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# Are There Revolutions in Mathematics

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Thomas Kuhn's Theory of the Structure of Scientific Revolutions, first published in 1962, heralded both a renaissance and a shift in the philosophy of science. The main tendency had been towards Logical Positivism and its successor Logical Empiricism, with an emphasis on the logical structure of scientific theories, shown in the work of Carnap, Frank, Hempel, Nagel and others. This was revitalized with the English publication of Popper's Logic of Scientific Discovery in 1959. It was not until after the impact of Kuhn that the philosophy of science became thoroughly cognizant of developments in the history of science (although there were precursors, such as Hanson). Kuhn offered a powerful new synthesis of pre-existing elements (some perhaps unknown to him) such as Wittgenstein's notion of a 'paradigm' and Bachelard's concept of 'epistemological rupture' in the history of ideas. He constructed profound theory in the philosophy of science, the influence of which, controversy notwithstanding, had reverberated through many other fields of enquiry since.

According to Kuhn, science does not grow by a simple accumulation of knowledge. Instead, it alternates between periods of 'normal' and 'revolutionary' science in its development. During a period of 'normal' science, new knowledge is accumulated by accretion, as a dominant theory and paradigm of inquiry are followed and used as a model. Anomalies and contradictions in the dominant paradigm lead to a period of revolution in which competing camps of scientists promote alternative theories (including the falsified old theory). A new theory comes to be accepted and gradually becomes the new paradigm of explanation and enquiry. In the shift to the new theory many of the concepts involved change meaning (e.g. mass and length in the transition from Newtonian Mechanics to Relativity Theory). Kuhn's controversial claim is that the old and new theories are 'incommensurable', and that their supporters may not be able to understand each other.

The Kuhn-Popper debate in the philosophy of science hinged on the issue of rational versus irrational criticism of scientific theories. Popper's position is prescriptive, and he posits falsification as a rational criterion for the rejection of a scientific theory. Kuhn, on the other hand, proposes a more descriptive philosophy of science, which while treating the growth of objective knowledge acknowledges that rational features are neither necessary nor sufficient to account for theory acceptance or rejection.

Although it is beside the point, there is a fascinating analogy between Kuhn's theory of normal and revolutionary development, and Piaget's theory of assimilation and accommodation

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in cognitive growth, respectively. This lends some support to the thesis that individual conceptual developments mirrors that of humankind as a whole (the Phylogenetic Law). It represents the application of the evolutionary maxim 'ontogenesis recapitulates phylogenesis' to the intellectual plane. This is a strongly heuristic analogy which provides a rationale for the use of history in the teaching of mathematics and science (although ultimately the analogy breaks down).

The claim is made in this and earlier issues of the newsletter that the philosophy of mathematics is currently undergoing a Kuhnian revolution, with the rationalist Euclidean paradigm of mathematics as an absolute, incorrigible and logically and hierarchically organized body of knowledge increasingly under question. A number of mathematicians, philosophers and educators are taking mathematical practice and history as central to any account of mathematics, in place of the traditional narrow focus of the philosophy of mathematics on the foundations of pure mathematical knowledge and the existence of mathematical objects. This new 'maverick' tradition, as Kitcher terms it, regards mathematics as quasi-empirical and fallible, a view which is supported by an examination of the history of mathematics.

A key question concerns the applicability of Kuhn's theory of scientific revolutions to mathematics. Is this theory applicable to mathematics? Does mathematics have revolutions? H. B. Griffiths (1987:71) questions the applicability of the notion of revolutions to mathematics, and argues 'it is doubtful whether Kuhn"s notion of a paradigm applies to mathematics in the same way that it does to other

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sciences'. Griffiths makes this point in the context of an extended review of a book on mathematics education. He argues that incompatible theories and indeed paradigms can coexist in mathematics, unlike in science, where all the theories purport to describe the same underlying objective reality. This is a point well made. Any over-facile parallel with Kuhn must fall foul on this issue. He argues that the 'overthrow' of the paradigm of Euclidean geometry by that of non-Euclidean geometry does not force mathematicians to reject it, as physics reject Newtonian theory in favour of Relativity theory.

On this basis, it must be accepted that not all major changes or developments of new theories in mathematics deserve the epithet of 'revolutionary'. Nevertheless, I still want to argue that some radical changes or global restructuring of the background epistemological and scientific context of mathematics can be described as Kuhn-type revolutions. Such changes result in a profound re-orientation of mathematics, which can lead to as much 'incommensurability' as is found in science.

Some possible candidates for mathematical revolutions are the following. First of all, infinitesimal based proofs in analysis were universally accepted, despite Berkeley's (1734) pungent criticism, until they were banished by new standards of mathematical rigour in analytic proofs introduced by Cauchy, Weierstrass and Heine in the nineteenth century. This change reflects a shift in the nature and standards of proof from those based on geometric intuition, to those of arithmetical argument (Boyer, 1968). Another chapter in this story is the re-introduction of infinitesimal based arguments in the proofs of non-standard analysis (Robinson, 1966). This reflects a further change in the nature and standards of proof accepted in analysis, from those based on arithmetic to those of axiomatic first-order logic (Lakatos, 1978; Robinson, 1967). Many other such examples can be sighted. These include in the late nineteenth century, the shift of geometric demonstrations from those relying on spatial intuition to a reliance on an axiomatic logical basis (Hilbert, 1899; Richards, 1989); the move to an axiomatic basis in arithmetic proofs (Peano, 1889); an the axiomatic rigorization of deductive logic itself (Frege, 1879).

To dwell a little longer on an example, a further example of a 'revolution in mathematics' is the shift of standards of proof in algebra in the nineteenth century. These changed dramatically from intuitive generalizations of arithmetic to a deductive axiomatic basis (Richards, 1987). The conceptual difficulties in making this transition should not be underestimated. The rigid attachment to the field-structure of number, crystallized in such laws as Peacock's 'principle of the permanence of equivalent forms' constituted what Bachelard terms an 'epistemological obstacle' to reconceptualizing the nature and epistemological basis of algebra. It took the mathematician Hamilton over ten years to overcome this obstacle in inventing his non-commutative ring of Quaternions. In doing so, he enabled a reconceptualization which heralded a revolution in the nature of algebra and the basis of proof in the subject.

This and the above examples illustrate a global restructuring of a branch of mathematics that might in my view legitimately be termed a 'revolution in mathematics'. What they illustrate is not the replacement of one mathematical theory by another. Instead they record a revolutionary shift in the background scientific and epistemological context, its constituent proof criteria and paradigms, and the associated meta-mathematical views. Changes in the background context can involve a changed pool of problems, concepts, methods, informal theories, the language and symbolism of mathematics, proof criteria and paradigms. It will also include a shift in the meta-mathematical views accepted by the mathematical community, including accepted standards for proof and definition, views of which types of inquiry are valuable, and views concerning the scope and structure of mathematics. Such changes can result in a profound re-orientation of mathematics.

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The outcome of this radical restructuring is a new or revised scientific and epistemological context for mathematics. In particular, it represents a global restructuring of the epistemology underlying mathematics, and the way truth, proof and meaning are conceptualized by the mathematical community. In the examples cited, not only did the standards of proof change. In addition the criteria for evaluating mathematical theories changed, for these themselves are largely based on the proof and definition standards employed in the formulations of the the theory. Such shifts do seem to correspond well to Kuhn's notion of scientific revolution, and would not appear to admit multiplicity as in the case with mathematical theories. In other words, like scientific theories, multiple epistemological frameworks cannot consistently coexist in mathematics, justifying the extension of Kuhn's theory to mathematics.

A number of other authors have also suggested that there are revolutions in mathematics, including Kitcher (1984), Gillies (forthcoming), and McCleary (1989). Overall, whilst agreeing with Griffiths that Kuhn's Theory of Scientific Revolutions cannot be directly applied to mathematics, my claim is that a transformation of it directed at the underlying epistemological contest, instead of just at mathematical theories, does offer a valuable insight to the history and philosophy of mathematics.

A final aside is that the above argument offers grounds for a criticism of Lakatos (1976). Lakatos' Logic of Mathematical Discovery only treats mathematical innovations at the micro-level, and does not accommodate macro-level changes such as the mathematical revolutions described above. Elsewhere, in recognition of this deficiency, I propose a Generalized Logic of Mathematical Discovery, see Social Constructivism as a Philosophy of Mathematics, forthcoming, SUNY Press.)

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