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An Exhibition of Exponential Sums: Visualizing Supercharacters

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An Exhibition of Exponential Sums: Visualizing Supercharacters

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Abstract

We discuss a simple mathematical mechanism that produces a variety of striking images of great complexity and subtlety. We briefly explain this approach and present a selection of attractive images obtained using this technique.

In this short note we discuss "supercharacters" on abelian groups, which provide a simple mathematical mechanism that produces a variety of striking images of great complexity and subtlety. We briefly explain this approach and present a selection of attractive images obtained using this technique.

Our original motivation stems from the recent (2008) work of P. Diaconis and I. M. Isaacs [3] in combinatorial representation theory. Here we only consider supercharacters on abelian groups. In this setting, the details are simpler and we do not require any knowledge of representation theory. A more general approach would require too much machinery and take us too far afield. A recent treatment of supercharacter theory on abelian groups can be found in [2], while some useful remarks on the general case can be found in [5].

In what follows, we consider an abelian group $G = (\mathbb{Z}/n\mathbb{Z})^d$ whose typical elements will be denoted $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, $\mathbf{y} = (y_1, y_2, \ldots, y_d)$, and so forth. A homomorphism $\varphi : G \to \mathbb{C}^\times$ is called a *character* of G; that is, φ is a nonzero, complex-valued function on G that satisfies $\varphi(\mathbf{x}+\mathbf{y}) = \varphi(\mathbf{x})\varphi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in G$. If $\zeta_n = e^{2\pi i/n}$, then the characters of G are precisely those functions of the form $\varphi(\mathbf{x}) = \zeta_n^{\mathbf{x}\cdot\mathbf{y}}$ for some $\mathbf{y} \in G$. Here $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_dy_d$ denotes the formal dot product of $\mathbf{x}, \mathbf{y} \in G$.

The values assumed by a character are nth roots of unity; plotting these values in the complex plane yields little of interest. However, if one considers certain sums of characters, more interesting patterns often emerge. We are concerned here with several families of character sums that display some truly remarkable graphical behavior.

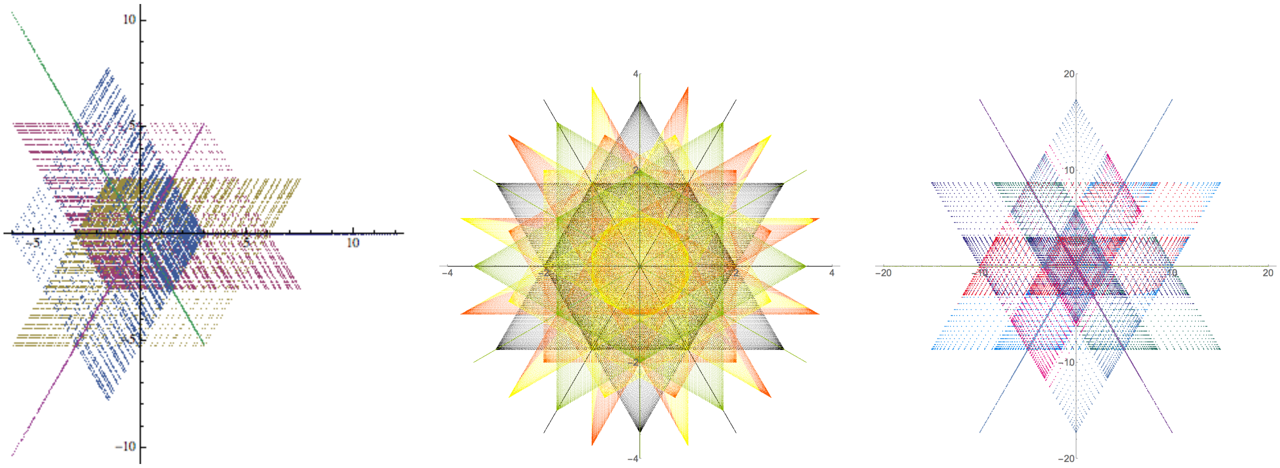
To construct our character sums, we need to select a group of automorphisms of G. Suppose that Γ is a group of $d \times d$ invertible matrices with entries in $\mathbb{Z}/n\mathbb{Z}$. For certain technical reasons, we insist that Γ is closed under the transpose operation. In the images that follow, Γ is constructed (in whole or in part) using the cyclic subgroup $\langle \omega \rangle$ of the unit group $(\mathbb{Z}/n\mathbb{Z})^\times$ generated by some unit $\omega \in \mathbb{Z}/n\mathbb{Z}$.

The natural action $(A, \mathbf{x}) \mapsto A\mathbf{x}$ of Γ on G yields a partition of G into Γ-orbits. For instance, the orbit of $\mathbf{x} \in G$ is the set $\{A\mathbf{x} : A \in \Gamma\}$ and the orbit of $\mathbf{0} = (0, 0, \ldots, 0)$ is $\{\mathbf{0}\}$. For each Γ-orbit X in G, the associated *supercharacter* is the function $\sigma_X : G \to \mathbb{C}$ defined by

$$\sigma_X(\mathbf{y}) = \sum_{\mathbf{x} \in X} \zeta_n^{\mathbf{x}\cdot\mathbf{y}}. \tag{1}$$

It can be shown that a supercharacter is constant on each Γ-orbit in G; that is, $\sigma_X(\mathbf{y}') = \sigma_X(\mathbf{y})$ if $\mathbf{y}' = A\mathbf{y}$ for some $A \in \Gamma$. Thus, to plot the values $\{\sigma_X(\mathbf{y}) : \mathbf{y} \in G\}$ in the complex plane, we need only select one representative from each Γ-orbit.

We now present a gallery of supercharacter plots obtained by choosing suitable G, Γ, and X. The resulting images are often visually striking and exhibit a variety of unexpected phenomena. Colors may be



(A) $n = 1767, \omega = 88$ (B) $n = 1776, \omega = 295$ (C) $n = 2442, \omega = 73$

Figure 1: The images above depict the values in $\mathbb{C}$ of the supercharacter $\sigma_X : G \to \mathbb{C}$ defined by (1). Here $G = (\mathbb{Z}/n\mathbb{Z})^2$ is the direct sum of two copies of $\mathbb{Z}/n\mathbb{Z}$. We let $\Gamma = S_2 \times \langle\omega\rangle$ be the direct product of the symmetric group S_2 and a cyclic subgroup $\langle\omega\rangle \subseteq (\mathbb{Z}/n\mathbb{Z})^\times$ generated by some unit ω. That is, the group Γ that acts on G consists of all multiples of 2×2 permutation matrices by a power of the generator ω. Finally, $X = \{A\mathbf{x} : A \in \Gamma\}$ is the Γ-orbit of $\mathbf{x} = (0, 1)$ in G. The color schemes are ad hoc; the color of the point $\sigma_X(\mathbf{y})$ depends upon some particular arithmetic properties of the input $\mathbf{y}$.

added to these images in an ad-hoc manner by considering various arithmetic properties of the input $\mathbf{y}$. For instance, one might assign one of five different colors to $\sigma_X(\mathbf{y})$ depending upon the value of $y_1+y_2+\cdots+y_d$ modulo five. The choice of coloring scheme, while largely an aesthetic consideration, often clarifies the details of our plots and hints at underlying arithmetic properties of the corresponding supercharacters.

The proofs that certain combinations of parameters result in particular types of images (e.g., the nested five-cusped hypocycloids in Figure 2D) typically involve nontrivial tools from algebraic or analytic number theory. In addition, the ad hoc analysis of some associated multivariate Laurent polynomials is often required. We are therefore unable to give a satisfactory account of the mechanism behind any of these patterns in the confines of this short note. However, the interested reader is invited to consult the papers [1, 4, 6] in which the details are fully worked out.

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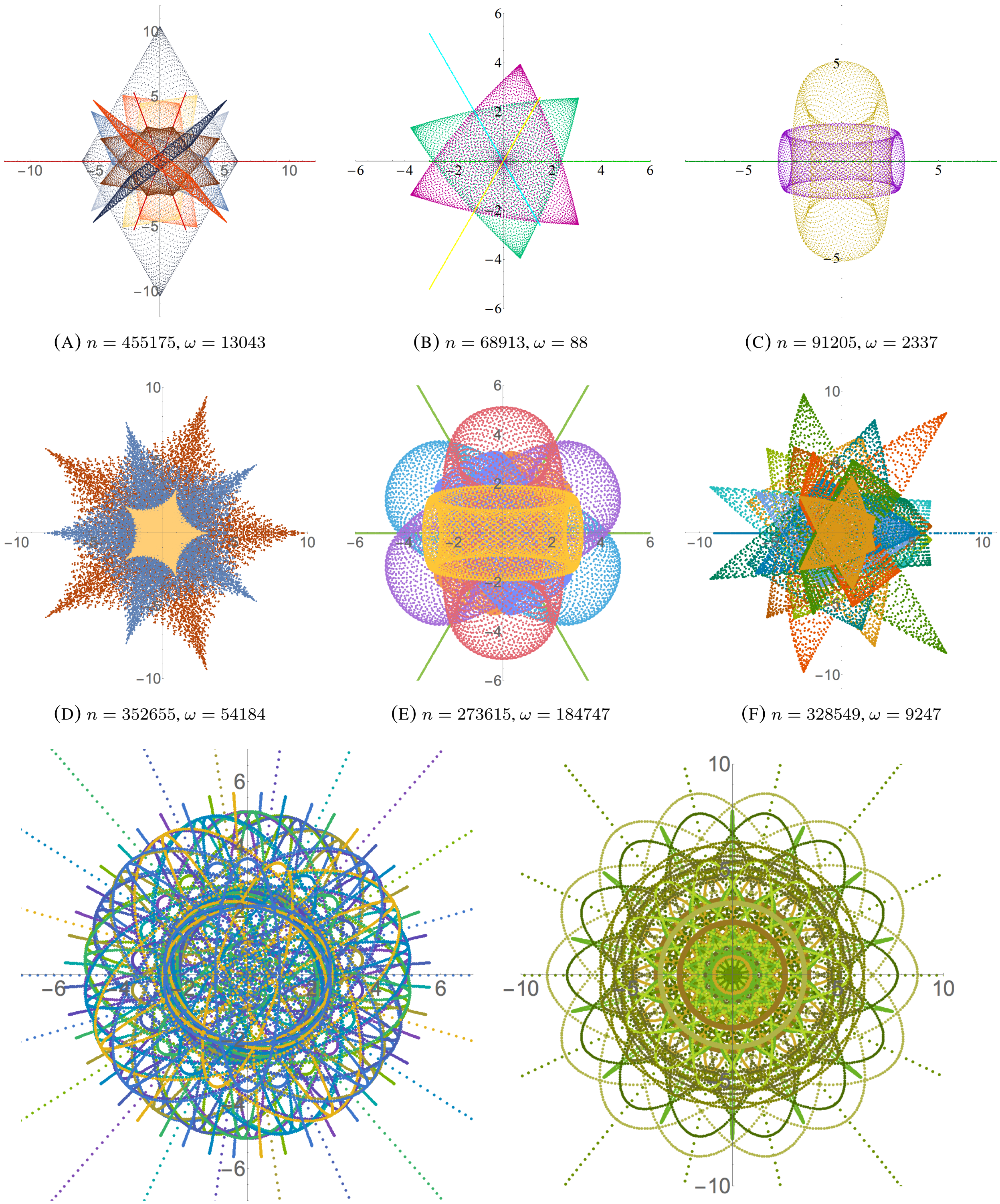
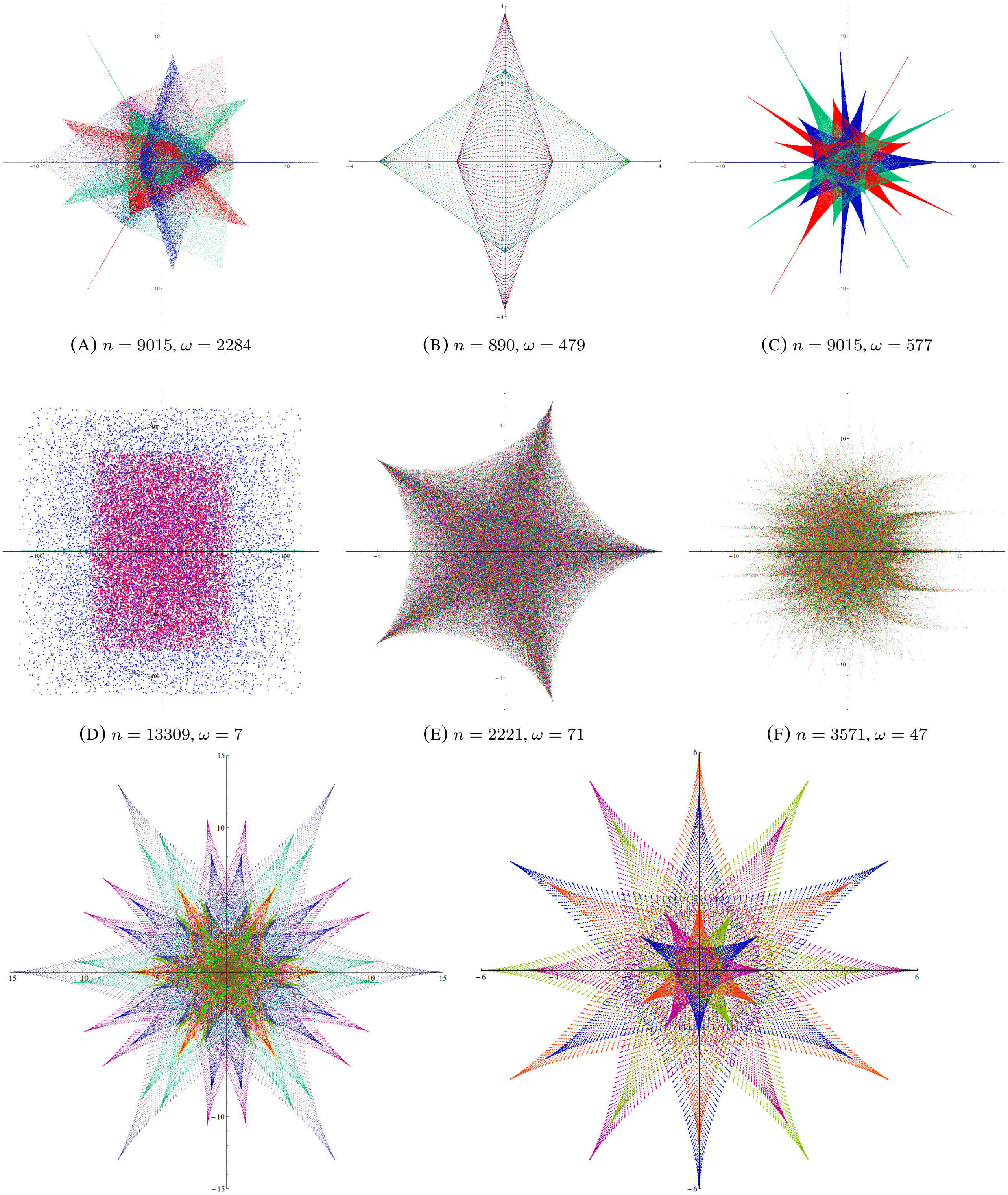


(A) $n = 455175, \omega = 13043$ (B) $n = 68913, \omega = 88$ (C) $n = 91205, \omega = 2337$

(D) $n = 352655, \omega = 54184$ (E) $n = 273615, \omega = 184747$ (F) $n = 328549, \omega = 9247$

(G) $n = 255255, \omega = 254$ (H) $n = 455175, \omega = 3599$

Figure 2: The images above depict the values in $\mathbb{C}$ of the supercharacter $\sigma_X : G \to \mathbb{C}$ defined by (1). Here $G = \mathbb{Z}/n\mathbb{Z}$ is a cyclic group (i.e., $d = 1$) and $\Gamma = \langle \omega \rangle$ is the cyclic subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ generated by an invertible element ω in $\mathbb{Z}/n\mathbb{Z}$. The group Γ acts on G by multiplication. We let $X = \Gamma$ denote the orbit of 1 under the action of Γ. The color schemes are ad hoc; the color of the point $\sigma_X(\mathbf{y})$ depends upon some particular arithmetic properties of the input $\mathbf{y}$.



(A) $n = 9015, \omega = 2284$

(B) $n = 890, \omega = 479$

(C) $n = 9015, \omega = 577$

(D) $n = 13309, \omega = 7$

(E) $n = 2221, \omega = 71$

(F) $n = 3571, \omega = 47$

(G) $n = 4378, \omega = 291$

(H) $n = 3020, \omega = 1089$

Figure 3: The images above depict the values in $\mathbb{C}$ of the supercharacter $\sigma_X : G \to \mathbb{C}$ defined by (1). Here $G = (\mathbb{Z}/n\mathbb{Z})^2$ is the direct sum of two copies of $\mathbb{Z}/n\mathbb{Z}$. We let $\Gamma = \{\operatorname{diag}(u, u^{-1}) : u \in \langle \omega \rangle\}$ be a group of diagonal matrices that acts on G by multiplication. Here ω is an element of the unit group $(\mathbb{Z}/n\mathbb{Z})^\times$ that generates the cyclic subgroup $\langle \omega \rangle$ of $(\mathbb{Z}/n\mathbb{Z})^\times$. Finally, $X = \{A\mathbf{x} : A \in \Gamma\}$ is the orbit in G of $\mathbf{x} = (1, 1)$ under the action of Γ. The color schemes are ad hoc; the color of the point $\sigma_X(\mathbf{y})$ depends upon some particular arithmetic properties of the input $\mathbf{y}$.