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Four Quotient Set Gems

Bryan Brown, Michael Dairiko, Stephan Ramon Garcia,
Bob Lutz, and Michael Someck

Abstract. Our aim in this note is to present four remarkable facts about quotient sets. These observations seem to have been overlooked by the MONTHLY, despite its intense coverage of quotient sets over the years.

INTRODUCTION. If A is a subset of the natural numbers $\mathbb{N} = \{1, 2, \dots\}$, then we let $R(A) = \{a/a' : a, a' \in A\}$ denote the corresponding *quotient set* (sometimes called a *ratio set*). Our aim in this short note is to present four remarkable results, which seem to have been overlooked in the MONTHLY, despite its intense coverage of quotient sets over the years [3, 4, 5, 9, 10, 11, 14]. Some of these results are novel, while others have appeared in print elsewhere but somehow remain largely unknown.

In what follows, we let $A(x) = A \cap [1, x]$ so that $|A(x)|$ denotes the number of elements in A which are $\leq x$. The *lower asymptotic density* of A is the quantity

$$\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{|A(n)|}{n},$$

which satisfies the obvious bounds $0 \leq \underline{d}(A) \leq 1$. We say that A is *fractionally dense* if the closure of $R(A)$ in \mathbb{R} equals $[0, \infty)$ (i.e., if $R(A)$ is dense in $[0, \infty)$).

Our four gems are as follows.

1. The set of all natural numbers whose base- b representation begins with the digit 1 is fractionally dense for $b = 2, 3, 4$, but not for $b \geq 5$.
2. For each $\delta \in [0, \frac{1}{2})$, there exists a set $A \subset \mathbb{N}$ with $\underline{d}(A) = \delta$ that is not fractionally dense. On the other hand, if $\underline{d}(A) \geq \frac{1}{2}$, then A must be fractionally dense [15].
3. We can partition \mathbb{N} into three sets, each of which is not fractionally dense. However, such a partition is impossible using only two sets [2].
4. There are subsets of \mathbb{N} that contain arbitrarily long arithmetic progressions, yet that are not fractionally dense. On the other hand, there exist fractionally dense sets that have no arithmetic progressions of length ≥ 3 .

BASE- b REPRESENTATIONS. In [5, Example 19], it was shown that the set

$$A = \{1\} \cup \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\} \cup \{100, 101, \dots\} \cup \dots$$

of all natural numbers whose base-10 representation begins with the digit 1 is not fractionally dense. This occurs despite the fact that $\underline{d}(A) = \frac{1}{9}$, so that a positive proportion of the natural numbers belongs to A . The consideration of other bases reveals the following gem.

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Gem 1. *The set of all natural numbers whose base- b representation begins with the digit 1 is fractionally dense for $b = 2, 3, 4$, but not for $b \geq 5$.*

To show this, we require the following more general result.

Proposition 1. *Let $1 < a \leq b$. The set*

$$A = \bigcup_{k=0}^{\infty} [b^k, ab^k) \cap \mathbb{N} \tag{1}$$

is fractionally dense if and only if $b \leq a^2$. Moreover, we also have

$$\underline{d}(A) = \frac{a-1}{b-1}.$$

Proof. We first compute $\underline{d}(A)$. Since the counting function $|A(x)|$ is nondecreasing on each interval of the form $[b^k, ab^k)$ and constant on each interval of the form $[ab^k, b^{k+1})$, it follows that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{|A(n)|}{n} &= \liminf_{n \rightarrow \infty} \frac{1}{b^n} \sum_{k=1}^{n-1} |[b^k, ab^k) \cap \mathbb{N}| \\ &= \lim_{n \rightarrow \infty} \frac{1}{b^n} \sum_{k=1}^{n-1} (a-1)b^k \\ &= \lim_{n \rightarrow \infty} \frac{(a-1)(b^n-1)}{b^n(b-1)} \\ &= \frac{a-1}{b-1}. \end{aligned} \tag{2}$$

Note that in order to obtain (2), we used the fact that the difference between $ab^k - b^k$ and $|[b^k, ab^k) \cap \mathbb{N}|$ is ≤ 1 . Having computed $\underline{d}(A)$, we now turn our attention to fractional density.

Case 1. Suppose that $a^2 < b$. By construction, each quotient of elements of A belongs to an interval of the form $I_\ell = (a^{-1}b^\ell, ab^\ell)$ for some integer ℓ . If $j < k$, then $a^2 < b^{k-j}$, whence $ab^j < a^{-1}b^k$ so that every element of I_j is strictly less than every element of I_k . Therefore, $R(A)$ contains no elements in any interval of the form $[ab^\ell, a^{-1}b^{\ell+1}]$, each of which is nonempty since $a^2 < b$. Thus A is not fractionally dense.

Case 2. Now let $b \leq a^2$, noting that

$$(0, \infty) = \bigcup_{j \in \mathbb{Z}} \left[\frac{b^j}{a}, b^j \right) \cup [b^j, ab^j).$$

Suppose that ξ belongs to an interval of the form $[b^j, ab^j)$ for some integer j . Given $\epsilon > 0$, let k be so large that $1 < b^k \epsilon$, and observe that

$$b^{j+k} \leq b^k \xi < ab^{j+k}.$$

Let ℓ be the unique natural number satisfying

$$b^{j+k} + \ell \leq b^k \xi < b^{j+k} + \ell + 1 \quad (3)$$

and

$$0 \leq \ell \leq (a-1)b^{j+k} - 1. \quad (4)$$

From (3), we find that

$$0 \leq b^k \xi - (b^{j+k} + \ell) < 1,$$

from which it follows that

$$0 \leq \xi - \frac{b^{j+k} + \ell}{b^k} < \epsilon.$$

Since (4) ensures that $b^{j+k} + \ell$ belongs to the interval $[b^{j+k}, ab^{j+k})$, we conclude that $(b^{j+k} + \ell)/b^k$ belongs to $R(A)$. A similar argument applies if ξ belongs to an interval of the form $[\frac{b^j}{a}, b^j)$. Putting this all together, we conclude that A is fractionally dense. ■

To see that Gem 1 follows from the preceding proposition, observe that if $a = 2$ and $b \geq 2$ is an integer, then the set A defined by (1) is precisely the set of all natural numbers whose base- b representation begins with the digit 1. The inequality $b \leq a^2$ holds precisely for the bases $b = 2, 3, 4$ and fails for all $b \geq 5$.

CRITICAL DENSITY. Having seen that there exist sets that are not fractionally dense, yet whose lower asymptotic density is positive, it is natural to ask whether there exists a critical value $0 < \kappa \leq 1$ such that $\kappa \leq \underline{d}(A)$ ensures that A is fractionally dense. The following gem establishes the existence of such a critical density, namely $\kappa = \frac{1}{2}$.

Gem 2. *If $\underline{d}(A) \geq \frac{1}{2}$, then A is fractionally dense. On the other hand, if $0 \leq \delta < \frac{1}{2}$, then there exists a subset $A \subset \mathbb{N}$ with $\underline{d}(A) = \delta$ that is not fractionally dense.*

Establishing this result will take some work, although we are now in a position to prove the second statement (originally obtained by Strauch and Tóth [15, Thm. 1] and by Harman [8, p. 167] using slightly different methods). If $0 \leq \delta < \frac{1}{2}$, then we may write $\delta = \frac{1}{2+\epsilon}$ where $\epsilon > 0$. Now let $a = 1 + \frac{\epsilon}{2}$ and $b = 1 + \epsilon + \frac{1}{2}\epsilon^2$ so that $1 < a^2 < b$. For these parameters, Proposition 1 tells us that the set A defined by (1) satisfies $\underline{d}(A) = \delta$ and fails to be fractionally dense. If $\delta = 0$, then we note that the set $A = \{2^n : n \in \mathbb{N}\}$ is not fractionally dense and has $\underline{d}(A) = 0$.

Thus, we can construct sets having lower asymptotic density arbitrarily close to $\frac{1}{2}$, yet which fail to be fractionally dense. On the other hand, it is clear that fractionally dense sets with $\underline{d}(A) = \frac{1}{2}$ exist. Indeed, one such example is $A = \{2, 4, 6, \dots\}$. However, the question of whether a non-fractionally dense set can have lower asymptotic density equal to the critical density $\frac{1}{2}$ is much more difficult.

In the late 1960s, Šalát proposed an example of a set $A \subset \mathbb{N}$ that is not fractionally dense and such that $\underline{d}(A) = \frac{1}{2}$ [12, p. 278]. However, Šalát's example was flawed, as

pointed out in the associated corrigendum [13]. In 1998, Strauch and Tóth established a more general result [15, Thm. 2], which implies that a set satisfying $\underline{d}(A) \geq \frac{1}{2}$ must be fractionally dense (fortunately, the lengthy corrigendum [16] to this paper does not affect the result in question). For $\underline{d}(A) > \frac{1}{2}$, the fractional density of A also follows from a sophisticated theorem in metric number theory [8, Thm. 6.6] (although Harman was kind enough to show us an elementary proof in this case). The proof below, which covers the critical case $\underline{d}(A) = \frac{1}{2}$, is essentially due to Strauch and Tóth.

Theorem 2. *If $\underline{d}(A) \geq \frac{1}{2}$, then A is fractionally dense.*

Proof. Let $\underline{d}(A) \geq \frac{1}{2}$ and suppose toward a contradiction that $0 < \alpha < \beta \leq 1$ and $R(A) \cap (\alpha, \beta) = \emptyset$. Noting that A must be infinite, we enumerate the elements of A in increasing order $a_1 < a_2 < \dots$. Let k be a natural number, which is so large that $k\alpha > 1$, and let $0 < \theta < 1$. For all m and n in \mathbb{N} , let

$$J_m^n = (\alpha a_{k(\lfloor \theta n \rfloor + m)}, \alpha(a_{k(\lfloor \theta n \rfloor + m)} + k)).$$

For each n in \mathbb{N} , we claim that the intervals

$$J_0^n, J_1^n, \dots, J_{n - \lfloor \theta n \rfloor - 1}^n, (\alpha a_{kn}, \beta a_{kn}), \tag{5}$$

used below in a delicate counting argument, are pairwise disjoint. Indeed, since the a_i are integers, it follows that

$$\alpha(a_{k(\lfloor \theta n \rfloor + m)} + k) \leq \alpha(a_{k(\lfloor \theta n \rfloor + m) + k}) = \alpha(a_{k(\lfloor \theta n \rfloor + m + 1)}),$$

so that the right endpoint of J_m^n is at most the left endpoint of J_{m+1}^n . A similar argument shows that the right endpoint of $J_{n - \lfloor \theta n \rfloor - 1}^n$ is at most αa_{kn} .

Next, let n be so large that $\lfloor \theta n \rfloor \geq \frac{\alpha}{\beta - \alpha}$. We claim that

$$J_m^n \subseteq (\alpha a_{k(\lfloor \theta n \rfloor + m)}, \beta a_{k(\lfloor \theta n \rfloor + m)}) \tag{6}$$

for each $m = 0, 1, \dots, n - \lfloor \theta n \rfloor - 1$. To see this, note that

$$\frac{k\alpha}{\beta - \alpha} \leq k\lfloor \theta n \rfloor \leq a_{k\lfloor \theta n \rfloor} < a_{k(\lfloor \theta n \rfloor + m)},$$

which implies that $\alpha(a_{k(\lfloor \theta n \rfloor + m)} + k) \leq \beta a_{k(\lfloor \theta n \rfloor + m)}$ so that (6) holds.

Since $(\alpha a_n, \beta a_n) \cap A = \emptyset$ by assumption, it follows now that the intervals (5) are contained in the complement $[0, \infty) \setminus A$. Letting $B = \mathbb{N} \setminus A$, a naïve counting argument gives

$$|B(\beta a_{kn})| \geq ((\beta - \alpha)a_{kn} - 1) + (n - \lfloor \theta n \rfloor)(k\alpha - 1).$$

Dividing through by βa_{kn} and taking limits superior yields

$$\begin{aligned} 1 - \underline{d}(A) &= \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n} \\ &\geq \limsup_{n \rightarrow \infty} \frac{|B(\beta a_{kn})|}{\beta a_{kn}} \end{aligned}$$

$$\begin{aligned}
&\geq \limsup_{n \rightarrow \infty} \left(\frac{(\beta - \alpha)a_{kn} - 1}{\beta a_{kn}} + \frac{(n - \lfloor \theta n \rfloor)(k\alpha - 1)}{\beta a_{kn}} \right) \\
&\geq \frac{\beta - \alpha}{\beta} + \liminf_{n \rightarrow \infty} \frac{(n - \lfloor \theta n \rfloor)(k\alpha - 1)}{\beta a_{kn}} \\
&\geq 1 - \frac{\alpha}{\beta} + (1 - \theta) \liminf_{n \rightarrow \infty} \frac{\alpha kn - n}{\beta a_{kn}} \\
&\geq 1 - \frac{\alpha}{\beta} + (1 - \theta) \left(\frac{\alpha}{\beta} \liminf_{n \rightarrow \infty} \frac{kn}{a_{kn}} - \frac{1}{\beta k} \limsup_{n \rightarrow \infty} \frac{kn}{a_{kn}} \right) \\
&\geq 1 - \frac{\alpha}{\beta} + (1 - \theta) \left(\frac{\alpha}{\beta} \underline{d}(A) - \frac{1}{\beta k} \right).
\end{aligned}$$

Taking the limit as $\theta \rightarrow 0$ and $k \rightarrow \infty$ and rearranging gives

$$\left(1 + \frac{\alpha}{\beta} \right) \underline{d}(A) \leq \frac{\alpha}{\beta}. \tag{7}$$

If $\underline{d}(A) \geq \frac{1}{2}$, then the preceding inequality implies that $\beta \leq \alpha$, a contradiction. \blacksquare

The astute reader will note that we have actually shown that if $(\alpha, \beta) \cap R(A) = \emptyset$, then (7) must hold. In fact, with only a little more work, we can show that

$$\underline{d}(A) \leq \frac{\alpha}{\beta} \min \{ \bar{d}(A), 1 - \bar{d}(A) \}$$

and $\bar{d}(A) \leq 1 - (\beta - \alpha)$, where

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A(n)|}{n}$$

denotes the *upper asymptotic density* of A [15, Thm. 2]. Moreover, it is also known that $\underline{d}(A) + \bar{d}(A) \geq 1$ implies that A is fractionally dense [15, p. 71].

PARTITIONS OF \mathbb{N} . Clearly, \mathbb{N} itself is fractionally dense since $R(\mathbb{N}) = \mathbb{Q} \cap (0, \infty)$, the set of positive rational numbers. An interesting question now presents itself. If \mathbb{N} is partitioned into a finite number of disjoint subsets, does one of these subsets have to be fractionally dense? The following result gives a complete answer to this problem.

Gem 3. *We can partition \mathbb{N} into three sets, each of which is not fractionally dense. However, such a partition is impossible using only two sets.*

This gem is due to Bukor, Šalát, and Tóth [2]. These three, along with P. Erdős, later generalized the second statement by showing that if the set A is presented as an increasing sequence $a_1 < a_2 < \dots$ satisfying $\lim_{n \rightarrow \infty} a_{n+1}/a_n = 1$, then for each $B \subseteq A$, either B or $A \setminus B$ is fractionally dense [1].

The proof of our third gem is contained in the following two results.

Proposition 3. *There exist disjoint sets $A, B, C \subset \mathbb{N}$, none of which are fractionally dense, such that $\mathbb{N} = A \cup B \cup C$.*

Proof. Let

$$A = \bigcup_{k=0}^{\infty} [5^k, 2 \cdot 5^k) \cap \mathbb{N},$$

$$B = \bigcup_{k=0}^{\infty} [2 \cdot 5^k, 3 \cdot 5^k) \cap \mathbb{N},$$

and

$$C = \bigcup_{k=0}^{\infty} [3 \cdot 5^k, 5 \cdot 5^k) \cap \mathbb{N},$$

so that A , B , and C consist of those natural numbers whose base-5 expansions begin, respectively, with 1, with 2, and with 3 or 4. We consider only C here, for the remaining two cases are similar (in fact, the set A is already covered by Proposition 1). Observe that each quotient of elements of C is contained in an interval of the form $I_\ell = (\frac{2}{3}5^\ell, \frac{5}{3}5^\ell)$ for some integer ℓ . If $j < k$, then every element of I_j is strictly less than every element of I_k , since $\frac{5}{3} \cdot 5^j < \frac{2}{3} \cdot 5^k$ holds if and only if $\frac{25}{9} < 5^{k-j}$. Thus, $R(C) \cap [\frac{2}{3}5^\ell, \frac{5}{3}5^{\ell+1}] = \emptyset$ for any integer ℓ . ■

Theorem 4. *If A and B are disjoint sets such that $\mathbb{N} = A \cup B$, then at least one of A , B is fractionally dense.*

Proof. Without loss of generality, we may assume that both A and B are infinite. Suppose, toward a contradiction, that neither A nor B is fractionally dense. Thus, there exists $\alpha, \beta > 1$, and $\epsilon > 0$ such that $(\alpha - \epsilon, \alpha + \epsilon) \cap R(A)$ and $(\beta - \epsilon, \beta + \epsilon) \cap R(B)$ are both empty. Now, let $n_0 \in \mathbb{N}$ be such that

$$\frac{1 + \alpha + \beta + 2\alpha\beta}{n_0} < \epsilon.$$

Since A and B are both infinite, there exists $n > \alpha\beta(n_0 + 1)$ such that $n \in A$ and $n + 1 \in B$. If $s = \lfloor \frac{n}{\alpha\beta} \rfloor - 1$ belongs to A , then setting $t = \lfloor \alpha s \rfloor$ yields

$$\left| \frac{t}{s} - \alpha \right| = \left| \frac{\lfloor \alpha s \rfloor - \alpha s}{s} \right| < \frac{1}{s} \leq \frac{1}{n_0} < \epsilon. \quad (8)$$

Since $(\alpha - \epsilon, \alpha + \epsilon) \cap R(A) = \emptyset$, we conclude that t belongs to B . Now, observe that

$$\begin{aligned} \left| \frac{n+1}{t} - \beta \right| &= \frac{n+1 - \beta \lfloor \alpha s \rfloor}{t} \\ &< \frac{n+1 - \beta(\alpha s - 1)}{t} \\ &= \frac{1 + \beta + \alpha\beta + n - \alpha\beta \lfloor \frac{n}{\alpha\beta} \rfloor}{t} \\ &< \frac{1 + \beta + 2\alpha\beta}{t} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1 + \beta + 2\alpha\beta}{n_0} \\ &< \epsilon. \end{aligned}$$

Hence, $\frac{n+1}{t}$ belongs to $(\beta - \epsilon, \beta + \epsilon) \cap R(B)$, which is a contradiction. Therefore, it must be the case that s belongs to B .

Assuming now that $s = \lfloor \frac{n}{\alpha\beta} \rfloor - 1$ belongs to B , we now let $t = \lfloor \beta s \rfloor$. Proceeding as in (8), we can show that $|\frac{t}{s} - \beta| < \epsilon$ and hence that t belongs to A . Moreover,

$$\begin{aligned} \left| \frac{n}{t} - \alpha \right| &= \left| \frac{n - \alpha \lfloor \beta s \rfloor}{t} \right| \\ &< \frac{n - \alpha(\beta s - 1)}{t} \\ &= \frac{\alpha + n - \alpha\beta s}{t} \\ &= \frac{\alpha + n - \alpha\beta(\lfloor \frac{n}{\alpha\beta} \rfloor - 1)}{t} \\ &= \frac{\alpha + \alpha\beta + (n - \alpha\beta \lfloor \frac{n}{\alpha\beta} \rfloor)}{t} \\ &< \frac{\alpha + 2\alpha\beta}{n_0} \\ &< \epsilon. \end{aligned}$$

Thus, $\frac{n}{t}$ belongs to $(\alpha - \epsilon, \alpha + \epsilon) \cap R(A)$, a contradiction which implies that s belongs to neither A nor B , an absurdity, since $\mathbb{N} = A \cup B$. ■

Although we might be tempted to postulate a relationship between Theorems 2 and 4, lower asymptotic density has no bearing on the preceding result, since it is possible to partition \mathbb{N} into two subsets both having lower asymptotic density zero.

Proposition 5. *There exist disjoint sets $A, B \subset \mathbb{N}$ such that $\mathbb{N} = A \cup B$, $\underline{d}(A) = \underline{d}(B) = 0$, and $\overline{d}(A) = \overline{d}(B) = 1$.*

Proof. We require the Stolz–Cesàro Theorem [6], which tells us that if x_n and y_n are two increasing and unbounded sequences of real numbers, then

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L \implies \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = L.$$

Now put the first $1!$ natural numbers into A , then the next $2!$ into B , the next $3!$ into A , and so forth, yielding the sets

$$A = \{1, 4, 5, 6, 7, 8, 9, 34, \dots\}, \quad B = \{2, 3, 10, 11, 12, \dots, 32, 33, 154, 155, \dots\}.$$

By construction, $A \cap B = \emptyset$ and $\mathbb{N} = A \cup B$. Let $x_n = \sum_{k=1}^{2n-1} k!$ and $y_n = \sum_{k=1}^{2n} k!$, observing that $|A(x_n)| = |A(y_n)| = \sum_{k=1}^n (2k-1)!$. Since

$$\frac{|A(y_{n+1})| - |A(y_n)|}{y_{n+1} - y_n} = \frac{(2n+1)!}{(2n+2)! + (2n+1)!} = \frac{1}{2n+3} \rightarrow 0$$

and

$$\frac{|A(x_{n+1})| - |A(x_n)|}{x_{n+1} - x_n} = \frac{(2n+1)!}{(2n+1)! + (2n)!} = \frac{1}{1 + \frac{1}{2n+1}} \rightarrow 1,$$

it follows from the Stolz–Cesàro Theorem that $\underline{d}(A) = 0$ and $\overline{d}(A) = 1$. An analogous argument using $x_n = \sum_{k=1}^{2n} k!$ and $y_n = \sum_{k=1}^{2n+1} k!$ now reveals that $\underline{d}(B) = 0$ and $\overline{d}(B) = 1$ as well. ■

ARITHMETIC PROGRESSIONS. Our final gem concerns arithmetic progressions of natural numbers. Recall that an *arithmetic progression* with common difference b and length n is a sequence of the form $a, a + b, a + 2b, \dots, a + (n - 1)b$. Subsets of the natural numbers that contain arbitrarily long arithmetic progressions are often thought of as being “thick” or “dense” in some qualitative sense.

Gem 4. *There exists a subset of \mathbb{N} containing arbitrarily long arithmetic progressions that is not fractionally dense. On the other hand, there exists a fractionally dense set that has no arithmetic progressions of length ≥ 3 .*

The first statement of Gem 4 has already been proven. Indeed, Proposition 1 produces sets that are not fractionally dense and that contain arbitrarily long blocks of consecutive natural numbers. We therefore need only produce a fractionally dense set having no arithmetic progressions of length three (see also [9, Thm. 2]).

Proposition 6. *The set $A = \{2^j : j \geq 2\} \cup \{3^k : k \geq 2\}$ is fractionally dense and contains no arithmetic progressions of length three.*

Proof. First, recall that Kronecker’s approximation theorem [7, Thm. 440] asserts that if $\beta > 0$ is irrational, $\alpha \in \mathbb{R}$, and $\delta > 0$, then there exist $n, m \in \mathbb{N}$ such that $|n\beta - \alpha - m| < \delta$. Let $\xi, \epsilon > 0$ and note that $\beta = \log_2 3 > 0$ is irrational. By the continuity of $f(x) = 2^x$ at $\log_2 \xi$, there exists $\delta > 0$ such that

$$|\log_2 x - \log_2 \xi| < \delta \implies |x - \xi| < \epsilon. \quad (9)$$

Kronecker’s theorem with $\beta = \log_2 3$ and $\alpha = \log_2 \xi$ now yields $n, m \in \mathbb{N}$ so that

$$\left| \log_2 \left(\frac{3^n}{2^m} \right) - \log_2 \xi \right| = |n \log_2 3 - \log_2 \xi - m| < \delta.$$

In light of (9), it follows that $|3^n/2^m - \xi| < \epsilon$, whence A is fractionally dense.

We now show that A contains no arithmetic progressions of length three. Assume, toward a contradiction, that such an arithmetic progression exists. By the definition of A , the first term in this progression is either a power of 2 or a power of 3. We consider both cases separately, letting b denote the common difference in each progression.

Case 1. Suppose that $2^j, 2^j + b$, and $2^j + 2b$ belong to A . Since $2^j + 2b$ is even and belongs to A , it must be of the form 2^k for some $k > j$, whence $b = 2^{k-1} - 2^{j-1}$ is even since $j \geq 2$. Thus, $2^j + b$ must also be of the form 2^ℓ for some $\ell > j$ so that

$$2^\ell = 2^j + b = 2^j + (2^{k-1} - 2^{j-1}) = 2^{j-1}(2^{k-j} + 1).$$

Upon dividing the preceding by 2^{j-1} we obtain a contradiction.

Case 2. Now suppose that $3^j, 3^j + b, 3^j + 2b$ is an arithmetic progression in A of length three that begins with 3^j . In this case, $3^j + 2b$ is odd and hence must be of the form 3^k for some $k > j$. Therefore,

$$b = \frac{3^k - 3^j}{2} = 3^j \frac{(3^{k-j} - 1)}{2},$$

so that the second term $3^j + b$ in our progression is divisible by 3^j . Thus, $3^j + b = 3^\ell$ for some $\ell > j$, from which it follows that

$$3^\ell = 3^j + b = 3^j + 3^j \frac{(3^{k-j} - 1)}{2}.$$

Since this implies that $2 \cdot 3^{\ell-j} = 2 + (3^{k-j} - 1)$, which is inconsistent modulo 3, we conclude that A has no arithmetic progressions of length three. ■

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REFERENCES

1. J. Bukor, P. Erdős, T. Šalát, J. T. Tóth, Remarks on the (R) -density of sets of numbers, II, *Math. Slovaca* **47** (1997) 517–256.
2. J. Bukor, T. Šalát, J. T. Tóth, Remarks on R -density of sets of numbers, *Tatra Mt. Math. Publ.* **11** (1997) 159–165.
3. J. Bukor, J. T. Tóth, On accumulation points of ratio sets of positive integers, *Amer. Math. Monthly* **103** (1996) 502–504.
4. S. R. Garcia, Quotients of Gaussian primes, *Amer. Math. Monthly* **120** (2013) 851–853.
5. S. R. Garcia, V. Selhorst-Jones, D. E. Poore, N. Simon, Quotient sets and Diophantine equations, *Amer. Math. Monthly* **118** (2011) 704–711.
6. R. Gelca, T. Andreescu, *Putnam and Beyond*. Springer, New York, 2007.
7. G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*. Sixth edition. Oxford University Press, Oxford, 2008.
8. G. Harman, *Metric Number Theory*. London Mathematical Society Monographs. New Series. Vol. 18. Clarendon Press Oxford University Press, New York, 1998.
9. S. Hedman, D. Rose, Light subsets of \mathbb{N} with dense quotient sets, *Amer. Math. Monthly* **116** (2009) 635–641.
10. D. Hobby, D. M. Silberger, Quotients of primes, *Amer. Math. Monthly* **100** (1993) 50–52.
11. A. Nowicki, Editor's endnotes, *Amer. Math. Monthly* **117** (2010) 755–756.
12. T. Šalát, On ratio sets of sets of natural numbers, *Acta Arith.* **15** (1969) 273–278.
13. ———, Corrigendum to the paper “On ratio sets of sets of natural numbers,” *Acta Arith.* **16** (1970) 103.
14. P. Starni, Answer to two questions concerning quotients of primes, *Amer. Math. Monthly* **102** (1995) 347–349.
15. O. Strauch, J. T. Tóth, Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$, *Acta Arith.* **87** (1998) 67–78.
16. ———, Corrigendum to Theorem 5 of the paper: “Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$,” *Acta Arith.* **103** (2002) 191–200.

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**100 Years ago this Month in *The American Mathematical Monthly*
Edited by Vadim Ponomarenko**

That there has been and still is an unrest among thoughtful high school teachers of mathematics is manifest. This problem certainly exists: How are we to adapt the courses in algebra and geometry to the experiences of high school students, in order that these may contribute to their broader development in both a cultural and a practical way? On the one hand we have the student who is preparing for the university and on the other hand the student whose school life ends with the high school or who, going on to the university, will not continue the subject of mathematics. The one who would solve this problem must have these two classes in view, and the solution for the one must be the solution for the other since it is only in the largest schools that distinction can be made by having separate classes, even if it were advisable.

—Excerpted from “Miscellaneous Questions” **21** (1914), 239.