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Notes on Formal Constructivism

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ABSTRACT

Our aim is to sketch some ideas related to how we (as in, we two) think we (as in, we humans) think.

"That theory is useless. It isn't even wrong."
- Wolfgang Pauli.

Our hope in this paper is to provide a theory, admittedly somewhat vague, of how we think about mathematics. We also hope our ideas do not cause the reader to be reminded of Pauli's quote above.

These notes were motivated by the interesting book by Changeaux and Connes [CC].

REALISM VS CONSTRUCTIVISM

Realism: Mathematical objects exist independently of experience (or "physical reality") which we process using our senses (smell, touch, sight, ...) and interpret using our brain. For example, Descartes speaks of a triangle as an "immutable and eternal" figure whose existence is independent of the mind which imagines it. Similar statements are made regarding God by many religious experts.

Constructivism: Mathematical objects exist *solely* in the mind as a certain electro-chemo-biological pattern of neurons, synapses, chemicals, ... in the brain. For an extreme example, Hume believed that ideas are merely copies of sense impressions.

Examples: Alain Connes (and probably most mathematicians) are realists. For example, the famous quote of Kronecker's, "The integers are made by God, all else is made by man," indicates a realist point-of-view. On the other hand, the biologist Jean Pierre Changeux and philosopher David Hume are constructivists (though Hume is the more extreme). Poincaré was possibly a constructivist in this sense (see [D], chapter 9).

The realist position might be roughly summarized as follows: The physical world is modeled as much as possible by mathematics. Mathematicians merely discover what is already in existence.

The constructivist position might be summarized as follows: Models for the physical world are constructions of the mind (only) and all such mental constructs exist solely as electro-chemo-biological patterns of neurons, ... in the brain. To the question, "Why is mathematics so well-suited to the description of physics?", the constructivist might counter that physicists tend to examine reproducible phenomena which tend to have "universal" characteristics. Hence mathematics, which is also universal, is admirably suited for physical description.

Points of Agreement

- Mathematics provides a "universal language", i.e., a grammar and set of terms which can be understood by anyone (sufficiently trained), independently of their cultural background.

- There is a "physical world" independent of our mind (which, however, we sense using our brain and sensory organs).

- Mathematical objects can be represented as a certain electro-chemo-biological pattern of neurons, synapses, chemicals, ... in the brain.

- A given mathematical construction can be represented as a program in a "Turing machine".

  (Using an over-simplification, these representations are modeled using neural networks, which are related to Turing machines [M].)

FORMAL CONSTRUCTIVISM

Assorted Thoughts of Our Own

- Though mathematics may indeed be universal, its development and current state is inspired and influenced by culture and the human experience.

There may be experiences outside the human realm (the realm includes other methods of detection, such as computers, microscopes, cyclotrons, ...) which might lead to mathematics which we humans might never "discover/invent/realize".

- The brain is capable of translating (or "producing") from sensual patterns certain "grammar" (or, loosely speaking, patterns of patterns). These may be thought of as rules that mental objects satisfy, though they are more intuitive feelings than rigorous laws. (For example, one never expects to see a mouse turn into an elephant, so such a concatenation of mental objects in our senses would be regarded as ungrammatical.)

- The brain is capable of translating certain universal mental objects into symbolic objects (such as translating sounds into written words). Grammar satisfied by the mental objects can be translated into grammar for the symbols.

- Suppose that we do indeed perceive "reality" via our brain and senses. Suppose, in the extreme, that all mental objects are merely copies of sense impressions. ("Copy" is being loosely interpreted here, as it is assumed that a sense impression is an electro-chemo-biological pattern in the brain and the brain may be less reliable than a camera or xerox machine!) These objects may possess properties (at least as far as we may sense them). Assume that we may posulate (using our imagination) new properties for these objects. Define a "model" to be a logically coherent collection of objects and their properties. It seems reasonable to hypothesize that all of mathematics belongs to some such model.

  If these assumptions are accepted, one may then create a sort of "platonic model" of mathematics, even though the actual objects may exist only as mental constructs (this idea which you have just read about may be such a "model"!). Call this **formal constructivism**.

- In formal constructivism the mind-body dualism problem might be regarded as follows. The mind is merely a collection of mental objects, constructions and models. The body is a collection of nerves, bones, synapses, ... Constructive realism defines the model to be a collection of electro-chemo-biological patterns in the brain. Thus the "mind" is part of the functioning of the "body" and hence there is no dualism.

- If the "realm of human experience" consists of all human ideas, experiences, and activities, then within this realm statements may be divided into two classes, T (testable) and NT (non-testable):

  (T) Those statements regarding experiences which are in principle testable by some physical device or thought experiment. We assume that the thought experiment is one which tests the truth or falsity of a well-defined statement within an axiomatically presented (logical, mathematical or philosophical, for example) internally consistent universal model. In this case the validity of the test would, of course, only be relative to the axioms assumed. For example, "The person reading this sentence is a human being," is both testable and probably true!

  (NT) Those statements regarding experiences which are not testable in the sense above. For example, "Triangles existed 2 billion years ago," is not testable.

  All statements in the second class cannot be knowable in the sense that they can be tested. However, depending on one's axioms, many models are testable. For example, if one hypothesizes the existence of God in a consistent model, then of course the statement, "God exists," is axiomatically true.

A mathematical analogy of this testable/non-testable idea: mathematical statements may be divided into two classes, D (decidable) and UD (undecidable):

(D) Those statements which may be proven true or false logically from the axioms of some mathematical model with a recursive set of axioms (say Zermelo-Fraenkel set theory).

(UD) Those statements which cannot be proven true or false logically from the axioms of some mathematical model with a finite set of axioms. Such statements could be:

– known to be true (but "self-referential" and not provable, as in Godel's incompleteness theorem),

– known to be false (but not disprovable),

– poorly formed (using the grammar of the model),

– well-formed but independent of the axioms (such as the Continuum Hypothesis).

LANGUAGE AND GRAMMAR

So far, we have simply regarded the brain as a processing unit, which is capable of translating (or producing) certain patterns from sequences of mental objects. These patterns may be translated and represented (not necessarily faithfully) using more universal mental objects. Call these objects "symbols". This capability of the brain might be regarded as an "abstraction device": a machine which is capable of noticing patterns from sequences of inputs. Of course, some patterns are more relevant than others. The brain is also capable of distingishing, evaluating, and selecting patterns.

The brain also has a tendency towards using universal mental objects (constructing order from chaos, if you will). Therefore, it is natural for the brain to process stimuli in terms of symbols and grammar. This leads naturally to language, which is useful for processing even more information.

The point is that the way we think about mathematics falls into one of several categories. We either

- formally manipulate symbols (such as algebraic expressions), following grammatical rules,
- formally manipulate mathematical objects (such as knots), following grammatical rules,
- experimentally determine grammatical rules using sequences of mathematical objects,
- select mathematical objects from sequences using some evaluation procedure (possibly for the purpose of manipulating them or determining grammatical rules for them).

All these involve one of the brain's capabilities discussed above.

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Calculus

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I tell my students the story
of Newton versus Leibniz,
the war of symbols,
lasting five generations,
between The Continent
and British Isles,
involving deeply hurt
sensibilities, grievous
blows to national pride;
on such weighty issues
as publication priority
and working systems
of logical notation:
whether the derivative
must be denoted by a "prime",
an apostrophe atop
the right hand corner
of a function, evaluated
by Newton's fluxions method
or by a formal quotient
of differentials dy/dx,
intimating future possibilities,
terminology that guides
the mind.