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Leibniz figures in the standard histories of mathematics mostly as sharing, with Newton, the main credit for the first significant formulation of the calculus. That is appropriate in the sense that there indeed lay his most vital and enduring contribution to the subject. But such a focus limits considerably the role of mathematics in Leibniz' own life and thought. Mathematical considerations also suggested, crystallized, governed in many pivotal ways the metaphysical system that places him among the West's supreme philosophers. What follows is an attempt to outline some features of this broader picture, to correct the sometime fragmentations in our estimate of his work, to see his mathematical activity as a whole.

I

We can not hope to understand him except against the background of his age. In particular his famous (or notorious) optimism, though doubtless grounded partly in personal makeup, had discernible contemporary roots. His unquestioning faith in the existence and supreme benevolence of the God of Christianity mirrored a climate in which atheism was widely equated less with wickedness than with mere stupidity. He lived in the heady days when the homely apparatus of the Royal Society's "sooty empiricks" promised to unlock the last secrets of nature, and when his great rival Newton brought the universe itself under the sway of mathematical law. In his time these advances in physical science, and many of the great issues of philosophy, remained close enough to common modes of thought that many inquirers, Leibniz among them, could address their speculations to duchesses and kings — who occasionally joined in the game. It has been called the "Age of Reason" and the "Age of Genius," but an equally valid tag would be the "Age of Confidence." And of course much of the pervasive euphoria was born of the visible power and promise of mathematics, its application (as by Galileo and Newton) to physical understanding, the conviction (as in Hobbes and Spinoza) that its methods could bring unprecedented improvement in other fields. Not surprising then that Leibniz, himself superbly skilled in mathematics and steeped (as we shall see) in a view of the subject calculated to encourage bold extrapolations, yielded to no one in that exuberant age in his hopes both for the human understanding of nature and for the scientific amelioration of social ills.

He came relatively late in life to mathematics — probably the latest "bloomer" among all the subject's most gifted creators. His formal education in mathematics was slight and superficial; his fundamental work on the calculus awaited his historic sojourn in Paris (1672–76), that began when he was already 26 years old. His earlier training and preoccupations recall the biologists' old notion that "ontogeny recapitulates phylogeny" — that the individual's development retraces its species' evolutionary course. For like the post-medieval western mind in general, Leibniz came to an awareness of the power and beauty of mathematics from an immersion in the modes and vocabulary of scholasticism, behind which in turn rested the gigantic figure of Aristotle; and Leibniz' own philosophy retained this imprint to the end. But just as E. T. Bell declared that the scholastic philosophers were mathematicians *manqués,* so Leibniz in his youth groped instinctively toward mathematical forms and procedures that his education had not revealed to him. As a teenager, he told a correspondent, he wondered whether, "since simple terms or concepts are ordered through the known categories" (Aristotle's word for the basic organizing concepts of all thinking), "one could not set up categories and ordered series for complex terms or truths as well . . . at that time I did not know that mathematical demonstrations were what I was seeking."[1] The triumphs of his Paris phase "hooked" him forever on mathematics, and his mature writings sing its praises countless times. Mathematical studies, he declared in 1686, have a twofold use and value, "partly as an example of more rigorous judgment, partly for the knowledge of harmony and the idea of beauty."[2] These ideals are of course Greek; Leibniz fell in love with the spirit of Hellenism a century ahead of its "rediscovery" for the German mind by Winckelmann, Lessing and Goethe.

Greek too was Leibniz' conception of the ontological status, the "reality," of mathematical concepts and forms. His unchallengeable place among the subject's "modern" creators has masked the fact that his own view of it was profoundly traditional. Mathematics was for him a collection of timeless, necessary truths. These are binding even on God, who (for example) could not, even if he wished, create a triangle with an angle sum different from 180 degrees.[3] We reach the primary truths and concepts of mathematics by observation, by induction, and by the aid of the "natural light," that higher intuitive faculty which Aristotle called *nous* and which Leibniz took over from a European tradition ranging from Augustine to Descartes. Thus mathematics, on this ancient view, describes an idealized but objective order, grounded in our physical experience. In particular its axioms, so far from being arbitrary, are exceptionally certain truths, which are in principle provable — and Leibniz himself undertook on at least two occasions to demonstrate from still more basic assumptions the Euclidean postulate that the whole is greater than the part.[4] One must stress again that this whole conception of mathematics was standard already in antiquity, part of the vast corpus of thought codified for the western heritage by Aristotle and representing at bottom a kind of enormously intelligent and deeply reflective common sense.

It is true that Leibniz, for his part, stood near the beginning of the eventual replacement of this traditional view of mathematics by another. That tremendous change, the transition to a modern mathematics far richer and stranger but increasingly divorced from experienced reality and stripped of its claim to absolute truth — non-Euclidean geometry is the central symbol — is surely the pivotal watershed in all the subject's long history, a revolution much more profound even than the rise of axiomatic and deductive methods in classic Greece. The 17th century debate over the status of infinitesimals formed one episode in that historic passage, for, as Leibniz wrote, these mysterious entities have no counterpart in "nature," no validating presence in our experience. His own response was in part pragmatic: the fruitful use of infinitesimals in the calculus, he urged, does not require that these be "real," nor that the philosophical dilemmas besetting them be resolved. But he grappled with those dilemmas himself, and ended by seeing infinitesimals as consistent with the ancient tradition of mathematical realism. He linked them explicitly with other novelties which contemporary mathematicians were contemplating with diverse degrees of uneasiness — with imaginary numbers, with dimensions beyond the third, with "powers whose exponents are not ordinary [i.e., natural] numbers." All of these are useful "to shorten our reasoning," and may indeed be essential. But they are not — he insisted — merely fictions. Demonstrably, for example,

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6},$$

so that our use of imaginary numbers ultimately returns to, is justified by, a foundation in objective reality (*fundamentum in re*); and so with our conceptions of the infinite and of infinitesimals.[5] Thus even Leibniz' own groundbreaking work in the calculus wrought no essential change in his tradition-sanctioned vision of the objective, Platonist character of mathematical ideas.

And after all, that same perspective was precisely the necessary condition for the hopes of Leibniz and others who would extend the methods of mathematics to other fields. The clarity of mathematical concepts (infinitesimals notwithstanding), and the rigor of mathematical demonstrations, had been paradigmatic in western thought since Euclid. In Leibniz' mind mathematics joined with religious faith in fostering a conception of metaphysics and ethics as realms of potentially sure knowledge, of eternal truths underwritten by God and accessible to human understanding, and therefore as natural candidates for cultivation *more geometrico*. But like Thomas Hobbes (who as a young man he much admired) Leibniz regretted that the Euclidean method had not yet been applied with sufficient zeal and subtlety outside of its home domain — "we have demonstrations about the circle, but only conjectures about the soul."[6] At one time in his life, he tells us, he tried his own hand at such metaphysical geometry, in the loftiest of all spiritual enquiries, the study of God. "I often actually played the mathematician in theology, incited by the novelty of the role; I set up definitions and tried to deduce from them certain elements which were not inferior to those of Euclid in clarity but far exceeded them in the magnitude of their consequences."[7] In such philosophical adventures, he felt, the strict deductive chains characteristic of geometry are not only possible and appropriate but vital, lest deep and difficult truths elude our reasoning. In fact demonstrative rigor is actually more urgent in metaphysics than in mathematics itself, where errors are easier to detect.[8]

Hence Leibniz' lifelong goal of a "universal characteristic," a calculus that would allow the extension of logical and mathematical reasoning to other fields. He took his cue, and his hopes, from contemporary algebra, the still excitingly novel symbolic manipulations pioneered above all by Francois Viète. Without that example, wrote Leibniz, he "could hardly have attained" his own more grandiose schemes.[9] Algebra indeed offered the most "beautiful" existing example of the possibilities, but Leibniz groped toward an "art of combinations" that would far exceed algebra in power and applicability. He had the modern insight that algebra is empty of content, that any calculus is "nothing but operation through characters" and hence can in principle be brought to bear in very diverse spheres. His "characteristic" would generalize algebra in the sense in which, in geometry, the concept of similarity generalizes the concept of equality; it would be a universal science of forms rather than merely a calculus of numbers and magnitudes. And just as algebra operates on arbitrary letters of the alphabet, so (Leibniz urged) appropriate combinations and manipulations of letters can be made to mirror all human thought. We can even hope to calculate, by tallying such combinations, "the number of truths which men are able to express," and hence "the size of a work which would contain all possible human knowledge"[10]; here again speaks the authentic voice of the Age of Confidence. The universal characteristic would replace confusion of thought by clarity, and would allow reasoning as exact in metaphysics or morality as in mathematics. Hence it promised to end forever the clash of differing opinions, the endless and futile debates and disagreements, that had chronically plagued mankind. Leibniz had found a seductive hint of this last benefit in the Aristotelian logic of his scholastic training. Caught up in philosophical controversy with another scholar, "I proposed the syllogistic form, which was agreeable to my opponent. We carried the matter beyond the twelfth prosyllogism, and, from the time we began this, complaints ceased, and we understood each other, to the advantage of both sides."[11] But a full development of the *ars combinatoria* promised much more, held out the hope that the parties to any dispute whatever might be able to say merely "let us calculate," and all contention would be resolved.

II

It will be obvious that the sine qua non of such optimism was the certainty that the areas of potential dispute — metaphysics, politics, ethics, theology, law — are, like mathematics itself, realms of necessary truth, which need only be elucidated to convince. To the study of such truths Leibniz often returned. A proposition is "necessary," by his definition, if its denial is (or entails) a contradiction. But how, in practice, does one identify necessary propositions as such? Leibniz' examples are always statements of the subject-copula-predicate form that dominates Aristotelian logic — statements interpretable as comparing the memberships of sets or the ranges of concepts. A proposition is necessary in Leibniz' sense if it can be "resolved," by analysis of its subject and predicate, to an "identity" — that is, a statement with the property that of the two sets or concepts involved in the subject and predicate respectively, one can be shown to contain, by definition, the other. (For example, the statement "a red rose is a rose" is an identity in this sense.) Leibniz more than once illustrated his technique of analyzing necessary propositions with a sentence like "A duodenary number [i.e., one divisible by 12] is a quaternary number" (i.e., one divisible by 4). Interestingly, the passionate champion of algebraic manipulation does not prove this with the trivial observation that $m=12n$ implies $m=4(3n)$, but undertakes instead a cumbersome dissection of the ungainly adjectives that define the respective sets. A duodenary is (by definition) a "binary binary ternary," hence (by definition) a "quaternary ternary," hence a quaternary, "q. e. d."[12] It is to be noted that "analysis" and (equivalently) "resolution" are in this context technical terms whose meanings stem from the mathematics and philosophy of classic Greece: they describe the familiar problem-solving strategy that seeks to reduce the complex to the simple, the secondary to the fundamental, the derived to the axiomatically true.

Now propositions which are not necessary are said by Leibniz to be "contingent." They are statements which can be denied without contradiction, like "Leibniz attended the University of Leipzig." The 17th century's Scientific Revolution threw into sharp relief the philosophical issues raised by the ubiquitous presence of such contingent facts in everyday life. How could these be reconciled with the deterministic world-view emerging from the new physics? What sense could be made of the unnecessitated, of the apparently random and accidental, what scope remained for human choice and freedom, in a world bound by mathematically provable "laws" (that powerful metaphor!), in a climate of thought that soon would evoke the mechanistic philosophy proclaimed by La Mettrie, the cosmic predestinationism voiced by Laplace? For his part Leibniz reached a justification of contingency that could occur only to a mind profoundly molded by mathematics. The resolution of *necessary* propositions, described above, can always be accomplished in a finite number of steps. A contingent proposition, by contrast, has the property (according to Leibniz) that the same sort of analysis *does not terminate*. Thus a full understanding of such propositions is beyond human capacity: we can not perform the infinite sequence of reductions which alone would show that the concept "Leibniz" actually *includes* attendance at the University of Leipzig. But God, on the other hand, can take in the whole of this infinite act of analysis in, so to say, a single glance. Leibniz' thought here reflects, no doubt, the limitless powers ascribed to God by Christian tradition; but it echoes contemporary mathematics as well. We recall that in his time mathematicians were increasingly comfortable with the "completed" infinite that had so spooked their Greek predecessors—witness Newton's famous declaration that our reasoning is "no less sure" in the context of infinite series than when applied to finite sums, though in the former case our minds can not embrace all the terms. Human mathematicians, wrote Leibniz in the same spirit, "even have demonstrations about infinite series"; how much more readily, then, are "contingent or infinite truths subject to the knowledge of God."[13]

But his study of contingent propositions drew on mathematics in another and much more specific way. He found a wonderfully illuminating analogy in a celebrated piece of ancient geometry. The "Euclidean algorithm," in Euclid's original conception (*Elements,* VII, 2), sought the greatest common measure of two magnitudes by the repeated subtraction of the smaller remaining magnitude from the larger, a process guaranteed to terminate if the magnitudes are commensurable — if, to put the matter in our terms though not in Euclid's, the ratio of the measures of the original magnitudes is a rational number. In the case of two magnitudes which are *not* commensurable — whose ratio is, for us, irrational — the process of reciprocal subtraction does not terminate. This contrast became for Leibniz the guide and touchstone of his distinction between necessary and contingent propositions. The subject and predicate in a necessary proposition are (he argued) like commensurable magnitudes, in that their shared range of reference, revealed by a finitary analysis, is like the magnitudes' greatest common measure, computed by the Euclidean algorithm; correspondingly, contingent propositions resemble surds. Leibniz conceded that the analogy is not perfect, for one can calculate the true (irrational) ratio of two incommensurable magnitudes with arbitrarily small error, whereas no such narrowing of the gap between human and divine understanding of contingent truths is possible. Nevertheless he rejoiced in having discovered through mathematics the key to a riddle "which had me perplexed for a long time; for I did not understand how a predicate could be in a subject, and yet the proposition would not be a necessary one. But the knowledge of geometry and the analysis of the infinite lit this light in me, so that I might understand how notions too could be resolved to infinity."[14]

In such ways — and more tellingly, perhaps, than in any other mind of which we have record — mathematical ideas constantly informed and colored Leibniz' entire vision of the world. Many other thinkers, of course, have drawn inspiration from the same source; Aristotle, for one, anticipated Leibniz' way of reaching at every turn for mathematical illustrations of philosophical arguments, resorting naturally to the best founded and most richly developed science of his age. But in Leibniz the transference of ideas went deeper. For his work on the calculus put him at the frontier of contemporary advance, and he brought from mathematics a technical knowledge and sophistication, a grasp of precise and particular detail, which he applied in philosophy with a specificity that remains unique. We cannot know — perhaps Leibniz himself could not have reconstructed — the full course of this creative borrowing, the complex interplay of mathematical examples and their metaphysical analogues in the final shaping of his thought. Sometimes, as in his study of necessary and contingent propositions, mathematical considerations might seem merely to have provided him with a convenient model, that might be imperfect though deeply suggestive. But often, reading him — and remembering always his image of mathematics as a collection of eternal truths, and of concepts perceived with matchless clarity — one cannot resist the feeling that he seized on certain of those ideas as not merely suggesting or confirming metaphysical points but as offering sure signposts to the very contours of existential possibility, the very scope and direction of God's creative design of the world.

It is fascinating to see how much of his metaphysics can be expounded in such terms. "In the very origination of things," he wrote, "a certain Divine mathematics or metaphysical mechanics is employed," which ensured the maximum production of all desirable things; we see the same optimizing principle in the operation of nature even now, in (for example) the fact that "when several heavy bodies are operating against one another, the result is that movement which secures the greatest descent on the whole."[15] In the act of creation, said Leibniz, God acted "like the greatest geometer, who prefers the best constructions of problems." That is to say, just as a geometer will seek a proof or construction that combines maximum range and power with supreme economy of argument, so God, in choosing among the infinitely many potential orders of existence, opted for the one which would yield "the greatest effect" — the maximum of goodness and happiness — from "the simplest means."[16] Leibniz lived too far in advance of saddle-point calculus — not to mention the modern theory of games — to make much mathematically of such "minimax" considerations, but they remained basic to the optimistic tenor of his philosophy. For once, indeed, the catch-phrase that has filtered to popular perception from the complex thought of a great mind is wholly accurate: Leibniz really did believe that this is, strictly and absolutely, the best of all possible worlds — whence, of course, the brilliant, bitter mockery directed against his system by Voltaire.

Further details of Leibniz' cosmic vision were bred or reinforced by specific features of contemporary analytic geometry and calculus — their achievements and their limitations alike. To him the order detectable in the universe was like the unity imposed on a plane curve by a single algebraic expression that describes and governs all its features. He seems to have shared with at least some of his fellow analysts a remarkably bullish sense of the possibilities of curve-fitting; he related that Johann Hudde claimed the ability to find an algebraic equivalent for the profile of any human face, and Leibniz himself agreed that this is possible.[17] More strikingly still, he held that, given any set of randomly scattered points in a plane, one can find a curve "whose notion is constant and uniform, following a certain rule" — meaning, apparently, the graph of a continuous function given everywhere by a single formula — which not merely passes through all the given points but does so in the order in which they were laid down. Similarly — and the analogy is of course made fully explicit — God could fashion a harmonious universe from any original chaos of potential existents, for "no way of creating the world can be conceived which is so disordered that it does not have its own fixed and determinate order."[18]

This mathematically sustained faith in the world's ultimate rationality and goodness went further still. Undeniably, we seem to perceive many irregularities and inequities in the physical and moral fabric of things. Likewise (said Leibniz) every curve has points — singularities, extrema, points of inflection — which seem to stand out as different from the others. But in fact the seemingly anomalous nature of such points is shown by the new calculus to follow from, to conform to, the "equation or general nature of the whole" curve, which thus remains, on a broader perspective, "perfectly ordered" after all; and similarly for the seeming imperfections in the world around us.[19] And as in the universe as a whole, so also in our individual lives. All the seemingly exceptional events that befall us, even our very births and deaths, are only, as it were, peaks or valleys or cusps on the trajectories of our immortal souls; they are not outside the uniformity of nature, they violate no general laws.[20] In one especially confident passage Leibniz declared that the world's overall perfection obtains also in all its smallest component parts — even as the shortest-descent property of the cycloid arc which solves the brachistochrone problem holds between any two points, however close.[21]

As is well known, Leibniz' philosophy is suffused by a deep organicism, which saw each of the world's smallest parts as related to all of the others through constant "intercourse" and mutual influence. It is an idea which, as Joseph Needham urged, echoed more vividly the Chinese sages whom Leibniz studied than the prevailingly mechanistic outlook of contemporary Europe. But it owed something to his mathematics too. We have seen his belief that to any arbitrary set of points can be fitted a curve "whose notion is *constant and uniform*" (emphasis here added). Leibniz scarcely knew — or at any rate scarcely considered — discontinuous functions; and this prevailing tendency of his mathematics encouraged him to find, everywhere in nature, continuous passages from one state of affairs to another. The "Law of Continuity" became one of the most fruitful guiding principles of his thought. Ellipses, parabolas and hyperbolas, for example, seem from "external shape" to be entirely different from one another, yet we know that in fact each of these passes into the others by gradations so "intimate" as to bar the insertion of any different kind of curve in the sequence. "Therefore," said Leibniz, making one of his grandest leaps, "I think I have good reasons for believing" that in like manner all the world's endlessly varied species of organic creatures form a single continuous chain, "like so many ordinates of the same [continuous!] curve whose unity does not allow us to place some other ordinates between two of them because that would be a mark of disorder and imperfection."[22] This ladder of organic life is of course the "Great Chain of Being," a staple of the western intellectual tradition since the time of Plato (and the subject, long after Leibniz, of one of the most absorbing and seminal books ever written on the history of ideas).[23] Leibniz' tendency to find continuities everywhere assured him that "when the essential determinations of one being approximate those of another . . . all the properties of the former should also gradually approximate those of the latter" — or, as we should say, any biological character is a continuous function of position on the Chain. Certain creatures with unusual traits, like the "zoophytes" that seem to bridge the plant and animal kingdoms, may be viewed as occupying, "so to say," the Chain's "regions of inflection or singularity."[24] The Great Chain of Being was hoary with antiquity when Leibniz described it, but never before or since was it conceived in such specifically mathematical terms.

Every particle of matter, said Leibniz, teems with an infinity of living creatures — a notion that plausibly owed much to the wonders discovered in his time, by Leeuwenhoek and others, with the first microscopes. At the very bottom of the organic hierarchy are the simple soul-like substances that Leibniz called "monads." Leibniz used mathematical ideas in wrestling with the notoriously difficult problem of relating these elementary souls to physical matter. Material bodies, he proposed, are aggregates of these substances in precisely the way that geometrical lines are aggregates of points. A point, that is to say, is not actually *part* of a line, for "a part is always of the same nature as the whole;" rather, "a line in which there is a point is a part of a larger line, and similarly "a soul is not a part of matter, but a body in which there is such a soul is such a part of matter."[25] In Leibniz' organicist vision of nature every monad, though absolutely simple and without parts, has nevertheless a multiplicity of relations with things outside itself, just as "in a center or point, in itself perfectly simple, are found an infinity of angles formed by the lines which meet there."[26]

This survey of the mathematical bases of Leibniz' thought could be supplemented by other examples. But no case is here made for the notion that the whole of his philosophy is so describable. He would have been the first to scorn such a claim as grotesque, for in fact he insisted repeatedly that much in nature is not to be explained by mathematics.[27] The present account has set aside, as not so palpably tied to mathematics, such fundamental and characteristic of Leibniz' preoccupations as the nature of substance, the relation of "efficient" and "final" causes, the case for immortality, and many more. I hope only to have shown that the role of mathematics in shaping his philosophy was very considerable, and that it took surprisingly detailed, crucial and sophisticated forms. This side of the great philosopher has been underappreciated — perhaps above all by mathematicians. To speak of him merely as a co-founder of the calculus is doubtless to set him correctly in the history of technical progress — but at the price of a limited perspective on the whole man and on the splendid originality and power of his thought.

NOTES

1. Leibniz, *Philosophical Papers and Letters*, tr. and ed. Leroy E. Loemker (Chicago: University of Chicago Press, 1956), pp. 757–58. (Cited below as "Loemker.")
2. Leibniz, *Selections*, ed. Philip P. Wiener (New York: Charles Scribner's Sons, 1951), p. 62. (Cited below as "Wiener.")
3. Loemker, p. 833.
4. Leibniz, *Philosophical Writings*, ed. G. H. R. Parkinson (London and Melbourne: Dent, 1973), p. 87. (Cited below as "Parkinson.")
5. Loemker, p. 883.
6. Wiener, p. 59.
7. Ibid.
8. Loemker, p. 297.
9. Ibid., p. 763
10. Wiener, p. 75.
11. Loemker, p. 763.
12. Parkinson, pp. 96–97; cf. pp. 108–9.
13. Ibid., pp. 110–11; Leibniz, *Philosophical Essays*, tr. Roger Ariew and Daniel Garber (Indianapolis and Cambridge: Hackett, 1989), pp. 98–101. (Cited below as "Ariew and Garber.")
14. Parkinson, p. 97.
15. Ibid., p. 139.
16. Ibid., pp. 76 (note), 200.
17. Wiener, p. 185.
18. Parkinson, p. 78.
19. Wiener, p. 189.
20. Ibid.
21. Loemker, p. 780.
22. Wiener, p. 187.