atrix functions and Darlington synthesis*, Acta Sci. Math. (Szeged) **34** (1973), 61–67.
7. Douglas, R. G., Shapiro, H. S., Shields, A. L., *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. 1, 37–76.
8. Duren, P. L., *Theory of H^p Spaces,* Pure and Applied Mathematics, Vol. 38, Academic Press, New York—London, 1970.
9. Garcia, S. R., Putinar, M., *Complex symmetric operators and applications.* (submitted)
10. Garcia, S. R., *Conjugation, the backward shift, and Toeplitz kernels* (to appear in Journal of Operator Theory).
11. Garcia, S. R., Sarason, D., *Real outer functions*, Indiana Univ. Math. J. **52** (2003), 1397–1412.
12. Helson, H., *Large analytic functions.* II, Analysis and Partial Differential Equations, Lecture Notes in Pure and Appl. Math., Vol. 122, Dekker, New York, 1990, pp. 217–220.
13. Krein, M. G., Naimark, M. A., *The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations*, Linear and Multilinear Algebra **10** (1981), no. 4, 265–308.
14. Ross, W. T., Shapiro, H. S., *Generalized Analytic Continuation* University Lecture Series, Vol. 25, American Mathematical Society, Providence R.I., 2002.

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