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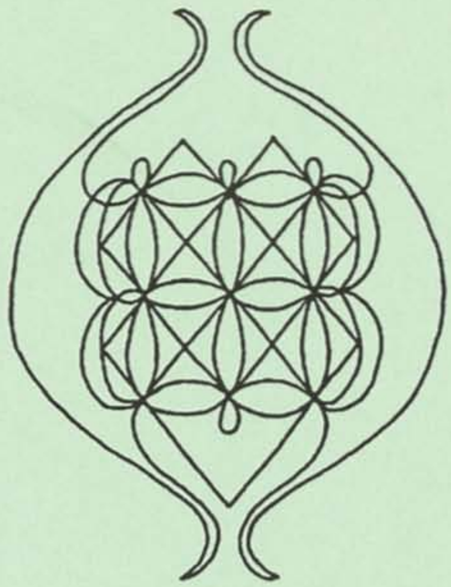

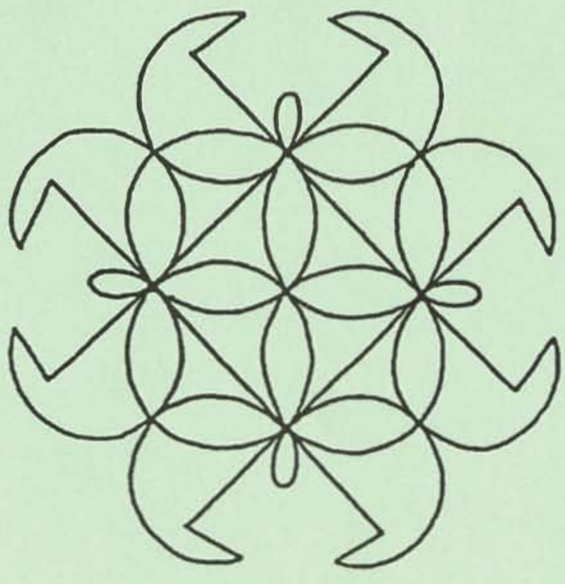

INVITATION TO AUTHORS

Essays, book reviews, syllabi and letters are welcome. Two copies, double spaced should be sent to Alvin White, HUM. MATH. NET., Harvey Mudd College, Claremont, CA 91711. If possible, avoid footnotes and put references and bibliography at the end using a consistent style. If you use a word processor please send a diskette in addition to the typed paper. *The Newsletter* is assembled using Microsoft Word 4.0 and PageMaker 4.0 on a Macintosh. It is possible, however, to convert from other word processing systems. Clean typed copy can be scanned (but not dot matrix). Your essay should have a title, your name and address, and a brief summary. Your telephone number (not for publication) would be helpful. Essays and communications may be transmitted by electronic mail to the editor at AWHITE@YMIR.BITNET.

COVER

The figures on the cover were traced in the sand by the Malekula who live in Vanuatu in the South Pacific. They are two of at least ninety figures intimately related to the Malekula religious beliefs associated with death. Their stipulation is that each figure must be traced by covering every edge once and only once and, if possible, beginning and ending at the same point. The Malekula sand-tracing tradition, involving this challenge and the systems and procedures used to trace the figures, is but one example of mathematical ideas in a traditional culture. [For the Malekula procedures used to trace the cover figures and others, see *Ethnomathematics: A Multicultural View of Mathematical Ideas*, M. Ascher, Brooks/Cole, Belmont, 1991.]

Supported by a grant from the EXXON EDUCATION FOUNDATION

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FROM NEWSLETTER #1

Dear Colleague, August 3, 1987

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by the Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings."

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other, and how they might better come to understand mathematics as a meaningful rather than an arbitrary discipline were among the ideas of the first theme.

The second theme was focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, relate discovery to verification, mathematics to science, truth to utility, and in general, to relate mathematics to the culture in which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical."

b) An appreciation for the human dimensions that motivate discovery — competition, cooperation, the urge for holistic pictures.

c) An understanding of the value judgments implied in the growth of any discipline. Logic alone never completely accounts for *what* is investigated, *how* it is investigated, and *why* it is investigated.

d) There is a need for new teaching, learning formats that will help wean our students from a view of knowledge as certain, to-be-received.

e) The opportunity for students to think like a mathematician, including a chance to work on tasks of low definition, to generate new problems and to participate in controversy over mathematical issues.

f) Opportunities for faculty to do research on issues relating to teaching, and to be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures, . . . , the network might formally support writing, team-teaching, exchanges, conferences, ...

Please send references, essays, half-baked ideas, proposals, suggestions, and whatever you think appropriate for this quarterly newsletter. Also send names of colleagues who should be added to the mailing list. All mail should be addressed to

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This issue contains some papers and excerpts of papers that were presented at the conferences.

FROM THE EDITOR

Alvin White
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Humanistic Mathematics is alive and flourishing. The CUNY Mathematics Discussion Group chose Humanistic Mathematics as the theme of their first meeting of 1989-90. Guest panelists were Anneli Lax (NYU Courant Institute), Richard Schwartz (College of Staten Island, CUNY), and Leonard Saremsky (La Guardia Community College, CUNY).

Mathematics: A Humanistic Discipline was the title of a conference organized by Mary Sapienza (Newton North High School) at Emmanuel and Simmons Colleges, Boston 26 April 1990. The second conference — Mathematics: A Humanistic Perspective was held 25 April 1991. Philip Davis (Brown University) was the keynote speaker.

The Fall-Winter 1990 issue of *Mathematics in College* published by the CUNY Mathematics Discussion Group devoted its Forum section to Humanistic Mathematics. Authors are Alvin White, Reuben Hersh, Alan Schoenfeld, Deborah Hughes Hallet, Solomon Garfunkel, Anneli Lax, Dorothy Buerk and Jorge Perez. Copies are available by writing to Editor, Mathematics in College, Instructional Resource Center, CUNY, 535 E. 80th Street, New York, NY 10021.

Contributed paper sessions on Humanistic Mathematics in Phoenix 1989 and San Francisco 1991 at the MAA-AMS meetings each lasted over eight hours. The panel presentation on Humanistic Mathematics in Louisville 1990 attracted a standing room crowd.

The Humanistic Mathematics Network-Movement is sponsoring a Poetry Reading at the annual meetings in Baltimore 1992. Dan Kalman, Elena Marchisotto and JoAnne Growney are organizing these sessions. See *FOCUS* for details.

As part of the poetic-artistic celebration, calculus text author Louis Leithold and his friend, artist-poet d'Arcy Hayman will present "The Calculus Virgin." As indicated in the short note in this newsletter, Leithold and Hayman each had a peak experience in the encounter with the other mode of thinking and perceiving.

Fewer people are still asking, "What is Humanistic Mathematics?" More people are doing Humanistic Mathematics in their research, writing, teaching and organizing (conferences, seminars, etc.). Phil Davis gives a description of Humanistic Mathematics in Newsletter #5.

In her recollections of Hassler Whitney (1907–1989) in Newsletter #4, Anneli Lax mentioned the report by L.P. Benezet of an educational experiment on the formal teaching of arithmetic that Hassler distributed widely. With permission from the Journal of the National Educational Association, Benezet's report is reprinted here in response to several requests and as a memorial to Hassler Whitney.

The variety of essays in this issue indicates the scope of Humanistic Mathematics. History, philosophy, poetry, hermeneutics, integrity, teaching, . . . , are all part of the movement. This number of the newsletter is a double issue. Please send letters, essays, reviews, etc. for inclusion in a future issue.

THE CALCULUS VIRGIN

Louis Leithold
Author of "The Calculus with Analytic Geometry," sixth edition, published by Harper Collins.

d'Arcy Hayman, artist and poet, is a former faculty member of the University of California at Los Angeles and Columbia University. For twenty years from 1960 through 1980, she was head of the International Arts Program for the United Nations in Paris.

In March of 1988, my good friend d'Arcy attended as an observer a seminar I conducted for teachers of advanced placement calculus that consisted of a discussion of the theory behind some important calculus topics as well as techniques of teaching them. I did not encourage her presence because she had no background in mathematics.

Nothing in my many years of teaching calculus had prepared me for her reaction to the language of calculus she heard at this seminar. She made associations with the literary content of this language and brought to the words the cultural references that have meaning to her as an artist and world traveler. She characterized her response to the seminar as both passionate and thrilling.

I was so excited about this revelation that the following month I gave a lecture about it to a group of four hundred calculus teachers at the annual meeting of the National Council of Teachers of Mathematics. Since then by popular demand I have repeated this lecture to other groups of calculus teachers. The enthusiasm generated by these presentations persuaded d'Arcy to make drawings of images she associates with the vocabulary of calculus. She has also written an explanation of the symbolism of each drawing.

In January, 1992 at the annual meeting of the Mathematical Association of America in Baltimore I shall conduct a session describing d'Arcy's experience at my seminar. At this session, titled THE CALCULUS VIRGIN, I will show some of d'Arcy's drawings and read the accompanying explanations. To quote d'Arcy, "Perhaps these images will bring you another view of the language of calculus to illustrate one of the wonderful things you say in that language: 'both sides exist.'"

THE TEACHING OF ARITHMETIC I
THE STORY OF AN EXPERIMENT

L. P. Benezet
Superintendent of Schools, Manchester, New Hampshire
Originally published in the November 1935 edition
of The Journal of the National Education Association

In the spring of 1929 the late Frank D. Boynton, superintendent of schools at Ithaca, New York, and president of the Department of Superintendence, sent to a number of his friends and brother superintendents an article on a modern public-school program. His thesis was that we are constantly being asked to add new subjects to the curriculum [safety instruction, health instruction, thrift instruction, and the like], but that no one ever suggests that we eliminate anything. His paper closed with a challenge which seemed to say, "I defy you to show me how we can cut out any of this material." One thinks, of course, of McAndrew's famous simile that the American elementary-school curriculum is like the attic of the Jones' house. The Joneses moved into this house fifty years ago and have never thrown anything away.

I waited a month and then I wrote Boynton an eight-page letter, telling him what, in my opinion, could be eliminated from our present curriculum. I quote two paragraphs:

> In the first place, it seems to me that we waste much time in the elementary schools, wrestling with stuff that ought to be omitted or postponed until the children are in need of studying it. If I had my way, I would omit arithmetic from the first six grades. I would allow the children to practise making change with imitation money, if you wish, but outside of making change, where does an eleven-year-old child ever have to use arithmetic?
>
> I feel that it is all nonsense to take eight years to get children thru the ordinary arithmetic assignment of the elementary schools. What possible needs has a ten-year-old child for a knowledge of long division? The whole subject of arithmetic could be postponed until the seventh year of school, and it could be mastered in two years' study by any normal child.

Having written the letter, I decided that if this was my real belief, then I was falling down on the job if I failed to put it into practise. At this time I had been superintendent in Manchester for five years, and I had already been greatly criticized because I had dropped practically all of the arithmetic out of the curriculum for the first two grades and the lower half of the third. In 1924 the enrollment in the first grade was 20 percent greater than the enrollment in the second, because, roughly, one-fifth of the children could not meet the arithmetic requirements for promotion into the second grade and so were forced to repeat the year. By 1929 the enrollment of the first grade was no greater than that of the third.

Meanwhile, I was distressed at the inability of the average child in our grades to use the English language. If the children had original ideas, they were very helpless about translating them into English which could be understood. I went into a certain eighth-grade room one day and was accompanied by a stenographer who took down, verbatim, the answers given me by the children. I was trying to get the children to tell me, in their own words, that if you have two fractions with the same numerator, the one with the smaller denominator is the larger. I quote typical answers.

"The smaller number in fractions is always the largest."

"If the numerators are both the same, and the denominators one is smaller than the one, the one that is the smaller is the larger."

"If you had one thing and cut it into pieces the smaller piece will be the bigger. I mean the one you could cut the least pieces in would be the bigger pieces."

"The denominator that is smallest is the largest."

"If both numerators are the same number, the smaller denominator is the largest — the larger — of the two."

"If you have two fractions and one fraction has the smallest number at the bottom. it is cut into pieces and one has the more pieces. If the two fractions are equai, the bottom number was smaller than what the other one in the other fraction. The smallest one has the largest number of pieces — would have the smallest number of pieces, but they would be larger than what the ones that were cut into more pieces."

The average layman will think that this must have been a group of half-wits, but I can assure you that it is typical of the attempts of fourteen-year-old children from any part of the country to put their ideas into English. The trouble was not with the children or with the teacher; it was with the curriculum. If the course of study required that the children master long division before leaving the fourth grade and fractions before finishing the fifth, then the teacher had to spend hours and hours on this work to the neglect of giving children practise in speaking the English language. I had tried the same experiment in schools in Indiana and in Wisconsin with exactly the same result as in New Hampshire.

In the fall of 1929 I made up my mind to try the experiment of abandoning all formal instruction in arithmetic below the seventh grade and concentrating on teaching the children to read, to reason, and to recite — my new Three R's. And by reciting I did not mean giving back, verbatim, the words of the teacher or of the textbook. I meant speaking the English language. I picked out five rooms — three third grades, one combining the third and fourth grades, and one fifth grade. I asked the teachers if they would be willing to try the experiment. They were young teachers with perhaps an average of four years' experience. I picked them carefully, but more carefully than I picked the teachers, I selected the schools. Three of the four schoolhouses involved [two of the rooms were in the same building] were located in districts where not one parent in ten spoke English as his mother tongue. I sent home a notice to the parents and told them about the experiment that we were going to try, and asked any of them who objected to it to speak to me about it. I had no protests. Of course, I was fairly sure of this when I sent the notice out. Had I gone into other schools in the city where the parents were high school and college graduates, I would have had a storm of protest and the experiment would never have been tried. I had several talks with the teachers and they entered into the new scheme with enthusiasm.

The children in these rooms were encouraged to do a great deal of oral composition. They reported on books that they had read, on incidents which they had seen, on visits that they had made. They told the stories of movies that they had attended and they made up romances on the spur of the moment. It was refreshing to go into one of these rooms. A happy and joyous spirit pervaded them. The children were no longer under the restraint of learning multiplication tables or struggling with long division. They were thoroly [sic] enjoying their hours in school.

At the end of eight months I took a stenographer and went into every fourth-grade room in the city. As we have semi-annual promotions, the children who had been in the advanced third grade at the time of the beginning of the experiment, were now in the first half of the fourth grade. The contrast was remarkable. In the traditional fourth grades when I asked children to tell me what they had been reading, they were hesitant, embarrassed, and diffident. In one fourth grade I could not find a single child who would admit that he had committed the sin of reading. I did not have a single volunteer, and when I tried to draft them, the children stood up, shook their heads, and sat down again. In the four experimental fourth grades the children fairly fought for a chance to tell me what they had been reading. The hour closed, in each case, with a dozen hands waving in the air and little faces crestfallen, because we had not gotten around to hear what they had to tell.

For some years I had noted that the effect of the early introduction of arithmetic had been to dull and almost chloroform the child's reasoning faculties. There was a certain problem which I tried out, not once but a hundred times, in grades six, seven, and eight. Here is the problem: "If I can walk a hundred yards in a minute [and I can], hour many miles can I walk in an hour, keeping up the same rate of speed?"

In nineteen cases out of twenty the answer given me would be six thousand, and if I beamed approval and smiled, the class settled back, well satisfied. But if I should happen to say, "I see. That means that I could walk from here to San Francisco and back in an hour" there would invariably be a laugh and the children would look foolish.

I, therefore, told the teachers of these experimental rooms that I would expect them to give the children much practise in estimating heights, lengths, areas, distances, and the like. At the end of a year of this kind of work, I visited the experimental room which had had a combination of third- and fourth-grade children, who now were fourth and fifth graders. I drew on the board a rough map of the western end of Lake Ontario, the eastern end of Lake Erie, and the Niagara River. I asked them to guess what it was, and was not surprised when they identified the location. I then labeled three spots along the river with the letters "Q," "NF," and "B." They identified Niagara Falls and Buffalo without any difficulty, but were puzzled by the "Q." Some thought it was Quebec but others knew it was not. I finally told them that it was Queenstown. I then drew a cross section of the falls, showing the hard layer of rock above and the soft layer eating out underneath, and they told me what it was and why it was that the stone was falling, little by little, from the edge. They told me how this process was going on. I then made the statement that in 1680, when white men had first seen the falls, the falls were 2500 feet lower down than they are at present. I then asked them at what rate the falls were retreating up-stream. These children, who had had no formal arithmetic for a year but who had been given practise in thinking, told me that it was 250 years since white men had first seen the falls and that, therefore, the falls were retreating upstream at the rate of ten feet a year. I then remarked that science had decided that the falls had originally started at Queenstown, and, indicating that Queenstown was now ten miles down the river, I asked them how many years the falls had been retreating. They told me that if it had taken the falls 250 years to retreat about a half mile, it would be at the rate of 500 years to the mile, or 5000 years for the retreat from Queenstown. The map had been drawn so as to show the distance from Niagara Falls to Buffalo as approximately twice the distance from Queenstown to Niagara Falls. Then I asked these children whether they had any idea how long it would be before the falls would retreat to Buffalo and drain the lake. They told me that it would not happen for another ten thousand years. I asked them how they got that and they told me that the map indicated that it was twenty miles from Niagara Falls to Buffalo, or thereabouts, and that this was twice the distance from Queenstown to Niagara Falls!

It so happened that a few days after this incident I was visiting a large New England city with five of my brother superintendents. Our host was interested in my description of this incident and suggested that I try the same problem on a fifth grade in one of his schools. With the other superintendents as audience, I stood before an advanced fifth grade in what was known as the Demonstration School, the school used for practise teaching and to which visitors were always sent.

The home superintendent: Boys and girls, would you like to have Superintendent Benezet of Manchester, New Hampshire, ask you some questions about Niagara Falls?

The children express pleasure at the idea.

Mr. Benezet: [Drawing a map on the board] Children, what is this that I have drawn on the blackboard?

Children: The Great Lakes.

Mr. B.: Good. What lakes?

A child: Lake Ontario and Lake Erie.

Mr. B.: Good. What is this river?

Child: The St. Laurence River.

Mr. B.: That is really correct. It is the St. Laurence River. But they call it by a different name here. They call it the Niagara River. What have you heard in connection with the Niagara River?

Another child: Niagara Falls are there.

Another child: Niagara Falls are connected with Niagara River.

Mr. B.: Oh! How are they connected?

Child: The water trickles down the Falls and goes into the Niagara River.

Mr. B.: I should call that quite a trickle. Have any of you children seen Niagara Falls?

Three raise their hands.

Mr. B.: How high are the falls? Have you any idea? Are they higher than this room?

Children: Yes [dubiously].

Mr. B.: Well, how high is this room?

Its height is guessed anywhere from 11 feet to 40 feet. The room is actually about 16 feet high. The question of the height of the falls is finally dropped.

Mr. B.: Well, never mind how high the falls are. On this map here I have indicated one spot and marked it "NF." and another spot and marked it "B." What does "NF" mean?

Children: Niagara Falls.

Mr. B.: What does "B" stand for?

Another child: Bay.

Mr. B.: No. Remember that Niagara Falls is not only the name of the Falls, but the name of a city.

Child: Baltimore.

After considerable pause, the home superintendent, in the back of the room, tells the class that the name of the city is also the name of an animal.

Child: Buffalo.

Mr. B.: Yes. Now there is another town here that I am going to mark "Q." It is not Quebec; it is Queenstown. People who have studied this carefully tell us that once upon a time the falls were at Queenstown. Tell me now. What does it mean if I say that I show you the cross section of an apple?

Class is uncertain.

Mr B.: Suppose that you cut an apple in half with a knife. What do I show you if I hold up one-half?

Child: Half the apple.

Another child: The core of the apple.

Third child: The inside of an apple.

Mr. B.: Tell me. Is the word "section" a new word to the majority of you?

Enthusiastic chorus of "No."

Mr. B.: Well, a cross-section of an apple means a cut right thru an apple. Why have I said this to you?

Meantime he has drawn on the board a cross-section of Niagara Falls.

Child: Because that is a cross-section of the falls.

Mr. Benezet now explains the two kinds of rock and asks which is the harder. They finally decide that the rock above is the harder. He then shows how the underneath rock rotted away, and that finally there was a shelf of hard rock overhanging. This became too heavy and fell off; and the falls have thereby moved back some ten feet.

Mr. B.: Now, when white men first saw the falls in 1680 [placing this date on the board], the falls were further down the river than they are now, and it is estimated that since that time they have moved back upstream about 2500 feet. Now how long ago was it that white men first saw the falls?

Child: Four hundred years.

Another child: Two hundred years.

Third child: Three hundred years.

Guesses range anywhere between 110 years and 450 years. One boy says it was about the time that Columbus sailed to America; another says that it was about the time of the Pilgrims and the Puritans.

Mr. B.: Well, how are we going to find out?

General bewilderment for a while. Finally:

Child: Take 1930 and subtract it from 1680.

Mr. B.: Fine.

He writes on the blackboard: 1680
<u>1930</u>

Mr. B.: Now take a look and tell me how many years that was. See if you can tell me before we subtract it, figure by figure.

It is to be noted that not one child called attention to the wrong position of the two sets of figures. They guess 350 years, 200 years, 400 years.

Mr. B.: Well, let's subtract it figure by figure.

Child: Zero from 0 equals 0. Three from 8 equals 5. Nine from 6 equals 3. Three hundred fifty years is the answer.

Mr. B.: How many think that 350 years is right?

About two-thirds of the hands go up. Finally two or three think that it is wrong.

Mr. B.: All right, correct it.

Child: It should have been 9 from 16 equals 7.

Mr. Benezet thereupon puts down 750 for the answer. When he asks how many in the room agree that this is right, practically every hand is raised. By this time the local superintendent was pacing the door at the rear of the room and throwing up his hands in dismay at this showing on the part of his prize pupils. After a time, as Mr. Benezet looks a little puzzled, the children gradually become a little puzzled also. One little girl, Elsie Miller, finally comes to the board, reverses the figures, subtracts, and says the answer is 250 years.

Mr. B.: All right. If the falls have retreated 2500 feet in 250 years, how many feet a year have the falls moved upstream?

Child: Two feet.

Mr. Benezet registers complete satisfaction and asks how many in the class agree. Practically the whole class put hands up again.

Mr. B.: Well, has anyone a different answer?

Child: Eight feet.

Another child: Twenty feet.

Finally Elsie Miller again gets up, and says the answer is ten feet.

Mr. B.: What? Ten feet? [Registering great surprise]

The class, at this, bursts into a roar of laughter. Elsie Miller sticks to her answer, and is invited by Mr. Benezet to come up and prove it. He says that it seems queer that Elsie is so obstinate when everyone is against her. She finally proves her point, and Mr. Benezet admits to the class that all the rest were wrong.

Mr. B.: Now, what fraction of a mile is it that the falls have retreated during the last 250 years?

Children guess 3/2, 3/4, 2/3, 1/20, 7/8 — everything except 1/2. The bell for dismissal rings and the session is over.

It will be noted that the local superintendent gave them a little hint at the outset, that was not given to the Manchester children, when he said, "Niagara Falls." They were prepared to identify my map. Also, the Manchester children who had not learned tables but had talked a great deal about distances and dimensions, recognized the fact that 2500 feet was about a half a mile, while the children in the larger city who were fresh from their tables, had little conception of the distance.

I was so delighted with the success of the experiment so far that in the fall of 1930 we started six or seven other rooms along the same line. The formal arithmetic was dropped and emphasis was placed on English expression, on reasoning, and estimating of distances.

One day I tried an experiment having to do with English expression. I hung before a 7-B class a copy of a painting by Frederick Waugh, representing a polar bear floating on a small berg of ice. This was a traditionally taught room in a school where there were very few children of foreign extraction. I asked the children to write anything which they felt inspired to put down as a result of seeing the picture. Three-quarters of an hour later I hung the same picture before another 7-B grade, one of the experimental groups this time, in a school where not more than three children in the room came from homes where English was the language of the parents. I then called the seventh-grade teachers of the city together and read them the ten best papers from one room and the ten best from the other. I asked them if they saw any difference. One teacher remarked that one group was about a year and a half or two years ahead of the other in maturity of expression, and there was general assent to this statement. I said to the teachers, "If I should tell you that one group came from the 'A' school and the other from the 'B,' from which school would you guess the better group of papers came?"

"Oh, the 'A' school, undoubtedly," said they, naming the school whose patrons speak English in their homes.

"Well," I said, "it was just the other way," and there was a murmur of incredulity. Then we analyzed the papers and counted the number of adjectives used by the traditionally taught pupils. There were forty all told: nice, pretty, blue, green, cold, etc. We then counted the adjectives used by the other group [the number of papers was approximately the same] and we found 128, including magnificent, awe-inspiring, unique, majestic, etc. The little Greeks, Armenians, Poles, and French-Canadians had far surpassed their English-speaking opponents.

I next tried a rather similar test. I hung the same picture — a landscape representing a river scene in the vicinity of Manchester — before ten different fifth-grade rooms. Five of them had been brought up under the old traditional curriculum and five of them were of the experimental group. It was the same story: the experimental rooms far excelled the others in fluency of expression. They used words that the others had never heard of. Nevertheless, when we came to test the papers for spelling, the poorest of the experimental rooms exactly tied the record of the best of the traditional groups. The most surprising result came in a certain room in which there was housed a 5-B grade and a 5-A. The younger pupils, the 5-B's, had been brought up under the experimental curriculum, without arithmetic, while the other half of the room were traditional. The 5-A's made the poorest record of all the ten groups while the 5-B's, the younger group, were next to the top. For four months they had been taught by the same teacher but by different methods.

Now we were ready to experiment on a much larger scale. By the fall of 1932 about one-half of the third-, fourth-, and fifth-grade rooms in the city were working under the new curriculum. Some of the principals were a little dubious and asked permission to postpone formal arithmetic until the beginning of the sixth grade instead of the beginning of the seventh. Accordingly, permission was given to four schools to begin the use of the arithmetic book with the 6-B grade. About this time Professor Guy Wilson of Boston University asked permission to test our program. One of our high school teachers was working for her master's degree at Boston University and as part of her work he assigned her the task of giving tests in arithmetic to 200 sixth grade children in the Manchester schools. They were divided fairly evenly, 98 from experimental rooms and 102 from the traditional groups, or something like that. These were all sixth graders. Half of them had had no arithmetic until beginning the sixth grade and the other half had had it thruout [sic] the course, beginning with the 3-A. In the earlier tests the traditionally trained people excelled, as was to be expected, for the tests involved not reasoning but simply the manipulation of the four fundamental processes. By the middle of April, however, all the classes were practically on a par and when the last test was given in June, it was one of the experimental groups that led the city. In other words these children, by avoiding the early drill on combinations, tables, and that sort of thing, had been able, in one year, to attain the level of accomplishment which the traditionally taught children had reached after three and one-half years of arithmetical drill.

THE TEACHING OF ARITHMETIC II
THE STORY OF AN EXPERIMENT

L. P. Benezet
Superintendent of Schools, Manchester, New Hampshire
Originally published in the December 1935 edition of The Journal of the National Education Association

This is the second instalment of an article describing an experiment which has been carried out in Manchester, New Hampshire, since 1929. In the preceding section, which appeared in the November JOURNAL, *Mr. Benezet explained that: In some schools of Manchester, the only arithmetic in the first six grades was practise in estimating heights, areas, and the like; formal arithmetic was not introduced until the seventh grade. In tests given to both the traditionally and experimentally taught groups, it was found that the latter had been able in one year to attain the level of accomplishment which the traditionally taught children had reached after three and one-half years of arithmetic drill. In addition, because the teachers in the experimental group had had time to concentrate on teaching the children to "read, reason, and recite," these children developed more interest in reading, a better vocabulary, and greater fluency in expression.*

In the fall of 1933 I felt that I was now ready to make the big plunge. I knew that I could defend my position by evidence that would satisfy any reasonable person. Accordingly, a committee of our principals drew up a new course of study in arithmetic. I would have liked to go the whole route and drop out all the arithmetic until we reached the seventh grade, for we had proved, in the case of four rooms, that this could be done without loss, but the principals were more cautious than I was and I realized, too, that I would now have to deal with the deeply rooted prejudices of the educated portion of our citizens. Therefore, a compromise was reached. Accordingly, on September 1, 1933, we handed out the following course of study in arithmetic:

Grade I — There is no formal instruction in arithmetic. In connection with the use of readers, and as the need for it arises, the children are taught to recognize and read numbers up to 100. This instruction is not concentrated into any particular period or time but comes in incidentally in connection with assignments of the reading lesson or with reference to certain pages of the text.

Meanwhile, the children are given a basic idea of comparison and estimate thru [sic] the understanding of such contrasting words as: more, less; many, few; higher, lower; taller, shorter; earlier, later; narrower, wider; smaller, larger; etc.

As soon as it is practicable the children are taught to keep count of the date upon the calendar. Holidays and birthdays, both of members of the class and their friends and relatives, are noted.

Grade II — There is no formal instruction in arithmetic.

The use of comparatives as taught in the first grade is continued.

The beginning is made in the telling of time. Children are taught to recognize the hours and half hours.

The recognition of page numbers is continued. The children are taught to recognize any numbers that they naturally encounter in the books used in the second grade. If any book used in this grade contains an index, the children are taught what it means and how to find the pages referred to. Children will naturally pick up counting in the course of games which they play. They will also easily and without formal instruction learn the meaning of "half," "double," "twice," or "three times." The teacher will not devote any formal instruction to the meaning of these terms if the children do not pick them up naturally and incidentally.

To the knowledge of the day of the month already acquired is added that of the name of the days of the week and of the months of the year.

The teacher learns whether the children come in contact with the use of money at all in their life outside the school. If so, the meaning of "penny," "nickel," "dime," and "dollar" is taught. In similar fashion, and just incidentally, the meaning and relation of "pint" and "quart" may be taught.

Grade III — While there is no formal instruction in arithmetic, as the children come across numbers in the course of their reading, the teacher explains the significance of their value.

Before the year is over the children will be taught that a "dime" is worth 10 cents, and a "dollar" 10 dimes or 100 cents, a "half dollar" 5 dimes or 50 cents, etc. They will learn that 4 quarters, or 2 halves, are worth as much as one dollar.

They add to their knowledge of hours and half hours the ability to tell time at any particular moment. *The first instruction omits such forms as 10 minutes to 4; or 25 minutes to 3.* They are first taught to say 3:50; 2:35; etc. In this connection they are taught that 60 minutes make one hour.

It is now time, also, for them to know that 7 days make a week and that it takes 24 hours to make a day. They are also taught that there are 12 months in a year and about 30 days in a month.

The instruction in learning to count keeps pace with the increasing size of the textbooks used and the pages to which it is necessary to refer. Games bring in the recognition of numbers. Automobile license numbers are a help in this respect. For example, the teacher gives orally the number of a car [of not over four digits] which most of the children are likely to see, and later asks for the identification of the car. Children are encouraged to bring to class their own house numbers, automobile license numbers, or telephone numbers and invite the class to identify them.

The use of comparisons is continued, especially those involving such relations as "half," "double," "three times," and the like.

Grade IV — Still there is no formal instruction in arithmetic.

By means of foot rules and yard sticks, the children are taught the meaning of inch, foot, and yard. They are given much practise in estimating the lengths of various objects in inches, feet, or yards. Each member of the class, for example, is asked to set down on paper his estimate of the height of a certain child, or the width of a window, or the length of the room, and then these estimates are checked by actual measurement.

The children are taught to read the thermometer and are given the significance of 32 degrees, 98.6 degrees, and 212 degrees.

They are introduced to the terms "square inch," "square foot," and "square yard" as units of surface measure.

With toy money [or real coins, if available] they are given some practise in making change, in denominations of 5's only. All of this work is done mentally. Any problem in making change which cannot be solved without putting figures on paper or on the blackboard is too difficult and is deferred until the children are older.

Toward the end of the year the children will have done a great deal of work in estimating areas, distances, etc., and in checking their estimates by subsequent measuring. The terms "half mile," "quarter mile," and "mile" are taught and the children are given an idea of how far these different distances are by actual comparisons or distances measured by automobile speedometer.

The table of time, involving seconds, minutes, and days, is taught before the end of the year. Relation of pounds and ounces is also taught.

Grade V-B — There still is no formal instruction in arithmetic except that the children are asked to count by 5's, 10's, 2's, 4's, and 3's. This work is done mentally at first with no written figures before them, either on paper or on the blackboard. This leads naturally to the multiplication tables of 5's, 10's, 2's, 4's, and 3's which, in this order, are given to the children before the end of the semester.

With toy money, or with real coins if available, the children practise making change in amounts up to a dollar, involving, this time, the use of pennies.

The informal work of previous grades in the estimating of distance, area, time, weights, measure of capacity, and the like, is continued. The ability to guess and estimate by games is developed. Each child in the class writes his estimate before these are checked up by actual measurement.

The children compare the value of fractions and discover for themselves that 1/3 is smaller than 1/2 and greater than 1/4; i. e., that the larger the denominator the smaller the fraction. This is illustrated concretely or by pictures.

Toward the end of the semester the children are given the book, *Practical Problems in Mental Arithmetic,* grade IV. The solution of these problems involves a knowledge of denominations which the children have not had and the use of tables and combinations which have not yet been taught to them. Nevertheless, children with a natural sense of numbers will be able to give the correct answers. *The teacher will not take time to explain by formula or tables the solution of any problem to those who do not grasp it quickly and naturally.* The purpose of the mental arithmetic book is to stimulate quick thinking and to get children away from the old-time method of using the fingers to do the work of the head. If some of the children do not grasp the problems easily and quickly, the teacher simply passes on, knowing that the power to reason will probably develop in them a year or two subsequently. The one thing which is avoided is that children shall get the idea that a fixed method or formula can be used as a substitute for thinking. The problems listed under September, October, and November are covered before the end of the semester.

Grade V-A — The children are asked to count by 6's, 7's, 8's, and 9's. *This work is done mentally without written tables before them, either upon paper or on the blackboard.* After a time this leads naturally to the multiplication tables of 6's, 7's, 8's, and 9's. The attention of the children is called to the fact that in the table of 9's the second digit is always diminished by one [18, 27, 36, etc.] and the reason is explained that adding 9 is the same as adding 10 and taking away 1. In similar fashion it is shown that adding 8 is the same as adding 10 and taking away 2, so that in the table of 8's the second digit of each successive product is 2 less than the second digit of the product above it [48, 56, 64]. In similar fashion it is shown that adding 7 is the same as adding 10 and taking away 3. After the tables have been learned the teacher makes sure that the children know the products in any order; i.e., that it is not necessary for the child to start at the beginning of the table and run thru [sic] until he reaches the product which he is asked to give. They learn that 2 times 3 is always equal to 3 times 2.

Children are given a little idea about the relative value of the fractions 1/2, 1/4, 1/5, and 1/10. Concrete examples assist in this; e.g., when the children remember that 2 quarters are worth one half dollar, it is easy to show them that twice 1/4 equals 1/2 or that twice 1/10 equals 1/5.

The problems listed under December to June, inclusive, in the book *Practical Problems in Mental Arithmetic,* grade IV, are covered in the course of the semester. If the children do not grasp the problem quickly and easily, the teacher does not stop to explain the method or prescribe any formula for solution. Of course as new terms occur in the problems [pecks, gallons, etc.] the teacher explains, incidentally, what they mean.

Grade VI-B [*20 to 25 minutes a day*] — At this grade formal work in arithmetic begins. The first 108 pages of the Strayer-Upton Arithmetic, book III are used as a basis.

The processes of addition, subtraction, multiplication, and division are taught. Care is taken to avoid purely mechanical drill. Children are made to understand the reason for the processes which they use. This is especially true in the case of subtraction. Problems involving long numbers which would confuse them are avoided. Accuracy is insisted upon from the outset at the expense of speed or the covering of ground, and where possible the processes are mental rather than written. *Before starting on a problem in any one of these four fundamental processes, the children are asked to estimate or guess about what the answer will be and they check their final result by this preliminary figure.* The teacher is careful not to let the teaching of arithmetic degenerate into mechanical manipulation without thought.

Fractions and mixed numbers are taught in this grade. Again care is taken not to confuse the thought of the children by giving them problems which are too involved and complicated.

Grade VI-A [*25 minutes a day*] — The work of this grade is based upon Chapter II [pages 109 to 182] of the Strayer-Upton Arithmetic, book III, and the first 50 pages of book IV.

Multiplication tables and tables of denominate numbers, hitherto learned, are reviewed. The teacher keeps in mind that the objectives to be gained are first of all reasoning and estimating, rather than mere ease in manipulation of numbers.

Again, as in the previous grade, the children before beginning any problem make an estimate [individually] as to what the answer ought to be and check the final result by the preliminary guess.

Grade VII-B [*25 minutes a day*] — The assignment in the text is the latter part of Strayer-Upton, book IV, beginning with page 51.

Tables of denominate numbers, including United States money, found in the rear of book IV are reviewed. In addition to the table of linear measure, as given, it is taught that there are 1760 yards in a mile, 880 yards in a half mile, 440 yards in a quarter mile, etc.

The teacher will omit any problems in the book which, because of the length of numbers involved, cause the child in using the four fundamental processes to lose sight of the reasoning process which, after all, is the main purpose of the problem.

There is a great deal of work in mental arithmetic, involving the solution of problems without reference to paper or blackboard. This is far more important than than accuracy in the four fundamental processes.

Grade VII-A [*30 minutes a day*] — The assignment in the text is the first one hundred pages of Strayer-Upton, book V, omitting the following pages: 1–10, 28, 71–77. Wherever possible the work is done mentally.

Note that most of the pages omitted in this grade reoccur in book VI.

The practise of estimating the probable answer and checking the result with this preconceived estimate is constantly followed.

Again the teachers remember that ability to reason the problem correctly is far more important than errorless manipulation of the four fundamental processes.

Grade VIII-B [*30 minutes a day*] — The assignment in the test is the latter part of Strayer-Upton, book V, beginning with page 101 [omitting pages 127–34] and the first 32 pages of book VI.

The practise of making a preliminary estimate or an approximation at the answer before attacking the problem is continued. The ability to guess closely and promptly what the answer will be is one of the most important objectives to be gained from the study of arithmetic.

Tables of denominate numbers are kept fresh in the minds of the children. The practise of estimating lengths, heights, and areas of familiar objects and the checking up by actual measurement is constantly kept up.

Grade VIII-A [*30 minutes a day*] — The text for the grade is book VI of the Strayer-Upton series, beginning with page 35 and omitting the following pages: 36, 46–8, 57–9, 80–2, 92–3, 104, 158–188, 194, 203–4, 206–8.

The work of this grade must necessarily be a summary of everything that has been learned in arithmetic, but, above all, the ability to approximate and estimate in advance the probable answer is kept as the important objective.

The children are shown reasons for the various processes employed; why it is that a correct answer is obtained in the division of fractions by inverting the divisor and multiplying, etc. The ability to read problems intelligently and explain how they should be attacked is far more important than the ability to add large columns of figures without an error.

The teacher will bear in mind that a great deal of work in mensuration [pages 88 to 100 inclusive] will be difficult for some pupils to understand. Of course this work is really using geometrical formulas without giving the geometrical reasons why they work, and some children will be unable to grasp the meaning of it all. It will be found worthwhile to have models in class and to perform experiments like filling a cylinder with water from 3 times the contents of a cone of equal base and altitude, etc.

Again as much of the work as possible is done mentally. Problems are chosen to illustrate principles and give practise in reasoning rather than practise in the manipulation of large figures or complicated fractions.

THE TEACHING OF ARITHMETIC III
THE STORY OF AN EXPERIMENT

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This is the third and final instalment of an article by Superintendent L. P. Benezet, in which he describes an experiment in arithmetic in the Manchester, New Hampshire, schools. The first instalments [November 1935, p 241–4 and December 1935, p 301–3] have aroused many favorable comments. William McAndrew calls the material "powerful good reading, a scientific article free of the common dullness of such." Helen Ives Schermerhorn, of New Jersey, writes that upon returning to teach in junior high-school after many years in the adult education field, she "was appalled at the changes which had taken place, the great number of new activities which had developed, each good in itself, but nevertheless cluttering up the time of the children. The weakness in English seemed inexcusable; too little time had been given to its mastery. I hope great things from the influences of Mr. Benezet's article." A letter from C. E. Birch, superintendent of schools, Lawrence, Kansas, indicates that the Lawrence schools have been revising the arithmetic program for the past two years. Mr. Birch has recommended the discussion in faculty meetings of the Benezet articles and their possible application in the light of the local situation.

Is your school making similar use of these articles? It would be an interesting thing to call some of the leading citizens in your community together around the table and read the articles to them to see what their attitude would be.

It must be understood that I knew very well what my hardest task was ahead. I had to show my more conservative teachers what we were trying to do and convert them to the idea that it could be done. I went into room after room, day after day, testing, questioning, giving out examples.

We had visitors. Two Massachusetts superintendents, a superintendent of a large Massachusetts city with five of his principals, and two instructors in the Boston Normal School came. They saw what we were trying to do and were surprised at the ability to reason and to talk, shown by children whose minds had not been chloroformed by the dull, drab memorizing of tables and combinations. But there were murmurs thruout [sic] the city. It finally broke out in a board meeting. A motion was made that we throw out the new course of study in arithmetic and go back to the old. It was defeated by a vote of nine to four, but a committee of three was appointed to study the problem carefully. Taking with me two members of the committee and a stenographer, I visited four different schools in our own city and three in a city not thirty miles away.

The most convincing test was in connection with the problem which I tried out in not less than six different rooms. Four of these rooms were made up of children who learned their arithmetic in the old formal way, whereas the other two were groups who had been taught according to the new method. In every case it was an advanced fifth grade, within one month of promotion to the 6-B.

I give verbatim accounts of two of these recitations, the first from a traditional room and the other from one of the experimental groups. I drew on the board a little diagram and spoke as follows: "Here is a wooden pole that is stuck in the mud at the bottom of a pond. There is some water above the mud and part of the pole sticks up into the air. One-half of the pole is in the mud; 2/3 of the rest is in the water; and one foot is sticking out into the air. Now, how long is the pole?"

First child: "You multiply 1/2 by 2/3 and then you add one foot to that."

Second child: "Add one foot and 2/3 and 1/2."

Third child: "Add the 2/3 and 1/2 first and then add the one foot."

Fourth: "Add all of them and see how long the pole is."

Next child: "One foot equals l/3. Two thirds divided into 6 equals 3 times 2 equals 6. Six and 4 equals 10. Ten and 3 equals 13 feet."

You will note that not one child saw the essential point, that 1/2 the pole was buried in the mud and the other half of it was above the mud and that 1/3 of this half equaled one foot. Their only thought was to manipulate the numbers, hoping that somehow they would get the right answer. I next asked, "Is there anybody who knows some way to get the length?"

Next child: "One foot equals 3/3. Two-thirds and 1/2 multiplied by 6."

My next question was, "Why do you multiply by 6?"

The child, making a stab in the dark, said, "Divide."

It may be that he detected in my voice some stress on the word "multiply." I then gave them a hint which, had they been able to reason at all, should have shown them how to solve the problem. "How much of the pole is above the mud?" said I. The answer which I had hoped for was, of course, "One-half of it is above the mud."

The first child answered: "One foot and 2/3."

I looked dubious, so the second child said, "One foot and 1/3."

I then said, "I will change my question. How much of the pole is in the mud?"

"Two-thirds," said the first child.

"One-half," said the second.

"One-half," said the third.

"Then how much of the pole is above the mud," said I, thinking that now the answer was plainly indicated as one-half.

"Two-thirds," said the next child.

"One foot and 2/3," said the next.

"One-half of the pole is in the mud," said I. "Now, how long is the pole?" and the answers given were "Two feet." "One and one-half feet." "One-half foot." "One foot." "One foot." "One foot," and I gave it up.

I gave the same problem the same week to a fifth grade in our city which had been brought up under our new curriculum, with no formal drill in addition, multiplication, and division of big numbers but with much mental work in reasoning. I drew the diagram again and said, "Here is a pond with a rock bottom and mud and water, with a pole sticking in the mud. One-half of the pole is in the mud; 2/3 of the rest of the pole is in the water; one foot of the pole sticks up in the air above the water. How long is the pole? How would you go to work to do that problem?"

First child: "You would have to find out how many feet there are in the mud."

"And what else?" said I.

Another child: "How many feet in the water and add them together."

"How would you go to work and get that?" said I to another child.

"There are 3 feet in a yard. One yard is in the mud. One yard equals 36 inches. If 2/3 of the rest is in the water and one foot in the air [one foot equals twelve inches] the part in the water is twice the part in the air so that it must be 2 feet or 24 inches. If there are 3 feet above the mud and 3 feet in the mud it means that the pole is 6 feet or 72 inches long. Seventy-two inches equals 2 yards."

It amazed me to see how this child translated all the measurements into inches. As a matter of fact, to her, the problem was so simple and was solved so easily, that she could not believe that she was doing all that was necessary in telling me that the pole was 6 feet long. She had to get it into 72 inches and 2 yards to make it hard enough to justify my asking such a problem.

The next child went on to say, "One-half of the pole is in the mud and 1/2 must be above the mud. If 2/3 is in water, then 2/3 and one foot equals 3 feet, plus the 3 feet in the mud equals 6 feet."

The problem seemed very simple to these children who had been taught to use their heads instead of their pencils.

The committee reported to the board and the board accepted their report, saying that the superintendent was on the right track. They merely suggested that, to quiet the outcry of some of the parents, the teaching of the tables should be begun a little earlier in the course.

The development of the ability to reason is one of the big results of the new course of study in arithmetic. Not long ago, hearing that a complaint had been made by the mother of a child in a 5-B room, regarding the teaching of arithmetic, I visited the room with the principal and tried to discover just what the youngsters could and could not do. I gave them several problems to test their ability to do mental arithmetic, and was surprised at the accuracy and speed with which they answered me. I then tried them on a problem which involved a little reasoning. I drew a picture of two faucets and of a pail placed beneath them. Stating that either one of the faucets could fill the pail alone in two minutes, I asked how long it would take to fill it if the two were running at the same time. Confidently expecting that the children would tell me four minutes, I was much gratified to receive the answer, one minute, from three-fourths of the class. I next changed the problem by stating that I would replace one of the faucets by a smaller one, which could fill the pail in four minutes. I then asked about how long it would take to fill the pail, if the two faucets ran together. A few told me three minutes, but the great majority guessed between one minute and two, the popular answer being about a minute and a half. I next asked what part of the pail would be filled at the end of one minute, and the children told me, without any difficulty, that it would be three-quarters full. My next question was, "How long exactly would it take then, to fill the pail?" The first child that I called upon gave me the correct answer, one minute and twenty seconds. The principal expressed his astonishment and asked me to try the same problem on the eighth grade. I did so. These children, brought up under the old method of formal arithmetic, did not do nearly as well as did their younger brothers and sisters.

I have recently tried, in several parts of the city, a test involving five simple problems. Here it is:

[1] Two boys start out together to race from Manchester to West Concord, a distance of 20 miles. One makes 4 miles an hour and the other 5 miles an hour. How long will it be before both have reached West Concord?

[2] A man can row 4 miles an hour in still water. How long will it take him to row from Hill to Concord [24 miles one way] and back, if the river flows south at the rate of 2 miles an hour?

[3] The same man again starts rowing from Hill to Concord in the spring when the water is high and the current is twice as swift as it was before. How long will it now take him to make the round trip?

[4] Remus can eat a whole watermelon in 10 minutes. Rastus in 12. I suggest a race between them, giving each half of a melon. How long will it be before the melon is entirely gone?

[5] The distance from Boston to Portland by water is 120 miles. Three steamers leave Boston, simultaneously, for Portland. One makes the trip in 10 hours, one in 12, and one in 15. How long will it be before all 3 reach Portland?

It looks easy enough, but I advise you to try it. I will guarantee that high-school seniors, preparing for College Entrance Board Examinations in Mathematics, will not average 70 percent. I had some rather ridiculous results. I tried the fourth and fifth examples on a second grade the other day and had an almost perfect score, while a ninth-grade class in arithmetic, which had been taught under the old arithmetical curriculum, made a sorry showing. Out of twenty-nine in the class only six gave me the correct answer to problem five.

We have already seen results of our new course of study. The head of the English Department in our Central High-school [enroling [sic] 2450 pupils] tells me that in the English classes made up of pupils who entered on February 1, 1935, there is a fluency and a readiness with the mother tongue that is surprising. The old-time diffidence is gone. Children are no longer tongue-tied and unable to put a new idea into words.

I am not surprised. I had expected a report like this. You will recall the terrible English used in one of our eighth grade rooms, taken down as it was spoken, which I have quoted in the first article. I went into the same room five years afterwards. The same teacher was in charge, and some of the children in the room were younger brothers and sisters of the previous group, but the methods of teaching had radically changed. With the stenographic report of the previous recitation in my hand, I asked this latter day group the same questions which I had propounded five years before to their older brothers and sisters. I pick out typical answers, and I assure you that I am not giving you the top of a "deaconed" barrel of apples.

"When the numerators of any two fractions remain the same, the fraction with the smaller denominator is the largest."

"The principle that we have proved is that the smaller the denominator gets — no, the larger the denominator gets, the smaller the fraction."

"The larger the denominator is, the smaller the fraction would be if the numerator is the same."

"The smaller the numerator gets, if the denominator remains the same, the smaller the fraction is."

"The larger the denominator gets, the smaller the fraction will be, provided that the numerator remains the same."

"The larger the denominator gets, provided the numerator remains the same, the smaller the fraction becomes."

I then tried an experiment which to me was the most conclusive of all. I read from the account in my hand typical answers which had been given in that same room five years before [of course they were not told that it was the same room] and these present-day eighth graders shouted with laughter at statements which had not provoked a smile five years before. I asked them why they laughed and they proceeded to pick out the flaws in the reasoning and choice of words of their predecessors. To me it was the most heartening sign yet, and a prophecy of what we may expect when this present eighth grade shall have become seniors in our high-schools.

LEIBNIZ — BEYOND THE CALCULUS

Hardy Grant
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Leibniz figures in the standard histories of mathematics mostly as sharing, with Newton, the main credit for the first significant formulation of the calculus. That is appropriate in the sense that there indeed lay his most vital and enduring contribution to the subject. But such a focus limits considerably the role of mathematics in Leibniz' own life and thought. Mathematical considerations also suggested, crystallized, governed in many pivotal ways the metaphysical system that places him among the West's supreme philosophers. What follows is an attempt to outline some features of this broader picture, to correct the sometime fragmentations in our estimate of his work, to see his mathematical activity as a whole.

I

We can not hope to understand him except against the background of his age. In particular his famous (or notorious) optimism, though doubtless grounded partly in personal makeup, had discernible contemporary roots. His unquestioning faith in the existence and supreme benevolence of the God of Christianity mirrored a climate in which atheism was widely equated less with wickedness than with mere stupidity. He lived in the heady days when the homely apparatus of the Royal Society's "sooty empiricks" promised to unlock the last secrets of nature, and when his great rival Newton brought the universe itself under the sway of mathematical law. In his time these advances in physical science, and many of the great issues of philosophy, remained close enough to common modes of thought that many inquirers, Leibniz among them, could address their speculations to duchesses and kings — who occasionally joined in the game. It has been called the "Age of Reason" and the "Age of Genius," but an equally valid tag would be the "Age of Confidence." And of course much of the pervasive euphoria was born of the visible power and promise of mathematics, its application (as by Galileo and Newton) to physical understanding, the conviction (as in Hobbes and Spinoza) that its methods could bring unprecedented improvement in other fields. Not surprising then that Leibniz, himself superbly skilled in mathematics and steeped (as we shall see) in a view of the subject calculated to encourage bold extrapolations, yielded to no one in that exuberant age in his hopes both for the human understanding of nature and for the scientific amelioration of social ills.

He came relatively late in life to mathematics — probably the latest "bloomer" among all the subject's most gifted creators. His formal education in mathematics was slight and superficial; his fundamental work on the calculus awaited his historic sojourn in Paris (1672–76), that began when he was already 26 years old. His earlier training and preoccupations recall the biologists' old notion that "ontogeny recapitulates phylogeny" — that the individual's development retraces its species' evolutionary course. For like the post-medieval western mind in general, Leibniz came to an awareness of the power and beauty of mathematics from an immersion in the modes and vocabulary of scholasticism, behind which in turn rested the gigantic figure of Aristotle; and Leibniz' own philosophy retained this imprint to the end. But just as E. T. Bell declared that the scholastic philosophers were mathematicians *manqués,* so Leibniz in his youth groped instinctively toward mathematical forms and procedures that his education had not revealed to him. As a teenager, he told a correspondent, he wondered whether, "since simple terms or concepts are ordered through the known categories" (Aristotle's word for the basic organizing concepts of all thinking), "one could not set up categories and ordered series for complex terms or truths as well . . . at that time I did not know that mathematical demonstrations were what I was seeking."[1] The triumphs of his Paris phase "hooked" him forever on mathematics, and his mature writings sing its praises countless times. Mathematical studies, he declared in 1686, have a twofold use and value, "partly as an example of more rigorous judgment, partly for the knowledge of harmony and the idea of beauty."[2] These ideals are of course Greek; Leibniz fell in love with the spirit of Hellenism a century ahead of its "rediscovery" for the German mind by Winckelmann, Lessing and Goethe.

Greek too was Leibniz' conception of the ontological status, the "reality," of mathematical concepts and forms. His unchallengeable place among the subject's "modern" creators has masked the fact that his own view of it was profoundly traditional. Mathematics was for him a collection of timeless, necessary truths. These are binding even on God, who (for example) could not, even if he wished, create a triangle with an angle sum different from 180 degrees.[3] We reach the primary truths and concepts of mathematics by observation, by induction, and by the aid of the "natural light," that higher intuitive faculty which Aristotle called *nous* and which Leibniz took over from a European tradition ranging from Augustine to Descartes. Thus mathematics, on this ancient view, describes an idealized but objective order, grounded in our physical experience. In particular its axioms, so far from being arbitrary, are exceptionally certain truths, which are in principle provable — and Leibniz himself undertook on at least two occasions to demonstrate from still more basic assumptions the Euclidean postulate that the whole is greater than the part.[4] One must stress again that this whole conception of mathematics was standard already in antiquity, part of the vast corpus of thought codified for the western heritage by Aristotle and representing at bottom a kind of enormously intelligent and deeply reflective common sense.

It is true that Leibniz, for his part, stood near the beginning of the eventual replacement of this traditional view of mathematics by another. That tremendous change, the transition to a modern mathematics far richer and stranger but increasingly divorced from experienced reality and stripped of its claim to absolute truth — non-Euclidean geometry is the central symbol — is surely the pivotal watershed in all the subject's long history, a revolution much more profound even than the rise of axiomatic and deductive methods in classic Greece. The 17th century debate over the status of infinitesimals formed one episode in that historic passage, for, as Leibniz wrote, these mysterious entities have no counterpart in "nature," no validating presence in our experience. His own response was in part pragmatic: the fruitful use of infinitesimals in the calculus, he urged, does not require that these be "real," nor that the philosophical dilemmas besetting them be resolved. But he grappled with those dilemmas himself, and ended by seeing infinitesimals as consistent with the ancient tradition of mathematical realism. He linked them explicitly with other novelties which contemporary mathematicians were contemplating with diverse degrees of uneasiness — with imaginary numbers, with dimensions beyond the third, with "powers whose exponents are not ordinary [i.e., natural] numbers." All of these are useful "to shorten our reasoning," and may indeed be essential. But they are not — he insisted — merely fictions. Demonstrably, for example,

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6},$$

so that our use of imaginary numbers ultimately returns to, is justified by, a foundation in objective reality (*fundamentum in re*); and so with our conceptions of the infinite and of infinitesimals.[5] Thus even Leibniz' own groundbreaking work in the calculus wrought no essential change in his tradition-sanctioned vision of the objective, Platonist character of mathematical ideas.

And after all, that same perspective was precisely the necessary condition for the hopes of Leibniz and others who would extend the methods of mathematics to other fields. The clarity of mathematical concepts (infinitesimals notwithstanding), and the rigor of mathematical demonstrations, had been paradigmatic in western thought since Euclid. In Leibniz' mind mathematics joined with religious faith in fostering a conception of metaphysics and ethics as realms of potentially sure knowledge, of eternal truths underwritten by God and accessible to human understanding, and therefore as natural candidates for cultivation *more geometrico*. But like Thomas Hobbes (who as a young man he much admired) Leibniz regretted that the Euclidean method had not yet been applied with sufficient zeal and subtlety outside of its home domain — "we have demonstrations about the circle, but only conjectures about the soul."[6] At one time in his life, he tells us, he tried his own hand at such metaphysical geometry, in the loftiest of all spiritual enquiries, the study of God. "I often actually played the mathematician in theology, incited by the novelty of the role; I set up definitions and tried to deduce from them certain elements which were not inferior to those of Euclid in clarity but far exceeded them in the magnitude of their consequences."[7] In such philosophical adventures, he felt, the strict deductive chains characteristic of geometry are not only possible and appropriate but vital, lest deep and difficult truths elude our reasoning. In fact demonstrative rigor is actually more urgent in metaphysics than in mathematics itself, where errors are easier to detect.[8]

Hence Leibniz' lifelong goal of a "universal characteristic," a calculus that would allow the extension of logical and mathematical reasoning to other fields. He took his cue, and his hopes, from contemporary algebra, the still excitingly novel symbolic manipulations pioneered above all by Francois Viète. Without that example, wrote Leibniz, he "could hardly have attained" his own more grandiose schemes.[9] Algebra indeed offered the most "beautiful" existing example of the possibilities, but Leibniz groped toward an "art of combinations" that would far exceed algebra in power and applicability. He had the modern insight that algebra is empty of content, that any calculus is "nothing but operation through characters" and hence can in principle be brought to bear in very diverse spheres. His "characteristic" would generalize algebra in the sense in which, in geometry, the concept of similarity generalizes the concept of equality; it would be a universal science of forms rather than merely a calculus of numbers and magnitudes. And just as algebra operates on arbitrary letters of the alphabet, so (Leibniz urged) appropriate combinations and manipulations of letters can be made to mirror all human thought. We can even hope to calculate, by tallying such combinations, "the number of truths which men are able to express," and hence "the size of a work which would contain all possible human knowledge"[10]; here again speaks the authentic voice of the Age of Confidence. The universal characteristic would replace confusion of thought by clarity, and would allow reasoning as exact in metaphysics or morality as in mathematics. Hence it promised to end forever the clash of differing opinions, the endless and futile debates and disagreements, that had chronically plagued mankind. Leibniz had found a seductive hint of this last benefit in the Aristotelian logic of his scholastic training. Caught up in philosophical controversy with another scholar, "I proposed the syllogistic form, which was agreeable to my opponent. We carried the matter beyond the twelfth prosyllogism, and, from the time we began this, complaints ceased, and we understood each other, to the advantage of both sides."[11] But a full development of the *ars combinatoria* promised much more, held out the hope that the parties to any dispute whatever might be able to say merely "let us calculate," and all contention would be resolved.

II

It will be obvious that the sine qua non of such optimism was the certainty that the areas of potential dispute — metaphysics, politics, ethics, theology, law — are, like mathematics itself, realms of necessary truth, which need only be elucidated to convince. To the study of such truths Leibniz often returned. A proposition is "necessary," by his definition, if its denial is (or entails) a contradiction. But how, in practice, does one identify necessary propositions as such? Leibniz' examples are always statements of the subject-copula-predicate form that dominates Aristotelian logic — statements interpretable as comparing the memberships of sets or the ranges of concepts. A proposition is necessary in Leibniz' sense if it can be "resolved," by analysis of its subject and predicate, to an "identity" — that is, a statement with the property that of the two sets or concepts involved in the subject and predicate respectively, one can be shown to contain, by definition, the other. (For example, the statement "a red rose is a rose" is an identity in this sense.) Leibniz more than once illustrated his technique of analyzing necessary propositions with a sentence like "A duodenary number [i.e., one divisible by 12] is a quaternary number" (i.e., one divisible by 4). Interestingly, the passionate champion of algebraic manipulation does not prove this with the trivial observation that $m=12n$ implies $m=4(3n)$, but undertakes instead a cumbersome dissection of the ungainly adjectives that define the respective sets. A duodenary is (by definition) a "binary binary ternary," hence (by definition) a "quaternary ternary," hence a quaternary, "q. e. d."[12] It is to be noted that "analysis" and (equivalently) "resolution" are in this context technical terms whose meanings stem from the mathematics and philosophy of classic Greece: they describe the familiar problem-solving strategy that seeks to reduce the complex to the simple, the secondary to the fundamental, the derived to the axiomatically true.

Now propositions which are not necessary are said by Leibniz to be "contingent." They are statements which can be denied without contradiction, like "Leibniz attended the University of Leipzig." The 17th century's Scientific Revolution threw into sharp relief the philosophical issues raised by the ubiquitous presence of such contingent facts in everyday life. How could these be reconciled with the deterministic world-view emerging from the new physics? What sense could be made of the unnecessitated, of the apparently random and accidental, what scope remained for human choice and freedom, in a world bound by mathematically provable "laws" (that powerful metaphor!), in a climate of thought that soon would evoke the mechanistic philosophy proclaimed by La Mettrie, the cosmic predestinationism voiced by Laplace? For his part Leibniz reached a justification of contingency that could occur only to a mind profoundly molded by mathematics. The resolution of *necessary* propositions, described above, can always be accomplished in a finite number of steps. A contingent proposition, by contrast, has the property (according to Leibniz) that the same sort of analysis *does not terminate*. Thus a full understanding of such propositions is beyond human capacity: we can not perform the infinite sequence of reductions which alone would show that the concept "Leibniz" actually *includes* attendance at the University of Leipzig. But God, on the other hand, can take in the whole of this infinite act of analysis in, so to say, a single glance. Leibniz' thought here reflects, no doubt, the limitless powers ascribed to God by Christian tradition; but it echoes contemporary mathematics as well. We recall that in his time mathematicians were increasingly comfortable with the "completed" infinite that had so spooked their Greek predecessors — witness Newton's famous declaration that our reasoning is "no less sure" in the context of infinite series than when applied to finite sums, though in the former case our minds can not embrace all the terms. Human mathematicians, wrote Leibniz in the same spirit, "even have demonstrations about infinite series"; how much more readily, then, are "contingent or infinite truths subject to the knowledge of God."[13]

But his study of contingent propositions drew on mathematics in another and much more specific way. He found a wonderfully illuminating analogy in a celebrated piece of ancient geometry. The "Euclidean algorithm," in Euclid's original conception (*Elements*, VII, 2), sought the greatest common measure of two magnitudes by the repeated subtraction of the smaller remaining magnitude from the larger, a process guaranteed to terminate if the magnitudes are commensurable — if, to put the matter in our terms though not in Euclid's, the ratio of the measures of the original magnitudes is a rational number. In the case of two magnitudes which are *not* commensurable — whose ratio is, for us, irrational — the process of reciprocal subtraction does not terminate. This contrast became for Leibniz the guide and touchstone of his distinction between necessary and contingent propositions. The subject and predicate in a necessary proposition are (he argued) like commensurable magnitudes, in that their shared range of reference, revealed by a finitary analysis, is like the magnitudes' greatest common measure, computed by the Euclidean algorithm; correspondingly, contingent propositions resemble surds. Leibniz conceded that the analogy is not perfect, for one can calculate the true (irrational) ratio of two incommensurable magnitudes with arbitrarily small error, whereas no such narrowing of the gap between human and divine understanding of contingent truths is possible. Nevertheless he rejoiced in having discovered through mathematics the key to a riddle "which had me perplexed for a long time; for I did not understand how a predicate could be in a subject, and yet the proposition would not be a necessary one. But the knowledge of geometry and the analysis of the infinite lit this light in me, so that I might understand how notions too could be resolved to infinity."[14]

In such ways — and more tellingly, perhaps, than in any other mind of which we have record — mathematical ideas constantly informed and colored Leibniz' entire vision of the world. Many other thinkers, of course, have drawn inspiration from the same source; Aristotle, for one, anticipated Leibniz' way of reaching at every turn for mathematical illustrations of philosophical arguments, resorting naturally to the best founded and most richly developed science of his age. But in Leibniz the transference of ideas went deeper. For his work on the calculus put him at the frontier of contemporary advance, and he brought from mathematics a technical knowledge and sophistication, a grasp of precise and particular detail, which he applied in philosophy with a specificity that remains unique. We cannot know — perhaps Leibniz himself could not have reconstructed — the full course of this creative borrowing, the complex interplay of mathematical examples and their metaphysical analogues in the final shaping of his thought. Sometimes, as in his study of necessary and contingent propositions, mathematical considerations might seem merely to have provided him with a convenient model, that might be imperfect though deeply suggestive. But often, reading him — and remembering always his image of mathematics as a collection of eternal truths, and of concepts perceived with matchless clarity — one cannot resist the feeling that he seized on certain of those ideas as not merely suggesting or confirming metaphysical points but as offering sure signposts to the very contours of existential possibility, the very scope and direction of God's creative design of the world.

It is fascinating to see how much of his metaphysics can be expounded in such terms. "In the very origination of things," he wrote, "a certain Divine mathematics or metaphysical mechanics is employed," which ensured the maximum production of all desirable things; we see the same optimizing principle in the operation of nature even now, in (for example) the fact that "when several heavy bodies are operating against one another, the result is that movement which secures the greatest descent on the whole."[15] In the act of creation, said Leibniz, God acted "like the greatest geometer, who prefers the best constructions of problems." That is to say, just as a geometer will seek a proof or construction that combines maximum range and power with supreme economy of argument, so God, in choosing among the infinitely many potential orders of existence, opted for the one which would yield "the greatest effect" — the maximum of goodness and happiness — from "the simplest means."[16] Leibniz lived too far in advance of saddle-point calculus — not to mention the modern theory of games — to make much mathematically of such "minimax" considerations, but they remained basic to the optimistic tenor of his philosophy. For once, indeed, the catch-phrase that has filtered to popular perception from the complex thought of a great mind is wholly accurate: Leibniz really did believe that this is, strictly and absolutely, the best of all possible worlds — whence, of course, the brilliant, bitter mockery directed against his system by Voltaire.

Further details of Leibniz' cosmic vision were bred or reinforced by specific features of contemporary analytic geometry and calculus — their achievements and their limitations alike. To him the order detectable in the universe was like the unity imposed on a plane curve by a single algebraic expression that describes and governs all its features. He seems to have shared with at least some of his fellow analysts a remarkably bullish sense of the possibilities of curve-fitting; he related that Johann Hudde claimed the ability to find an algebraic equivalent for the profile of any human face, and Leibniz himself agreed that this is possible.[17] More strikingly still, he held that, given any set of randomly scattered points in a plane, one can find a curve "whose notion is constant and uniform, following a certain rule" — meaning, apparently, the graph of a continuous function given everywhere by a single formula — which not merely passes through all the given points but does so in the order in which they were laid down. Similarly — and the analogy is of course made fully explicit — God could fashion a harmonious universe from any original chaos of potential existents, for "no way of creating the world can be conceived which is so disordered that it does not have its own fixed and determinate order."[18]

This mathematically sustained faith in the world's ultimate rationality and goodness went further still. Undeniably, we seem to perceive many irregularities and inequities in the physical and moral fabric of things. Likewise (said Leibniz) every curve has points — singularities, extrema, points of inflection — which seem to stand out as different from the others. But in fact the seemingly anomalous nature of such points is shown by the new calculus to follow from, to conform to, the "equation or general nature of the whole" curve, which thus remains, on a broader perspective, "perfectly ordered" after all; and similarly for the seeming imperfections in the world around us.[19] And as in the universe as a whole, so also in our individual lives. All the seemingly exceptional events that befall us, even our very births and deaths, are only, as it were, peaks or valleys or cusps on the trajectories of our immortal souls; they are not outside the uniformity of nature, they violate no general laws.[20] In one especially confident passage Leibniz declared that the world's overall perfection obtains also in all its smallest component parts — even as the shortest-descent property of the cycloid arc which solves the brachistochrone problem holds between any two points, however close.[21]

As is well known, Leibniz' philosophy is suffused by a deep organicism, which saw each of the world's smallest parts as related to all of the others through constant "intercourse" and mutual influence. It is an idea which, as Joseph Needham urged, echoed more vividly the Chinese sages whom Leibniz studied than the prevailingly mechanistic outlook of contemporary Europe. But it owed something to his mathematics too. We have seen his belief that to any arbitrary set of points can be fitted a curve "whose notion is *constant and uniform*" (emphasis here added). Leibniz scarcely knew — or at any rate scarcely considered — discontinuous functions; and this prevailing tendency of his mathematics encouraged him to find, everywhere in nature, continuous passages from one state of affairs to another. The "Law of Continuity" became one of the most fruitful guiding principles of his thought. Ellipses, parabolas and hyperbolas, for example, seem from "external shape" to be entirely different from one another, yet we know that in fact each of these passes into the others by gradations so "intimate" as to bar the insertion of any different kind of curve in the sequence. "Therefore," said Leibniz, making one of his grandest leaps, "I think I have good reasons for believing" that in like manner all the world's endlessly varied species of organic creatures form a single continuous chain, "like so many ordinates of the same [continuous!] curve whose unity does not allow us to place some other ordinates between two of them because that would be a mark of disorder and imperfection."[22] This ladder of organic life is of course the "Great Chain of Being," a staple of the western intellectual tradition since the time of Plato (and the subject, long after Leibniz, of one of the most absorbing and seminal books ever written on the history of ideas).[23] Leibniz' tendency to find continuities everywhere assured him that "when the essential determinations of one being approximate those of another . . . all the properties of the former should also gradually approximate those of the latter" — or, as we should say, any biological character is a continuous function of position on the Chain. Certain creatures with unusual traits, like the "zoophytes" that seem to bridge the plant and animal kingdoms, may be viewed as occupying, "so to say," the Chain's "regions of inflection or singularity."[24] The Great Chain of Being was hoary with antiquity when Leibniz described it, but never before or since was it conceived in such specifically mathematical terms.

Every particle of matter, said Leibniz, teems with an infinity of living creatures — a notion that plausibly owed much to the wonders discovered in his time, by Leeuwenhoek and others, with the first microscopes. At the very bottom of the organic hierarchy are the simple soul-like substances that Leibniz called "monads." Leibniz used mathematical ideas in wrestling with the notoriously difficult problem of relating these elementary souls to physical matter. Material bodies, he proposed, are aggregates of these substances in precisely the way that geometrical lines are aggregates of points. A point, that is to say, is not actually *part* of a line, for "a part is always of the same nature as the whole;" rather, "a line in which there is a point is a part of a larger line, and similarly "a soul is not a part of matter, but a body in which there is such a soul is such a part of matter."[25] In Leibniz' organicist vision of nature every monad, though absolutely simple and without parts, has nevertheless a multiplicity of relations with things outside itself, just as "in a center or point, in itself perfectly simple, are found an infinity of angles formed by the lines which meet there."[26]

This survey of the mathematical bases of Leibniz' thought could be supplemented by other examples. But no case is here made for the notion that the whole of his philosophy is so describable. He would have been the first to scorn such a claim as grotesque, for in fact he insisted repeatedly that much in nature is not to be explained by mathematics.[27] The present account has set aside, as not so palpably tied to mathematics, such fundamental and characteristic of Leibniz' preoccupations as the nature of substance, the relation of "efficient" and "final" causes, the case for immortality, and many more. I hope only to have shown that the role of mathematics in shaping his philosophy was very considerable, and that it took surprisingly detailed, crucial and sophisticated forms. This side of the great philosopher has been underappreciated — perhaps above all by mathematicians. To speak of him merely as a co-founder of the calculus is doubtless to set him correctly in the history of technical progress — but at the price of a limited perspective on the whole man and on the splendid originality and power of his thought.

NOTES

1. Leibniz, *Philosophical Papers and Letters*, tr. and ed. Leroy E. Loemker (Chicago: University of Chicago Press, 1956), pp. 757–58. (Cited below as "Loemker.")
2. Leibniz, *Selections*, ed. Philip P. Wiener (New York: Charles Scribner's Sons, 1951), p. 62. (Cited below as "Wiener.")
3. Loemker, p. 833.
4. Leibniz, *Philosophical Writings*, ed. G. H. R. Parkinson (London and Melbourne: Dent, 1973), p. 87. (Cited below as "Parkinson.")
5. Loemker, p. 883.
6. Wiener, p. 59.
7. Ibid.
8. Loemker, p. 297.
9. Ibid., p. 763
10. Wiener, p. 75.
11. Loemker, p. 763.
12. Parkinson, pp. 96–97; cf. pp. 108–9.
13. Ibid., pp. 110–11; Leibniz, *Philosophical Essays*, tr. Roger Ariew and Daniel Garber (Indianapolis and Cambridge: Hackett, 1989), pp. 98–101. (Cited below as "Ariew and Garber.")
14. Parkinson, p. 97.
15. Ibid., p. 139.
16. Ibid., pp. 76 (note), 200.
17. Wiener, p. 185.
18. Parkinson, p. 78.
19. Wiener, p. 189.
20. Ibid.
21. Loemker, p. 780.
22. Wiener, p. 187.

AN HISTORICAL APPROACH TO PRECALCULUS AND CALCULUS

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As a college teacher of mathematics I receive many new texts each year in precalculus and calculus, each one trumpeting its virtues and its new ideas. But a study of the actual material presented shows that not only are there few new ideas but that chapter for chapter and almost section for section each such book is a repeat of every other one. In fact, I amazed my daughter one day by telling her the titles of the first ten chapters in the calculus text she was using, without ever having seen the book itself. Since all such texts are the same, one could assume that there is a general agreement in the mathematical community that there is one correct way to teach precalculus and calculus. With all the conferences and position papers of the past several years, first on the desirability of discrete mathematics and more recently on the lean and lively calculus, it appears, however, that large numbers of faculty members are dissatisfied with the way these courses are presented. As a matter of fact, the high failure rate in calculus seems to indicate that students too are dissatisfied.

As a possible answer to this dissatisfaction, and as a new way of organizing these two courses, I have experimented over the past several years with an historical approach to both precalculus and calculus — considered together as a four or five term sequence. Not only does this approach help integrate the discrete algorithmic material with the continuous analytic mathematics, since in fact much of the former was developed alongside of the latter, but it also helps to introduce our science and engineering students to the relationship between mathematics and the rest of our culture. As it stands now, many students are sadly lacking in an awareness of the place which mathematics occupies in our culture. They are interested in the mastery of technique to the exclusion of learning the reasons that the ideas were developed and the use of mathematics in the world. And without the intellectual content behind the mathematical techniques, it appears that in large measure the students fail to grasp even the techniques. An historical approach to these courses helps to provide a solid motivation for the learning of mathematics as it ties together much of the students' backgrounds in history and literature with their scientific studies. It also encourages the student at every stage of his/her studies to explore the ramifications of scientific work as it relates to the world around them. And it seems to me that the prospective scientists and engineers I am teaching will more than ever need a sense of how their own highly technical work fits in with the needs of society as they make decisions which will affect the fate of the world.

By an historical approach to mathematics teaching, I do not mean simply giving the historical background for each separate topic or giving a biographical sketch of the developers of various ideas. I do mean the organizing of the desired topics in essentially their historical order of development. I do mean discussing the historical motivations for the development of each of these topics, both those within mathematics and those from other fields. I also mean connecting the development of each of the mathematical topics with the development of the other sciences and with the other things which were going on in the world. Naturally, I cannot always stick precisely to the historical record. Many seemingly good ideas led to dead ends or to methods which are too difficult for the level of these particular courses. And one should make use of today's technology of calculators and computers to perform tedious computations rather than have the students repeat their ancestors' hand calculations. Nevertheless, using history as a general guide does provide an organizing tool for both precalculus and calculus which helps to motivate the students and shows them that mathematics has always been an important part of the overall culture, not only in the West but also in other parts of the world.

A reasonable question at this point is how we can do all this when we can barely cover all the material in the syllabus as it is. I do not have a simple answer to that question. My experience is that to a large extent, this historical approach is more time-efficient than a more standard approach and that the historical connections drawn help to motivate and excite the students, enabling them to do more work independently. Of course, we cannot neglect the development of technical proficiency or of problem solving skills. But again, both of these aspects can be set into the historical context. And if, in fact, this approach forces us to drop a few topics in the current syllabus, this will only be in line with current recommendations in any case.

Let me now describe the mathematics course I have in mind. In the context of the school at which I teach, it is necessary to begin with precalculus material. But it would not be difficult to begin this course at a later point. As will be clear, virtually all of the standard precalculus and calculus topics will be dealt with, but often in an order and a context differing from the standard ones. The course description will emphasize the novel aspects of my approach and how it better serves the students than the usual method.

The course begins with a description of the common mathematical knowledge of various ancient civilizations, namely, basic algebra through quadratic equations, basic geometric formulas, the Pythagorean theorem, and the calculation of square roots. We discuss the nature of these ancient societies, the Babylonian, the Egyptian, the Chinese, and what it means for a certain society to "know" mathematical ideas. Who in the society knew these ideas? Why did they know them? What kinds of problems did they need to solve? In particular, we discuss the value of pi. What does it mean to approximate the ratio of the circumference of a circle to its diameter and how good an approximation is necessary? In the same mode, we also discuss the square root algorithm and its geometric origins. In this context, an introductory discussion of the nature of an algorithm is warranted along with a discussion of accuracy. A review of the quadratic formula is also useful along with the Chinese numerical method of solving quadratic equations.

The major change from these ancient civilizations to the classical Greek in terms of mathematics was in the introduction of deductive proof. Thus we deal briefly with the nature of Greek civilization and its differences from those of Egypt and Babylonia. Then, even though the students have generally had some course in geometry, a discussion of some salient points of Euclid's *Elements* is warranted. Among the topics included are the nature of logical argument and the idea of an axiomatic system, the basic triangle and parallelogram theorems of Book I, the circle theorems of Book III, the construction of the pentagon in Book IV (for its later use in trigonometry), the similarity results of Book VI, and finally, Euclid's result on the area of a circle from Book XII.

Curiously, there is no biographical data available on the author of the world's most famous mathematics text except a few stories dating from some 700 years later than the time of its writing. Since the text is a compilation of several different strands of Greek mathematics and since it was written in Alexandria, a city whose inhabitants came from many different backgrounds, some historians have speculated that it was written by a committee. Among the possibilities for members of this committee would be native Africans, Jews, and even women. Though this particular idea on the authorship of the *Elements* is pure speculation, it should be made clear to students that though certain groups have traditionally been excluded from mathematical knowledge, there were always individuals who somehow managed to buck the trend and make their own contributions. Unfortunately, sometimes the history books ignore these contributions or possibilities, denying a sense of "ownership" to large parts of our population.

Euclid's work is followed in the course by an introduction to conic sections in terms of sections of a cone and in the context of the problem of doubling the cube. I generally stretch the historical record a bit here and interpret Apollonius' work in terms of coordinate geometry as I develop the equations and some of the elementary properties of these important curves. I also introduce the beginnings of mathematical physics in the work of Archimedes, emphasizing in particular the idea of a mathematical model.

The idea of a model is further developed with the study of trigonometry, since that subject originated as a mathematical tool for astronomy, which in turn, in the Greek world, was based on the two-sphere model of the universe. Thus I introduce the students to some ideas in astronomy, using the model having the earth fixed in the center of the heavens. It is interesting, in fact, to ask the students for any evidence that the earth is not stationary. I introduce trigonometry itself in essence as Ptolemy treated it, though rather than deal with his chords I give the modern ratio definitions of the sine, cosine, and tangent. But the sum, difference, and half-angle formulas are done following Ptolemy's geometric proofs and these are used to begin the actual calculation of values of the trigonometric functions. The values for 30, 45, and 60 degrees come from simple right triangle geometry, while those for 36 and 72 degrees come from Euclid's pentagon construction. I think it is important for the students actually to perform some of these calculations themselves, so that they learn that the sine function on their calculator is not magically generated nor does it come from actually measuring triangles. Among the benefits of these calculations is an appreciation of interpolation and approximation as well as the realization that the sine function is nearly linear for small angles. In fact, it was that realization which enabled Ptolemy to complete the calculation of his table. I even recalculate the sine using radian measure, since, in effect, radian measure was used both in Greece and in India early on. It was realized that for small angles, using this measure, sine x is very nearly equal to x itself.

The students naturally use their calculators rather than the calculated table in applying trigonometry to solve problems. These problems include not only plane problems but also spherical ones, since it was the latter which were most important to the solving of astronomical problems. With a brief discussion of some astronomical concepts and the introduction of the important spherical trigonometry formulas, the students can easily solve problems such as determining the length of daylight on a given day at a given location or the exact direction in which the sun rises or sets.

The next major topic, as we move out of the Greek period. is that of equation solving. I begin this section with a geometric justification of the quadratic formula taken from the work of the Arab algebraist al-Khwarizmi. (That is, completing the square means exactly what it says.) This justification needs a geometric version of the binomial theorem $(a+b)^2 = a^2 + 2ab + b^2$. Continuing in this vein, I ask the natural question of how to solve a cubic equation. There are several medieval Arab works which seek to answer this question, although the answers they give are probably not what students expect. We therefore discuss what it means to "solve" a cubic (or any) equation. The Chinese in the thirteenth century certainly could solve such an equation numerically, while the Arabs of the same time period knew how to do it geometrically. One interesting method to discuss is that of al-Tusi, who used techniques related to the calculus to decide what types of solutions to a cubic were possible.

To find an algebraic solution, however, one must turn to sixteenth century Italy and the story of Tartaglia and Cardano, a story that should certainly be shared with the students. The students should also be made aware of the Renaissance background here and learn why there was a renewed urge in Europe to do mathematics. In particular, it was often the merchants who brought mathematics back to Europe through their travels to Africa and Asia. In any case, the algebraic solution of the cubic begins with the binomial theorem in degree three and, at least in the simple cases, is not difficult to understand. I do not give a complete treatment of cubics, but only enough for the students to get the general idea and see why, in the irreducible case, complex numbers are necessary. I then can introduce these numbers, as Bombelli did late in the sixteenth century, and use them in solving not only cubic but also quadratic equations.

It is now worthwhile to continue the study of solutions of equations from other points of view, once the complexity of the cubic and, perhaps, the quartic formulas are understood. For example, I use the work of Descartes to develop the factor and remainder theorems as well as the methods of finding rational solutions to polynomial equations. If no rational solutions exist, the Chinese method already discussed, as well as Newton's method, itself an adaptation of earlier work, can be used to approximate solutions. So we return to the notion of an algorithm and by using it can explore the idea of convergence. Again, even though modern computer software will enable any polynomial equation to be solved numerically with the touch of a button, it is always important for the students to understand how the algorithms built into the software were developed.

Having already considered the binomial theorem in the cases $n = 2$ and $n = 3$, it is now time to give a more detailed treatment of combinatorics concentrating on the work of Pascal and his medieval Chinese, Arabic, and European predecessors. First, the binomial theorem for any positive integral exponent needs to be developed. Naturally, this is the place to introduce and study the notion of mathematical induction. A consideration of the background and motivation for Pascal's triangle is useful as an introduction to the ideas of probability and the calculations of permutations and combinations. A second major result is the formula for the sums of powers of integers. This is often associated with Bernoulli, but the basic ideas date from somewhat earlier. A third important idea treated here is the general idea of arithmetic and geometric sequences. These latter provide a gentle introduction to the idea of an infinite series and its sum, an idea already understood in some sense by Archimedes.

Once we have geometric and arithmetic sequences, it is time to investigate logarithms. I discuss the scientific need for logarithms as an aid in the tedious computations necessary for the preparation of astronomical tables. So I also need to discuss the need for such tables in the age of European exploration and discovery. In any event, the relationship between geometric and arithmetic sequences and the desire to convert multiplication to addition by using this relationship leads to the necessity for the choice of a good base. Adapting the ideas of Napier and Briggs, I show the students why "natural" logarithms as well as common logarithms were both developed. In fact, the question of the "naturalness" of natural logarithms leads via the work of Napier himself to some important ideas of differential calculus as well as to the idea of the exponential function.

Another idea being developed in the 17th century, through the work of Fermat and Descartes, was that of analytic geometry. We therefore return to Apollonius' conic sections and show algebraically that any quadratic equation in two variables leads to a conic section. The important tangent and focal properties of these curves are then treated as a further introduction to ideas of calculus. Kepler's laws are then discussed as a continuation of our earlier notion of a mathematical model. Of course, whenever one discusses Kepler, one also must discuss Galileo and the idea that the scientific success of a particular mathematical model does not always mean its acceptance. But we also deal with Galileo as a mathematical physicist as we treat the motion of projectiles and the general idea of a function. Though functions have been discussed earlier on an ad hoc basis, it is here that I make the first attempt to develop the idea in detail, first in terms of physical phenomena and then as a purely mathematical idea. In particular, we consider and graph polynomial functions and some easy rational functions, including one we will use often later, $y = 1/(x+1)$. And again the idea of a mathematical model occurs, this time in earthly terms rather than in the heavens. Finally, the graphs of the logarithm, exponential, and trigonometric functions are briefly discussed, leaving more details to the development of the calculus of these functions.

Certain calculus ideas having been introduced earlier, it is now time for a detailed discussion, using the work of Fermat, of the two basic problems of calculus, areas and extrema (or tangents). I first solve these two problems for the curves $y = x^2$ and $y = x^3$ and then proceed to generalize to $y = x^n$. For derivatives, I use the binomial theorem, and for integrals, I use either the material on Bernoulli numbers or the work on geometric sequences. In both cases, we need the basic idea of a limit, so an intuitive discussion of limits is warranted here. The elementary results on power functions can then be extended to a procedure for finding derivatives and integrals of polynomials. Now although Fermat essentially had these results, he never noticed what Newton did, the Fundamental Theorem. In any case, we have now reached the central figure in the scientific revolution. A statement and at least a sketch of a proof of the Fundamental Theorem can now be given, probably in terms of velocity and distance. In fact, with the right preparation, the students can "discover" this theorem for themselves. But I also discuss to some extent the scientific revolution itself and Newton's part in it. In particular, throughout the remainder of the course I deal with some of the problems the calculus enabled Newton and others to solve.

It is at this point in the syllabus that I believe the historical approach makes its most important contribution to the study of calculus, namely the early introduction of the notion of a power series. Power series were one of Newton's earliest discoveries in calculus and one which he used constantly. In fact, I also note that for the sine and cosine, power series had been developed even earlier in India. It is quite effective pedagogically to introduce series at the earliest possible moment, namely, as soon as the basic derivative and integral algorithms are known and the fundamental theorem is proved. Power series can then be used as a theme throughout the rest of the course. They provide examples of algorithms, explicit calculations of certain integrals, ideas on the relationships among various functions, a further introduction to the fundamental idea of convergence, and a prime example of the method of discovery through analogy.

As a beginning to the study of power series, we simply deal with them as generalized polynomials which we can add, subtract, multiply, and divide. We then discuss convergence by trying to use power series to represent functions. In particular, we can think of power series as generalizing infinite decimal expansions of numbers. As a first nice example of a power series, I show the generalization of the binomial theorem to negative and fractional exponents. I then use this to calculate square and cube roots, for example. I also note that, since the power series can be considered as generalized polynomial functions, we can take the derivative and anti-derivative term by term.

We do need other techniques in calculus. So we proceed to the basic approximation theorem, $f(x+h) \approx f(x) + f'(x)h$ and then the ideas of Leibniz and his followers, in particular the idea of a differential. The approximation theorem gives us straightforward proofs of the product rule, quotient rule, and chain rule. We can then deal easily with integration by parts and by substitution. On the other hand, we can integrate $\sqrt{1-x^2}$ and $\sqrt{1+x^2}$ by using power series since the other methods do not apply. Of course, physical problems, particularly those centered around Newton's laws of motion, must be dealt with here now that the basic tools are available. Maximum-minimum problems are among the more important of these types of problems.

As in any calculus course, it is now necessary to deal with the transcendental functions. The natural logarithm can be developed as is standard, but is also historically justified, via the integral of $1/(1+x)$. But by considering the power series for that function, we can also get the series for the logarithm. And this, of course, enables us to calculate values. The series for the exponential function can then be developed and discussed by inverting the series for the logarithm, by Euler's method of using the binomial theorem, or, perhaps, by solving the differential equation $y' = ky$, essentially introduced in the earlier treatment of these functions. There are many interesting problems involving the exponential and logarithm functions in early calculus textbooks. It is interesting for today's students to see what kinds of problems earlier students solved, so I show them some from the calculus text of Maria Agnesi, probably the best of those before the works of Euler and evidence that women could and did study mathematics.

The calculus of the trigonometric functions is developed via Newton's idea of relating them to arcs of circles. This leads again to the "naturalness" of radian measure. Thus I begin with arc length and the definition of the inverse functions. Again, these are done in terms of power series. The series for sine, cosine, and tangent can easily be developed by various methods as can the basic rules for derivatives. In fact, the latter is best done by a geometric argument using differentials rather than via the limit argument commonly used. The power series representations of e^x, sin x, and cos x then provide a natural question of how these functions are related. The answer leads into a renewed discussion of complex numbers and also into a treatment of the hyperbolic functions. Finally, the notion of simple harmonic motion and its associated differential equation $y'' = -ky$ is dealt with and shown to result in the trigonometric functions. Thus the basic periodicity of these functions can be understood in terms of an important physical idea.

Other types of physical problems are treated in the process of dealing with various techniques of integration. Not too much time is spent on the techniques themselves, however, since power series methods are available to give results and since new computer algebra systems will generate these results in any case. But we do need to deal with such ideas as arc length, volume, and center of gravity and see what integrals are necessary to solve these problems. At the same time, some of the elementary ideas of differential equations, including the separation of variables and the integration of exact equations, are covered in the context of the physical and mathematical problems which led to their study in the first place.

It is at the end of the one-variable section of the calculus course that I give a detailed treatment of the notions of limit and convergence. It is only after the experience of dealing with these ideas on an intuitive basis for many months that I can expect the students to understand their theoretical underpinnings and the use of epsilons and deltas. After all, the work of Cauchy and Bolzano occurred fully 150 years after that of Newton and Leibniz.

An historical approach to the study of precalculus and calculus can provide valuable insights to the students. With appropriate examples, it can also serve to show that women and minorities have been involved in mathematics in the past and can certainly expect to make further contributions in the future. I am convinced that this approach to this important segment of mathematics is better than the current method which, in calculus at least, begins quite unhistorically, and quite unsuccessfully, with the abstract notion of a limit. Unfortunately, the publishers of mathematics texts hesitate to publish a text which deviates in any major respect from the current model. It is therefore not very easy to get new curriculum ideas out into the mathematics community. If the readers of this essay try out some of the ideas expressed in it, however, and exchange their findings, the mathematical community as a whole may eventually see the benefits of this approach to the teaching of the most popular courses in the mathematics curriculum.

AN ALTERNATIVE APPROACH TO THE HISTORY OF MATHEMATICS

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Many scholars have described the value of teaching the history of mathematics. Some, such as Andre Weil, argue that studying the history of mathematics is primarily for current or future mathematicians since it can give us insights into how to tackle contemporary mathematics problems. Others agree that the history of mathematics is of particular interest to mathematicians, not so much for its utilitarian value, but rather for the enrichment one feels in understanding ones past. Still others argue that the history of mathematics is of interest to non-mathematicians and mathematicians alike. It is part of our cultural heritage, and has influenced many other aspects of our society, therefore we should study the history of mathematics, just as we study the history of art or literature. In this vein, Judith Grabiner, in her article "The Centrality of Mathematics in the History of Western Thought," argues that mathematics has influenced religion, philosophy, economics, and even the Declaration of Independence. It would seem, then, that there are many reasons to include the history of mathematics in a liberal arts undergraduate curriculum.

However history of mathematics courses are relatively rare, and when they do exist they are usually housed in mathematics departments and seen as peripheral to the core mathematics curriculum. Nor are students organizing massive protests due to the absence of such courses, which often conjure up images of dull, rote memorization of big names, dates and theorems, lacking the connective tissue that would sustain students' interests. This article is meant to explore an alternative approach to the history of mathematics, one that would be of interest to potential mathematicians and non-mathematicians alike.

History of mathematics courses are more important today than ever before. As we become aware of the necessity of recruiting more students to study mathematics, particularly groups that are traditionally under-represented, women and minorities, we must look at mathematics with new eyes, and with a wider perspective.

The traditional approach to the history of mathematics reinforces the alienation that many women and minorities feel about mathematics. They see no people they can identify with, if indeed they see people at all. How the questions of mathematics arose, and what the connections were between mathematics and the larger culture are often neglected. Someone who is not already engaged in mathematics is not likely to be inspired by such a course.

One first step therefore in the re-vision of the history of mathematics is to look for the participation of women and minorities. This includes studying the lives of women such as Hypatia (the little we can find about her), Sofia Kovalevskaia, Marie Agnesi, Ada Byron Lovelace, Emilie du Chatelet, and Emmy Noether. Simultaneously, we can begin our history of mathematics not with the Greeks but with early Egypt and Babylonia, and, in looking at the development of mathematics in Africa, consider how our Western perspective and values influence the way we evaluate the mathematics of other cultures. But it is not enough to just add the few women and minority mathematicians we can find and think we now have a more inclusive curriculum. We must continue our inquiry and investigate why there have been so few women and minorities to include. To some extent this involves studying the obstacles that have prevented the entrance of many people into the world of mathematics. But it also involves a deeper analysis of what we *define* as mathematics and how this is culturally dependent, as well as how the values of a culture shape what is considered important mathematics.

Philip Davis considers some of these issues. He reminds us of the very useful words of William James, "The community stagnates without the impulse of the individual; the impulse dies away without the sympathy of the community," and Davis goes on to ask "Is it possible to write history of mathematics along the lines suggested by this quotation?" In this article, I discuss the ways in which we can begin to think about teaching such a history of mathematics.

This past Fall, I had an opportunity to teach a history of mathematics course at Middlebury College as a freshman seminar. Since I had total freedom in designing the course I decided to structure the course to examine these questions, in particular the relationship between mathematics and its cultural context.

The Body of the Course

The course was divided into three parts, each a different way of approaching the history of mathematics. The first part focused on individual mathematicians, the second on how mathematics has developed in different cultures and the third focused on one area of mathematics, namely geometry. The goal of the course was to have the students grapple with the fundamental questions "what do we mean by mathematics?" and "what do we mean by history?" By taking three different approaches to the history of mathematics, students began to realize that answers to these questions are intimately connected to understanding the relationship between mathematics and the culture in which it is practiced.

In looking at the lives of individual mathematicians, we read biographies and autobiographies of selected people from various times and cultures. One of the first assignments the students had for the course was to write their own mathematical autobiography (an informal and ungraded assignment). This gave me an opportunity to get to know them better and to get some indication of their writing skills. It also served as an excellent device for priming their interest in the lives of other mathematicians. They had to think about what aspects of their own mathematical experience were important and why, and to begin to think about the extent to which they were influenced by environmental or cultural factors. It was natural therefore to investigate the role of mentors, family, and friends in mathematicians' development.

We considered questions such as who had the opportunity to pursue mathematics? Why did they choose to become mathematicians? What field(s) did they pursue and why? What kind of contact did they have with other mathematicians (if any)? How were they received by the larger community? How did they think about their work and the nature of mathematics? Who were the women in mathematics, and in what ways, if any, were there lives different? What has been the role of institutions (social, political, educational, and religious) in shaping mathematicians' lives and mathematics in general?

We read "The Ideal Mathematician" from *The Mathematical Experience* which was useful when reading about the lives of various mathematicians and considering the extent to which their lives agreed or disagreed with the one portrayed in "The Ideal Mathematician." The article "The Individual and The Culture" was important in raising the question of the interplay between the individual and society which was a central theme of the course.

Some articles, for example Koblitz's on Sofia Kovalevskaia, address the cultural influences directly, such as the ways in which the early Nihilist movement opened doors for Kovalevskaia by creating a social climate in which it was not only acceptable for women to pursue mathematics and science, but it was seen as a progressive mission which helped promote liberation for the people. And yet at the same time it was Kovalevskaia's very own country, Russia, that denied her access to higher education and employment, forcing her to go abroad for both. However, other readings on individual mathematicians, maintain the traditional split between mathematician's lives and the social factors that helped or hindered them. This made it more challenging for students to investigate those connections. In some cases, reading different accounts of a person's life revealed how much of a vested interest there can be in describing the mathematician's sources of inspiration, or vision of mathematics. This was most vivid in the case of Ramanujan where Hardy, Berndt, and Ranganatham present varying points of view on issues such as the role of intellect versus intuition, the role of religion, ability as a mathematician, and how a mathematician is to be judged or evaluated.

The focus in this part of the course was primarily on well known mathematicians such as Pythagoras, Euclid, Newton, Leibniz, Euler, Gauss, etc., but in looking at more contemporary mathematicians students were able to read about some less famous mathematicians which gave rise to interesting discussions about "rating" mathematicians, and thinking about whose name gets remembered and why (for example, why is the Pythagorean Theorem attributed to Pythagoras when other cultures such as Babylonia and China seem to have been familiar with it more than 1,000 years earlier). We looked at the many factors that contribute to a person pursuing mathematics, both the factors that encourage and those that discourage success. For example, until recently, most mathematicians came from upperclass backgrounds, giving them the time and resources to pursue mathematics. We considered also the similarities and differences for women in mathematics, the various obstacles they might encounter (e.g. family or community resistance, lack of access to educational institutions or professional societies, internalizing cultural attitudes that women cannot do mathematics, inability to get jobs), and how some of them overcame these obstacles. Whenever possible we tried to examine how the various mathematicians thought about their work and the nature of mathematics, and to what extent their philosophy of mathematics agreed with their practice of mathematics.

In the second part of the course we compared and contrasted the mathematics of different cultures. In particular we looked at periods in the early history of Babylonian, Egyptian, Chinese, Greek, African, and Islamic mathematics. We tried to see in what ways the culture might influence the mathematics. What were the different cultures' concept of proof, in what ways did that meet the needs of the society? What was the status of mathematics in different cultures? What constituted mathematics in these societies, for example were art and music considered branches of mathematics, was there a focus on geometric or algebraic mathematics, why? What was considered important mathematics? In what ways did these cultures influence each other? [See the essay by Pryor and Pellett at the end of this article].

One of the outcomes of this part of the course was that students began to see what a variable concept mathematics is, that it is not something fixed in stone, defined by some external force, but rather a changing, evolving activity responding to the needs of the culture. By looking at the connection between mathematics, music, art, games, etc., students also saw the many ways mathematics is relevant to their own lives. They began to see how even such bedrock concepts as mathematical truth and proof are evolving over time and are culturally dependent.

Finally in the third part of the course we focused on one particular area of mathematics, namely geometry. Since the history of geometry is so old and rich, this part of the course tied the whole semester together quite nicely. We looked at the development of geometry from its early beginnings in Egypt to its formalization in Euclid's elements to its surprising and profound expansion in the 19th Century with the introduction of non-Euclidean geometry. We considered what the introduction of non-Euclidean geometry does to the Euclid myth, and began to think about other ways of conceiving of the nature of mathematics. Non-Euclidean geometry returned us to the philosophical questions that we grappled with at the beginning of the course, such as "What is mathematics" and as such made a wonderful ending to the class.

Central Questions

In the following sections I will elaborate on how we investigated the fundamental questions that were discussed at the beginning of the course but continued to be central throughout: "what is mathematics" and "what is history."

A. "What is Mathematics?"

Before doing any readings, the students wrote a short informal essay on "What is mathematics?" For many students this was the first time they had asked this question, though they had been doing mathematics for more than twelve years. Is mathematics discovered or created? If it is created, who creates it? Are there ground rules that everyone agrees on about what constitutes a proof? We discussed three philosophies of mathematics, Platonism, Formalism, and Constructivism (see Snapper in [7]), as students began to formulate their own opinion on the nature of mathematics. It was interesting to see how students' opinions matured (though did not necessarily change) as the class went on.

We considered the traditional popular answer to the question "What is mathematics" that Davis and Hersh label the "Euclid Myth," that mathematics is a body of truths which are derived from a set of self evident truths (the axioms). The rules of logic, which are chosen to preserve truth, are used to derive theorems from the axioms. Such a view of mathematics implies that mathematics is certain (since we start out with certain truths, i.e. the axioms), objective (it does not depend on human beings since the rules of logic firmly establish what theorems can be derived) and eternal (since it reflects truths about the universe, yet is not dependent on sense experience). This is quite a firm foundation for mathematics to rest on. As we went through the course, we considered whether this is an accurate description of mathematics over time and in different cultures.

Other questions we explored in discussing "What is mathematics" are: Is mathematics a science or an art? In what ways could it be beautiful? Is it important that it be useful? How do we decide what is important in mathematics? I had the students imagine that they were in charge of a large funding organization like the National Science Foundation. They had to think about what criteria they would use in allocating funds (tackling not only the question "what is mathematics?," but also "what is *important* mathematics?"). Finally, we discussed the different metaphors that mathematicians use to describe mathematics, and what that reveals about their vision of mathematics, as well as how their vision can affect the mathematics that they do.

B. What Is History?

The next major question that must be tackled at the beginning of the course is "What do we mean by history?" To introduce this question we did the following exercise at the beginning of a class. I had the students write for ten minutes on what happened in Tiannemen Square (any major event that most of the students knew about would do). Many students looked rather puzzled, but agreed to entertain me for such a short period of time. [When I asked who had heard of Tiannemen Square, all but one student raised their hands. I told the one student to wait and he would learn about it momentarily. Later he told me that he had indeed known about Tiannemen Square, but he couldn't believe that that is what I was referring to since it had nothing to do with mathematics. He assumed Tiannemen Square was something like the Pythagorean square or the golden rectangle!] Then I asked a number of students to read what they had written. There was a wide variety of responses. One student who had been to China and knew quite a bit about its history wrote a long fairly technical summary of political and military factors involved in the tragic event. Another student wrote a very simple synopsis, " a bunch of students were protesting for democracy at the square. The government sent in tanks and killed many of the protestors . . ." Still others took very different points of view. The point, of course, was to see how each person could have a very different perspective of a given historical event. We discussed how 2,000 years from now, one of their sheets of paper might be the only surviving document of what happened in Tiannemen Square. In the same way, when we look back 2,000 years to ancient Greece, it is difficult to piece together what really happened, or the nature of mathematical activity at the time, or what Pythagoras discovered (if he did in fact exist) from the little we have to work with. And the further back we go, the more difficult it is to gather evidence or information. For this reason it is important to take all recorded history with a grain of salt. In addition to limited records, human bias can also be a major factor in shaping our image of the past. This becomes particularly clear later in the course in looking at Western descriptions of African mathematics [see Zaslavsky].

Organization of Class

The class met two days a week, Tuesday and Thursday for an hour and a half. It was part of the freshman seminar program which had several advantages. The class size was small (enrollment is limited to 15). The purpose of the freshman seminars is to have interdisciplinary, discussion oriented classes in which faculty and students get to know each other well. There was also some funding to bring in guest speakers. The freshman seminars are writing intensive courses, hence the large amount of writing for the course.

Because this was the first time this seminar was being offered, and enrollment was not full, we did allow several upperclassmen to join the class. The benefit of having these students is that their additional experience in mathematics made the class much richer for everyone, and made for livelier discussions. One thing that became quite clear from having these older students, is that this course would be extremely successful as a junior or senior level course as well, and of course there would be much more mathematical experience to draw on.

As the syllabus indicates, the course was driven by questions. One of the major goals of the course was to have students generate their own questions; this is one of the most important aspects of learning and one that is often neglected in the classroom. In the beginning of the course students were given specific questions such as "What is Platonism?" or "What is Euclid's myth?," but as the students matured the questions became more open ended such as "What are the values underlying African mathematics?" or "What are similarities and differences between ancient Greek and Babylonian mathematics?" or "What might account for those differences?" As the course progressed, they came up with more and more questions of their own.

Assignments

Throughout the course students developed both their informal and formal writing voice. The informal voice was developed in journals which they kept throughout the term. The formal voice was used in two research papers and a final exam. The journals were an important part of the course. Students used them to take notes on the readings, give a brief synopsis of each article, critique the readings and record their own questions and thoughts about the course in general. They served as a vehicle for learning how to do close and critical readings. Having these detailed records not only made class discussions much better, it was also extremely useful as a reference for their papers and exams. I would collect them approximately once a month, and give comments.

The two research papers corresponded to the first two parts of the course. The first was on a topic pertaining to mathematical people, the second was on mathematics in different cultures. I encouraged students to find topics that were of particular interest to them. This was an opportunity for students to delve more deeply into a topic we covered in class, or to go in another direction.

The final was the culminating experience of the course. It involved two parts. The first part was to write two essays. One was to return to the original question of the course "What is mathematics," and use readings and discussions to support their position. I was interested to see how the course had affected their thoughts on this question. The second essay was "Discuss the ways in which society can influence mathematics." This was also meant to pull the whole course together and to reflect on the central theme of the course. [See comments in section on evaluation].

The second component of the final was to create a History of Mathematics timeline. This was a smashing success, and I was delighted by the results. Students were encouraged to be as imaginative as they could in creating a time line of any form. I wanted them to think about questions that were very important in the course. What will they choose to include in the timeline? What criteria will they use in making their selection? Where will they begin the timeline, how does this effect the story that is presented? They were asked to include a written rationale/explanation for their timeline, addressing these questions. They were also allowed to work in pairs if they wanted to.

The projects were indeed quite creative and varied. Two students made a History of Mathematics video, one wrote a history of mathematics children's book, some did various types of posters and maps. One excellent project was done by two women in the class. It was a mathematical quilt. Because it speaks for itself so well, I will include their essay describing the quilt in this article. I found this project significant for a number of reasons. We had spent time in class discussing the metaphors that are used for mathematics, and how one's metaphor influences the way one approaches mathematics. I found it very interesting that these two students chose the metaphor of a quilt for the history of mathematics. A quilt, being an archetypically female symbol, was a new way of conceiving of the history of mathematics, one which lends itself quite naturally to discussions of relationships: relationships between cultures (corresponding to the different colors in the quilt), relationships between different time periods (the positions on the quilt), and relationships between individual people or facts (represented by a patch) and the whole matrix of the quilt. I think it is not a coincidence that this project emerged in a course in which gender was recognized as a legitimate and significant factor of analysis. This created an atmosphere in which the students could choose a form of expression that might not have otherwise surfaced, or that they might have stifled for fear that it would be inappropriate.

Evaluation

This article is meant to generate ideas for alternative ways to approach the history of mathematics. Overall I thought the course was quite successful and most of the students seemed to agree. But as always there is endless room for improvement. One thing that must be acknowledged from the very start is that this was a kind of survey course, though it was not intended as a comprehensive survey. Perhaps a more appropriate description comes from the title of Asger Aaboe's book "Episodes in the Early History of Mathematics." It was meant to give students a glimpse at the deep and rich history of mathematics, but even more importantly to have them begin to think for themselves about the nature of mathematics, the evolution of mathematics, and the *humanness* of mathematics. My hope was that it would stimulate their interest in at least some aspect of the history of mathematics that they might pursue on their own. And at very least, that they might begin to see mathematics with new eyes, as an organic enterprise.

Because there was so much material to cover it was difficult to do justice to all aspects of the course. The third part of the course became much shorter than I had first envisioned. These kinds of balances can be played with depending on the interests of the students and the teacher.

In retrospect, I would revise the final so that there was at most one essay question (the second one). This is plenty of opportunity to pull the course together, and allows more time for the timeline projects. I would also alter some of the readings, delete some, add others. But those are the kind of changes and decisions that keep teaching stimulating.

Summary

As the demographics of our society change, we must be responsive to the changes in our population. The most notable change in higher education is the increased representation of various ethnic groups, and the now equal representation of women. In all disciplines, it behooves us to reevaluate how we teach our material, what we consider important and how we tell the story of the past.

The first step in opening the world of mathematics to other people is to find ways to make it more relevant to their lives. One way to do this is to look to the past to see how mathematics emerged in different cultures, and why. What made mathematics relevant to the lives of the people who developed it? For example, it may be that music or art or games are the appropriate vehicles to introduce mathematical concepts of symmetry and combinatorics. If, as Steen suggests, mathematics is the study of patterns, we must decide where we look for patterns. Certainly music and art are rich sources of patterns. Simultaneously, it may be that we need to emphasize that the concept of proof has evolved over time, and to a certain extent is culturally dependent.

Most of all we need to emphasize that mathematics is a process, and to de-emphasize the pervasive image of mathematics as an external, eternal and objective truth that has little to do with human beings. By seeing the connections between the lives of individual mathematicians and the mathematics that they (or a culture) produces, students gain a sense of confidence in their own ability to use mathematics for their own needs, or to discover mathematics both as a tool and a language that we use to learn more about the world around us.

OUR MATHEMATICAL QUILT

Kathy Pryor and Anne Pellett

In attempting to determine how to structure our History of Mathematics time line, we had to face the awesome task of deciding, from the vast array of events, peoples and discoveries, which we considered most important. What we have created is largely a symbolic work. We have chosen a quilt format because we believe that, over the centuries, mathematics has developed in a variety of colors and patterns which, when all pieced together, constitute its history.

The green diamonds at the center of the quilt represent the contributions of the ancient Egyptian civilization, and the red diamonds represent the mathematical accomplishments of the Babylonians. We assigned each culture a separate color, for there is no historical evidence that they ever exchanged mathematical information. We did, however, place them side-by-side, intermingled the hues, because these primitive peoples both developed their civilizations during the period from approximately 4000 B. C. – 600 B. C. The yellow squares which surround the diamonds contain some of the mathematicians and mathematical accomplishments of the Greeks from Pythagoras through Euclid. These are followed by the blue trapezoids which consist of the mathematical advances of the Muslim and Hindu civilizations during the period from approximately the 7th through the 12th centuries A. D. We constructed the center in this fashion in order to show how the developments of each separate civilization sprang in some degree from that which preceded it. For example, the ancient Greeks gathered rules for the determination of areas and volumes from the Egyptians and advanced the process one step further by establishing symbolic proofs of these methods. Likewise, the Muslim civilization acted as the caretaker for Greek mathematical documents and added new innovations of its own such as algebra. Both the Hindu and Muslim cultures were assigned the same color because they existed geographically, and are said to have shared mathematical information with each other.

The Egyptian, Babylonian, Greek, Muslim, and Hindu civilizations were established as the focus of the quilt because they represent the foundation of the Western mathematics with which we are so familiar today, which has played such an important role in our own personal mathematical development and in the scientific and technological advances which have occurred through the ages.

At each of the four corners of this epicenter is a group of 5 squares representing a century of "Western" (that associated with Europe and the United States) mathematical development. In the upper left hand corner is the 17th century; in the upper right hand corner is the 18th century; in the lower right hand corner is the 19th century; and in the lower left hand corner is the 20th century. Each square within these four divisions consists of three sheets of construction paper. The green square represents the fact that the development belongs to "Western" mathematics, or what we traditionally *call* "Western Mathematics." The color of the square immediately on top of this represents the century (17th century = red; 18th century = yellow; 19th century = orange; 20th century = white). Finally, the hue of the topmost square represents a particular person and his mathematical contribution. We selected this structure because all of mathematical history, all of the advances which are made, take their shape from a unique combination of the advances which are made, take their shape from a unique combination of the attitudes and conditions of society during a given period of time, cultural events, whether positive or negative, and the special circumstances of each mathematician's life and work. To chronologically list theorems, etc., and to simply name their discoverers would fail to provide an adequate insight into the complex forces which together culminate in a mathematical advancement.

The manner in which we selected the mathematicians who would represent the various time periods also had a symbolic intention. Within each century, we chose both individuals who are often hailed as the "Great Mathematicians" and some "little ones" whose contributions the world tends to pass over. For example, in the 19th century, we included Gauss, whose successes in the field of mathematics still can't be properly estimated, since he did not publish much of his work during this lifetime. He was responsible, among other things, for establishing number theory as an organic branch of mathematics. At the same time, however, we also devoted a portion of our quilt to Ada Byron Lovelace, for she was the first person to detail the process now known as computer programming, although it is only recently that she has begun to get the credit she so justly deserves. We sought to give a voice to mathematicians who are not as well known or perhaps are not as well-esteemed for their work — names that are not on the tip of the tongue when one is asked to list significant mathematicians. Although their efforts may not be regarded as gigantic achievements, they were nevertheless important because they constituted some sort of advancement, an attempt, however small, to move the field of mathematics on to greater development, not to allow it to stagnate. Besides, very often the smaller works ultimately facilitated the greater discoveries. The black squares and triangles in the quilt represent those who perhaps made mathematical innovations or worked diligently to solve some mathematical enigma, though possibly without great success, whom the history books don't even *mention*. To us, these unknowns are equally valuable as the knowns, for it is the spirit of all working mathematicians that keeps the flame of mathematical knowledge burning brightly into the future.

The squares which are half the color of one century and half the color of another are used to show that the mathematical developments which occur in one time period, ultimately have an impact on the advancements of later eras. Achievement is not made in a vacuum, but relies on the knowledge of times long gone by; it is a cumulative entry.

Finally, the red semi-circles and the orange triangles represent the mathematics developed by African and Chinese civilizations, respectively. We assigned these cultures a different shape because the mathematics "invented" by them is not historically believed to be linked with the progression of traditionally Western mathematics. To us, however, their mathematics *is* significant and valuable, and should be considered part-and-parcel of the history of this subject. It widens one's view of what specifically constitutes mathematics. The skills of pattern recognition and gesture counting, for example, which the Africans have cultivated and their proficiency in which we could not equal, show that it is possible to look at math from a different perspective. More importantly, by including these cultures in our quilt, we wanted to stress that just because their outlook is alien to ours, that doesn't mean it is any the less mathematical than our systems of formalized proof, etc.

Our quilt has been an effort to symbolize the complex forces at work in the development of mathematics. Indeed, the history of this subject contains much more than can be represented by a two-dimensional time line (_|_|_|_|_). It is a story of people of all genders, races, nationalities, etc., influenced by the times, by their cultures, and by their own unique personalities, striving to break new ground, to further the cause of mathematics. The combination of all of these motley patches together with the ones to be "sewed on" in the future (shown by the black fringe) comprises the history of mathematics. This is what we have learned and this is what we will take with use from this course.

HISTORY OF MATHEMATICS SYLLABUS FALL 1989 — HENRION

9/12

Topic:
General introduction to the course.

Assignment:
Hand in math autobiography.

9/14 – 9/19

Topic:
What is mathematics?

Readings:
1) *Mathematical Experience*, pp. 319–331, 391–399
2) "Three Crises in Mathematics" by Snapper in *Mathematics: People, Problems and Results, Vol. 2* [MPP&R, V2]
3) "The Science of Patterns" by L.A. Steen
4) "A Dialogue on the Applications of Mathematics" [MPP&R, V1], pp. 255–263
5) "Music of the Spheres" [MPP&R, V1], pp. 61–71

Discussion Questions:
1) What do we mean by mathematics? (What areas are included?)
2) Is mathematics discovered or created?
3) Is mathematical knowledge certain, i.e. always true?

4) If you were on an isolated island with other people, what kind of mathematics would you develop, if any?
5) What is the difference between pure and applied mathematics?
6) How do we decide what is important mathematics?
7) How do we decide what is true in mathematics?
8) Have the answers to any of these questions changed over time?

Assignments:
1) BEFORE doing the readings, write a short 1–2 page essay on "What is mathematics?." This is an informal essay on your reflections of what mathematics is all about. There is no right answer, I am interested in what *you* think. (DUE: Thursday, September 14)
2) Keep notes on the readings in your journal. Also write down your thoughts about the discussion questions in your journal.

9/21

Topic:
What is history? What is the history of mathematics?

Readings:
1) "History of Mathematics: Why and How?" by Andre Weil
2) "Reflections on Writing the History of Mathematics" by Philip Davis
3) "The Centrality of Mathematics in the History of Western Thought" by Judith Grabiner

Discussion Questions:
1) What is history?
2) What is the history of mathematics?
3) Is there such a thing as true history?
4) How do we decide what counts as important history?
5) Should the history of mathematics be approached differently than other types of history?
6) What would be interesting to *you* in the history of mathematics?
7) Discuss how you would go about looking for evidence of mathematical activity in an ancient culture. What would you look for? Where would you look for it?

9/26

NOTE:
Guest Speaker

Topic:
Mathematical People (Introduction)

Readings:
1) *Mathematical Experience* : "The Current Individual and Collective Consciousness," "The Ideal Mathematician," "The Individual and the Culture."
2) "Career and Home Life in the 1880's: The Choices of Mathematician Sofia Kovalevskaia" by Ann Koblitz in *Uneasy Careers and Intimate Lives.*

Discussion Questions:
1) What kinds of people do mathematics (is it possible to find similarities/themes)?
2) What motivates them (initially, sustains them)?
3) What kinds of family background (does this have bearing on their work)?
4) Personality. (Does this have bearing on work?)
5) What are the social conditions of the time? Does this effect their work?
6) How did they gain access to mathematics?
7) How do they view the nature of mathematics?
8) To what extent do mathematicians influence each other, to what extent do they work alone (i.e. role of the community)?
9) How have answers to these questions changed over time?

9/28

Topic:
Early Mathematicians (Pythagoras, Archimedes, Euclid)

Readings:
1) *A Short Account of the History of Mathematics* by W.W. Rouse Ball (on reserve in Starr Library). Chapter 2, p. 13–32 (especially material on Pythagoras); p. 50–77 (especially Euclid and Aristotle).
2) Boyer, Chapter 4 "Ionia and the Pythagoreans."
3) Chapters 7 and 8 of Boyer.

10/3

Topic:
Mathematicians 17th–18th Century (Descartes, Newton, Leibniz, Euler)

Readings:
1) [MPP&R V.1] Newton p. 113–124.
2) "The Life of Leonard Euler" by Rudolph Langer.
3) "Descartes" by Oliver Wendell Holmes.
4) "The Great Mathematicians" by Darrah.
5) Look through Chapter 19 of Boyer (many of you may not be familiar with much of the mathematics in this

chapter — that's okay, just read through what you can).
6) Boyer, p. 367–371 (Descartes).

10/5

Topic:
Mathematicians in the 19th-early 20th Century

Readings:
1) [MPP&R, V1] Gauss p. 125–133.
2) Hardy's *A Mathematician's Apology* (on reserve in Starr Library) p. 61.
3) [MPP&R V1] Mordell piece of Hardy, p. 155–159.
4) Ramanujan material.
5) [MPP&R, V1] Hamilton, p. 134–144.
6) [MPP&R, V1] Littlewood, p. 145–154.

10/10

Topic:
Modern Mathematicians 20th Century

Readings:
Choose 3 from *Mathematical People* (on reserve) — present one

10/12

Topic:
Women in Mathematics

Readings:
1) *Math Equals* by Teri Perl
2) Read whole book including summary.

Discussion Questions:
1) Who were the women in math?
2) Why so few?
3) What are the obstacles they have had to overcome to get into Mathematics and then to stay in?
4) What is the role of the larger community in their lives (support and discouragement)?
5) Are their motivations, sources of support any different than the men we've considered?
6) How do they view the nature of mathematics?
7) What do they like about math?
8) Are there historical periods in which there were more women doing math? What factors are important?

10/16

Guest Speaker

10/17

Topic:
Finish up and summary of "People in Mathematics."

10/19

Paper #1 Due
5 minute Presentations on papers

Topic: Mathematics in Different Cultures

10/26–10/31

Topic:
Early African Mathematics

Readings:
1) *Africa Counts* by Claudia Zaslavsky — Sections 1 & 2 (p. 1–58), Section 5 (p. 153–196).
2) Zaslavsky, Sections 3 and 4.

11/2

Topic:
Babylonian Mathematics (Ancient Mesopotamia = modern Iraq)

Readings:
1) Asger Aaboe, *Episodes from the Early History of Mathematics*, Chapter 1 (p. 1–33)
2) Boyer, Chapter 3.

11/7

Topic:
Early Egyptian Mathematics

Readings:
1) [MPP&R, V1] p. 3–17.
2) Boyer, Chapter 2.

11/9

Topic:
Early Greek Mathematics

Readings:
1) [MPP&R, V1] p. 18–27.
2) Boyer, Chapters 4 and 5.

11/14

Topic:
Early Chinese and Indian Mathematics

Readings:
1) [MPP&R, V1] p. 28–37.
2) Boyer, Chapter 12.

11/16

Topic:
Muslim Mathematics

Readings:
1) [MPP&R, V1] p. 38–46.
2) Boyer, Chapter 13

Topic: Non-Euclidean Geometry

11/28

Reading:
Chapter 1 "Euclid's Geometry" from Greenberg *Euclidean and Non-Euclidean Geometry.*

11/30 and 12/5

Readings:
1) Read Chapter 5 (p. 121–129), Chapter 6 (p. 140–147) from Greenberg.
2) Reread "Non-Euclidean Geometry" from *The Mathematical Experience* by Davis and Hersch.

12/7

Reading:
Read Chapter 8 from Greenberg.

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STUDENT SEMINARS ON "FAMOUS EQUATIONS"

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Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.
— Johannes Kepler [1571–1630]

Treasures often lie obscure in mathematics programs constrained by rigid syllabi and taxing workloads. This report describes a practical way for students and faculty to examine some of the golden threads and sparkling jewels which are woven into our mathematical heritage.

Each winter term a small troop digs through the library stacks for information about famous equations. These diggers are participants in our Famous Equations Seminar-course. Each is looking for the general lore and specific features of a personally chosen equation, the makings of a seminar talk and term paper.

This course is overtly intended to counter the impression that mathematics somehow sprang full-grown into the bindings of textbooks; and that the only way to learn math is to study the text and listen to an instructor.

Credit participants meet once each week for two hours. The first session emphasizes the collegium nature of a seminar, puts in a plug for history, and discusses the broad meaning of "an equation." During the next few sessions, faculty talks are presented on a famous equation or two, providing reasonable models for the students as they prepare their own presentations. The student talks follow, one each week. All talks are open to the public. The final session recaps and gathers common threads from the series' talks, and returns to the philosophic issues raised in the opening session. It is also the time for students to exchange final versions of the papers they have written; each student departing with an anthology of "famous equations."

The theme of "famous equations" has proven ideal; it is catchy, permits well-focused individual investigations, and is nicely comprehensive. As our hallway poster expounds, "Over the centuries certain 'truths' have been discovered and (almost too) neatly packaged into now famous equations such as:

$$x^n + y^n = z^n \qquad \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

$$\overline{Q} = R$$

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} \qquad V - E + F = 2$$

$$A = \pi r^2$$

$$c = 2^{\aleph_0} \qquad C = R[\sqrt{-1}]$$

and $e^{i\pi} + 1 = 0$. The ideas captured by these germinal equations are body and soul for much of mathematics. The struggle to formulate these equations is a tribute to human persistence in the search for understanding." A student tracking down the heritage of any one of these equations is quickly exposed to the humanistic side of mathematics evolution.

Novel to many students is the idea that they are central in the information gathering and sharing process. A letter to prospective participants sets the cooperative tone of the series, and the course "Guidelines for Papers and Talks" pointedly reminds them that their audience is themselves. The Guidelines also indicate the appropriate mix of lore and rigor, and the expected level of performance. (A copy of the Guidelines is appended.)

The two-hour meetings allow for an hour talk, a break, and an open discussion about the newly presented information and earlier talks. The post-talk discussions are extremely valuable; into them often pop bits of information not included in the prepared talks. And it is especially during these informal exchanges that participants come to know and respect one another.

In conclusion, the seminar-course format recommends itself for several reasons:

(1) It is practical. Participants readily fit the series into their schedules, and are able to do much of the required work well in advance. The instructor needs to select participants, compile a partial list of possible equations with a few initial references, give a couple of talks, and carefully critique eight 15–20 page papers.

(2) It is adaptable. The level of equations may range from 1+1=2 to $\beta X = X$.

(3) It is immersive. The students together discover cross-threads of history. ("What, Euler again!" or "Your Fibonacci is in my Pascal triangle!")

(4) It is self-regulatory. Participants have automatically striven to do quality papers and talks for the benefit of their peers.

(5) It is naturally fortifying. It provides genuine (albeit fledgling) experiences in library research and in professional writing and speaking.

(6) It is fun!

Most importantly, the format fosters learning without a textbook and brings people together for the purpose of sharing information about a common interest — surely treasures of humanistic learning.

APPENDIX — Famous Equations
GUIDELINES FOR PAPERS AND TALKS
MATH 399 HONORS: FAMOUS EQUATIONS
WINTER 1987

PAPER

First draft due February 23.
Final paper due March 9.

This is to be an exposition written for your fellow students. As you write you should imagine the typical reader to be an upper-division mathematics student with some ability and curiosity. Your job is to organize, summarize, and verbalize what you find out about your equation.

The paper should contain both general lore and specific features. By "general lore" we mean such things as historical appearances, significance at that time and later, personages involved, and interesting uses. By "specific features" we mean the statement and details of interesting theorems, mathematical arguments, unexpected applications, variations or generalizations. On proofs, use your judgement about how much detail should be included in the text itself. Sometimes the details are the heart of understanding; other times they should be relegated to an appendix.

Be sure your paper includes those things which, when first encountered by you, evoked reactions such as: "That's neat!," "Clever!," "I don't believe it!," "Hmm..." or "Curious!"

It is expected that an Honor's paper will conform to proper English practices. The paper should contain an introduction and a conclusion. References tagged to the text are essential and a briefly annotated bibliography is very desirable.

The use of ugly sentence construction and sloppy punctuation are cause for flogging.

You do not need to be flowery with words. It is important to be direct, uncluttered and clear.

The paper should be 15–20 pages (not counting appendices and bibliography).

TALK

Your talk is another way to tell your friends about your equation and its curiosities. It should not be treated as a quick reading of your paper; it is an entirely different medium.

Spend some time telling us (the audience) about your equation and its "general lore." Then take one "specific feature" which you found particularly interesting and sketch details with enough comment and discussion to illuminate its significance and finer points.

Prepare well ahead of time! Think about what we (the audience) don't know, then plan a progression to enlighten us. Once you start your talk, treat it as a casual, but directed, chat with your friends.

THE HUMAN/COMPUTER INTERFACE: THEIR SIDE OR OURS?

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The much touted user-friendliness of computers, like any other aspect of popular culture, has presuppositions underlying it. In particular, it presupposes that there is a human/computer interface and that humans are on the side opposite to the computers. This essay is concerned with this possibly erroneous presupposition.

Because I can think of no better way to introduce my subject, I am going to approach it chronologically. Two things happened to me at the beginning of February that prompted the considerations I am sharing with you. Let me tell you about them.

I was shown an examination question that was well worded but about unfamiliar material. It had to do with positions on the surface of the earth and the position of the rising sun on the horizon. The careful wording was spoiled by the accompanying diagram, which included a circle apparently representing a sphere. The sphere was not the surface of the earth, but rather the celestial sphere viewed either from an unnatural position outside it or from the almost equally impossible position on it opposite the zenith. The labels 'equator' and 'north pole' did nothing to distinguish the diagram from one of the earth. We are all familiar with badly posed problems, but I was struck forcefully by this one because I had not posed it badly myself.

Posing a problem badly is a standard way to make a problem difficult. It is notorious that problems that problem solvers are called upon to solve in the so-called real world are badly posed, but I do not offer this fact as an excuse for unintentionally making problems hard by posing them ineptly. Other reasons that one finds difficulty in interpreting a problem are that the mathematics or the area of application is unfamiliar or that one does not grasp what the problem states or asks. The student too can be inept. It is equally notorious that 'if Johnny has five marbles and loses two, how many marbles has Johnny left?' is more likely to produce an incorrect answer than '5 - 2 = ?.' Even when there is familiarity with the subject matter and the mathematics, the problem is well posed, and the student understands the problem, the hated 'word problems' are more difficult than five minus two.

My second jolt came from two students appealing grades of C and F in an applied-mathematics course. The student appealing the grade of C enclosed with her appeal a transcript of her high-school marks. It revealed steadily and substantially dropping marks in mathematics and low marks in English. She complained, as did the student appealing the grade of F, that she had worked very hard at the mathematics (induction, sequences, equations, trigonometry, and complex numbers) but that she had been hindered in obtaining the grade her effort deserved by her marks on term tests that had not been fair tests of 'mathematical principles' but instead had required 'interpretation.' I am enormously grateful for these students' causing me to focus on what precisely they were complaining of, which was that they — both native speakers of English — were required to understand a couple of English sentences, see what mathematics in the course was involved, and do it. Term tests in other sections of the course, they alleged, asked questions of a purely computational character, and these two students felt that they had been disadvantaged by the disparity in the term tests, having written a common final examination with the other sections.

These students were insisting — with some asperity — that it was unfair that I had demanded that they think as well as calculate. Not original creative thought, not even the less original creative thought of problem solving, but merely the thought of perceiving in some words an intelligible structure from a small list of intelligible structures on which they were being tested. They did not claim that it was not obvious what to do once they understood what the problems were about. They were claiming immunity on account of what I called above 'student ineptness.' They were claiming as a grievance that I had asked them to do the translation from Johnny's marbles to five minus two. This jarred me into considering seriously whether this was unfair.

If one takes the process that these students were unsuccessfully engaged in as being:

(1) extracting an intelligible structure from a context,
(2) calling upon a prior knowledge of that intelligible structure,
(3) engaging in routine ways of dealing with that structure,

then one can see one of the differences between teaching applied mathematics and teaching pure mathematics. In the latter, the structure is foremost and the others are there for the sake of learning about it; in the former, the structure is there to supply the necessary framework for the processes of extraction and solution.* In both cases, teaching is primarily about the structure, since the structure is logically prior to its extraction and to ways of dealing with it. If our tests and examinations test only the routine ways of performing calculations (3), perhaps intended to test a knowledge of content (2), but ignore 'applications' (1), then we are testing only what the students will do — after the examination is over — only by calculator or computer. We will be testing them solely on what they do not need to do and ignoring what it is increasingly important that they be able to do if they are not to be replaced by machines.

My students were complaining that I put them on the wrong side of the human/computer interface. At least I did! But I was not being up-front about it, just doing it automatically because they were my students. You can't get away from those presuppositions of popular culture.

Having returned now to the human/computer interface, I should say the little I want to say about computers: my subject is human. In the past decade, there has been a movement to take account of the availability of computers, especially in calculus and especially in the U.S.A. There has been a ICMI conference on the topic [1, 7], and a number of books have been written that make a gesture or more toward the fact that some students of calculus have access to a computer. This is inevitable, and with time it may become more generally not just a marketing gimmick but something more substantial, as for instance with David A. Smith's *Interface: Calculus and the computer*. Not being in the U.S.A. and not teaching much calculus, I have been more concerned with getting students on top of the capabilities of their pocket calculators and have been thinking that the availability of computers is far more significant to algebra than to calculus. It is in algebra particularly that Jon Barwise [2] has drawn attention to the problem of Miles, namely 'that symbolic mathematics packages may make it even harder for our students to understand the meaning of mathematics.' As Miles put it [9],

> Use of an algebra utility can eliminate the need to know the words and usage of algebra — the core of the language of applicable mathematics. Unquestionably one can persevere in calculus on this basis — many students already do so without benefit of algebra utilities. Whether one can find meaning in doing so is doubtful. And it is a serious question whether colleges can prosper without imparting a greater sense of meaning to their curricula.

In the terms I introduced above, computer power renders one's routine ways of dealing with mathematical structure possible without knowing that intelligible structure, but without that knowledge one cannot seek and find the structures in their non-mathematical contexts. This renders the structures invisible as well as meaningless. Applied mathematics becomes impossible to a human for the same reason as it is impossible to a computer: the mathematics has been reduced to software. The human has slipped across the human/computer interface. I see this as a danger to be combatted. (On meaning in mathematics, see [8] and [12].)

On a more humane side, another educational movement has spawned meetings and now a book, *Writing to learn mathematics & science* [4]. Both the Humanistic Mathematics Network and other organizations have been exploring ways of engaging students in the learning of mathematics, including writing about it. Three recent papers [6, 10, 11] have drawn attention to the benefits — even if only to their ability to write — of having students write about what they are doing when they are doing mathematics. By embedding mathematics in prose a large step is taken toward making it meaningful and something that can be recognized outside the classroom. In the context of teaching mathematics to first-year engineering students at the University of Manitoba, it might be possible to combine efforts with their technical-writing course in a way not wholly unlike Duke University's course, Introductory calculus with digital computation, which, as the title indicates, involves computers, but also involves weekly lab reports including from one to three pages of expository writing along with the data and graphs [6]. The possibility of benefits to both courses — and ultimately the students — merits investigation.

More universally, my students' complaint has brought home to me, as well intentioned things I have read have not, that we need to encourage the hated interpretation.

I can fairly claim that I have always done this, and I have the student complaints to prove it. But I have done it only on tests and examinations. I have never talked about it, warned them of it, pressed them to practise it, helped them with it (except individual difficulties). As Clement and Konold demonstrate with the scarcely mathematical problem ([3] adapted from [14]),

> What day precedes the day after tomorrow if four days ago was two days after Wednesday?

the difficulties are enormous even without any mathematical complexity at all. In the above taxonomy, difficulties with this are purely student ineptness, and whose job is it to help them with it but ours? Not only have I been remiss in expecting interpretation only under testing circumstances, but also I have neglected to influence my colleagues not to pose trivially mathematical questions on their tests and examinations. What I have done has been seen as my way of doing things and therefore tolerated (by colleagues) or complained of (by students). I have now realized that I think that what I have been doing is right — though far too limited — and I am prepared to defend it. (I am not prepared to defend wording questions badly.) The terms in which I defend it are those of the human/computer interface. It is easier for students to respond to keystroking than to the presentation of what is intelligible but not yet converted into ASCII codes. Students, like the rest of us humans, prefer what is easier. But computers respond to keystrokes far more dependably, powerfully, and quickly than they can; they cannot compete. What they must learn to do is extract intelligible structure and frame it in such a way that they can do the keystroking. In order to do this, they need help.

As a first step toward influencing my colleagues, I have suggested three things that I think I and others should do:

> shun meaningless manipulation,
> engage students in verbal expression of meaning,

and

> insist that students cope with verbal presentation,

all to teach them some mathematics usefully and by contributing to their education to keep them from slipping across the human/computer interface.

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* Two quotations from respondents to the survey reported on in [5] illustrate this.

"Applied departments use math as a tool. An individual topic is analogous to a hammer perhaps. They wish to 'hammer' with it. On the other hand, math departments often become more interested in its description and generalization of the 'hammer' itself."

"I cannot take it for granted that [students from calculus] are able to use their mathematical skills in problem solving. What appears to be . . . lacking is the ability to formulate a problem quantitatively and then to solve it using the tools they learned in their calculus course."

AUGSBURG'S HUMANISTIC CURRICULUM PROJECT

Larry Copes and Beverly Stratton

Summary: The Department of Mathematics at Augsburg College has embarked on a project to replace the traditional calculus/linear algebra sequence for mathematics and science majors with a curriculum more representative of the ideas and humanistic processes of mathematics.

Background and goals

This project grew out of several frustrations our mathematics faculty has had:

- We are not preparing most lower-division mathematics students to decide whether or not to become mathematics majors. They do not become aware of the breadth of mathematical ideas, and we do not teach them the logical or creative mathematical thinking mathematicians use.
- Nor are we preparing most lower-division mathematics students to major in the sciences. We do not give them a sufficiently deep understanding of the mathematical concepts they do encounter, nor do we introduce them to the breadth of mathematical topics now being used in the sciences.
- We are not teaching most lower-division mathematics students to read mathematics well enough to fill in these gaps in their mathematical knowledge. Nor do we teach them to write mathematics even well enough to communicate their mathematical results clearly, much less to use writing as a tool for better thinking.
- Numerous good, creative mathematics students drop out of our calculus sequence expressing a personal distaste for calculus, without realizing how broad the field of mathematics is.
- After one term, the non-science students in our "Mathematics for Liberal Arts" course know more about mathematics than our majors do. They are more aware of the processes involved in doing mathematics, and they understand more about the historical connections between mathematics and the rest of culture.

We want a curriculum that alleviates these frustrations, a curriculum to replace our traditional calculus/linear algebra curriculum for prospective mathematics and science majors.

What have we done toward that end? First, we talked a lot among ourselves, and with our science colleagues. We found ourselves in the unusual situation of having an entire mathematics faculty willing to work at this, and a science faculty supportive of experimentation in this direction.

Then, based on our conversations, we drew up a list of overall goals. We decided that the goals of this project are that science and mathematics majors

- achieve a deeper understanding of calculus and linear algebra concepts than they do now;
- encounter more breadth of mathematical ideas than the current sequence provides;
- think more logically about mathematics than have students in recent years;
- read and write mathematics better than they do now; and
- be more aware of the cultural roots and influence of mathematics.

Then we acted. We applied to the NSF calculus reform program twice, with negative results. The breadth of topics in our proposed program means that it is not just calculus. So we turned to FIPSE, the "Fund for the Improvement of Post-Secondary Education." It's the only part of the Federal government that we know of that prides itself on sponsoring innovative, cutting-edge programs and on getting them institutionalized.

FIPSE funded a three-year project, starting this past fall. Most of the support is for released time for four of the mathematics faculty to prepare the new curriculum. We're in the first year of that project, getting ready to teach the first year of the sequence next year. Next year we'll prepare the second year of the sequence, and during the third year of the project we'll teach both years of the sequence, prepare teaching materials for others, and host a dissemination conference. All along we'll be evaluating the effectiveness of the results.

Implementation

What have we done so far? First we had to determine where to start. Should we begin by deciding on the mathematical topics? How? Should we start with the current sequence and decide what to eliminate, or should we build from scratch? Or should we come up with particular objectives first? Should we decide on an overall organizational approach to give continuity to the courses? Should we each draw up a proposal and then merge them, or should we work as a group? How should we make decisions? These were some of the many questions we had to deal with initially. Some of them are still being discussed.

Fortunately, FIPSE encourages groups to pay a lot of attention to process, reasoning that even if our results don't fit well at another institution, our process might inform that institution's faculty in designing its own curriculum.

We consciously decided that group ownership of the project was extremely important, perhaps more important than sticking strictly to the details of the proposal we made to FIPSE. Striking a balance is still difficult, however.

Helping us gain ownership was our common experience in teaching our "Mathematics for Liberal Arts" course. Working against us were some differences: in the goals we wanted to stress, in visions of the final courses, in length of teaching careers, in preferences for involvement in group work, and in teaching styles. We've spent a great deal of time getting to know each other better and learning to work with those commonalities and differences.

Through this process we've reached some decisions:

- We've come to accept that each of us will take a different approach to the ideas, some more historically-based than others, so that we will not be specifying a single day-by-day sequence, but rather several.

The disadvantage of this approach is that our list of topics and the written materials will have to be compatible with several sequences. The advantages, however, are that we won't have to come up with a single sequence of topics with which we all can live, and that the results should be more widely adaptable.

- We even have some tentative lists of mathematical topics for the first year. At this point it appears that about half of the class sessions will be spent on calculus ideas, with about the scope of (but with more depth of understanding than) a short calculus course for non-science majors. The other half will range through geometry, probability, combinatorics, number theory, matrices, graph theory, and simple algebraic structures. We expect that the third term will be a more abstract approach to many of the same ideas, with more of the calculus details.

Right now we envision a fork in the road after the third term. Replacing our differential equations course will be a course in applied mathematics, including not only differential equations but also, for example, more of the vector calculus used by scientists. For the more theoretically-oriented will be a course with more abstraction and rigor, answering many of the "how do you know you can do this?" questions that arose earlier in the program.

- Although we've listed traditional categories of topics above, and we'll be flexible in allowing a variety of approaches to these topics, none of us expects to consider the topics in traditional chunks. We ourselves are excited about connections among mathematical ideas, and we want our students to encounter many of those relationships. So each of expects to interweave the categories in some way.

This spring we plan to gather written materials from a variety of sources, get permissions to use those materials, and write a study guide to make connections among those various materials. We know that we'll have to write some materials ourselves, but we hope that we won't have to write too much, at least this year.

A plea for help

This all has been leading up to a plea for help along three lines:

- We want to know of excellent writing, expository and technical, about any mathematical ideas at all, but especially those we've listed above. We're very interested in writing that stresses the mathematical processes involved in developing ideas, not just the results.

- Along the same lines, we want to know your own ideas about approaches to mathematical topics that illustrate how scientists or mathematicians do mathematics.

- Finally, are you personally interested in eventually providing a section or two of a course that would follow these general ideas? If so, what kinds of evidence from the evaluation of our program would it take to convince you, or your department or dean or whoever, that this approach is worthwhile enough to try?

To give suggestions or receive more information, please contact the project director, Larry Copes, at Augsburg College, Minneapolis, MN 55454, 612/330-1064, or through e-mail at copes@augsburg.edu.

ETHICS IN MATHEMATICS: A REQUEST FOR INFORMATION

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As part of a recent reform of general education, Longwood College initiated a requirement that all students take an ethics course at the junior or senior level. Each academic department was encouraged to develop an ethics course designed specifically for its majors. It fell my lot to design such a course for the Mathematics and Computer Science Department. This note is an appeal for help.

My research indicates that a good deal of work has been done on ethics in computer science. The Association for Computing Machinery (ACM) and the Institute of Electrical and Electronics Engineers (IEEE), the two major professional organizations, have professional codes of ethics. There is an active professional group, Computer Professionals for Social Responsibility, whose mission is to develop ways to deal more effectively with ethical problems. Numerous computer science departments offer ethics courses, and I have contacted some of the teachers. Course syllabae, materials, and texts exist.

Very little appears to have been done on ethics in mathematics. I can find no systematic treatment of the subject, not even a thorough bibliography. Many pure mathematicians appear to feel that ethical concerns are not pertinent to their discipline. I have found little evidence that applied mathematicians have thoughtfully considered the issues.

Important ethical considerations do occur in mathematics, however. Here are two of which I am aware. First, should algorithms be patentable? In 1988 Bell Labs was granted a patent on Karmarkar's improved linear programming method. What are the ethical implications of patenting such results of mathematical research? Second, should mathematicians submit to censorship? What if the censorship were voluntary and in the interests of national defense? The National Security Agency posed this dilemma when it requested researchers in coding theory to voluntarily submit prepublication versions of papers to NSA for review.

I am concerned with educating undergraduate math and computer science majors. I want to design and implement a course that presents them with ethical issues they will face as professionals. Do you know of researchers working in the field of mathematical ethics? Are there schools that offer such courses? Can you suggest additional case studies of ethics in mathematics?

I will appreciate any help you can give me. Please direct your responses to:

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HOW MATHEMATICS TEACHERS USE "WRITING TO LEARN"

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A Review of *Writing to Learn Mathematics and Science*, ed. Paul Connolly and Teresa Vilardi (Teachers College, Columbia University, New York and London: Teachers College Press, 1989) 307 pp., $32.95.

The essays collected in *Writing to Learn Mathematics and Science* present new ideas about how teachers are using writing to enable their students' conceptual learning in mathematics and science classes. Their students are not merely writing about topics in these disciplines; instead students are actually writing to learn mathematics and science. A number of features distinguish this book as practical and thought-provoking for writing teachers like me who'd like to affect our students beyond the freshman composition classroom, as well as for those of you who teach mathematics as one of the humanities. In the 23 essays collected here, not composition specialists, but mathematicians and scientists who have used it in their classrooms present the pedagogy of "writing to learn." Thirteen of the essays are by mathematicians who describe how they have used natural written language as an integral part of their teaching from the elementary to the college level. These teachers offer practical advice and examples of assignments that I believe you'll want to experiment with in courses at your institutions. Some assignments may resemble those you already use, and here you'll read the theories behind why they work and how they can be made to work even better. These teachers and their assignments show how the "writing across the curriculum" movement has affected mathematics programs across the country.

ABOUT THE EDITORS AND THE BOOK'S ORGANIZATION

Originating from the Institute for Writing and Thinking at Bard College, and co-edited by the Institute's director, Paul Connolly, and associate director, Teresa Vilardi, each essay in this volume shows how informal writing can transform passive students into active ones, able to understand, not copy, ideas conveyed in lectures and textbooks. Influenced by Bruner, Freire, Polya, and Vygotsky, these teachers offer practical ways to use ordinary language to enable conceptual learning in mathematics classrooms. Leon Botstein's foreword, Paul Connolly's introduction, and mathematics professor Barbara Rose's bibliographic essay provide background for the six parts of the collection:

1) Defining Problems, Seeing Possibilities
2) Writing as Problem Solving
3) Classroom Applications: What Works and How
4) Programmatic Policies and Practices
5) The Context of Learning
6) Responses to the collection as a whole from scholars Vera John-Steiner and Reuben Hersh.

JOHN DEWEY AND THE POLITICAL AGENDA OF "WRITING TO LEARN"

The "Foreword: The Ordinary Experience of Language," by Leon Botstein, President of Bard College, is well worth reading. There Botstein sets the political agenda for mathematics education as we head into the 21st century. He observes that as our passive, customary daily reliance on various technologies increases in the closing decades of the twentieth century, "the more distant and irrelevant the motivation to understand [mathematics and science] seems to have become" (xiv). This collection of essays, according to Botstein, goes a long way toward addressing this dilemma. Botstein notes how this collection is connected to John Dewey's appeal in *Experience and Education* (1938) to the role of "ordinary experience" in education because these essays "take language and writing . . . as elements of 'ordinary experience' that can be used to enhance the teaching of science and mathematics" (xii).

Further, understanding mathematics and science is becoming crucial to the enlightened political participation by all citizens which Dewey espoused. Perhaps the most grandiloquent claim Botstein makes for this volume follows here:

> The use of ordinary language can help break the cultural barriers that have prevented minorities and women from achieving well in proportionate numbers in these fields. By encouraging motivation and understanding through a method that connects the subject matter to the pupil's initial

frames of reference, the pedagogical strategies outlined in this volume can help rectify the distorted selection process within the school system through which a minority, mostly white males, emerges as sufficiently trained to consider careers in science and mathematics (xvi).

According to Botstein, then, this collection of original essays rests squarely in the progressive tradition: the authors combine faith in education with substantive expertise and the willingness to develop new pedagogical strategies.

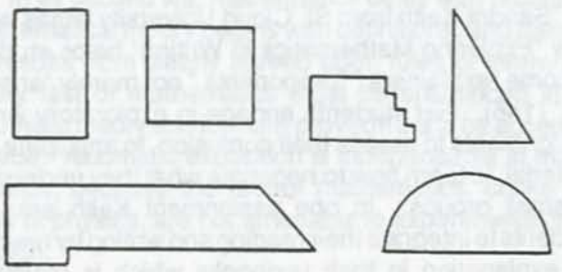
WRITTEN LANGUAGE: A HEURISTIC OF LEARNING

What do we mean by "writing to learn," "natural language," and "informal" writing? In the opening essay, Paul Connolly establishes the theoretical underpinnings of informal writing and "writing to learn" which mathematics teachers provide examples of in subsequent essays. Informal writing is done in and out of class to help students acquire ideas and concepts as their own. "Written language becomes . . . an invaluable heuristic of learning. It develops students' abilities (to read, define, hypothesize), inculcates methods (of problem solving), increases knowledge (particularly, metacognitive awareness), recognizes attitudes, and promotes collaboration" (11). For example, focused freewriting allows a student to begin exploring a term, issue, or problem. Metacognitive process writing helps the student to record her own learning behavior. Creating problems of his own rather than just answering others questions draws the student into the conversation of an expert community.

To give you a glimpse of what is possible in your classrooms, I'll tap the wealth of material from this collection that shows how informal writing meets cognitive and affective goals and enables theoretical mathematical understanding. Some examples I'll discuss apply the cognitive power of writing to conceptual learning. Others use journals to develop metacognitive awareness about the learning process. The contributors to this volume do not mandate these new approaches but they do encourage them, not just for non-technical majors but for the quantitatively adept as well. Still, they realize writing is not the panacea for all the problems that arise in teaching mathematics.

CLASSROOM APPLICATIONS: INFORMAL WRITING ENABLES PROBLEM SOLVING

In "Using Writing to Assist Learning in College Mathematics Classes," Marcia Birken describes the kinds of informal and formal writing her students do at Rochester Institute of Technology. She claims that by " . . . having my students learn mathematics through writing . . . I've learned a great deal about students' mathematical misconceptions, and I can usually pinpoint exactly where their thinking went wrong and help to redirect it" (41–42). Her in-class writing and homework assignments require students to interpret and analyze answers and to reflect on concepts. She responds to this writing and gives credit for it, but she does not grade it. Here's an informal writing assignment she calls the "Logical Order Question":



Instructions for Sheet 1. On this sheet of paper construct a geometric design using the six shapes given above. You must use all of the shapes, without repeating any shape, and keep their relative scale. You may turn, translate, rotate, reflect, or otherwise move the given shapes in any manner you feel satisfies your artistic desires.

Instructions for Sheet 2. On this sheet of paper write down, in English, the steps that are necessary to create your design. No drawing should take place on this sheet of paper — only instructions given in English sentences. Be explicit enough that someone else can follow your instructions and recreate your artistic masterpiece. (43)

Birken does grade essay questions on exams like this one that she asks in Calculus II: "We have just finished studying the Fundamental Theorem of Calculus. Write one to two paragraphs explaining why this theorem is so named and how it links the indefinite integral (antiderivative) and the definite integral" (44).

At Southern Connecticut State University and Colby College, William P. Berlinghoff helped non-technical majors to do "Locally Original Mathematics Through Writing." He has transformed such writing assignments as "Write about a famous mathematician or mathematical event" or "Report on an article" to emphasize the

"... process of solving a particular problem or examining a particular mathematical object, a problem or object assigned to that student alone" (89). For example, to develop a mathematical way of thinking among non-science students, Berlinghoff assigned a mathematical research paper on numbers that involved original research and collaboration with the teacher. His students searched for patterns of numbers from a table they had made while they were learning prime factorization. The papers they wrote were descriptions of their investigations.

Sandra Keith from St. Cloud University writes about how "Exploring Mathematics in Writing" helps students become "explainers," "proponents," not merely "answerers" (146). Her students engage in exploratory writing assignments to assess their confusion, to anticipate new material, to learn how to negotiate what they understand in small groups. In one assignment Keith asks her students to integrate their reading and writing by rewriting an explanation in their textbooks which is difficult to understand. Unlike journal or process writing, her assignments introduce the idea of context and audience for writing. For example, students produce a "crib sheet" for a friend who is behind in the course or write the author of their book a critique of a section in a chapter.

One of the most persuasive cases for writing to learn is made in this volume by Arthur B. Powell and his student Jose A. Lopez. They report a case study for "Writing as a Vehicle to Learn Mathematics . . . " in Developmental Mathematics I at Rutgers University's Newark College of Arts and Sciences. Like many students in college today, the students in this course thought of mathematics as an "abstruse symbol system" and a "fixed body of knowledge whose secrets will not be revealed" (161). To promote critical reflection on mathematical experiences, Powell asked students to write daily in journals about any topic or questions related to their learning the math in the course and to the way they felt about it — a learning log. As you'll see from the excerpts of Jose Lopez's journal that Powell includes in this essay, the entries moved from summaries of class to personal reflection on learning math. They also revealed to Powell misconceptions and gaps in Lopez's understanding.

THE CONTEXT OF LEARNING

In Part V of *Writing to Learn Mathematics and Science*, Anneli Lax and the late Hassler Whitney are among those who claim that if mathematics education is to support risk, invite experiment, and allow error, students must be encouraged to use their own language to form their own understanding of mathematical concepts. This section of the volume is concerned less with techniques and assignments than with a philosophy of learning that recognizes the need to change what continues to happen in most math programs today.

In "On Preserving the Union of Numbers and Words: The Story of an Experiment," writing teacher Erika Duncan describes the benefits of combining math and writing which she and mathematics professor Anneli Lax discovered. On their way toward designing a math course for New York University freshmen who were not expected to be very good at math, Duncan and Lax held to their

> . . . shared belief that, as different as our disciplines might appear on the surface, in both, one must not think of fixed methods for finding the solutions to a given problem, but rather one must learn to conceptualize a wide variety of converging and diverging possibilities, forever being refined as each student let her or his own beginnings shape and set up logical boundaries for each new forward-reaching step (232).

Duncan and Lax encouraged oral discussion and collaboration which allowed students to hear the process of solving problems from "the first spoken conception to the present stage" (242). Students' mathematical autobiographies revealed their individual problem-solving methods. A composition about the virtues and drawbacks of open-endedness and imaging in math and writing led one of their students to the following reflection — a reflection which captures the shared message of all the essays in the volume:

> Open-endedness has the connotation of something being incomplete and therefore not finished. But open-endedness can also mean that there is space left for further questioning and stimulation of thought. ... Images can be placed in people's minds, but will they incite a person to search for or create other images? (246)

I'm convinced by the testimony of these mathematics teachers and their students that had "writing to learn" been used in the mathematics classrooms of the '60's, I still might not be in a technical field today, but I would certainly be a mathematically literate citizen. As a writing specialist, I am encouraged that the math and science teachers who contributed to *Writing to Learn Mathematics and Science* believe all students can learn mathematical ways of thinking.

MATHEMATICS AND PHILOSOPHY: THE STORY OF A MISUNDERSTANDING

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We shall argue that the attempt carried out by certain philosophers in this century to parrot the language, the method, and the results of mathematics has done harm to philosophy. Such an attempt results from a misunderstanding of both mathematics and philosophy, and has done harm to both subjects.

1. The Double Life of Mathematics

Are mathematical ideas invented or discovered? This question has been repeatedly posed by philosophers through the ages, and will probably be with us forever. We shall not be concerned with the answer. What matters is that by asking the question, we acknowledge the fact that mathematics has been leading a double life.

In the first of its lives, mathematics deals with facts like any other science. It is a fact that the altitudes of a triangle meet at a point, it is a fact that there are only seventeen kinds of symmetry in the plane, it is a fact that there are only five non-linear differential equations with fixed singularities, it is a fact that every finite group of odd order is solvable. The work of a mathematician consists in dealing with these facts in various ways. When mathematicians talk to each other, they tell the facts of mathematics. In their research work, mathematicians study the facts of mathematics with a taxonomic zeal similar to that of the botanist who studies the properties of some rare plant.

The facts of mathematics are as useful as the facts of any other science. No matter how abstruse they may appear at first, sooner or later they find their way back to practical applications. The facts of group theory, for example, may appear abstract and remote, but the practical applications of group theory have been numerous, and they have occurred in ways that no one might have anticipated. The facts of today's mathematics are the springboard for the science of tomorrow.

In its second life, mathematics deals with proofs. A mathematical theory begins with definitions, and derives its results from clearly agreed upon rules of inference. Every fact of mathematics must be ensconced in an axiomatic theory and formally proved if it is to be accepted as true. Axiomatic exposition is indispensable in mathematics, because the fact of mathematics, unlike the facts of physics, are not amenable to experimental verification.

The axiomatic method of mathematics is one of the great achievements of our culture. However, it is only a method. Whereas the facts of mathematics, once discovered, will never change, the method by which these facts are verified has changed many times in the past, and it would be foolhardy not to expect that it will not change again at some future date.

2. The Double Life of Philosophy

The success of mathematics in leading a double life has long been the envy of philosophy, another field which also is blessed — or maybe we should say cursed — to live in two worlds, but which has not been quite as comfortable with its double life.

In the first of its lives, philosophy sets to itself the task of telling us how to look at the world. Philosophy is effective at correcting and redirecting our thinking. It helps us do away with glaring prejudices and unwarranted assumptions. Philosophy lays bare contradictions that we would rather avoid facing up to. Philosophical descriptions make us aware of phenomena that lie at the other end of the spectrum of rationality, phenomena which science will not and cannot deal with.

The assertions of philosophy are less reliable than the assertions of mathematics, but they run deeper into the roots of our existence.

The philosophical assertions of today will be part of the common sense of tomorrow.

In its second life, philosophy, like mathematics, relies on a method of argumentation that seems to follow the rules of some logic or other. But the method of philosophical reasoning, unlike the method of mathematical reasoning, has never been clearly agreed upon by philosophers, and much philosophical discussion since the beginnings in Greece has been spent on discussions of method. Philosophy's relationship with Goddess Reason is closer to a forced cohabitation than to the romantic liaison that has always existed between Goddess Reason and mathematics.

The assertions of philosophy are tentative and partial. It is not even clear what it is that philosophy deals with. It used to be said that philosophy was "purely speculative," and this used to be an expression of praise. But lately the word "speculative" has become a Bad Word.

Philosophical arguments are emotion-laden to a greater degree than mathematical arguments. Philosophy is often written in a style which is more reminiscent of a shameful admission than of a dispassionate description. Behind every question of philosophy there lurks a gnarl of unacknowledged emotional cravings which act as powerful motivation for conclusions in which reason plays at best a supporting role. To bring such hidden emotional cravings out into the open, as philosophers have felt it their duty to do, is to call for trouble. Philosophical disclosures are frequently met with the anger that we reserve for the betrayal of our family secrets.

This confused state of affairs makes philosophical reasoning more difficult, but far more rewarding. Although philosophical arguments are blended with emotion, although philosophy seldom reaches a firm conclusion, although the method of philosophy has never been clearly agreed upon, nonetheless, the assertions of philosophy, tentative and partial as they are, come far closer to the truth of our existence than the proofs of mathematics.

3. The Loss of Autonomy

Philosophers of all times, beginning with Thales and Socrates, have suffered from the recurring suspicions about the soundness of their work, and have responded to them as best they could.

The latest reaction against the criticism of philosophy began around the turn of the century and is still very much with us.

Today's philosophers (not all of them, fortunately) have become great believers in mathematization. They have rewritten Galileo's famous sentence to read "The great book of philosophy is written in the language of mathematics."

"Mathematics calls attention to itself," wrote Jack Schwartz in a famous paper on another kind of misunderstanding. Philosophers in this century have suffered more than ever from the dictatorship of definitiveness. The illusion of the final answer, what two thousand years of Western philosophy had failed to accomplish, was thought in this century to have come at last within reach by the slavish imitation of mathematics.

Mathematizing philosophers have claimed that philosophy should be made factual and precise. They have given guidelines to philosophical argument which are based upon mathematical logic. They have contended that the eternal riddles of philosophy can be definitively solved by pure reasoning, unencumbered by the weight of history. Confident in their faith in the power of pure thought, they have cut all ties to the past, on the claim that the messages of past philosophers are now "obsolete."

Mathematizing philosophers will agree that traditional philosophical reasoning is radically different from mathematical reasoning. But this difference, rather than being viewed as strong evidence for the heterogeneity of philosophy and mathematics, is taken instead as a reason for doing away with non-mathematical philosophy altogether.

In one area of philosophy the program of mathematization has succeeded. Logic is nowadays no longer a part of philosophy. Under the name of mathematical logic, it is now a successful and respected branch of mathematics, one that has found substantial practical applications in computer science, more so than any other branch of mathematics.

But logic has become mathematical at a price. Mathematical logic has given up all claims to give a foundation to mathematics. Very few logicians of our day believe any longer that mathematical logic has anything to do with the way we think.

Mathematicians are therefore mystified by the spectacle of philosophers pretending to re-inject philosophical sense into the language of mathematical logic. A hygienic cleansing of every trace of philosophical reference had been the price of admission of logic into the mathematical fold. Mathematical logic is now just another branch of mathematics, like topology and probability. The philosophical aspects of mathematical logic are qualitatively no different from the philosophical aspects of topology or the theory of functions, aside from a curious terminology which, by an accident of chance going back to Leibinz's reading of Suárez, goes back to the Middle Ages.

The fake-philosophical terminology of mathematical logic has misled philosophers into believing that mathematical logic deals with the truth in the philosophical sense. But this is a mistake. Mathematical logic does not deal with the truth, but only with the game of truth. The snobbish symbol-dropping one finds nowadays in philosophical papers raises eyebrows among mathematicians. It is as if you were at the grocery store and you watched someone trying to pay his bill with Monopoly money.

4. Mathematics and Philosophy: Success and Failure

By all accounts, mathematics is the most successful intellectual undertaking of mankind. Every problem of mathematics gets solved, sooner or later. Once it is solved, a mathematical problem is forever finished: no later event will disprove a correct solution. As mathematics progresses, problems that were once difficult become easy enough to be assigned to schoolboys. Thus, Euclidean geometry is now taught in the second year of high school. Similarly, the mathematics that mathematicians of my generation have learned in graduate school has now descended to the undergraduate level, and the time is not far when it may be taught in the high schools.

Not only is every mathematical problem solved, but eventually, every mathematical problem is proved trivial. The quest for ultimate triviality is characteristic of the mathematical enterprise.

When we look at the problems of philosophy, another picture emerges. Philosophy can be described as the study of a few problems whose statements have changed little since the Greeks: the mind-body problem, or the problem or reality, to recall only two. A dispassionate look at the history of philosophy discloses two contradictory features: first, these problems have in no way been solved, nor are they likely to be solved as long as philosophy survives; second, every philosopher who has ever worked on any of these problems has proposed his own "definitive solution," which have been invariably rejected as false by his successors.

Such crushing historical evidence forces us to the conclusion that these two paradoxical features must be an inescapable concomitant of the philosophical enterprise. Failure to conclude has been an outstanding characteristic of philosophy throughout its history.

Philosophers of the past have repeatedly stressed the essential role of failure in philosophy. José Ortega y Gasset, for example, used to describe philosophy as "a constant shipwreck." However, the fear of failure did not stop him or any other philosopher from doing philosophy.

Philosophers' failure to reach any kind of agreement does not make their writings any less relevant to the problems of our day. We reread with interest the mutually contradictory theories of mind that Plato, Aristotle, Kant, and Comte have bequeathed to us, and we find their opinions timely and enlightening, even in problems of artificial intelligence.

Unfortunately, the latter-day mathematizers of philosophy are unable to face up to the inevitability of failure. Borrowing from the world of business, they have embraced the ideal of success. Philosophy had better be successful, or else is should be given up, like any business.

5. The Myth of Precision

Since mathematical concepts are precise, and since mathematics has been successful, they mistakenly infer that philosophy would be better off if it dealt with precise concepts and unequivocal statements. Philosophy will have a better chance at being successful, if it becomes precise.

The prejudice that a concept must be precisely defined in order to be meaningful, or that an argument must be precisely stated in order to make sense, is one of the most insidious of the Twentieth Century. The best known expression of this prejudice appears at the end of Ludwig Wittgenstein's *Tractatus*, and the author's later work, in particular the *Philosophical Investigations*, is a loud and repeated retraction of his earlier *gaffe*.

Looked at from the vantage point of ordinary experience, the ideal of precision appears preposterous. Our everyday reasoning is not precise, yet it is effective. Nature itself, from the cosmos to the gene, is approximate and inaccurate.

The concepts of philosophy are among the least precise. The mind, perception, memory, cognition, are words that do not have any fixed or clear meaning. Yet, they do have meaning. We misunderstand these concepts when we force them to be precise. To use an image due to Wittgenstein, philosophical concepts are like the winding streets of an old city, which we must accept as they are, and which we must familiarize ourselves with by strolling through them, while admiring their historical heritage. Like a Carpathian dictator, the advocates of precision would raze the city to the ground and replace it with a straight and wide Avenue of Precision.

The ideal of precision in philosophy has its roots in a misunderstanding of the notion of rigor. It has not occurred to our mathematizing philosophers that philosophy might be endowed with its own kind of rigor, a rigor that philosophers should dispassionately describe and codify, as mathematicians did with their own kind of rigor a long time ago. Bewitched as they are by the success of mathematics, they remain enslaved by the prejudice that the only possible rigor is that of mathematics, and that philosophy has no choice but to imitate it.

6. The Misunderstanding of the Axiomatic Method

The facts of mathematics are verified and presented by the axiomatic method. One must guard, however, against confusing the *presentation* of mathematics with the *content* of mathematics. An axiomatic presentation of a mathematical fact differs from the fact that is being presented as medicine differs from food. It is true that this particular medicine is necessary to keep the mathematician at a safe distance from the self-delusions of the mind. Nonetheless, understanding mathematics means being able to *forget* the medicine, and to enjoy the food. Confusing mathematics with the axiomatic method for its presentation is as preposterous as confusing the music of John Sebastian Bach with the the techniques for counterpoint in the Baroque age.

This is not, however, the opinion held by our mathematizing philosophers. They are convinced that the axiomatic method is a basic instrument for discovery. They mistakenly believe that mathematicians *use* the axiomatic method in solving problems and proving theorems. To the misunderstanding of the role of the method they have added the absurd pretension that this presumed method should be adopted in philosophy. Systematically confusing food with medicine, they have pretended to replace the food of philosophical thought with the medicine of axiomatics.

This mistake betrays the philosophers' pessimistic view of their own field. Unable or afraid as they are of singling out, describing and analyzing the structure of philosophical reasoning, they seek help from the proven technique of another field, a field that is the object of their envy and veneration. Secretly disbelieving in the power of autonomous philosophical reasoning to arrive at the truth, they have surrendered to a slavish and superficial imitation of the truth of mathematics.

The negative opinion that many philosophers hold of their own field has caused damage to philosophy. The mathematician's contempt at the philosopher's exaggerated estimation of a method of mathematical exposition feeds back onto philosophers' inferiority complex, and further decreases the philosophers' confidence.

7. "Define your terms!"

This old injunction has become a platitude in everyday discussions. What could be healthier than a clear statement, right at the beginning, of what it is that we are talking about? Doesn't mathematics start with definitions and then develop the properties of the objects that have been defined, by an admirable and inexorable logic?

Salutary as this injunction may be in mathematics, it has had disastrous consequences when carried over to philosophy. Whereas mathematics *starts* with a definition, philosophy *ends* with a definition. A clear statement of what it is we are talking about is not only missing in philosophy; such a statement would be the end of all philosophy. If we could define our terms, then we would dispense with philosophical argument.

Actually, the "define your terms" imperative is deeply flawed in more than one way. While reading a formal mathematical argument, we are given to believe that the "undefined terms," or the "basic definitions" have been whimsically chosen out of a variety of possibilities. Mathematicians take mischievous pleasure in faking the arbitrariness of definition. In actual fact, no mathematical definition is arbitrary. The theorems of mathematics motivate the definitions as much as the definitions motivate the theorems. A good definition is "justified" by the theorems one can prove with it, just like the proof of a theorem is "justified" by appealing to previously given definition.

There is thus a hidden circularity in formal mathematical exposition. The theorems are proved starting with definitions, but the definitions themselves are motivated by the theorems that we have previously decided ought to be right.

Instead of focusing on this strange circularity, philosophers have pretended it does not exist, as if the axiomatic method, proceeding linearly from definition to theorem, were endowed with a definitiveness which is instead, as every mathematician knows, a subtle fakery to be debunked.

Perform the following thought experiment. Suppose that you are given two formal presentations of the same mathematical theory. The definitions of the first presentation are the theorems of the second, and vice-versa. This situation frequently occurs in mathematics. Which of the two presentations makes the theory "true?" Neither, evidently. What we have is two presentations of the *same* theory.

This thought experiment shows that mathematical truth is not brought into being by a formal presentation. Rather, formal presentation is only a technique for displaying mathematical truth. The truth of a mathematical theory is distinct from the correctness of any axiomatic method that may be chosen for the presentation of the theory.

Mathematizing philosophers have missed this distinction.

8. The Appeal to Psychology

What will happen to the philosopher who insists on precise statements and clear definitions? Realizing after futile trials that philosophy resists such a treatment, said philosopher will proclaim that most problems previously thought to belong to philosophy are heretofore to be excluded from consideration. He will claim that they are "meaningless," or at best that they can be settled by an analysis of their statements that will eventually show them to be vacuous.

This is not an exaggeration. The classical problems of philosophy have become forbidden topics in many philosophy departments. The mere mention of one such problem by a graduate student or by a junior colleague will result in raised eyebrows, followed by severe penalties. In this dictatorial regime, we have witnessed the shrinking of philosophical activity to an impoverished *problématique*, mainly dealing with language.

In order to justify their neglect of most the old and substantial questions of philosophy, our mathematizing philosophers have resorted to the ruse of claiming that many questions formerly thought to be philosophical are instead "purely psychological," and that they should be dealt with in the psychology department.

If the psychology department of any university were to consider only one tenth of the problems that philosophers are pawning off on them, then psychology would without question be the most fascinating of all subjects. Maybe it is. But the fact is that psychologists have no intention of dealing with problems abandoned by philosophers who have been derelict in their duties.

One cannot do away with problems by decree. The classical problems of philosophy are now coming back with a vengeance in the forefront of science. For example, the Kantian problem of the conditions of possibility of vision, after years of neglect, is now again rearing its old head in brain science.

Experimental psychology, neurophysiology and computer science may turn out to be the best friends of traditional philosophy. The awesome complexities of the phenomena that are being studied in these sciences have convinced scientists (well in advance of the philosophical establishment) that progress science will crucially depend on philosophical research of the most classical vein.

9. The Reductionist Concept of the Mind

What does a mathematician do when trying to work on a mathematical problem? An adequate description of this event might take a thick volume. We shall be content with recalling an old saying, probably going back to the mathematician George Pólya: "Few mathematical problems are ever solved directly."

Every mathematician will agree that an important step in solving a mathematical problem, perhaps the most important step, consists in analyzing other attempts, either attempts that have been previously carried out or else attempts that one imagines might have been carried out, with a view to discovering how such "previous" attempts were misled. In short, no mathematician will ever dream of attacking a substantial mathematical problem without first becoming acquainted with the *history* of the problem, whether the real history or an ideal history that a gifted mathematician might reconstruct. The solution of a mathematical problem goes hand-in-hand with the discovery of the inadequacy of previous attempts, with the enthusiasm that sees through and does away with layers of irrelevancies inherited from the past which cloud the real nature of the problem. In philosophical terms, a mathematician who solves a problem cannot avoid facing up to the *historicity* of the problem. Mathematics is nothing if not a historical subject *par excellence* .

Every philosopher since Heraclitus has stressed with striking uniformity the lesson that all thought is constitutively historical. Until, that is, our mathematizing philosophers came along, claiming that the mind is nothing but a complex thinking machine, not to be polluted by the inconclusive ramblings of bygone ages. Historical thought has been dealt a *coup de grace* by those who today occupy some of the chairs of our philosophy departments. Graduate school requirements in the history of philosophy have been dropped, together with language requirements, and in their place we find required courses in mathematical logic.

It is important to single out the myth that underlies such drastic revision of the concept of mind. It the myth that believes the mind to be a mechanical device. This myth that has been repeatedly and successfully attacked by the best philosophers of our time (Husserl, John Dewey, Wittgenstein, Austin, Ryle, to name only a few).

According to this myth, the process of reasoning is viewed as the functioning of a vending machine which, by setting into motion a complex mechanism reminiscent of those we saw in Charlie Chaplin's film *Modern Times*, grinds out solutions to problems, like so many Hershey bars. Believers in the theory of the mind as a vending machine will rate human beings according by "degrees" of intelligence, the more intelligent ones being those endowed with bigger and better gears in their brains, as can of course be verified by administering I.Q. tests.

Philosophers believing in the mechanistic myth believe that the solution of a problem is obtained in just one way: by thinking hard about it. They will go as far as asserting that acquaintance with previous contributions to a problem may bias the well-geared mind. A blank mind, they believe, is better geared up to initiate the solution process than an informed mind.

This outrageous proposition originates from a misconception of how mathematicians work. Our mathematizing philosophers behave like failed mathematicians. They gape at working mathematicians in wide-eyed admiration, like movie fans gaping at posters of Joan Crawford and Bette Davis. Mathematicians are superminds who turn out solutions of one problem after another by dint of pure brain power, simply by staring at a blank piece of paper in intense concentration.

The myth of the vending machine that grinds solutions out of nothing may perhaps appropriately describe the way to solve the linguistic puzzles of today's impoverished philosophy, but this myth is far off the mark in describing the work of mathematicians, or any other serious work.

The fundamental error is one of reductionism. The *process* of the working of the mind, which may be of interest to physicians but is of no interest to mathematicians, is confused with the *progress* of thought that is required in the solution of any problem.

This catastrophic misunderstanding of the nature of knowledge is the heritage of one hundred-odd years of pseudo-mathematization of philosophy.

10. The Illusion of Definitiveness

The results of mathematics are definitive. No one will ever improve on a sorting algorithm which has been proved best possible. No one will ever discover a new finite simple group, now that the list has been drawn, after a century of research. Mathematics is forever.

We could classify the sciences by how close their results come to being definitive. At the top of the list we would find the sciences of lesser philosophical interest, such as mechanics, organic chemistry, botany. At the bottom of the list we would find the more philosophically inclined sciences, such as cosmology and evolutionary biology.

The old problems of philosophy, such as mind and matter, reality, perception, are least likely to have "solutions." In fact, we would be hard put to spell out what might be acceptable as a "solution." The term "solution" is borrowed from mathematics, and tacitly presupposes an analogy between problems of philosophy and problems of mathematics that is seriously misleading. Perhaps the use of the word "problem" in philosophy raised expectations that philosophy could not fulfill.

Philosophers of our day go one step farther in their mis-analogies between philosophy and mathematics. Driven by a misplaced belief in definitiveness measured in terms of problems solved, and realizing the futility of any attempt to produce definitive solutions to any of the classical problems, they have had to change the problems. And where do they think to have found problems worthy of them? Why, in the world of facts!

Science deals with facts. Whatever it is that traditional philosophy deals with, it is not facts in the scientific sense. Therefore, traditional philosophy is worthless.

This syllogism, wrong on several counts, is predicated on the assumption that no statement is of any value, unless it is a statement of fact. Instead of realizing the absurdity of this assumption, philosophers have swallowed it, hook, line and sinker, and have busied themselves in making their living on facts.

But previous philosophers had never been equipped to deal directly with facts, nor had they ever considered facts to be any of their business. Nobody turns to philosophy to learn facts. Facts are the domain of science, not of philosophy. And so, a new slogan had to be coined: philosophy *should* be dealing with facts.

This "*should*" comes at the end of a long line of other "*should's*." Philosophy *should* be precise, it *should* follow the rules of mathematical logic, it *should* define its terms carefully, it *should* ignore the lessons of the past, it *should* be successful at solving its problems, it *should* produce definitive solutions.

"Pigs *should* fly," as the old saying goes.

But what is the standing of such "*should's*," flatly negated as they are by two thousand years of philosophy? Are we to believe the not so subtle insinuation that the royal road to right reasoning will at last be found if we follow these imperatives?

There is a more plausible explanation of this barrage of *should's*. The reality we live in is constituted by myriad contradictions, which traditional philosophy has taken pains to describe with courageous realism. But contradiction cannot be confronted by minds who have put their salvation in axiomatics. The real world is filled with absences, with absurdities, with abnormalities, with aberrances, with abominations, with abuses, with *Abgrund*. But our latter-day philosophers are not concerned with facing up to these unpleasant features of the world, nor, to be sure, to any real features whatsoever. They would rather tell us what the world *should* be like. They find it safer to escape from distasteful description of what is into pointless prescription of what isn't. Like ostriches with their heads in the ground, they will meet the fate of those who refuse to acknowledge the lessons of the past and to meet the challenge of our difficult present: increasing irrelevance followed by eventual extinction.

MATHEMATICS: CONTRIBUTIONS BY WOMEN

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Summary

Neither history nor a liberal arts education gives much recognition to mathematicians — regardless of their sex. Therefore it is not surprising that women have had almost no recognition in a field where men have had so little. It has been argued that this helps perpetuate the impression that math is a male domain. To combat this myth the author has developed a course for liberal arts students that includes the study of the biographies of 12–14 women mathematicians and of mathematical topics related to their work. In addition, math anxiety, math avoidance and sex-related differences in mathematics learning are investigated. At Loyola Marymount University this course can count toward the science core curriculum requirement or as a core course in the women's studies program. This paper will describe the course and provide information, resources, and an annotated bibliography useful for making students more aware of women's contributions to mathematics.

Neither history nor a liberal arts education gives much recognition to mathematicians — regardless of their sex. College students taking calculus can rarely identify Gauss, Cauchy, Euler, or Hilbert, although they are mathematical equivalents to Tolstoy, Beethoven, Rembrandt, Darwin, and Freud. Very few mathematicians have fame comparable to that of their counterparts in other disciplines. Usually, when asked to name some famous mathematicians, college calculus students can only manage to recall 2 or 3 of Einstein, Euclid, Pascal, and Newton. Therefore it is not surprising that women have had almost no recognition in a field where men have had so little.

There are a number of women who have made substantial contributions in mathematics. Yet they are rarely mentioned in history of math texts. Often when they are mentioned, it is for their non-mathematical activities which involved famous men. That so few women receive credit for their accomplishments in math helps perpetuate the myth that math is a male domain.

To combat this myth, the author developed a college level course for liberal arts students that includes the study of the biographies of 9 women mathematicians born before the twentieth century and 4 twentieth century women mathematicians along with mathematical activities related to their work. The course fulfills a science core requirement for liberal arts students and is a recognized elective for the women's studies program. It is designed to:

(1) give non-science majors a new insight into mathematics as a creative art and science;
(2) give students some experience of the kinds of investigations that make mathematics so fascinating to mathematicians;
(3) improve attitudes towards mathematics;
(4) change the impression that math is a male domain.

The course, entitled *Mathematics: Contributions by Women*, alternates math activities with readings of the biographies of women mathematicians and discussions of the causes of math anxiety and math avoidance. Through activities students survey a broad range of topics in mathematics, including conic sections, functions, limits, velocity, Venn diagrams, the cycloid curve, finite differences, modular arithmetic, and groups. (See the Appendix for a more complete list of the mathematical topics covered.) The presentations are given at a level requiring minimal math background. In addition, through readings and discussion, students examine questions, such as:

Why are so few women mathematicians known?
Can one delineate common experiences in the lives of women who have been successful in a stereotypically male field such as mathematics?
Are males better at math than females are?
Do males like math better than females do?
Has math education and counseling been different for females?

The course has been advertised as being a good choice for anyone who feels insecure with mathematics and usually attracts 2 or 3 re-entry women in a class of 20–25.

The workload is substantial: homework and weekly quizzes on mathematical content; 2 papers; 2 hour exams and a final exam; and a special project. The special project, chosen by the student in consultation with the instructor, involves one of the following: researching a mathematical topic, such as how honey bees navigate by polar coordinates; undertaking a program of study to improve basic math skills or decrease math anxiety; or reporting on psychological or sociological aspects of math avoidance or math anxiety.

There are two factors that make teaching this course difficult. One is that selecting interesting mathematical topics and presenting them at a level accessible to a group of students with a wide range of mathematical backgrounds is a challenge. The other has to do with the one or two males who enroll. It is not easy to get them truly engaged in the discussions.

Perhaps because of its challenges, there can be great satisfaction in teaching this course. Through the discussions, papers, and the selection and work on the special projects, the instructor gets acquainted with students on a more personal level than is usual in a typical mathematics class. The students also appear to benefit from this deeper involvement as is evidenced by the following quotes from evaluations. "I have enjoyed the class tremendously. Now, I don't hate math anymore like I did when I entered college." "I'm not as afraid and seem more relaxed taking math related exams." "I think now that anyone can learn to do math — it just takes some longer." "I feel that I have learned to appreciate the advances that have been made."

Try to imagine getting comments like these from students at the end of a traditional calculus class!

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APPENDIX

(The numbers in parentheses refer to the attached bibliography.)

0. The Problems of Math Anxiety and Math Avoidance (9, 11, 34, 35)

I. Hypatia 370–415 A. D. (17, 24, 27) (Greek mathematician, inventor, philosopher, teacher, textbook author)
   - A. Related mathematical topics
     1. Conic sections as the intersection of a plane and a cone. (27)
     2. Conic sections as paths of points subject to certain distance conditions. (27)
     3. Diophantine equations: How many ways can you make change for a dollar using nickels, dimes, and quarters? (23, 27)

II. Emile du Chatelet 1760–1749 (8, 21, 24, 27) (Expositor of Newton's *Principia*)
   - A. Related mathematical topics
     1. Function machines (27)
     2. Velocity — average and instantaneous (27)
     3. Limit concept (27)

III. Maria Agnesi 1718–1799 (24, 27) (Translator, textbook author, servant of the poor)
   - A. Related mathematical topics
     1. Cartesian coordinate system (27)
     2. Symmetry and graphing (27)
     3. Witch of Agnesi curve (10, 27)

IV. Sophie Germaine 1776–1831 (24, 27) (Researcher in number theory and mathematical physics, winner of grand prize in French Academy of Science Contest)
   - A. Related mathematical topics
     1. Number bases
     2. Clock arithmetic (27)

3. Minimal surfaces — Soap film demonstration (23, 27)
4. Related geometric constructions (27)

V. Mary Somerville 1780–1872 (24, 26, 27) (Popular science writer)
   A. Related mathematical topics
      1. Cycloid curve (2, 10, 27)

VI. Ada Byron Lovelace 1815–1852 (22, 27, 32) ("Mother" of computer programming)
   A. Related mathematical topics
      1. Functions (27)
      2. Difference Tables (27)
      3. Applications to various puzzles (27)

VII. Sonya Kovaleskaya 1850–1891 (5, 13, 18, 20, 24, 27, 29) (Researcher in applied mathematics, recipient of Prix Bordin by French Academy of Science, autobiographer)
   A. Related mathematical topics
      1. Infinite sequences (27)
      2. Geometric series (27)
      3. Chain letters

VIII. Grace Chisolm Young 1868–1944 (27) (First woman to receive a formal doctorate in any subject in Europe, geometer, textbook author)
   A. Related mathematics topics
      1. Paper-folding approach to geometry (37)
      2. Binary number system and "mind reading cards" (27)
      3. Regular polyhedra (23)
      4. Euler's formula (23)

IX. Emmy Noether 1882–1935 (3, 7, 16, 24, 27) ("Mother" of modern algebra)
   A. Related mathematical topics
      1. Groups (27)
      2. Flexagons (25)

X. Other Twentieth Century and Living Women Mathematicians (1, 4, 6, 12, 14, 15, 28, 29, 31, 33, 36, 38)
   Lenore Blum (28)
   Grace Hopper (38)
   Cathleen Morawetz (19)
   Mina Rees (6)
   Julia Robinson (30)
   Mary Ellen Rudin (I)
   Charlotte Angas Scott (15)

MATHEMATICS AND POETRY: ISOLATED OR INTEGRATED?

People Don't Want to Study Mathematics: Some Illuminations on the Status Quo and How to Change It

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Mathematics Subject Classification number: 00A99

Basic Premise: All Life is Art.

> . . . the whole universe was a work of art created by some Supreme Artist, in the way of artists, out of material that was practically nothing, . . . a method which, as children sometimes instinctively feel, is a kind of creative art.
> — Havelock Ellis, *The Dance of Life*

> Every child is an artist. The problem is how to remain an artist once he grows up.
> — Pablo Picasso

Observation 1A: Mathematics is a major Art Form.

Observation 1B: Poetry is a major Art Form.

Observation 1C: Mathematics and poetry are similar.

Observation 2: Major Art Forms in a culture each give clues to the key aspects of the culture as a whole.

Observation 3: In the United States today, people reject both mathematics and poetry as true Art Forms, i.e., as aspects of the Essence of Life.

Unsolved Problem: How to convince people that mathematics can be vital in their lives.

Partial Answers: slow down; open up; allow silence; open up; allow inconsistency; open up.

Is truth (a) logical or (b) episodic; is it (c) consistent or (d) inconsistent? Readers who strongly prefer (a) to (b) and (c) to (d) may dislike the manner of this article, outlined above. Herein I do not present a logical argument but an eclectic collection of statements, sometimes unsubstantiated yet real, sometimes contradictory, yet in their very contradictoriness hinting at truth. Consider, if you will, their bearing on a very large problem that faces mathematicians today: few people wish to study our subject in depth and succeed us as mathematicians and, moreover, few people want to study our subject at all, perhaps paying lip-service to the value of "quantitative literacy" but seeing no genuine benefit in it for themselves.

Some months ago I began an investigation into the similarities between mathematics and poetry, hoping to gain insights that would help me to reach reluctant students — English majors, elementary education majors, majors in studio art, and other assorted math-avoiders — who enroll in Math 101 ("Mathematical Thinking") at Bloomsburg University because they must fulfil a "quantitative reasoning" requirement to graduate. By pointing out analogies between mathematics and poetry, I would help students to see the beauty and power of mathematics. That investigation continues; this article is a by-product.

All life is art

Concerning this basic premise, there is little that I wish to say. It is a point of view that one may adopt or reject. However, as you read on, you will be more ready to consider my words if you can temporarily envision life as art: as a painting, a poem, a theory, or a dance — fashioned by the self, responsive both to the inner spirit and the outer world, striving for beauty while expressing old truths and new insights. Gathered in Appendix A are a variety of quotations that refer to the nature and purposes of art and artists, of mathematics and poetry. Several of the cited references address the basic premise at length.

Whether mathematics is, like poetry, a major Art Form may deserve debate, but this article will avoid that controversy. There are significant similarities between mathematics and poetry. Consider:

> Both mathematics and poetry are abstract languages that practitioners use in an attempt to express truth precisely and concisely.

Both mathematicians and poets identify key ideas and express them symbolically. A poet may, for example, use deep water to symbolize death and its surrounding mysteries. A mathematician uses the derivative to symbolize a rate of change and its surrounding mysteries.

Both mathematics and poetry are feared and shunned by most of the populace:

> they have no meaning,
> no relevance,
> no usefulness.

Both mathematicians and poets are often regarded as isolated and peculiar.

Which has the greater beauty, the greater symmetry, a sonnet or the expansion of a binomial? Dante's Divine Comedy or Pascal's triangle? Have we any more or any less wonder when we contemplate the convergence of the infinite series

$$1+\frac{1}{2^1}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}+\ldots$$

than when we envision "a host of golden daffodils?" Was Lord Byron thinking of mathematics or poetry when he wrote:

> The power of
> Thought — the magic of the mind.

The search for truth that pervades both mathematics and poetry gives each an uplifting quality. In the words of Sir Francis Bacon, "No pleasure is comparable to the standing upon the vantage ground of truth."

Mathematics and poetry are more than their objects of formula or stanza, proof or poem. Each has become a mind — not just a product of mind but a mind on its own — a mind that approaches omniscience as an ideal. Each goes on with endless invention, hungry and restless for more.

Appendix B contains a list of quotations in which a key word — *mathematics* or *poetry* or *mathematician* or *poet* or a variation of one of these — has been left out. If you do not use the name of a quotation's author as a clue, you will find difficulty in deciding whether it is *mathematics* or poetry to which the quotation refers. Upon all of this evidence, then, let us rest the hypothesis that mathematics and poetry are fundamentally similar.

Is mathematics, like poetry, properly (today, as well as in the past) one of the Humanities?

A powerful argument for this placement of mathematics is given by Cassius J. Keyser [7]. He points out that the identification of mathematics with science is too limiting: we are thinking not of mathematics but of its application to a particular subject matter. Mathematics, after all, is a way of thinking; it has an individuality of its own. *Mathematics discloses the essential nature of man* by revealing, more than any other subject, man's ability to pass on achievements of one generation to the next, providing living capital for the production of ever-greater achievements. *Mathematics is a guide to human life* in its role as a keeper of ideals: number systems, geometries, logical thinking — these and more are ideals that are kept by mathematics and guide human life. Mathematics also sheds light on the nature of an ideal: like a mathematical limit, it can be approached by an endless sequence of closer and closer approximations and yet (unless it is a specious ideal) is incapable of actually being attained.

Art forms are culture clues

Any major Art Form produced by a culture is a valuable clue to the key aspects of the culture. Keyser illustrates the application of this idea by using mathematics to identify significant differences between the Greek, or Classical, Culture and the Modern, or Western Culture [7]. The Greek mathematics was finite and bound to the concrete. Numbers were positive integers and were bound to geometric things; they counted finite groups of objects or were the lengths of line segments. Geometry was a highly developed study of the properties of finite figures but was not an exploration of Space. This is in accord with the Classicists' lack of consideration of perspective in their arts of painting, city planning, and garden designing. Their mathematics was functionless: objects without relations. Likewise, their physics was nothing but statics; their music had rhythm and melody but no harmony. Modern mathematics, on the other hand, is dynamic, relational, and includes the infinite. We can see similar characteristics in modern science, drama, religion, art, . . .

Next, let us consider the role of poetry in our culture and to use this information for clues about mathematics.

Today, both mathematics and poetry are rejected

> I think that one possible definition of our modern culture is that it is one in which nine-tenths of our intellectuals can't read any poetry.
> — Randall Jarrell

For most Americans, poetry is not vital, and they may scoff, "Why should it be?" Likewise, most envision no benefit from knowing mathematics. The people we meet at social gatherings are not interested in talking about our subject. Mathematics is not popular and even those who like it frequently see it as non-useful. Few people see value in mathematics beyond the arithmetic of the checkbook. Despairingly smaller numbers of Americans choose it as a field for graduate study and a career.

The unpopularity of poetry is similar to that of mathematics. Although some of us have sought and found poets whose work inspires us, members of the general public find little time for poetry, and place little value on the role it might play in a full and happy life.

If we were to speak to poets and to learn of their distress about the exclusion of their art, how would we advise them to connect with essentiality once again? How would we react to their claims that it was our early schooling — and not the present nature of poets and poems — that had turned us off to poetry? How would they react to our complaints that we find much of modern poetry without meaning: we cannot understand or approach poetry directly; a teacher or reviewer must go between as a translator. How would they or we explain the lack of status accorded to the new voices in poetry — the black voices, the female voices, the third world voices — many of whose words are not shared with large audiences, whose works are "minor" poetry, perhaps only because their images are different and because they speak with simplicity and clarity.

As I write, I can hear echoes of friends' voices scoffing at these comparisons between mathematics and poetry and at the view that both are vital to human life:

> *One-third* of them scoff at the notion that poetry ought to be as useful as mathematics. (I use "useful" broadly: that which inspires or gives pleasure is "useful.")
>
> > *One-third of this first third* actually use some mathematics.
>
> *A different one-third* scoff at the notion that mathematics ought to be as useful as poetry: poetry, after all, appeals to the emotions, making it real, whereas mathematics is merely a mind-game.
>
> > *One-third of this second third* actually read poetry as a habit, at least once a month. Some of them even write poetry although they would not call themselves "poets" for that is an elitist designation.
>
> Left over are a *final one-third* who scoff at both mathematics and poetry — both are abstract, esoteric; neither applies to the real world.

For the time being, ignore the scoffing. Imagine, if you will, that you are willing to be convinced of the hypothesis that poetry is a vital source of the energy and insight that give meaning to daily life; merely allow yourself to ever-so-slightly consider this notion and to accept it only if sufficient reason is given. What evidence would you require?

Here, then, is a full statement of our question: *What evidence would you require before you would accept the hypothesis that poetry is vital for the full living of daily life?* Would you require changes in schooling, in the attitudes of teachers? Might they show you tools for experiencing poetry directly rather than first trying to translate it into prose? Would you wish to see demonstration of how others use poetry as a guide and find it significant? Would you wish to find poetry readily available in the popular media? Would you like to be able to approach good poetry directly and to discover its beauty and meaning on your own, without the aid of a teacher-translator? Would you like to be shown how poetry is a process and not just a product?

Suppose we now have a few criteria for poetry and poets. Turn them around toward mathematics. If mathematics and mathematicians met these same criteria, mightn't people find them also vital?

Convincing people that mathematics is vital

When I apply my own criteria for acceptance of poetry as vital to the problem of convincing people that mathematics is vital, I come up with the following:

SLOW DOWN

Many voices have said,
we must slow down,
we must allow time to let students learn,

Forces seemingly beyond our control make this change a difficult one to implement, and our course syllabi are packed with long lists of topics that must be covered to prepare students for other courses that offer more of the same. Too many of our students do not see mathematics as a process as well as a product. Might we not envision mathematics as rather like the complex mind of a friend. If we would consider how we get to know a friend, how we come to understand the depths of that other, we see a suggestion for how one may learn mathematics. Facts and techniques are not sufficient; context and meaning must accompany them.

As George Cobb observes in [3], "the more attention you pay to technique, the less you have left for meaning . . . To learn technique quickly, you have to treat it abstractly; context and meaning just get in the way and slow you down."

Howard Nemerov in "Poetry and Meaning," an essay included in [10], characterizes poetry as "getting something right in language." He goes on to express his observation that there has been in poetry in this century "a slow collapse in the idea of meaning which progressed simultaneously with an imposing acceleration of the rate at which knowledge was accumulated . . . the slow collapse in the idea of meaning . . . did not happen in poetry alone. It happened even more conspicuously and at about the same time in physics, in painting, in music. The whole world suddenly became frightfully hard to understand."

If speed and emphasis on technique drive out meaning, then it is clear that if we want to bring meaning back, to mathematics or to poetry, we must slow down.

> Slowness is beauty.
>
> — Auguste Rodin

open up

> In general the teacher of mathematics has been the high priest of an occult ritual, the keeper in many senses of an esoteric doctrine which only his superiors or predecessors have understood.
>
> — Scott Buchanan [1], 35.

How do mathematicians react to remarks like those of Buchanan? Do some show scorn for those who cannot or will not appreciated the beauty of pure mathematics? Do some level contempt at those who have not the discipline to master obscurities? Does laughter deride the unfortunate student who dares to wonder, "When are we going to use this?" Sometimes the unpopularity of mathematics is taken by mathematicians to suggest that there are deficiencies in others but not in themselves. Is it possible that our dissatisfied customers (i.e., our reluctant students) are correct: they have not been given sufficient evidence that our product is worthy.

In [4], in an essay entitled "The Ideal Mathematician," Philip Davis and Reuben Hersh endeavor to describe the most mathematician-like mathematician:

> He rests his faith on rigorous proof . . . He is labeled by his field, by how much he publishes . . . He finds it difficult to establish meaningful conversation with that large portion of humanity that has never heard of [his research topic] . . . His writing follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being . . .

Is this Davis-Hersh creature an attractive one? How can the general public appreciate mathematics if it emerges from such sterility?

William Benjamin Smith — scholar, poet, mathematician and master teacher — wrote "The Merman and the Seraph," a poem that won the Poet Lore competition of 1906. In it he sings sadly of the separation between the Merman — perhaps a mathematician, isolated in his sterile world of thought, and separated from beauty, from feeling and desire — and an angel or Seraph, who represents the world of whatsoever is good. Here are the opening stanzas of Smith's poem, reviewed by Keyser in [7].

THE MERMAN AND THE SERAPH

Deep the sunless seas amid,
Far from Man, from Angel hid,
Where the soundless tides are rolled
Over Ocean's treasure-hold,
With dragon eye and heart of stone,
The ancient Merman mused alone.

And aye his arrowed Thought he wings
Straight at the inmost core of things —
As mirrored in his Magic glass
The lightning-footed Ages pass, —
And knows nor joy nor earth's distress,
But broods on Everlastingness.
"Thoughts that love not, thoughts that hate not,
Thoughts that Age and Change await not,

All unfeeling,
All revealing,
Scorning height's and depth's concealing,
These be mine — and these alone!" —
Saith the Merman's heart of stone.

As the poem unfolds, the Merman dreams of a beautiful angel who visits him, offering love and all that is good. Too soon she is driven to retreat, to leave him in his dark world of sterile thought.

Along with his consideration of Smith's poem, Keyser expresses his concern about "the narrow canalising of their mental energies" which he sees as prevalent among mathematicians. He introduces his concerns with a quote from David Swing, noted Chicago clergyman and author:

> Men trained in a profession come by degrees into the profession's channel, and flow only in one direction, and always between the same banks. The master of a learned profession at last becomes its slave. He who follows faithfully any calling comes at length to wear a soul of that calling's shape . . . We are all clay in the hands of that potter which is called a pursuit. A pursuit is seldom an ocean of water; it is more commonly a canal.

Although Swing believed that the lawyer was least likely to escape the influence of his pursuit, Keyser gave this honor to "those who addict themselves long and assiduously to the study and teaching of mathematics." He wondered if this is why the world in general regards mathematicians as a bit peculiar, admirable for their intelligence and knowledge, but very narrow in their interests and feelings. While "canalising" is not a bad or wrong choice for any individual, its prevalence in a profession may cause the profession to be unattractive to newcomers. In short, canalising by mathematicians may cause students not to be attracted to mathematics.

One of Keyser's antidotes to canalising in his own life was the reading and rereading of *The Dance of Life* by Havelock Ellis [6]. He ranked Ellis's book "among those rare ones that are to be honored and revered as emancipators of the human mind." Keyser's compares mathematics to the art and natural human activity of dancing and, by so doing, enriches his conception of the nature of mathematics. If our students would see mathematics as a dance, i.e., as an art in which freedom of expression joins with responsiveness to surroundings and to disciplined training to create beauty, how might they respond differently?

Cecil Day-Lewis, professor of poetry at Oxford, in his 1951 inaugural address, *The Poet's Task*, offered the following views:

> Describing the present position of poetry:
> . . . poetry is not primarily a vehicle of extrinsic truth but the generator of a kind of truth peculiar to itself.
>
> . . . the function of poetry as a game with words [looms] larger than its function as a search for truth, and the tendency be toward pure poetry.

Day-Lewis asserted:

> Poetry has a moral function; it has the duty to give pleasure. A poet has a duty to love and to praise, to be serious and honest, to be dissatisfied with past attempts and alive to what the future holds.
>
> A task that is badly needed for the poet to take up today is to incline our hearts toward what is lovable and admirable in mankind.

If poetry is a culture clue that reveals some truth about the nature of mathematics, then we might take the words of Day-Lewis to heart: it is badly needed for the mathematician to take as his or her task to point mankind toward what is lovable and admirable. Consider what great satisfaction we feel when one of our students experiences direct pleasure from mathematical thinking; let us get greedy for more of this feeling.

That the poet's task is shared by the mathematician is a conviction that is found in [12], in the writings of David Eugene Smith, American mathematician and a president of the Mathematical Association of America (1920).

> . . . the call of mathematics is something beyond the physical; it is the call of the soul, precisely as in the case of music, of painting, and of other fine arts, or of science, or of letters. It is this call that must be answered if mathematics, the fine arts, the sciences, and letters are to advance and make the world a better place in which each succeeding generation is to play its part in the progress of the race.
>
> The call of mathematics is, then, to our physical well being, and this is always recognized; but it is also to our spiritual well-being, and this we must not fail to recognize if our labors are not to be in vain.

allow silence

Many mathematicians are good teachers. Our students like us, like our classes, and seem to learn a lot. But the facts remain that many Americans are quantitatively illiterate and eschew mathematical thinking, that too few of our students go on to become mathematicians. Perhaps good teaching needs to change.

The poet Howard Nemerov has some thought-provoking suggestions about teaching [10]; he, of course, is referring to teaching poetry but, if we consider mathematics in its role as a language, his ideas apply to teaching mathematics as well.

> The method I suspect we all use exclusively, or almost so, may be called analytic, and has to do broadly with finding out the meanings of poems; if one wanted to be critical of that method one could call it, as a friend of mine did, 'how to turn poems into prose,' . . .
>
> The method I am going to propose as the complement to the first is both simple and difficult, though I hope not impossible. It has to do less with 'teaching poetry' than with 'being taught by poetry' . . .
>
> In short, given that poetry is a language, our way of showing pupils how to deal with it is to translate it out of that language into our own more familiar one. Suppose, however, another object, the one we ordinarily have in studying any language: to learn to speak it, and at last to learn to think in it.
>
> It is not hard to see why we teach as we do, analytically; and seeing to sympathize with our plight. For the teacher, as Ezra Pound tersely defined him, is a man who must talk for an hour. . . .
>
> For if you have to talk for an hour, you concentrate naturally enough on what is sayable . . .
>
> In conclusion, I stress once again that I am trying to picture our situation, not to criticize it. For the first move of the understanding ought to be the silent contemplation of what is, and of how it got to be the way it is. No doubt the teacher of English will always be 'a man who must talk for an hour.' But if his talk is really to do its work, if his pupils are truly to become possessed of some sense of what poetry is and why it is, his speech itself will have to contain much silence.

Nemerov has observed [10] that an implicit message often is given by the teacher of poetry who translates poetry for students; this message is, "Look how sensitive I am." One key difference between the teacher of mathematics and the poetry teacher is that the former's implicit message is likely to be, instead, "Look how smart I am!" We will not have got it right until the implicit message to our students is, "Look how smart you are!"

Dorothy Buerk has written of some high school students who have contemplated mathematics in a direct and personal way. Consider the following response of a student in an advanced placement mathematics course, when asked to complete the phrase, "For me math is like a . . ." [2]

> For me math is kind of like an incredible book that you have to read through an infinite number of times. The first time you get the general idea, but until you reach the end you really have no idea what's going on in relation to anything else. Each successive reading brings out more meaning and . . .

Thomas Rishel [11] encourages his students to use their intuitive knowledge of geometry to help them to understand a difficult poem without the aid of a teacher-translator. He gives them Wallace Stevens' poem, "The Idea of Order at Key West," and asks them to complete the following assignment:

> Read the poem through thoroughly twice.
>
> After the second reading, underline any geometric words you find, especially concentrating on the penultimate stanza.
>
> Then, perhaps in a group, draw a picture based on the geometric words chosen.
>
> Finally, consider what the picture may have to do with the poem's final stanza.

Rishel's assignment not only provides students with a framework for letting a poem speak to them; it also allows them to discover something of the aesthetic nature of geometry.

open up

OUTWITTED

He drew a circle that shut me out —
Heretic, rebel, a thing to flout.
But Love and I had the wit to win:
We drew a circle that took him in!

— Edwin Markham

Two of my three best teachers from graduate school were women; the third was a Japanese man. Among the most successful mathematics teachers are Jaime Escalante (Los Angeles high school teacher made famous by the movie, "Stand and Deliver") who is Hispanic and Clarence F. Stephens (under whose leadership the State University of New York at Potsdam has a highly successful mathematics program) who is black. Are gender, race, or ethnic background relevant in these cases? Perhaps so.

Perhaps the traditional mold of mathematician as researcher — who cares not a fig about the connections of his theories to the humanities, who worries not about his pedagogy — needs to be recast. Our own initiation into the mysteries and magic of mathematics may have involved the same tough challenges to the intellect that we now provide for our followers:

the details are left to the student;
all mathematics has applications — it is up to you to find them;
all knowledge is interrelated — discover these interrelationships for yourself, or accept this on faith.

Even though we learned joyously under these circumstances, today is not yesterday and we may have participated in a narrowing specialization that has prepared us poorly to reach new adherents. Moreover, those who have achieved the highest status in the elitist group of mathematicians may be the least-well-prepared to aid in reform.

Frequently listed among our best teachers but seldom among our most respected researchers, are humanists, members of minority groups and women. WHY?

In over 4000 categories of mathematics recognized in the list of Mathematics Subject Classification Numbers, none except 00A99, "Miscellaneous Topics," covers mathematics education or mathematics as Art. WHY?

OPEN UP, YOU GUYS!!! A bright future for mathematics may depend on enlarging the definitions of "mathematics" and "mathematician." If we will enlarge the boundaries of the class of mathematicians to include teachers at all levels, students of all cultures, math hobbyists, and anyone who will admit to a liking for mathematics, then perhaps we can start to see ways to work together to reestablish mathematics to a preeminence it deserves.

allow inconsistency

If your views differ from mine, must one of us be wrong? May we not both be correct, even though we see things differently? Is it not a long-standing tradition in mathematical thought to embrace paradox as a goad to understanding, rebuilding ideas to encompass apparent inconsistencies?

For example, even though we see that when we crowd course syllabi with more topics and the result is less student learning, must consistency prevent us from experimenting with the paradoxical "less is more"?

What other inconsistencies can we entertain? Can we allow ourselves to consider the value of the opposite of each of our current attitudes and practices? What about mathematics as a humanistic subject as well as a scientific one? What about applications of mathematics to poetry as well as to practical projects? What about mathematics as a way to advance brotherhood as well as technology? What about an inclusive definition of "mathematician" rather than an exclusive one?

No man bathes twice in the same stream . . .

— Heraclitus

The man who consistently — as he fondly supposes 'logically' — clings to an unchanging opinion is suspended from a hook which has ceased to exist . . . We change, and the world change, in accordance with the underlying organization, and inconsistency, so conditioned by truth to the whole, becomes the higher consistency of life.

— H. Havelock Ellis

open up

Allow that mathematics is an Art and a humanistic endeavor. Perhaps it is also a garden: this thought is suggested to me by "Poetry" by Marianne Moore. Here is a fragment of it; the poem in full [8] deserves your reading.

. . . things are important not because a high-sounding interpretation can be put upon them but because they are useful . . .
. . . the same thing may be said for all of us, that we do not admire what we cannot understand . . .
. . . [Not until we] can present for inspection, imaginary gardens with real toads in them, shall we have it.

APPENDIX A

What Is MATHEMATICS? What is POETRY? What is ART? What is the role of the MATHEMATICIAN? The POET? The ARTIST?

The artist has a special task and duty: the task of reminding men of their humanity and the promise of their creativity.

— Lewis Mumford

Wherever there is number, there is beauty.

— Proclus

The useful and the beautiful are never separated.

— Periander

This, therefore, is mathematics: she reminds you of the invisible form of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes oblivion and ignorance which are ours by birth.

— Proclus

It is true that a mathematician, who is not somewhat of a poet, will never be a perfect mathematician.

— Weierstrass

On poetry and geometric truth,
And their high privilege of lasting life,
From all internal injury exempt,
I mused; upon these chiefly: and at length,
My senses yielding to the sultry air,
sleep seized me, and I passed into a dream.

— Wordsworth
The Prelude, Book 5

Does it not seem as if Algebra has attained to the dignity of a fine art, in which the workman has a free hand to develop his conceptions, as in a musical theme or a subject for painting? It has reached a point in which every properly developed algebraic composition, like a skillful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas.

— J. J. Sylvester

We do not listen with the best regard to the verses of a man who is only a poet, nor to his problems if he is only an algebraist; but if a man is at once acquainted with the geometric foundation of things and with their festal splendor, his poetry is exact and his arithmetic musical.

— R. W. Emerson

The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

— Bertrand Russell

It is with mathematics not otherwise than it is with music, painting or poetry. Anyone can become a lawyer, doctor or chemist, and as such may succeed well, provided he is clever and industrious, but not everyone can become a painter, or a musician, or a mathematician: general cleverness and industry alone count here for nothing.

— P. J. Moebius

When you can measure what you are speaking about, and express it in numbers, then you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind.

— Lord Kelvin

Man is the measure of all things.

— Protagoras

While you and I have lips and voices which are kissing and to sing with who cares if some one-eyed son-of-a-bitch invents an instrument to measure spring with.

— e. e. cummings

Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them.

— Henri Poincare

The poet's vocation . . . is to discover for use what Shelley called 'the hitherto unapprehended relations' between things.

— C. Day-Lewis

The aim of art is to represent not the outward appearance of things, but their inward significance.

— Aristotle

APPENDIX B

Are MATHEMATICS and POETRY fundamentally similar?

If you doubt their intrinsic similarity, consider the following quotations. In each of the following, the key word (*mathematics* or *poetry* or *mathematician* or *poet* or a variation of one of these terms) has been left out, although the name of the author may provide a give-away clue. Can you guess which art form is being described in each case? The missing words are supplied at the end of the quotations.

(1) ____ is the art of uniting pleasure with truth.

— Samuel Johnson

(2) To think the thinkable — that is the ____'s aim.

— Cassius J. Keyser

(3) All ___ [is] putting the infinite within the finite.

— Robert Browning

(4) The moving power of ____ invention is not reasoning but imagination.

— A. DeMorgan

(5) When you read and understand ____, comprehending its reach and formal meanings, then you master chaos a little.

— Stephen Spender

(6) ____ practice absolute freedom.

— Henry Adams

(7) I think that one possible definition of our modern culture is that it is one in which nine-tenths of our intellectuals can't read any ____.

— Randall Jarrell

(8) Do not imagine that ___ is hard and crabbed, and repulsive to common sense. It is merely the etherealization of common sense.

— Lord Kelvin

(9) The merit of ____, in its wildest forms, still consists in its truth; truth conveyed to the understanding, not directly by words, but circuitously by means of imaginative associations, which serve as conductors.

— T. B. Macaulay

(10) It is a safe rule to apply that, when a ___ or philosophical author writes with a misty profundity, he is talking nonsense.

— A. N. Whitehead

(11) ____ is a habit.

— C. Day-Lewis

(12) . . . in ___ you don't understand things, you just get used to them.

— John von Neumann

(13) ___ are all who love—who feel great truths. And tell them.

— P. J. Bailey
Festus

(14) The ____ is perfect only in so far as he is a perfect being, in so far as he perceives the beauty of truth; only then will his work be thorough, transparent, comprehensive, pure, clear, attractive, and even elegant.

— Goethe

(15) . . . [In these days] the function of ____ as a game . . . [looms] larger than its function as a search for truth . . .

— C. Day-Lewis

(16) A thorough advocate in a just cause, a penetrating ____ facing the starry heavens, both alike bear the semblance of divinity.

— Goethe

(17) ____ is getting something right in language.

— Howard Nemerov

The words missing are: (1) Poetry, (2) mathematician, (3) poetry, (4) mathematical, (5) a poem, (6) Mathematicians, (7) poetry, (8) mathematics, (9) poetry, (10) mathematician, (11) Poetry, (12) mathematics, (13) Poets, (14) mathematician, (15) poetry, (16) mathematician, (17) Poetry.

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ULTIMATELY, MATHEMATICS IS POETRY

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In the fall of 1987 I got in touch with the animator of the European Poetry Festival.* We discussed the possibility of editing a volume of poems. He, being himself the son of a famous professor of physics at the University of Leuven, was not surprised to meet a mathematician in his office. Why not? Because the interaction between poetry and science, although a difficult one to discuss, often occurs in western culture. We mention for the moment only a few mathematician-poets: Blaise Pascal, Lewis Carroll, Alexander Solzhenitzyn, Raymond Queneau. (Last year, Prof. D. J. Uherka of North Dakota lectured on Solzhenitzyn as a mathematician.)

Mathematics intrigues the artist and gives rise to artistic creativity: remember Salvador Dali with his Corpus Hypercubes, trying to represent the fourth dimension. Take for instance the hypercube movie of Thomas Banchoff, its striking simplicity and amazement: a fine moment of mathematical poetry. If it is easy to enumerate a lot of events in which artists and scientists realize in a moment of godliness exact or abstract science and art, it is quite new to try to describe what is exactly going on when bridging these two verges of our culture. One does not even know how mathematical or artistic creativity is physiologically generated (Although some progress has been made recently, see [1] for further references.)

But first, I would like to tell you the story of a beautiful encounter between mathematics and poetry.

1. *The story of a mathematical-poetical-pictorial encounter*

The story begins with my editor, E. Van Itterbeek. Faced with the possibility of editing the poetry of a mathematician, he suggested that I look around for some pictorial work related to mathematics. In this context we have at our disposal an illuminating example: Maurice Escher, the Dutch graphic artist, "discovered" by mathematicians as the geometer Coxeter who encouraged him in his continuous search for forms (the Möbius-strip), transformations and symmetries (the wallpaper symmetries and the space symmetries, the crystallographic groups) and even new mathematical results from the field of p-adic numbers, non-euclidean geometry and analysis.[2] But nowadays one can hardly speak of originality when using again Escher's work as an illustration of poetry. Fortunately, in the Mathematical Intelligencer[3] I discovered an article on the Russian geometer and graphic artist Anatole Fomenko. His drawings and critical remarks on the interplay between his mathematical and graphical work completely corresponded with my own work in mathematics and poetry. I began a correspondence which eventually led to his sending me almost one hundred photographs of his drawings to illustrate my poetical work with. I sent him a copy of the book[4] and a delightful mathematical encounter was realized. Not only was I delighted by the material fact of our contact, but, most importantly also, by the great similarity between his "mathematical fantasy" and the images, the symbolism, the use of mathematical ideas and language is my work, as, for example in my opening poem.

Ik herinner me de cirkels
die samen-klinken
tot een ring
Banneling
die herbronning
in het innerlijke van de cirkel vindt

(Translated by R. Leigh-Loohuizen:)

I remember the circles
harmonizing
in a ring
Exile
finding a resource
in the internal of the circle

This poem is illustrated with a drawing by Fomenko which could suggest exactly the same ideas. So, without a pre-arranged collaboration or a subsequent revision in the direction of his art, I was able to link his pictorial art with my poems. We both use the circle as the materialization of ideas like perfection, loneliness, a mathematical grail.

Metamofose van de cirkel

In de geslotenheid
van de cirkellijn
lig ik gevangen

zonder begin
ontgin ik eindeloos
de doorlopende zin

omknel zinloos
in kromme lijn
het open zijn

de droom
mezelf doorsnijden
van het midden bevrijden

zonder straal
omvormen tot spiraal
naar binnen draaien
narr buiten expanderen

in de kronkels
van de losse lijn
langs de kegel omvezeld
dein ik uit

om in oneindigheid
begin en eind
zorgvuldig te vergeten

(Translated by R. Leigh-Loohuizen:)

Metamorphosis of the circle

I am imprisoned in
the closed-ness
of the circular line

without a beginning
I endlessly develop
the continuous sense

senselessly clasping
the open-ness
in a curved line

the dream:
splitting myself
liberating the middle

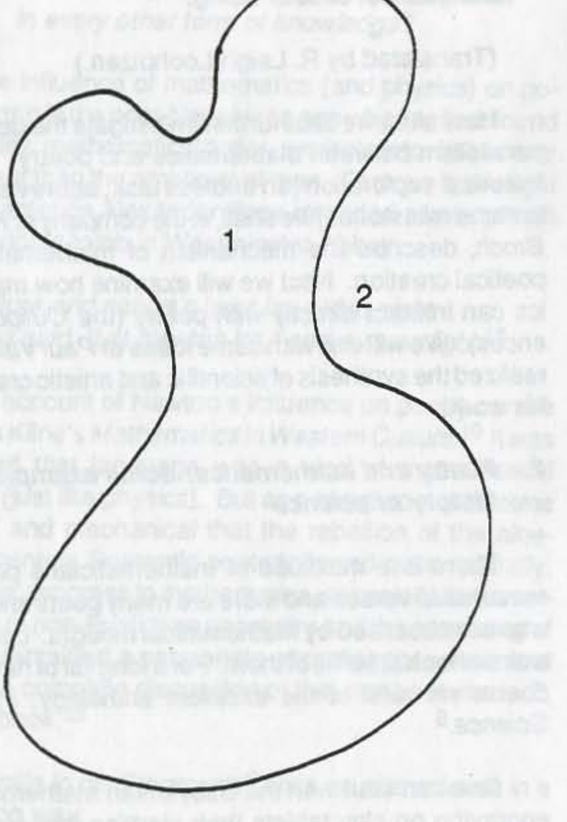
without a radius
transforming into a spiral
turning inward
expanding outward

in the coils
of the loose line
frayed around the cone
I am swaying out

in order to carefully
forget in infinity
beginning and end

A mathematical theorem can be a poetical muse:

Jordan's theorem. A simple closed curve divides the plane in two regions.



Proof

loose lines
originated
in my mind

where infinite
forces converge
to the point of uncertainties

know:
unity for the circle
is simple
unity is love

and that this plane
breaks into two parts
inseparably
united

inevitably alien
next to each other
linked by
the thinnest circular being.

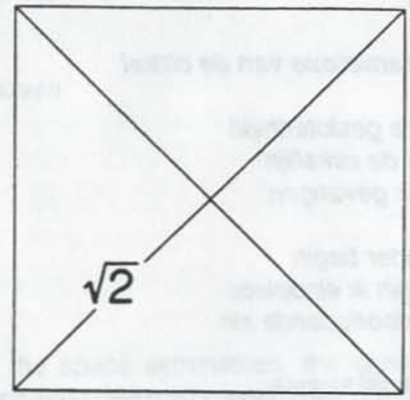
(Translated by R. Leigh-Loohuizen.)

Hereafter, we shall further investigate the question of parallelism between mathematics and poetry. After an historical exploration (an endless task, abbreviated here to some milestones) we shall, in the company of *Hermann Broch*, describe the mechanism of mathematical and poetical creation. Next we will examine how mathematics can interact directly with poetry (the *Oulipo* experiences). We will end with some ideas of *Paul Valéry*, who realized the synthesis of scientific and artistic creativity in his work.

2. *Poetry and mathematics: some examples in the history of science*

There is a multitude of mathematicians producing remarkable verses and there are many poets and writers who are obsessed by mathematical thought. Let us take a closer look at some of them. For a long list of names and poems we refer to the excellent anthology, *Poems of Science.*[5]

One can start with the Babylonian mathematicians, engraving on clay tablets their startling theorems, with nothing less than the first proof of the Pythagorean theorem in the following "poetical form":



(If we take the side of the square as unity we find the irrational number $\sqrt{2}$ on the diagonal, if we take half of the diagonal as unity then the side represents the same number $\sqrt{2}$.)

Or one can read in Plato the truly marvellous story of Menon and his slave to whom Socrates taught (in what was certainly an almost perfect example of mathematical teaching) a mathematical theorem in a language that I call poetical (just like the drawing of $\sqrt{2}$.) Euclid too, in the formulations of his definitions, axioms, and theorems used a highly poetical language, e.g. a point is what has no parts, a line is a breadthless length, a straight line is a line which lies evenly with the points on itself.

The history of science and thought contains plenty of examples of poetical expressions. The main reason for this is that, as pointed out by Peter Hilton,[6] a single mathematical invention can lead many times to a really profound and original astonishment. The creator, first and thus most intensely, has the sensation of having solved the problem, the joy of discovering a new theorem.

But every perceptive reader makes the same discovery. He can even develop the results and obtain new theorems: he can, better maybe than the original inventor, link the novelty to old and new theories. In this, I see a great parallelism with poetry where the reader of the poem has so often the impression that this poem was written specially for him, if not by him. The reader appropriates the poem.

Of course, we mostly meet geometers and astronomers on the ever-moving borderline between science, mysticism, and poetry. Let us always remember Giordano Bruno (1548–1600) who was perhaps the first to declare that the universe was infinite and to foresee the principle of relativity. He did it in a poem: "De immenso et innumerabilibus" where we find, see E. Maor[22] p. 198, a mixture of spiritualism and sound reasoning like:

The One Infinite is perfect; simply and of itself nothing can be greater or better than it. This is the one Whole everywhere, God, universal nature. Naught but the infinite can be a perfect image and reflection thereof, for the finite is imperfect.

Another geometer-alchemist-mystic was John Dee from whom we retain the beautiful mathematical poem (or hermetic geometry, translated from J. Dee by C. E. Josten[7]):

- *the first and simplest manifestation and representation of things, non-existent as well as latent in the folds of Nature, happens by means of the straight line and the circle;*
- *yet the circle cannot be artificially produced without the straight line, nor can the straight line be produced without the point.*

Dee's hieroglyph

In the Catholic tradition, I found a wonderful poem of St. John of the Cross (1542–1591) — in a French translation of Cyprian of the Nativity discovered by Paul Valéry — beginning with the stanza:

Je pénétrai où je ne savais
et je demeurai ne sachant
toute science dépassant

(Translated:)

I enter where I did not know
and remained ignorant
all science surpassed.[8]

This poem is a text, a programme, which serves as a symbolic illustration for his theological ideas. I rediscovered some reflections of the power of symbolism in the magic poetry of the Irish poet W. B. Yeats (1865–1939) who claimed: . . . "That this great spirit and this great memory can be evoked by symbols."

Interest in algebra and in the power of symbols is of course one of the aspects of the Romantic art in the nineteenth century. We read in the Fragments of Novalis[9] (1772–1801) aphorisms like:

Algebra is poetry. Each science becomes poetry.
The number system is the model of a true symbolic language. Our letters should become numbers, our language arithmetic.
Could God reveal himself in mathematics, just as in every other form of knowledge?

The influence of mathematics (and physics) on poetry began in the post-Newtonian age when writers found rationality, mathematical order, symbols, etc. to be more important than the emotions of man. Famous in this light is the sentence Alexander Pope intended as an epitaph for Newton's tomb in Westminster Abbey:

Nature and nature's laws lay hid in night
God said, "Let Newton be," and au was light.[10]

An account of Newton's influence on poetry can be found in Kline's *Mathematics in Western Culture.*[10] It was accepted that language was a kind of mathematical system (just like physics). But as a result poetry became so cold and mechanical that the rebellion of the nineteenth century Romantic poets followed quite naturally. However, progress in mathematics, especially the development of non-Euclidean geometry and the mastering of infinity, remained a passionate attraction for writers and poets. A complete discussion of this can be found in E. Maor's book.[22]

Novalis in his Fragments[9] links art and science in a surprising way:

I = not I. This is the highest principle of all art, and all science.

We cite another poem of Novalis where he links knowledge and consciousness:

> *Consciousness is a being outside being in being.*
> *Consciousness is thus an image of being in being.*
> *Needs to clarify images. Signs. Theory of signs.*[9]

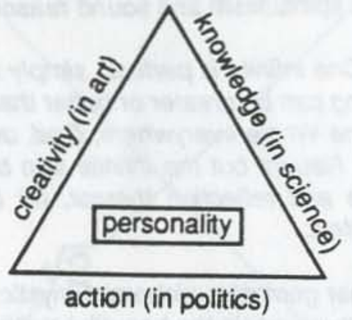
Let me end this part of my essay with a twentieth century mathematician-poet-artist: Alexander Zinoviev, born in 1922 in Russia. Zinoviev was professor at the University of Moscow. He is a logician, specializing in model theory. He wrote a lot of books on mathematical logic, e.g. *Philosophical Problems of Many-Valued Logic*; *Quantoren, Mödalitäten, Paradoxien*; *Foundation of the Logical Theory of Scientific Knowledge*; etc. He is the author of a remarkable novel translated in French under the title *Les hauteurs béantes.*[11] The book describes Stalinistic society and it is a diatribe against the homogeneous world and the programmed social machine. It takes the form of a platonic dialogue and a graphic poem about the ultimate end.

The history of science and thought is full of such examples of poetical effusions. There must be a parallelism between mathematical thinking and poetical creation. In the following section we shall try to unveil this mystery.

3. *Contemporary thinkers and writers on the relation between mathematics and poetry*

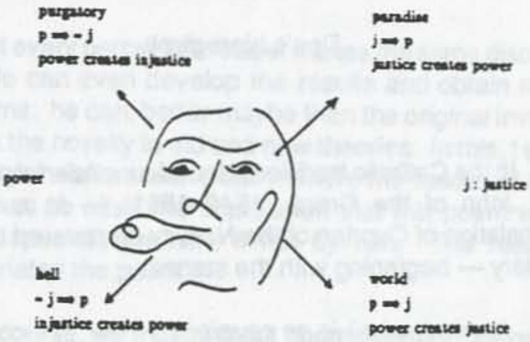
Hermann Broch (1886–1951)

F. Le Lionnais, writing on the beauty of mathematics in *Les grands courants de la pensée mathématique,*[12] frequently uses poetry to illustrate and emphasize his ideas. He cites Novalis ("Algebra is poetry"), but also Henri Michaux who claims that he cannot represent the beauty of mathematics: "Ce qu'il y a de plus intéressant dans ce pays, on ne le voit pas" (The most interesting in this country cannot be seen.). This statement illustrates how difficult it is to discover the poetical idea concealed in mathematics. With Hermann Broch[13] as explained by Hannah Ahrendt[14] in her illuminating introduction to the essays of Broch, I believe that there is a close link between poetry and mathematics. This link was present in the personality of Broch. In fact, every creator often agonizes between his logico-mathematical and poetico-visionary talents. These two poles of the creative personality can be closed by the third side of a Peircean Triangle:



This graph is a good description of Broch's personality, attracted as he always was by mathematics. Influenced by the Wiener Kreis** he even tried to find a system regulating nature, completely determined by group theory. In his view, not only geometry but all activities of the mind could be regulated by algebraic structures. In fact, in Hermann Broch's vision, poetry and science should be seen as the same kind of activity: both recreate the world by a removal of the frontiers of knowledge. He asks for poetry to become full of knowledge, for knowledge to become full of vision. However, Broch also pays attention to the latent tension between creativity in art and knowledge in science. He describes this in his powerful *The Death of Virgil*. H. Broch developed a theory of knowledge in which mathematics play a central role.

There is one constant in the theory: the human being, carrier of knowledge and vision, the intellectual. For this person, only the act can create the very moment when *the ability to create poetry meets the knowledge to handle science*. We give an example of a mathematical formulation stimulating with poetical power an act of knowledge and insight: the windrose of Hermann Broch.



Oulipo (Ouvroir de littérature potentielle - Workshop for Potential Literature)[15,16]

In the early sixties a remarkable experiment took place in Paris and Western Europe. A group, composed of writers, artists and mainly mathematicians, was formed around people like François Le Lionnais — editor of *Les grands courants de la pensée mathématique* (1948), containing the first non-technical paper of Nicolas Bourbaki, an introduction by Paul Valéry and so on — Raymond Queneau and several mathematicians (Claude Berge, Nico Kuiper, etc.). The aim of the group was to produce literature, especially poetry, with the aid of mathematics. They searched for procedures, techniques, potentially productive structures, often directly inspired by science and mathematics. Oulipo had something of a secret society that was in reality a serious workshop producing many publications and trying out a multitude of experimentations. In fact, their position in the cultural landscape of France became stronger and stronger during the last decade. They have had a real influence in literature (George Perec, Italo Calvino, Jacques Roubaud), in semiotics (structuralism) as well as in mathematics and science (Schutzenberger and his palindromes or the theory of free structures, information theory and algorithmic poems).

In this context, the word "potential" is important. Oulipo is empirical and tries to liberate new possibilities from existing mathematical objects, structures and techniques. Here are a few examples of its findings:

1° Applications to poetry of some surprising properties of the Möbius strip: write a poem on one side of a slip of paper; write a second one on the back side of the paper; construct the Möbius strip and realize a new poem this way.

2° The introduction of mathematical expressions in literature: set, class, $\in$ (is an element of), $\subset$ (inclusion), $\cup$ (union), $\cap$ (intersection), \ (complementation), etc. There is a volume of poems by J. Roubaud entitled $\in$; and we have discovered a book by Gaston Compère, a Belgian poet, in which each poem is illustrated and introduced by one of Euclid's theorem.

3° The use of permutations in literature. Combinatorial structures are widely employed in the attempts of the Oulipo-members to create new forms of poetry. Example: "Cent mille milliards de poèmes" by R. Queneau; try to find ten poems of fourteen verses in such a way that each verse can be replaced by one of the nine corresponding verses: you obtain 10^{14} poems. We came across the beautiful poem "Oeufs de Pâques" by Stéphane Mallarmé[17]: each verse was written on an egg numbered so that one could rebuild the poem.

1. Pâques apporte ses voeux
2. Toi vaine ne le déjoue
3. Au seul rouge de ces oeufs
4. Que se colore ta joue.

Only four even permutations (1 2 3 4, 2 1 4 3, 3 4 1 2, 4 3 2 1) are retained. Why?

The Oulipo people (R. Queneau, together with François Le Lionnais and J. Roubaud, was certainly the driving force of the group) discovered also the sextine of the troubadour Arnaut Daniel and tried to generalize this to a quentine, constructed by using the rule: a word ending verse p (or in place p), $p \leq n/2$, where n is the number of strophes in the poem, and the number of verses in the strophe, is put in place 2p and a word in place p, $p > n/2$, comes to the place $2n + 1 - 2p$.

Northrop Frye said it like this: "Both literature and mathematics proceed from postulates, not facts."

The Oulipo experiment in mathematico-literature is, of course, of high interest, often very enjoyable, and it puts new forms and surprisingly new viewpoints to old facts. Because of the highly technical aspect of its work, we can consider Oulipo as a primitive system of knowledge. The poetical act is, in general, absent from its work. But reading I. Calvino and G. Perec (his *La vie mode d'emploi*[18] is largely based on techniques elaborated by Oulipo), we now see that this kind of work can create the conditions that lead to an absolute system. The novel of G. Perec, entirely deductive and written within a fixed structural framework, is an example of this.

The work of R. Thom and E. C. Zeeman, two mathematicians, well-known as the fathers of catastrophe theory, is related to the Oulipo doctrine. In René Thom's book, *Stabilité structurelle et morphogénèse*,[19] a completely new approach to the explanation of the phenomena of acquisition of knowledge is explored. Thom shows how global regularities can be envisaged as geometric structures in a many-dimensional space. These forms have their own dynamics: and each form of life can be described in dynamic mathematical models. Thom was influenced by the English scientist D'Arcy Thomson[20] famous for his book *On Growth and Form*, from which Thom quotes:

The waves of the sea, the little ripples on the shore, the sweeping curve of the sanding bay *between the headlands, the outline of the hills, the shape of the clouds, all these are so many riddles of form, so many problems of morphology . . .*[20]

According to this theory, events in nature, in social or cultural life, in biology or any other science, in literature, etc. occur in points of catastrophe which can be studied by means of singularities of functions. Zeeman and Thom can describe in this way the splitting of a cell, the working of the brain, certain light phenomena, etc. Thus a highly poetical image of science was created using mathematical language as the vehicle of thought.

We can see a poem as a form endowed with its own dynamic; this form can change as a pure form (by techniques similar to the one of Oulipo), giving rise to different new poems; the poem also changes in form (and content) from one reader to the other, and has its own life! Poetry belongs to the field of structural morphogenesis of Thom: more, poems are themselves morphogenesis:

Pour nous autres Grecs
Toutes choses sont formes.
(For us, the Greek
All things are forms.)

— P. Valéry

Fire is in rest when changing. Fire is changing while in rest.

— Heraclitus

Paul Valéry (1871–1945): The Synthesis of Science and Literature

In his famous Cahiers,[21] twenty-nine volumes written between 1894 and 1945, and the *Variété I to V*,[21] Paul Valéry exposes the power of the mind and the force of language. The interplay between science and literature plays a special role in the development of his ideas. We quote:

It is remarkable that mathematics has in common with poetry and music the fact that the idea (le fond) becomes the act of the form: the truth depends on formal conditions.[21]

What poetry and mathematics has in common is said here loud and clear. Mathematics works around ideas — problems or paradigms, e.g. the twenty-three problems of Hilbert, the continuum hypothesis and its relations to the Zermelo-Fraenkel axioms, a property of lines in the plane, the conjecture of Goldbach, etc. The mathematician, faced with his creative work, has to make a formalism: some definitions, a system of axioms (satisfying conditions, creating new objects and so on), some nice examples, a few new theorems. Once these ingredients are found, the mathematician, guided by an idea, can put his formalism to work. The poet also has his raw material: the language (le langage ordinaire), the form of his poem (e.g. a sonnet), also called the space of the poem, the rhyme and rhythm, the examples, the metaphors and so on. Now by means of his idea the poet can force the form onto his idea.

Paul Valéry, in his essay "Poésie et pensée abstraite," Variété V, made a profound analysis of the possible contradiction between poetry and the abstract idea. There is no such contradiction, Valéry claims. In fact, poetry and science are complementary sides of an intellect motivated by the attempt to understand all the problems of man.

Valéry's main hope was that a mechanics of thinking would exist, using mathematics as formalism (see again H. Broch: the only knowledge is the logico-mathematical knowledge). In the case of poetry, Paul Valéry described how a number (he was very intrigued by the Pythagorean idea of number) came first in his mind and how afterwards a poem was based on this number (determining the number of syllables in a verse and the rhythm). He also claimed that analogies were more important than metaphors in poetry. For him, analogies were comparisons based on the structure allowing a certain type of reasoning, i.e. functional analogies. Metaphors and analogies are particular cases of general transformations: their general group (group in the mathematical sense) is the "nervous system."

Valéry was very unhappy with the distinction between the so-called "esprit géométrique" (geometrical thought) and "esprit de finesse" (poetical thought), made by Blaise Pascal.[23] If such ways of thinking existed there had to be a bridge between them, Valéry claimed: 1° "esprit géométrique" is dangerous because the geometer cannot study what he cannot define: he also automatically follows certain ways of thinking and has a tendency to always replace terms by values; 2° "esprit de finesse" is dangerous because it is reasoning by means of rather loose notions or badly defined items (emotions!).

The bridge consists of a sacred moment: 1° the geometer looks for a definition (or for a deep insight!) and wants to pass from imagination to structure and form; this is the moment of poetry: he has to choose, to delete, to adapt, to force the concrete towards the abstract, he has to engage his whole being; only after this, his formalism, guided by imagination and idea, can move into results using transformations of this formalism; 2° sometimes the poet wants to construct!

The geometer is the one who sees in a problem or in a situation what can be rendered by a system of definitions, axioms, symbols, operations, etc. in order to obtain results about the problem or situation which are straightforward, clear and proved — hence acceptable to others. The artist disposes of the means of expressions offered by the form of his art and of the subject (idea); but he disposes of form and idea as long and as freely as he wants. The goal of his art is the correspondence, the suggestion. Inspiration is for him a strong excitement directly caused by the object-subject relation of his art. But this object-subject situation (which causes and is the cause of artistic activity) is as free as possible, excited during the creation by profound knowledge of the means of expression as well as by the passionate desire to bring this knowledge alive.

Afterword

We are living in a time of bulkheads, in a society splintered into the many "holy chapels." Bridge-builders, like Paul Valéry, are solitary and scarce. It has become suspect to look over the wall. I believe in a renewed renaissance, neither because we want to adore the god of others, nor because we do not recognize our own god. Our god is the hidden treasure in our heart. This is the content of the following "Psaume" of Paul Valéry:

Tu n'adoreras pas les dieux des autres:
(Mais prends garde de te tromper sur le tien!)
Tu connaîtras le Tien à sa simplicité
Il ne te propogera pas des énigmes vides
Il ne s'entourera pas d'éternité
Il sort de toi comme tu sors de ton sommeil
Comme la fleur et le parfum sortent de la terre
confuse et du fumier qui se décompose, il
sort quelquefois de ta vie, un peu de Lui et
une idée de son énergie.
Cache ton dieu. Que ce dieu soit ton trésor —
que ton trésor soit ton dieu.

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* The 11th congress of the society took place from 24-30 November 1989 and had as main theme "Poetry and Science." The address of the society is Blijde Inkomststraat 9, 3000 Leuven, Belgium.

** Group of philosophers and scientists formed around Moritz Schlick in the twenties, amongst which people like K. Gödel, K. Reidemeister and later R. Carnap and A. Tarski. They promoted a new philosophy of science, called neo-positivism or logical positivism, characterised by the idea that theology and metaphysics are imperfect modes of knowledge and that only rigorous reasoning is valuable.

THE HERMENEUTICS OF MATHEMATICAL MODELING

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I. Introduction

Mathematics is a language. Those who speak this language frequently use it to describe the world around them. As in any language, signs (words, symbols, signifiers) are created to represent those objects of discussion in the language [20,23]. Depending on the existence of physical referents for the signs created, points of view may fall into two broad categories. There are those who believe philosophically that, physical referents are not necessary, that the only meaningful discourse in the language is through the signs and their relationships to one another. These are the "pure mathematicians" (or, one may call them "structuralists"). On the other hand, the non-structuralists, or "applied mathematicians," attempt to construct a "symbolic order" or sub-language of mathematics which, ideally, would be a perfect representation of some physical ("real world") phenomenon. This representation would be "perfect" in the sense that every change in the "real world" would be reflected by a corresponding change in the "symbolic world" and vice versa. In other words, the transformation relating the "real world" to the "symbolic world" would be explicitly known. This does not seem likely to occur. Yet, the predictive power and pragmatic application of mathematics has produced undeniable results in science and technology. The objective, then, of the applied mathematician is to minimize, in some way, the discrepancy between the behavior of the symbol used in the symbolic world and that of the object represented in the real world. In other words, applied mathematics is constantly evolving towards pure mathematics because the ultimate goal of the former is to ignore the discrepancy between sign and referent and to exist solely within the realm of the symbolic world.

This essay investigates some of the historical and philosophical background of the division between pure and applied mathematics. The "symbolic order" constructed by the pure mathematician and used by the applied mathematician to describe the "real world" is called a mathematical model. The nature, interpretation and limitations of the mathematical model are also discussed. An illustration of the means used by applied mathematicians to drive the above mentioned evolutionary process toward pure mathematics is given. This process, the "modeling cycle," is presented as a response to the "hermeneutic circle" of applied mathematics. The term "hermeneutic circle" (borrowed from the theory of literature [8,9]) refers to the dilemma that, before a model is developed, one must know the important factors contributing to the phenomenon under investigation but, in order to know these factors, one should first develop a model.

It is hoped that a better understanding of the objectives and limitations of the use of mathematical models will contribute to the increased acceptance of them as a means of providing additional information and perspectives in areas of research traditionally considered "non-quantitative." The crucial factor in this understanding is the analysis of the connection between philosophy and theory, pragmatism and application.

II. History and Theory

Plato is seen by many as one of the major figures contributing to the logocentric nature of western philosophy. Logocentrism sets forth the premise that there is a division between word and thought [21, p165ff]. Plato was consistent in maintaining this dichotomy and in the Republic, applied it to his view of mathematics:

> . . . those who deal with geometrics and calculations . . . take for granted . . . things cognate... in each field of inquiry; assuming these things to be known, they make them hypotheses, and . . . setting out from these hypotheses, they go at once through the remainder of the argument until they arrive with perfect consistency at the goal to which their inquiry was directed. . . . although they use visible figures and argue about them, they are not thinking about these figures but of those things which the figures represent.

(Strictly adhering to Platonic belief, the constructs of mathematics represent "disembodied eternal forms," or "archetypes" which are perceived only by the intellect.) For subsequent philosophers also, this was the prevailing view in mathematics — it always "represented" something. Mathematicians were constructing a language with which they could describe the world around them. This description was accomplished through what is today called a "mathematical model."

There are two general categories of models, *iconic* and *symbolic* [7]. An *iconic* model is one that is intended to resemble in some way the object modeled. A "model" train would be an example. Although the determination of "iconic" versus "symbolic" is often controversial, a road map or a schematic diagram of some electrical circuit may also be considered to be an iconic model. A *symbolic* model is one that is not iconic. This type of model often takes the form of some kind of equation whose variables represent some quantities in nature. Descartes introduced the association of iconic and symbolic models in modern mathematics. He represented algebraic relations between variables in a geometric "Cartesian coordinate system" (the type of coordinate system used in road maps). Thus an explicitly demonstrable relationship between the iconic and symbolic model contributed greatly to the facility with which later models could be built and analyzed.

The distinction "pure" and "applied" mathematics did not exist until the second half of the 19th century. Until then, the manipulation of the symbols of mathematics was simply a necessary part of the use of the "symbolic order" used to describe the non-mathematical world. Indeed, there was no need to study pure mathematics of itself until inconsistencies in certain predictions based on interpretations of the mathematics forced the consideration of the logical foundations of mathematics itself.

Thus, at the end of the 19th and beginning of the 20th centuries, about the time that Ferdinand de Saussure was searching to define the basic "signs" and "values" or "significations" of linguistics [20], Bertrand Russell and other philosophers and mathematicians were attempting to reduce language (and hence mathematics) to a fundamental class of irreducible objects, thus generating the entire spectrum of valid claims concerning the "language" [22, 23]. In mathematics this meant that for each area, a set of axioms was sought from which all valid theorems may be deduced. This property of a set of axioms is called "completeness." Perhaps a more desirable property of a set of axioms is that they be "consistent." This means that one must not be able to prove the validity of a proposition and its negation from the given axioms. Hilbert, Russell and Whitehead (see [22]) believed wholeheartedly in the possibility of establishing such axiomatic systems and spent a tremendous effort in attempting it. Kurt Gödel, in his article "Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I," [11, 16] finally settled the question in a most unsettling way. He first restricted his attention to the set of integers, that is, the usual whole numbers of arithmetic 0, ±1, ±2. He then proved that for this most primitive "world" the axiomatic method has inherent limitations in the following sense. If the axiomatic system is complete, then it will be inconsistent and if the system is restricted enough to be consistent, then there are propositions concerning the integers which cannot be proven from this consistent system. Since an inconsistent system is entirely unpalatable to the mathematician, consistent axiomatic systems are used at the expense of completeness. Thus the working mathematician is fully aware that there are most likely questions which may be asked but may not be resolved from within the system.

Russell had already happened upon what is now called "Russell's Paradox" [22, p124–125, p153] which foreshadowed Gödel's discovery. The most popular form of this paradox is to consider a barber in a town who shaves everyone who does not shave himself. Does the barber shave himself? Now, if the barber shaves himself, then he must be one of those who don't shave themselves. Therefore, he doesn't shave himself. On the other hand, if he doesn't shave himself, then he is one of those whom the barber shaves, so he must shave himself. Either way we answer, we arrive at a contradiction. In Set Theory, Russell's Paradox takes the following form: Let S be the set of all sets which are not elements of themselves. Is S an element of itself?

Thus, one of the most important consequences of the 19th and 20th century developments in the logical foundations of mathematics is that it is possible to prove the impossibility of proving something. Russell noted the impact of such an advance on philosophical discourse: "Those philosophers who have adopted the methods derived from logical analysis can argue with one another, not in the old aimless way, but cooperatively, so that both sides can concur as to the outcome." This, of course, refers to the "conditional" nature of modern mathematics (both pure and applied), which Russell [19] humorously expresses thus:

> We start, in pure mathematics, from certain rules of inference, by which we can infer that if one proposition is true, then so is some other proposition. These rules of inference constitute

the major part of the principles of formal logic. We then take any hypothesis that seems amusing, and deduce its consequences. If our hypothesis is about anything, and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Having examined some of the historical background and theoretical limitations of mathematics (and hence its applications), we address, in the next section some of the responses to the problems arising from the theory.

III. Pragmatism and Hermeneutics of Mathematical Modeling

Suppose one wishes to investigate some aspect of a particular field of inquiry and employs mathematics as a part of the analysis used to carry out the study. Variables (signs) are created which represent entities (referents). The method is to establish the behavior of the signs and make conclusions about them. Since mathematical symbols are not the objects they represent, the question then is in what way one could in applied mathematics assert that "only the behavior of the signs need be understood"? The answer is in the degree of *association* between sign and referent. In other words, a perfect association would mean that "events" taking place in the model would be perfectly reflected in the object or phenomenon being modeled and vice versa. Thus, the axioms governing the mathematical order would obtain for the model. This association and the degree of it are the two major goals (and problems) of mathematical modeling. The "pragmatic solution" to the problem of creating the association must, of course, begin with the construction of the model. One uses mathematics — or any method of analysis, for that matter — in order to understand something. So, in order to build a mathematical model, one decides first what the most influential factors governing the observed phenomenon actually are. One then represents them as variables and builds the model. But, in order to decide upon these "most influential factors" one must already understand the "observed phenomenon." The achievement of this understanding is, however, the original reason for building the mathematical model. This is a problem in general hermeneutics, the theory of interpretation which in literary terms Abrams [1] defines as "a formulation of the procedures and principles involved in getting at the meaning of all written texts." In discussing the theory of understanding texts, Dilthey labeled the problem of not understanding the whole without understanding its component parts and not understanding the parts without understanding the whole, the "hermeneutic circle" [1, p84]. Thus, anyone including mathematics as part of the process of understanding is confronted with the "mathematical hermeneutic circle" introduced above: how can the object of investigation be understood without a model, and how can a model be built without understanding the object already?

Dilthey [5,6] and, more specifically, Gadamer [8,9] advocated an approach to overcome the problem of the hermeneutic circle and, interestingly, mathematicians have arrived at the analogous "solution" in their own context. Gadamer's solution was that one establishes a "dialogue" between the "pre-understanding" brought to the text being read and the ideas expressed by the text itself. The reader must then modify the "preunderstanding" using a synthesis of the new ideas of the text and the "old understanding." Thus, one builds to an understanding of a given text through the spiraling process of reading, dialogue (comparison) and synthesis of ideas.

The analogous situation in applied mathematics is the "modeling cycle" [13]. The investigator observes the "natural phenomenon" and conjectures what the most significant factors affecting the observed behavior and the relationships among these variables are. A model is then developed, which can include virtually any object or technique considered to be within the realm of mathematics. For the sake of argument, we may assume the model takes the form of some kind of equation (this is usually the case, but not always). Then the model must be analyzed mathematically. This is the "pure" or "abstract" stage of the process. After a "solution" to the equation is obtained or the mathematical analysis has been otherwise completed the results are compared to the actual observations. This is part of the measurement of the association between sign and referent. If this comparison reveals that the representation is not adequate, then more observations must be made and also more conjectures as to the missing "important factors." Then the existing model is usually modified to improve the approximation to (association with) the observations. The process then continues until the researcher is satisfied that the representation is sufficient for the intended purpose. Ideally, the association between sign and referent would become "perfect," consistent with the rigors of the axiomatics of pure mathematics. However, if such a perfect correlation were possible, we would still be faced with the problem of the incompleteness of the axiomatic system. That is to say, it is possible that questions may be asked that cannot be answered from within the system itself. While the modeling cycle is a pragmatic answer to the problem of obtaining functional representations of physical systems, the problem of incompleteness is a limitation of the use of mathematics.

Despite the limitations, there are many attractive features of the mathematical modeling method. Since mathematicians have chosen to forego completeness in favor of consistency, results derived from the mathematics itself cannot be contradictory. The variables and relations of mathematics, i.e. the vocabulary, is free of connotation. Nagel and Newman [16, p12] attribute this to the fact that "the validity of mathematical demonstrations is grounded in the structure of statements, rather than in the nature of a particular subject matter." Implicit in this view is that the variables, their relations and the means of analysis are clearly and unambiguously defined. In other words, with a mathematical model one may create an idealized world in which all variables and factors influencing them are known and fully controlled. The point of view taken then is that if some particular behavior is observed in the idealized world, then one cannot exclude the possibility of it occurring in the "real" world and possibly for the same reasons. This can and will be expressed more strongly depending on the degree of the sign-referent association. In many cases, there is a way of measuring the extent of this association. This measurement is based on the simple fact that models have a certain predictive power. Thus, in many instances one may compare the predictions of the model with subsequent occurrences in the "world of referents" and formulate a sense of confidence or no confidence in the ability of the idealized world to reflect this behavior. Besides being predictive, models may indicate further areas of research, reveal fundamentals of the underlying dynamical processes observed (subject to the degree of sign/referent association), or, in some cases, discover previously unknown relationships between variables. Finally, a great advantage of mathematics is that its results are reproducible. That is to say, if two investigators accept the same axiomatic system and the same hypotheses concerning the phenomenon in question, both will obtain the same results. This is Russell's observation about "philosophers" arguing from "methods derived from logical analysis." The modelers or philosophers may argue about axiomatic systems or hypotheses but once these are fixed, so are the results.

IV. Illustration

To make the ideas discussed in the previous sections more concrete, consider the following examples from epidemiology. Among the first models of this type are those constructed by Kermack and McKendrick [15]. The book by Bailey [3] treats such equations but is also a comprehensive introduction to the subject matter with an extensive bibliography. The models presented here are from Hethcote [12]. They were selected for several reasons. First, they are from a field of investigation in which the analytic tool of mathematics has not yet been fully accepted. Second, they do involve typical modeling techniques. Third, they are qualitative in the sense that, while they involve parameters which cannot be measured (or have not yet been measured), they nevertheless may indicate significant characteristics of epidemics.

The objective of mathematical modeling in epidemiology is understanding better the dynamic factors influencing the spread and/or maintenance of a communicable disease throughout a population. This information may be useful in designing strategies for reducing the incidence of the disease or eliminating it altogether. Indeed, much mathematical research is now being done to understand the dynamics of AIDS. (See Jacquez. et. al. [14], and the references there.)

It is easy to posit many factors which could contribute to the transmission of a disease. For example, some diseases are incurable, some the body will eventually overcome; some confer immunity, some don't; some are preventable by immunization, others must run their course. Many diseases are transmitted from person to person, some from animal to person or vice versa and some even travel from person to animal to person. Sometimes it is possible for a person to be a carrier of the disease, i.e. to transmit the disease without demonstrating the symptoms. The population dynamics may also play a role. Individuals may enter or leave a population through birth and death or through emigration or immigration. The age of individuals in the population could be important as well as the presence and interference of other diseases. Sexual promiscuity could be important (even outside the context of epidemiology). Geographic location and spread among numerous other factors may affect the disease.

To "break into" the modeling cycle, many simplifying assumptions must be made. It may be validly argued that these assumptions are too restrictive to provide a realistic representation of the transmission of disease, but it must be kept in mind that this is simply the first step in the process. The intent is to improve the initial models. At the outset, the following definitions and assumptions will then be made:

1) A *susceptible* is an individual who does not have the disease in question but is capable of contracting it. The set of all susceptibles is the *susceptible class*. The fraction of the population that is susceptible is called the *susceptible fraction* and at time t will be denoted by S(t);

2) An *infective* is an individual who has and is actively transmitting the disease or at least contacting other individuals sufficiently to transmit the disease. Definitions for *infective class, infective fraction* and I(t) are analogous to those above;

3) A *removed* is an individual who, by any means (immunity, inoculation, isolation) is not involved in the susceptible-infective interaction. The removed class and fraction and R(t) are also defined as above.

4) Each individual in the population must be in one of the three classes described above. Thus, S(t) + I(t) + R(t) = 1 for all t.

5) Diseases will be classified by the epidemiological states through which an individual passes in the course of the disease. Thus, an SI disease is one in which the susceptible becomes infective and never recovers. Herpes simplex is an example. An SIS disease is one that can be cured, but confers no immunity. An example is gonorrhea. A disease that confers permanent immunity is an SIR disease. Measles is such a disease.

6) A *contact* is any interaction between an infective and any other individual in the population that is sufficient to transmit the disease if the other individual is susceptible. The *contact rate*, λ, is the average number of contacts per unit time per infective. We will assume that the contact rate is constant.

7) The population size, N, will be assumed to be large and constant. This assumption is largely mathematically motivated. It allows a tractable model to be developed. It is, however, biologically defensible if the disease is to be studied over a relatively short period of time.

8) The population is assumed to be *homogeneously mixing*. This means that the probability of any two individuals coming in contact with one another is the same. This is admittedly restrictive, but again, a tractable model is then possible. This restriction can then be removed by considering the population to be composed of several homogeneously mixing subpopulations, so the difficulty may be overcome.

9) The susceptible-infective interaction is assumed to follow the "law of mass action" from physics. This means that the rate of loss from the susceptible class (gain in the infective class) is proportional to the product of the susceptible and infective fractions. This is perhaps made clearer by the following development: NS(t) is the actual number of susceptibles. $(NS(t))'$ is simply the mathematical notation for the rate of change per unit time of the number of susceptibles. Each infective contacts λ individuals per unit time and there are NI(t) infectives, so a total of $\lambda NI(t)$ individuals are being contacted per unit time. However, not all those contacted are susceptible. In fact, only S(t) (the susceptible fraction) are susceptible, so the rate of loss from the susceptible class due to the susceptible-infective interaction is given by $-\lambda NI(t)S(t)$.

10) Recovery from the disease will be assumed to follow the "law of exponential growth and decay." That is, the rate of loss from the infective class due to recovery is proportional to the size of the class. This is the same assumption made in radioactive decay or, in another context, the calculation of interest compounded continuously (at 5.5% interest compounded continuously, the rate of change of the amount of money is .055 x Amount, or $A' = .055A$). The principle involved is that the rate of growth or decay is proportional to the amount present. Thus, the rate of loss from the infective class due to recovery is given by $-\gamma NI(t)$. γ is called the *recovery rate* (it is analogous to the .055 above).

Assumptions 6 through 10 are debatable. They do, however, allow a model to be created. Noting that $(NS(t))'$, $(NI(t))'$, $(NR(t))'$ represent the rates of change per unit time of the numbers of susceptibles, infectives and removeds respectively, the model becomes the following system of equations:

$$\begin{aligned}(NS(t))' &= -\lambda NI(t)S(t)\\ (NI(t))' &= \lambda NI(t)S(t) - \gamma NI(t)\\ (NR(t))' &= \gamma NI(t)\\ \text{and}\quad & NS(0) + NI(0) + NR(0) = N, \quad NI(0) > 0\end{aligned}$$

We will not actually do a detailed analysis of this system of equations; it is presented only for the sake of discussion and illustration. Notice that the individuals leaving the susceptible class (first equation) go into the infective class (second equation) and those leaving the infective class (second equation) go into the removed class. The last equation simply says that initially (t=0) everyone in the population falls into one of the three categories and that we do have some infectives (NI(0)>0). Notice also that i) all variables are explicitly defined, ii) all relationships between the variables are demonstrated, and iii) the only dynamic factors involved are the susceptible/infective interaction and recovery. Although the mathematical analysis is not pertinent to the present study, we can see from the following mathematical calculations that the model is inadequate for certain diseases, and hence by refining the model, we will illustrate the modeling cycle. Notice that if the disease is in an endemic equilibrium, that is to say, has stabilized at some persistent level in the community, then the rate of change of NS(t) and NI(t) must be zero. This yields, from the second equation, that

$$0 = \lambda IS - \gamma I$$

Factoring out the common factor of I, we obtain the equation $0 = I(\lambda S - \gamma)$. So, either $I = 0$ and the disease dies out (no infectives), or $I \neq 0$ in which case $\lambda S - \gamma$ must be zero, so $S = \gamma/\lambda$. If this is the case, then $-\lambda NIS$ cannot be zero, so the rate of change of the number of susceptibles cannot be zero and we would not be at an equilibrium. This is a contradiction. Therefore, the only possibility is that at equilibrium $I = 0$. The disease dies out. This is unsatisfactory based on physical observations. Measles is an SIR disease and therefore should have these dynamic characteristics, but has shown no tendency to die out. To follow the modeling cycle then, we must make new conjectures as to the important dynamical factors determining the spread of the disease. Since a disease following explicitly the old assumptions would eventually "run its course" and die out, perhaps the introduction of new individuals into the population would replenish the depleted pool of susceptibles. Following through on this conjecture we make the assumptions:

11) Births and deaths occur at the same rate, α (exponential growth and decay as above). Note that the assumption of exponential growth in all cases in which it is assumed is also subject of "verification" through the comparison phase of the modeling cycle.

12) There are no disease-related deaths. So, deaths occur at the same rate in each class.

13) Birthrate equals deathrate. This is a mathematical assumption to guarantee that the population remains constant; and

14) All newborns are susceptible. Since maternal antibodies confer temporary immunity, a "newborn" is defined to be a child of 12–15 months.

Building on the old model, the new model becomes

$$\begin{aligned} (NS(t))' &= -\lambda NI(t)S(t) + \alpha N - \alpha NS(t) \\ (NI(t))' &= \lambda(t)S(t) - \gamma NI(t) - \alpha NI(t) \\ (NR(t))' &= \gamma NI(t) - \alpha NR(t) \\ NS(0) + NI(0) + NR(0) &= N \end{aligned}$$

The *infectious contact number*, σ, is defined to be the average number of contacts per infective per infectious period. The analysis of the new model yields that $\sigma = \lambda/(\gamma + \alpha)$ and the result that if $\sigma > 1$, then the disease remains in the population. This seems to give a more realistic prediction of the behavior of the disease than the original model. One conclusion that may be drawn from this is that the "vital dynamics" (births and deaths) are important in creating the behavior in the model that is actually observed in human populations. It then may be the case that the introduction of new susceptibles into the population is essential in the transmission characteristics of some diseases. There are, of course, many questions and objections that may be raised, among which are:

1) Why should the contact rate be constant? In schools, for example, winter contact rates should be much higher than summer contact rates.

2) The disease may be affected by spatial (geographic) spread.

3) What happens if the population size is allowed to vary?

4) How might the effect of immunization programs be studied?

5) Is the assumption of homogeneous mixing too severe?

6) What about the effects of immigration and emigration?

Most of these questions can and have been addressed by researchers in this area. Many interesting possibilities for explanations of the occurrence and transmission of communicable diseases have been suggested along with indications of areas of investigation not considered before the introduction of the modeling method — another benefit of the modeling approach.

V. Conclusion

If the language of mathematics is considered as a symbolic order with a very precisely defined syntax (axiomatic system), the distinction between pure and applied mathematics may be drawn through the treatment of sign versus referent. Pure mathematics considers the symbolic order itself as the subject of investigation, thus referents are not necessary. Applied mathematics, on the other hand, must deal with referents, and through an association which always seems to be imperfect.

The goal of applied mathematics is to become "pure" in the sense of working toward a perfect association between sign and referent so that the syntax of pure mathematics may be applied and only the signs need to be analyzed. The problem in the realm of pure mathematics is that it cannot solve all problems that may arise. It is incomplete. Accepting this, the applied mathematician nevertheless works toward the goal of the perfect association through the modeling cycle — a process designed to understand the phenomenon to which the mathematics is being applied. The interpretation of the results obtained through a mathematical model must be taken in the sense that, if a certain behavior of the model is observed, one cannot exclude the possibility of it occurring in the observed phenomenon and possibly for the same reasons.

The illustration of the modeling cycle in epidemiology showed how conjectures about the dynamical factors affecting the spread of infectious disease could be represented by equations. Thus, a system of signs was developed to analyze the behavior of physical referents. First the equations proved to be inadequate, but upon improvement, "behaved" well while revealing an additional factor (vital dynamics) not initially considered.

Now, one may think that modeling is very fruitful in those areas of inquiry to which it is applicable but leaves the question of identification of these areas open. The identification question may also be approached through the philosophy presented here. Are there phenomena so complex that they cannot be analyzed through the mathematical method? This must be rephrased (generalized) to ask whether there are phenomena so complex that they cannot be understood by human beings. The answer is "probably." However, the mere fact that a subject is being investigated at all is admission that those carrying out the investigation believe that some understanding may be achieved. The most pertinent response to the question posed above is that the "sufficient" complexity of the phenomenon in question cannot be determined a priori. In this sense, the modeling cycle could actually result in the conclusion that the mathematical method is inadequate for the problem at hand. But this in itself would be a significant contribution to the understanding of the phenomenon (if only to understanding its complexity). The major result of our study is that mathematical modeling may be considered as a particular form of philosophical discourse and as such should not be discounted as an approach to understanding.

There are special cases in which the "validity" of a particular model in science has been "proven." The connotation of the word "proven" in this case means that the model has "pragmatic validity." For example, predictions of chemical reactions based on present atomic theory are very consistently correct. The question of whether matter actually *is* made up of atoms then becomes irrelevant. We have a model and a high degree of *association* between model and observations. The first atomic theories however were not entirely adequate. The model has undergone many changes in recent decades. In one sense, the modeling cycle assumes that models are accepted only until they are "refined" or "replaced." As situations arise in which a model has been reformulated, the "new" model replaces the old, thus guaranteeing the evolution of the field in a constructive direction.

In those areas under mathematical investigation where the "pragmatic validity" has yet to be proven or where a controversy exists concerning mathematical applications at all, modeling must be considered to be the type of philosophical discourse mentioned above. In this light, the boundaries between those traditional "sciences" and the "non-quantitative" subjects have been identified and they are vague. If the argument against the mathematical method is that it cannot provide us with "truth," then we must reject any means of discourse, since none yet have succeeded in providing "truth." On the other hand, as a form of discourse, the conclusions from the argumentation are always subject to human interpretation, acceptance or rejection.

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ON TEACHING IN THE MATHEMATICAL SCIENCES

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There are a thousand rules on how to teach mathematics well. Some teachers tend to absorb these rules naturally and other teachers can't understand them at all.

However, rules can be irrelevant. There are a few teachers, virtuosos of the classroom, who violate the most basic rules and teach brilliantly. More often, there are teachers who have what seems to be excellent technique but fail to teach even adequately. The reason for all of this is that teaching mathematics (and to some extent all teaching) is an interpersonal skill and not merely a matter of disseminating knowledge.

Teaching mathematics means to communicate mathematics from teacher to students. Teaching is communication, and it is largely a social skill.

This point is not trite but rather profound, as so many would-be teachers do not understand it. It is in fact quite common — an everyday occurrence — for a class to be taught or lecture to be given without one individual in the audience knowing at all what the teacher is talking about. This is why I am not addressing the difficult question of how do we tell a good teacher from a bad one. There are many teachers that are not borderline. These are teachers of whom no student can say anything good. Undergraduates, graduates, D-students, A-students; they all condemn the teacher. And amazingly, nearly every university department in the country (of any size) has one or more teachers of this quality. Teachers who communicate with no one at all are largely a phenomena of the mathematical sciences, and are a primary reason that mathematics and mathematical reasoning is so poorly integrated into our society at large. Teachers like these make the mediocre teachers look good.

The Personal Side of Teaching

The most important aspect of teaching as communication is that communication with students is a two-way process. To teach effectively you need to gauge accurately what is being digested and what isn't. A crude way to do this, when you are unsure, is by giving an unannounced quiz (which doesn't have to count). The best tool, when lecturing, for judging the students is body language: faces and eyes. This is the window to the student, and if you do not know how to use it, you teach at a disadvantage. When you ask the students if they understand a certain point, their faces will give more information than their voices. At first their faces will not necessarily reveal much. The first week, students tend to regard certain questions as rhetorical. However, when they realize that you are genuinely concerned with their understanding (just as many teachers are not) they loosen up and it is then that their faces become revealing. It is this technique of watching faces that enables the teacher to set the proper pace.

It is incumbent in all of this that you are teaching the majority of your students. Again, teaching is communication and to maximize communication, you teach to as many students as you can. In many classes there are students who go into the course knowing much of the material. Many teachers invariably identify these students as their *good* students. However, with these students there is less communication, since it has taken place for them already. They should be taking another class! Similarly, there are students whose background is so deficient that they also should be in another class. However, when we say that a student's background is deficient, we mean that it is deficient with respect to the class at large. As a rule, virtually none of the students will have mastered the prerequisites to the degree that you feel that they should have. This is a basic law of the universe, that falls right behind Newton's laws of motion: *The students never know what they are supposed to know.* The exceptions are generally those students we have just encountered who already know the material of the course as well as the prerequisites. They should be taking another class and learning new material. With respect to the class at large, how much of the prerequisite material you review is a matter of judgment. It is nearly always wise to reiterate some of that material, but is inappropriate to spend much time on it.

Teaching Style

Pedagogical studies and treatises on teaching tend to concentrate on technique and style. However, these things should be entirely subordinate to communication. Style will not turn a bad teacher into a good one or vice versa. In fact, it is almost irrelevant to good teaching.

A unique style of teaching mathematics is due to R. L. Moore and is known as the *Texas school of teaching*. Although this style is worth study, it is not in itself a subject for this essay. Moore used his style with great success and it has been used by his "descendants" and their "descendants" (Moore's *descendants* are the people who received Ph.D.'s under him). The Texas school of teaching has been the subject of much attention including documentary films. It is clear that he was an inspirational teacher. Many teachers slavishly follow the precepts of Moore's teaching method to the letter. Yet remarkably often they achieve the precise opposite result. R. L. Moore would have been a great teacher had he used any style at all. Conversely, and unfortunately, no style will turn a bad teacher into a good one. To be a good teacher, it is necessary to be sensitive to other people as human beings. This is particularly true with the Texas school of teaching.

That style will not turn a bad teacher into a good one, has a corollary with respect to the incorporation of computers into mathematics instruction. Yes, computers can enhance math education at all levels, from preschool through graduate school. However, the successful integration of computers into education will never obviate the need for competent instructors unless the need for instructors is eliminated altogether.

Developmental Psychology

There is an order by which students learn material. I do not mean the tautological order; for example, one must study what a derivative is before studying differential equations. That much is trivial. I am referring to a couple of psychological truths. The first psychological truth is that students need to learn concepts in an ordered manner. Unfortunately, there are individual differences here, but some generalities hold. Most people learn from the specific to the general. One studies the real number line before one studies fields. At first, the student understands fields as generalizations of the reals. Eventually, the perspective changes. The scholar thinks in terms of fields, and sees the real number line as a specific case. And it is precisely at this point that many teachers make a critical mistake. Because this latter point of view has become clearer for them, many mathematics teachers teach abstractions before they teach specifics. The mistake is this: people learn mathematics in a different order than which they (later) come to understand it!

The second psychological truth is that students need to learn material in a paced manner. That is, some material needs to be digested before new material is learned. This is true at several levels. At the macro level, we don't go straight from the definition of the derivative to differential equations. At the micro level, I have seen a teacher severely undermine his effectiveness, simply by not pausing enough during his lectures. Lectures can be like (well-told) jokes: you need to pause for a beat or two. Here again the best tool for judging pace is body language. Many students seem to light up when an idea sinks in. If the idea isn't sinking in everywhere, you need to reiterate. If the idea is sinking in nowhere, you probably need to back up. It is a tragedy that so many teachers will continue when no one at all is absorbing anything.

Motivation

The foremost tool for teaching mathematics is motivation. Generally, the less advanced the students are, the more the need for motivation. Conversely, a characteristic of strong students and professionals is that they find their own motivation. Frequently, a proof does *not* motivate a theorem. The student wants to know where the theorem came from. For example, Lagrange Multipliers are usually *explained* with a proof that is largely algebraic. But the insight for the technique comes from a simple picture (and undoubtedly it was this picture that inspired Lagrange). Pictures tend to motivate better than does algebra, but not always. Since textbooks too frequently fail to provide motivation, it is the job of the lecturer to provide this motivation. It is generally supplied in the form of pictures, examples, or simply an indication of the where the topic is heading. Many teachers more or less write a textbook on the board. However, students nearly always prefer a textbook in hand; the teacher exists to supplement the book.

Homework

At all levels, to learn mathematics, it is necessary to do mathematics. Most students understand this, but elementary students in particular do not have the discipline to do mathematics without "encouragement." I find that most of them appreciate a little coercion. Here I am going to deviate from the usual pattern of this essay, and I will tell you what I personally have found to work in assigning homework.

When I have a grader, I assign homework at the end of each lecture on the basis of what I cover that day (and how well the students handle it). I collect homework at the beginning of each class and go over the problems on the board. I ask my grader to give me total scores at the end of the semester, standardized to some preset maximum, say 50 points. This should work out to be about 10% to 15% of the total grade. It should count little enough that a missed assignment will make little difference (and little hassle). On the other hand, homework should count enough that a student that consistently does not do the homework loses a letter grade. Both of these points are impressed on the students so that they feel compelled to do the homework, but it is not a matter for hysteria. Also they are more or less free to copy homework on the principal that what they gain on the homework, they more than lose on the tests. I tell the students that the purpose of the homework is to prepare them for the tests. I often use old test problems for homework. This mitigates against any advantage some students attain by having old tests in their possession. Also, students are very much interested in old test problems. In fact, if you run off copies of old test pages to hand out as homework, you cannot keep the students from doing the problems. I favor certain homework problems as being particularly pedagogical and tend to give similar homework problems from one semester to another. However as a matter of basic ethics, I have never given any test twice.

If I do not have a grader, I am inclined to give a quiz each week. Frequent quizzes will motivate students to do their homework, since that is how they study for the quiz. I find quizzes much quicker to grade than homework.

Textbooks

As a rule students prefer courses that are textbook oriented, and they are keenly interested in having a text that they can read and understand. Given the abundance of texts in most areas the instructor's job is to wade through a sea of mediocre texts and retrieve the exceptional text. However, not all texts are at least mediocre. There are not only poorly written texts published, but even major publishers will publish incompetent texts. These texts occur at all levels and in late editions. That such texts get selected is shameful. However, many teachers will favor a poorly written text. Such texts are often like computer manuals: you can only understand them if you already know the material. Clearly, such a text will be readable to the teacher but will be frustrating and nearly useless to the student. That teachers will insist upon using bad textbooks (when good ones are available) is an indictment of their sensitivity to the students' needs. There are many rationales for this behavior, but not one of them is any good.

The state of calculus education is an indicator of all mathematics education. High school calculus is most often a wasted year; it almost never prepares the student for the second semester of college calculus. We will only concern ourselves here with college calculus.

Current calculus texts are incredibly alike. This is demonstrated in a wonderful review — essay actually — by Underwood Dudley [*American Mathematical Monthly*, Vol 95, November 1988]. Anyone interested in calculus texts should read this essay. (See also Professor Dudley's review of three calculus texts, to appear in *The UMAP Journal* probably in 1990, Vol 11, no. 2). Calculus texts tend to be far too long and far too formal. Remember, we are talking about college freshmen. Most of them do not understand proofs, and they are not yet ready for long proofs. *Which brings me to an important digression. Proofs are not something you either understand or not. It takes practice to learn to understand proofs, and as with learning most things, one should start out with the simple and work up to the more complex. This is an ongoing process requiring the entire undergraduate four years. Delta epsilon proofs are difficult for freshmen. Yes, the students need to be exposed to delta epsilon arguments, but it is a mistake to over-emphasize it (for example with choose-the-epsilon-as-a-function-of-delta problems). Time is important; with exposure now, the same students will find delta epsilon proofs easy when they are juniors. It is a mistake to emphasize formalism in freshman calculus. Doing so actually gets in the way of teaching the many difficult concepts of basic calculus.* Current calculus texts do a bad job of teaching calculus. For an example of what a calculus texts should be, see the text by Gilbert Strang [to be published in 1990 by Wellesley-Cambridge Press]. Professor Strang's text features some material which is current, but many of the other sections, such as the material on trigonometry, could have been written two hundred years ago, but likely has never been written as well. I think Strang's text is the best calculus text since Courant's first calculus text and possibly since Euler's (eighteenth century) text.

Preparation

The key to a good lecture is organization. The teacher should go into the classroom intending to expound on one or two ideas. He (or she) should have a predetermined sequence of points. These points are illustrated by examples. The teacher should also be aware of the examples in the text. Sometimes it is sufficient to tell the students that a point is illustrated by a particular example in the text. Yet the teacher should be prepared to be flexible. A well-prepared lecture is like a flow chart. The actual words are supplied during the lecture and are not prepared beforehand. A lecture should appear somewhat spontaneous.

Lectures

Lectures are a framework for the course. This is where you put your organizing into action. You motivate what is in the text, and you order the material. For example, given ten formulas, probably two are more important than others. Typically, I remember only a few formulas out of any group and I remember how to derive the others from what I have memorized. It is *very* appropriate for the lecturer to share with the students his (or her) mnemonic devices (at this point the students need these devices more than the teacher does). The teacher in effect structures the course through the lecturing process. Again, it is not the lecturer's job to provide another book on the board. Not only is that inappropriate, but it is a waste of the lecturing environment. The primary purpose of the board should be for emphasis and enumeration. The lecturer simultaneously augments the text and incorporates it into the lecture. (Don't get me wrong, I've been forced many times to teach without a text, or to make minimal use of my text, but except for advanced graduate courses, this is usually not desirable.)

The War of the Students and the Teachers

Bad students abound. By bad students I do not mean slow learners, but the lazy and the rude (a good teacher needs to know how to deal with rude students without letting them have an adverse affect on his style — however that is beyond this essay). But given a class of thirty students you should have a fair sample of most basic human characteristics. Unless a class requirement is that the students have been convicted of violent felonies, every class should contain some good students. This is true even at the weakest schools. You just have to open your eyes and look.

A student has a class on Monday and Thursday evenings. One day the instructor says that he can't make the following Thursday, and class will be held on Friday instead. Two students object that they have conflicts. "Tough" he says, if you want to come to class, you will attend on Friday. Fortunately, the administration cancels the class.

So what is unusual about the preceding story? Absolutely nothing. It is merely the most recent story of its type that I have heard. Too many teachers regard the students as the enemy. Students do not have real illnesses or deaths in the family. Or as in the story, they do not deserve the common decency ordinarily accorded to human beings.

Teachers control the teaching environment. A good teacher should be able to handle the bad apples, but students are completely vulnerable to bad teachers. This is true at all levels, but becomes absolutely critical at the graduate level and especially the Ph.D. level. More then one Ph.D. student has blown four years or more down the drain because he has offended a professor.

As we are going into the 1990's, students are paying more and more in tuition and other education expenses and getting less and less in return. Do you remember ever seeing the following notice?: *XYZ University recognizes that Professor Glck's class was not properly taught. All academic records related to that course are being deleted. Please find enclosed a refund of your tuition along with interest.*

Teaching well is not simple. There are many difficult issues to address such as philosophy of testing, philosophy of grading, and so on. However, these issues only become relevant given some competence to begin with. If the teacher does not understand the basics of human communication, then chances are that he is not going to appreciate the fine points of lecturing.

MATHEMATICS, TRUTH AND INTEGRITY

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Some years ago there was a scandal at the Institute for Advanced Study at Princeton. It was proposed to appoint a certain social scientist to permanent membership, but the recommendations for him were ambiguous to say the least, ranging from strong approval to contemptuous dismissal. Certain leading mathematicians at the Institute led the campaign to ensure that this individual was not appointed. The story was featured in the *New York Times,* and many mathematicians, reading the account, fell to wondering if the work of a mathematician, rather than a social scientist, could have received such widely divergent judgments. Our strong belief is that this couldn't have happened.

More recently, we have witnessed the (successful) campaign of Serge Lang (see *Chronicle of Higher Education,* February 3, 1988, p. B4) against the election of a certain Professor Huntington to the National Academy of Sciences. Huntington is a social scientist who had invented certain equations relating to such quantities as 'satisfaction indices', designed to provide insight into the state of contemporary society. Lang argued that mathematics was being misused; the dispute was carried further in the columns of *The Mathematical Intelligencer* by Neal Koblitz and Herbert Simon, acting as surrogates for the main protagonists (see the Winter, Spring, and Summer issues of 1988); and, once again, mathematicians asked themselves whether there could be such utterly conflicting views about the work of a leading mathematician. Once again, too, we concluded that there could not.

Why do we distinguish in this way between mathematics and the social sciences? It is because we believe that there is an objective aspect to an assessment of the quality of a piece of mathematics which — it seems to us and evidently to others — is not necessarily present in the assessment of research in the social sciences, so that peer evaluation of mathematical research at least has the potential to be fair and reliable.[1] There may be disagreements about the relative standing of different areas of mathematics (e.g., algebra vs. analysis, hard analysis vs. soft analysis, point set topology vs. algebraic topology, algebraic topology vs. geometric topology, and so on) but, within a given branch, there is general agreement as to who are the giants and what are their major contributions. Of the Fields Medalists with whose work we are familiar — suffice it to name Atiyah, Serre, Thom, Kodaira, Thompson, Donaldson, Freedman, Novikov, Grothendieck, Smale — there is absolutely no doubt of their eminence and of the seminal significance of their work and the stimulation which it currently affords. In this respect the Fields Medals differ from Nobel Prizes, which are usually awarded long after the relevant work was done, and where there are often strong disputes over the merits of the laureates and over certain singular omissions. It is an open secret that Graham Greene has been passed over for the Literature Prize because of the prejudice of a member of the selection committee, while the award of the Peace Prize to Henry Kissinger and Le Duc Tho continues to strike most reasonable people as utterly ludicrous.

Lysenko was able to fool a lot of people, including even some biologists, into believing that he had revived Michurinism and demonstrated the inheritance of acquired characteristics. By contrast, we claim that there can be no successful chicanery in mathematics; proofs must be clear and convincing, results must be applicable. There can be no conspiracy to believe something which is ideologically acceptable or socially convenient, such as the Nazi 'theories' of racial superiority. It is true that there has been controversy over the proof of the 4-color theorem by Appel, Haken and Koch; but the question at issue is not 'Is it true?' or 'Is it important?' but 'Has it been proved?'

There has also been controversy in connection with the development of fractal geometry — the reader is again referred to *The Mathematical Intelligencer* to get the flavor of this juicy dispute (see the Fall, 1989, issue) — and, at first sight, this may appear to concern the quality of the mathematics. We claim, however, that this appearance is illusory. In reality, what is in question is not the quality of the mathematics in the theory of fractals but whether there is a distinctive mathematical theory of fractals, distinguishable from a theory derivable from classical function theory. Inherent in the controversy, therefore, is a disagreement over who has priority for discovering the undoubtedly important Mandelbrot set. Such questions of priority, in their turn, inevitably raise ethical issues.

These examples serve, in fact, to reinforce our conviction that there is an inescapable ethical component to mathematics as a human activity. Truth and integrity play a key role in mathematical research and publication — one of us (PH) recalls Henry Whitehead's advice, which for him was a principle, never to accept in your own work a result which you could not yourself prove. Of course, this precept has a practical value, since one does not wish to act as a channel for the transmission of error; but Henry's basic point is that one must take responsibility for what one publishes. It is its relation to truth and to the integrity of its practitioners which is the humanistic aspect of mathematics which we wish to stress in this essay. We thus find ourselves in strong disagreement with the views of our friend and colleague Reuben Hersh [He] who denies that pure mathematics has an ethical component.

Before developing our theme, we should stress that we are not speaking of the ethical or humanistic aspects of *teaching* mathematics.[2] Our concern is with the humanistic aspects of mathematics itself. On the other hand, neither we nor Hersh would deny that all teaching of mathematics provides the opportunity — indeed, we would say, the obligation — to bring to our students' attention the ethical commitment which the proper practice of mathematics requires. This obligation, deriving as it does from the nature of mathematics itself, does fall within our purview. We regard it as especially urgent to emphasize it in view of the fact that, for easily comprehensible reasons, it is so often neglected. Let it therefore receive our immediate attention.

COMMUNICATING ETHICAL VALUES TO STUDENTS

We believe that most of the difficulty encountered by our students in trying to learn mathematics at the university level stems from the fact that they have never seen any *real* mathematics before. They have been 'taught mathematics' in such a way that they don't recognize its relationship to the real world and don't understand that it is much more than merely a game. They don't realize that the symbols they write must mean something and that that something should always make sense; they don't understand that there is an unbridgeable gap between truth and falsehood in mathematics, not a mere continuum of meaningless statements. They don't realize that each statement they write down should follow logically from its predecessor; and they don't appreciate why an argument is not complete unless every step does in fact follow from the previous one. In a word, they do not appreciate the integrity of the subject — but this is scarcely their fault. Their experience has left them blissfully unaware of the fact that mathematics involves any question of integrity at all!

Crucial to any attempt to repair this situation is the understanding that the students are not to blame for it. When they reach the university they find themselves in the position of desperately trying to learn material for which they have not been suitably prepared. The response of students to their pre-college mathematics education[3] is, we believe, perfectly natural and should have been expected. All too often, that education has consisted of being given, each day, *the rule of the day*, followed by a set of exercises for which this rule produces *an answer* (that may, for odd numbered problems, be looked up in the back of their textbook!). We claim that students who have been taught mathematics in this catechistic way have been doubly cheated. They have not been given the opportunity to learn what mathematics really is, and they are not able to use what they have supposedly learnt.

For obvious reasons — at least to anyone who either appreciates or uses mathematics — we believe that it is absolutely essential that the teaching of mathematics, at all levels, should embody Henry Whitehead's emphasis on understanding what you use. This has a very important long-term practical aspect in that what students understand they will continue to have at their disposal. Even though some details of a mathematical result may fade over time, if one really understands the underlying principles then it is very likely that one will be able to reconstruct the desired result when it is needed.[4] We believe an equally important, and more immediate, consequence is that students who are taught mathematics for understanding (even at the expense of speed) will have a much better opportunity to learn — and so to appreciate — the real nature of mathematical thinking and hence, as we have said, to make the ethical commitment which the proper practice of mathematics requires. A devotion to the pursuit of truth, in all its aspects, would bring students, and teachers, closer to an understanding of the essential content of a mathematical statement. Thus, for example, they would understand that not all wrong answers are equally wrong, and that being able to recognize whether or not an answer is plausible is much more important than memorizing meaningless formulas long enough to pass a test. However, herein lies a severe practical problem.

We believe that mathematics, when practiced properly by students, should incorporate the ethical commitment inherent in mathematics itself. But to achieve this is difficult. Students naturally want to make good grades. They have been systematically programmed to become successful grade-grubbers. We cannot change their need for good grades — and even if we could it might not be desirable — but we can change the way we test and the way we grade.

We can give credit to the student who recognizes an answer is wrong, says so, and explains why the answer is not a reasonable one. Moreover, recognizing that an answer is unreasonable is itself a sign of a maturing awareness of an important feature of mathematics itself. It is a sad fact that most people do not realize that it is perfectly possible to know something is wrong without knowing what the correct answer is. If they had learned, and understood, the technique of casting out nines (see [HP]) to check an arithmetic calculation, they would thereafter fully appreciate the fact that one can, in some situations, know for certain that a calculation cannot be correct. They would also realize that if the check "works", that doesn't guarantee that the calculation *is correct*.

We can also give credit to the student who begins a proof, knows how it should end and *admits* that the intervening steps are missing; so much the better, of course, if the student also states the nature of what should be filled in. But we would not give credit to the student who puts in a few steps at the beginning and just before the end, hoping the instructor won't notice that there is a gap in the middle. We believe that this kind of behavior, which we would call fundamentally dishonorable, should be strongly discouraged, and that students should be made aware of the fact that there is an internal structure to mathematics that should not be violated. By giving the student credit for the correct thinking he or she does, we encourage both honesty and effective mathematical thinking.

THE ETHICS OF MATHEMATICS IN EVERYDAY LIFE

If we are right in asserting that the pursuit of mathematics has an inescapable ethical content, should not that moral component transfer itself to other aspects of our lives, professional and personal? Should we not be more consciously aware of this moral component? Should we not extend our respect for truth and our concern for the probity of our research to some of our other activities?

We claim that we often do — to our serious disadvantage as advocates and opinion-formers! Let us elaborate.

- It is a feature of our professional work to distinguish sharply between what we know well and the rest of mathematics. Thus, in particular, we are very well aware of our areas of ignorance; this awareness is, indeed, as an important criterion of the educated person. Unfortunately, such awareness is all too often absent among those who exert an influence on public opinion; unfortunately, too, it is no advantage if one wishes to popularize one's cause, since a confession of ignorance is usually taken — quite erroneously — to be an admission of incompetence. How many politicians today admit their ignorance of European history?

- We are naturally liberal at a time when extreme views tend to command more support. We live in an age of single-issue fanatics (to borrow Bernard Levin's vivid phrase); mathematicians should find such fanaticism very distasteful, if not impossible.

- Mathematicians are opposed to the use of force, as it is not possible to establish a theorem by intimidation of the sceptics, or by the demonstration of superior strength. It is thus offensive to their sense of proper order in the universe that disputes should be resolved by means which pay no heed to the worthiness of the cause.

- Since rational thought, and hence reasonableness, are our research method, we tend to see the other person's point of view. Such reasonableness is scarcely conducive to the evocation of fanatical support — one does not persuade people to man (or woman) the barricades by arguing that one's point of view is in certain clearly defined respects superior to that of the enemy.

- We tend to believe in the reasonableness of others, especially of those with whom we are in dispute or whose opinions we wish to influence. This belief is, unfortunately, naive and often mistaken. In such cases we are at a serious disadvantage and are likely to be completely outmaneuvered.

However, we do have some conspicuous successes to our credit. As we have said, our awareness of the existence, within our store of knowledge, of significant areas of ignorance often causes us to be unduly reluctant to participate in deliberations which range over a broad front (university mathematicians are all too rarely active on university-wide committees); but some of us, while retaining our intellectual integrity and honesty — indeed, largely because we bring those qualities to bear — are outstandingly effective in public office. Let us cite two enormously successful university presidents, John Kemeny and Paul Olum, and the man who constitutes for us the supreme vindication of our argument, the Polish mathematician Janusz Onyckiewicz, sometime spokesman for Solidarity and later Deputy Minister of Defence in the Solidarity government of Mazowiecki.

Of course we do not deny that there are counterexamples to our claim that the discipline of mathematics imposes standards of integrity and truthfulness on its practitioners which should inform their activities outside mathematics — the name of Ludwig Bieberbach comes all too readily to mind. We must emphasize that we are only asserting that everyday life offers scope for the exercise of virtues which should have been developed by activities devoted to the understanding of mathematics and research in mathematics. But human frailty is a factor whose strength and ubiquity we recognize.

COLOPHON

We believe that we do a disservice both to mathematics and to education by failing to insist as teachers, explicitly but, of course, not constantly, on the potential role of mathematics in the development of character and morals. For the proper — and, hence, the successful — pursuit of mathematics requires a dedication to truth and integrity. We should always be modest in our claims for ourselves as mathematicians — but there is every reason for us *not* to be modest in our claims for the vast ambit of mathematics itself.

ENDNOTES

[1]One must express oneself cautiously. The impression is widespread that peer evaluation of NSF research proposals exemplifies the dictum, 'You scratch my back, I'll scratch yours.'

[2]We were surprised that the great majority of papers contributed to the *Symposium on Mathematics and Humanism* at the Winter meeting of the AMS in Louisville in January, 1990, were concerned with teaching and not with mathematics itself.

[3]It would be more accurate to describe it as *training*, rather than education. Unfortunately it is usually not even good training.

[4]In the very short term understanding can bring disadvantages because the time required to reconstruct an argument is almost certain to be greater than that required simply to regurgitate a formula learned by rote.

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MATHEMATICS FOR LIFE AND SOCIETY

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ABSTRACT

A course (3 credit) by the title above was developed and taught to adult evening students as an alternative to a Basic Skills and Elementary Algebra remedial course. Quantitative concepts were acquired by extracting these from concrete social, economic and political problems of direct interest to the students. Applications considered were, for instance: The Consumer Price Index; Optimizing mass transit fares; Estimating world food and energy production; Population growth and extinction; Keynesian multiplier effect etc.

The student body of University College, Rutgers at Newark, where I taught for some 25 odd years consisted of adult evening students of many different backgrounds and of all ages. Many were minority women who would get up at 5 a.m. to cook dinner, clean the house and send the kids off to school. They came to class after a day's work and then returned home to do their homework. As mothers they saw to it that their children did well in school. One of my students between her and her husband's, raised twelve children all of whom were in high school or college. They were remarkable people all!

Approximately two thirds of the students received a failing grade in mathematics or withdrew. A poor education in the lower schools, accumulated anxiety and a lack of conviction that mathematical skills were of much benefit to their lives, combined to create a block towards achievement in the required remedial mathematics courses. To make matters worse, for a part of the twenty odd years that I taught these courses (I nearly always taught a remedial math course per semester), the New Math fad raged and textbooks demanded long theoretical arguments to "prove" the validity of the most elementary algebraic manipulations — while doing little to enhance the students' ability and confidence in applying quantitative skills.

"What use is all this mathematics to us? Why do we have to learn about these x's and y's?" they complained. I argued in vain that a quantitative insight into social and economic problems and data is essential to each citizen, if he is not to be deceived by the powers-that-be in our present day society.

I decided to try a novel approach. Under a grant from the Rutgers Educational Development Foundation I developed a course entitled MATHEMATICS FOR LIFE AND SOCIETY. Concrete practical problems in the social, economic and political domain of direct interest to the students were presented; clusters of applicable mathematical skills and concepts were extracted in the process of solving these. The intention was to make the mathematics sufficiently simple so that the students could learn and see its use simultaneously.

Topics covered in the course of one semester were Estimation and Powers of Ten; Variables; Linear Equations and Systems of Equations; Relation Notation; Functions and Graphs; Mathematical Trees; Combinations and Permutations.

Our first class consisted of a pep talk. I told a story of a bailiff at King Arthur's court who used Roman numerals to tally the taxes collected. He was deposed and beheaded by the King in favor of a "mathematical genius" like one of my students, who used decimal notation to perform the same task in no time flat. In the same vein, skills accessible to many today — such as solving complex problems with the use of computer programs — were accessible only to a highly trained mathematician some thirty years ago. An ordinary programmer secretly using a computer would be a "mathematical genius" in the mathematician's opinion.

This pep talk led to a review of decimal notation. The use of a familiar skill eased the students into the course and recalled power of ten notation.

We followed up with a preliminary discussion of the Consumer Price Index and proceeded to "stretch" this socioeconomic construct to cover many mathematical concepts. (It has been said that "one can stretch a word to cover the world.") A perusal of the C. P. I. table from its inception in 1913 until today showed how ratios, rates of change, percentages, proportions can be extracted from the numbers of the table to reveal the economic pulse of various time periods. (Why, for instance, was the rate of change small in World War II as compared to World War I? Because of price controls!). The concept of base year calls for the use of variables and formulas. The purchasing power of the dollar marks the notion of reciprocal, by which we shall learn to divide fractions. A compilation of the consumer basket prices over the years appears as a matrix. The total cost of the basket, calculated by weighted items, introduces the summation notation. The presentation of data in units of one thousand or a million justifies the introduction of scientific notation and the estimation with powers of ten. The shifting nature of individual and national priorities in assembling the items in the basket lends meaning to the mathematics of combinations and permutations. Finally a historical discussion of the C. P. I. alerts us to the importance of quantitative data in the functioning of modern society.

The course now moved back and forth between the qualitative and the quantitative, precipitating the mathematics — not necessarily in sequential order of topics, but rather introducing and returning to whatever skill was relevant — from the applications rather than vice-versa as is usually done.

SOME EXAMPLES

An application of Powers of Ten and Quadratic Functions

The following excerpt of a letter which appeared in the Bergen Record under the rubric *Port Authority needs a Math Lesson,* is an example of how these concepts were introduced in class:

> . . . We used the technique of Powers of Ten to estimate the total of tolls collected at rush hour daily and yearly on the George Washington Bridge. We considered the number of cars passing through a toll booth per hour, multiplied by the number of toll collectors. We arrived at a figure of some $42 million per year. This estimate — when checked against the not easily available Port Authority data was some $8 million short. Assuming an average salary per toll booth collector of $30,000 a year for 300 employees - this would absorb less than a fifth of the $50 million. Maintenance surely does not absorb the other $40 million. . .
>
> Another technique, that of finding the maximum point of a parabola, was employed to determine the optimum fare in a situation where every increase diminishes the number of consumers (demand under free competition). We found that beyond this maximum increases were counterproductive and diminished the total revenue. Perhaps it would behoove the directors of the Port Authority to study this simple theory. . .

Another Application of Powers of Ten

On the basis of a personal experience of a burglary in the City of New York and some available data, an estimate was made of the total yearly loot collected by burglars in the City. On the assumption that 1 in 7 of such crimes result in incarceration, it was found on the basis of an estimate with powers of ten that it would be considerably cheaper to disburse the average take at break-ins directly to the perpetrators.

Variables

The much feared x's and y's were introduced via tracing the progressive symbolization leading to increased abstraction in the history of writing:

Symbol Pictograph, Ideograph, Syllabary, Letter, Pronoun, Variable, Rebus
Domain Concrete Idea, Object, Word, Sound, Consonant,Vowel, Name of Object or Person, Natural Number

We introduced the notion of *substitution* through considering the pronoun as a variable:

She was a physicist who won the Nobel Prize.
Domain: all women physicists.
She was the first woman physicist to win the Nobel Prize.
Unique solution: Eve Curie.

From here on we clarify:

x is an even number.
Domain: all numbers.
$2x = 4$
Unique solution: $x = 2$.

Relation and Function Notation were introduced via pairings such as (Husband, Wife), (State, Capital), (Corporation, Rank), (Capital, Interest, Return). The representation in this notation of complex interrelations between many variables such as, say, the money allocated to education, child health care, housing, prison construction etc. teaches how to view social problems in a more abstract quantitative framework. Relations were formalized similarly to link taxation and investment policies to the quality of life and the competitive position of the U. S. etc.

A social vs. private cost benefit *matrix* to analyze the effect of the repealed catastrophic health bill was computed.

Numerical Functions and their Graphs were singled out as special cases of relations.

Polynomials were exemplified by:

The linear relation between yield per acre vs. amount of fertilizer applied in various regions of the world. The slope marks the productivity rate; the intercept the original level of agricultural production.

Calculating the optimum fare so as to yield the highest revenue to a mass transit line was used as a vehicle to discuss quadratic functions.

The dependency ratio, i.e. the ratio of wage earners to the total population — a quantity on which Social Security budget projections are based — was approximated with a third degree polynomial.

Powers of Ten once again were applied to compare items in the National Budget and to focus on the *order of magnitude* of military expenditures. The principle of exponential growth and the graph of the *exponential function* were then related to the growth of military expenditures and world population. The *negative exponential function* was applied to animal population extinction (blue whales). *Step functions* were made meaningful via population pyramids in various geographical and time periods.

Periodic functions appeared in cyclical fluctuations in grain production and the ensuing population in Western Europe in the period 1660–1860; and in relation to predator and prey population data.

The concept of *mathematical tree* precipitated from the principle of "Each one Teach Two" applied to wipe out illiteracy; similarly it applied to the conclusion that "we are all one human family" in tracing back the tree of generations some 2,000 years. The *multiplier effect* was similarly discussed in relation to the Keynesian Multiplier and the Social Spending Dividend as applied to the Head Start Program. Quoting once again from a letter to the Bergen Record entitled *The Social Spending Dividend:*

When Franklin Delano Roosevelt was inaugurated as president in 1933, he sought the advice of Alexander Sachs of Lehamnn Brothers on how to fight the depression. Sachs, following the Keynesian theory of pump-priming, suggested that every dollar spent by the federal government on public works would *multiply* through increased economic activity, jobs, and incomes and eventually produce more than enough tax revenue to cover the initial outlay.

. . . The Head Start program, which makes early childhood education available to disadvantaged children, by contrast has proven itself to be a *self-multiplying* and *self-liquidating* subsidy. The program started in 1965, has served and average of 500,000 three and four year olds a year — at an average cost of $2,500. The total cost of the program has been on the order of $13 billion.

It has reliably been estimated that for every $1.00 invested in this program, the savings to society in expenditures for health, remedial education and crime related activities *multiplies* to some $4.75.

This suggests that we calculate the dividends produced by socially useful projects, as they *multiply* and generate their own returns over the long term. Such a model of accounting could profitably be applied to, say, subsidized housing, in preventive health care, and job training of unemployed youth . . . .

Combinations and Permutations revealed the huge number of alternative ways of arranging our national priorities. Or the number of alternative sibling configurations in a four child family. Or the number of dances based on the nine fundamental movements of the belly dance. Or the huge variety of phenotypes resulting from the combinations in various groups of some twenty odd amino acids in the genotype's proteins.

The students were asked to write a term paper applying some of the skills learned. Topics of papers were, for instance: Election Campaign Financing, Gambling, Divorce: How does it relate to Juvenile Delinquency?, Owning an Apartment Building, Running a University.

Students performed distinctly better than in the standard remedial courses and quite a few went on to more advanced mathematics courses. The level of enthusiasm in the course was very high.

Below are some typical comments:

> Most of the class had a fear of math, like I did, and Prof. Yevick was able, over a few classes, to rid us of that very real feeling. Within a few weeks I was able to argue political questions using concepts I had learned in class. I could not only read, but understand the graphs used in the *New York Times*. Within ten minutes in a class one night, I was able to prove that every person in the world can be fed by using available land. The list can go on and on. The point is that I am more secure in my political choices, stock options and general interactions. I even used what Prof. Yevick taught me in a sermon in my church. . .

> I found the course to be a refreshing change from other math courses I took. Thus far I've taken Algebra, Probability and Statistics and Calculus and all of them seemed so irrelevant to the world. This course taught math from a practical standpoint . . . I might not have gone through college hating math so much and having anxieties about the topic when it was mentioned.

> Math for Life studies has equipped me with excellent math skills, technical expertise in problem solving, and the capacity to successfully learn other higher mathematic principles. Algebra, analytic geometry, statistics and other mathematical principles were creatively presented in class. After learning these principles they were applied to many areas of daily living, increasing our value and understanding of math . . .

> . . . Presently I am studying pre-calculus and maintaining a B+ grade in this course. Many of the concepts and skills learned in the Math for Life course enabled me to do well and expand my math education . . .

> In conclusion, I hope that a new and continuing student body will be given the opportunity I had to see a fantastic view of the world of mathematics. . .

Chapters from my manuscript *Mathematics for the Billions* and supplementary class notes and exercises written by me were used as class materials. (This manuscript is looking for a publisher.)

Mathematics for life was abolished as a course offering when University College was merged into the Rutgers Day College. I am interested in disseminating the methods and materials of this course by collaborating with others engaged in similar undertakings or by talking to teachers. The (copyrighted) manuscript is presently in Xeroxed form. If there is a sufficient demand, it could be put into a cheaper format for distribution.

To end, let me quote from Lancelot Hogben's best seller *Mathematics for the Million*:

> Fruitful progress can and will be made in solving the economic and political problems of the day when a large number of people will be thinking together about the same thing.

This course is based on the belief that to do so people must understand the basic quantitative tools with which to think these problems through.

ADDENDUM
MATHEMATICS AS A CONSCIOUSNESS RAISER
STREET MATHEMATICS

On the two nights when I returned from the meetings in San Francisco, I neglected to take off my badge. I was addressed on both occasions on the subject of math: once by an adult student of nursing on the trolley car; by a cab driver the other time. The future nurse — an A student except for math — worried about how to master the subject which she liked in spite of her struggles with it. The cab driver had just spent the evening working fractions with his son and discussed excitedly how the subject fascinated him, even though it was difficult.

I told my students in the *Math For Life And Society* course to think about mathematics when driving home from class, before going to sleep, while scrubbing floors or cooking dinner as I did very often. I told them that the benefits of such thinking extend beyond solving some particular math problem. The half hour a day, say, spent removed from pressing day-to-day concerns and the sudden flash of insight revealing the solution to a problem, help gain a broader perspective and activate clear thinking. The concentration enriches one's inner life and enhances one's general self-confidence. It, so to say, develops another thinking cap with which to view one's personal problems in a more abstract and social context.

During the latter half of the last century in Western Europe as well as during the Depression years here and there, free evening lectures for working people by academic volunteers or others were quite common. I personally am familiar with the cultural revival brought by Hans Polak, the founder of the Dutch *Diamant Werkers*

Bond, the diamond workers' union, to an impoverished and brutalized proletariat. He developed — among other things — an intense interest in Opera. The singing of arias became an accompaniment to the grinding of polishing wheels and their musical expertise increased the self-respect of the workers.

The philosopher, Susanne Langer in her book *Mind. An Essay on Human Feeling*, wrote that the human brain evolved in such a way as to have an independent need for self-assertion. This hunger of the mind feeds on imagination and action. Why not try mathematics to feed this hunger?

There is a lot of appetite for math out there in the streets, as one finds out soon enough in casual conversations about the subject. ("I wish I had learned more math." "I was never good at math because the teacher didn't make it clear," etc.) Perhaps we could appease some of this hunger by teaching relevant math on street corners or other places. (Bring a placard saying perhaps: "Powers of Ten Explained Tonight To Help You See Our Budget Priorities") and bringing along enough "tutors" to help those who wish to understand. Nor let us forget the work we might do with even little children. I have seen a change in self-esteem in a five year old black boy to whom I taught addition during a long train ride. We could involve ourselves in Head Start programs and so help grow a generation of mathematically competent minority children.

If a large number of people are "to think together about the same thing," as stated by Lancelot Hogben, we can do no better than to help them acquire the tools and thereby the self-confidence to do so.

Let's rap mathematics. Mathematical consciousness raising can stir the imagination and free people's pleasure in and courage to think.

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