2015

Chromatic Polynomials and Orbital Chromatic Polynomials and their Roots

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May, 2015

Abstract

The chromatic polynomial, $P_\Gamma(x)$ of a graph Γ, is a polynomial that when evaluated at a positive integer k, is the number of proper k colorings of the graph Γ. We can then find the orbital chromatic polynomial $OP_{\Gamma,G}(x)$ of a graph Γ and a group G of automorphisms of Γ, which is a polynomial whose value at a positive integer k is the number of orbits of k-colorings of a graph Γ when acted upon by the group G. By considering the roots of the orbital chromatic and chromatic polynomials, the similarities and differences of these polynomials is studied. Specifically we work toward proving a conjecture concerning the gap between the real roots of the chromatic polynomial and the real roots of the orbital chromatic polynomial.

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Chapter 1

Introduction

This thesis shall focus on chromatic polynomials, orbital chromatic polynomials, and the study of the roots of these polynomials. For a graph Γ, the chromatic polynomial of Γ denoted $P_\Gamma(x)$, when evaluted at some positive integer k is the number of proper k-colorings of Γ. Chromatic polynomials satisfy a deletion-contraction formula, which establishes an inductive relationship where the chromatic polynomial of a graph can be expressed as the difference of chromatic polynomials of graphs with fewer edges (Birkhoff and Lewis, 1946). This fact together with bases cases establishes that chromatic polynomials are polynomials (Birkhoff and Lewis, 1946). Chromatic polynomials were first introduced by Birkhoff as a tool to solve the 4-Color Conjecture which states,

Problem 1. (*Birkhoff,* 1912) *For any planar graph* Γ, *the chromatic polynomial of* Γ *when evaluated at* 4, *is greater than* 0 (*ie* $P_\Gamma(4) > 0$).

When Birkhoff first introduced and used chromatic polynomials he used methods and techniques which were heavily algebraic. To this day, the method of studying these polynomials is strongly influenced by the algebraic techniques that Birkhoff used.

Although Birkhoff did not succesfully prove the 4-Color Conjecture, Birkhoff and Lewis (1946) did prove that for a planar graph Γ that $P_\Gamma(x) > 0$ for $x \in \mathbb{N}$ and $x \geq 5$. The study of the 4-Color Conjecture and development of chromatic polynomials led the mathematical community to have an interest in the roots of this polynomial, which in turn has led to many rich and interesting developments. Although planar graphs were the first to be studied there have been many developments concerning the roots of chromatic polynomials which were not specific to any type of graph. Sokal

(2004) proved that the complex roots of chromatic polynomials were dense in $\mathbb{C}$, while Thomassen (1997) proved, that real chromatic roots are dense in $[\frac{32}{27}, \infty)$.

While there has been much study of chromatic polynomials, chromatic polynomials seem to overcount the number of colorings of a graph in some respects, since it is possible to have two distinct colorings where one coloring can be obtained from another by a symmetry of the graph. The orbital chromatic polynomial of a graph Γ with respect to a group G of automorphisms of the graph, denoted $OP_{\Gamma,G}(x)$, is a way of counting k-colorings that does not distinguish between two k-colorings if one can be obtained from the other by an automorphism. We can say this more rigorously as, the orbital chromatic polynomial of a graph Γ and a group of automorphisms of the graph G, denoted $OP_{\Gamma,G}(x)$, when evaluated at a positive integer k is the number of G-orbits of proper k-colorings of Γ. In Chapter 2 we shall define both chromatic and orbital chromatic polynomials, and give examples of deriving the chromatic and orbital chromatic polynomials for certain graphs.

The study of the roots of orbital chromatic polynomials was the focus of a paper written by Cameron and Kayibi (2007) where it was proven that the real orbital chromatic roots were dense in $\mathbb{R}$. In this same paper the following problem was posed,

Problem 2. (*Cameron and Kayibi,* 2007) *Are the real roots of $OP_{\Gamma,G}(x)$ bounded above by the largest real root of $P_{\Gamma}(x)$?*

In 2013, Kim, Mun, and Omar (2014), proved that the real roots of $OP_{\Gamma,G}(x)$ were not bounded above by real roots of $P_{\Gamma}(x)$. The focus of this thesis it to work toward the conjecture posed by Kim, Mun, and Omar (2014) which states,

Conjecture 3. *For any $N > 0$, there exists a graph Γ and an automorphism group G of Γ for which $OP_{\Gamma,G}(x)$ has a root at least N larger than the real root of $P_{\Gamma}(x)$?*

Chapter 2

Introduction to the Orbital and Chromatic Polynomials

In order to discuss chromatic and orbital chromatic polynomials, we first need to understand some basics from graph and group theory. Let us begin by introducing some elementary concepts from graph theory.

Definition 4. *A graph $\Gamma = (V, E)$ is a set V called vertices and a set E called edges where each edge is a* 2*-element subset of V.*

Generally, when graphs are depicted, the set of vertices are depicted as points, and lines are drawn to connect two points if they form an edge of the graph. Consider the example of a graph in Figure 2.1. We see that *A,B, C, D, E, F* are the vertices of the graph and *{A,B}*, *{A, C}*, *{A, E}*, *{B, C}*, *{B, D}*, and *{C, F}* are the edges of the graph. Now that we have defined what a

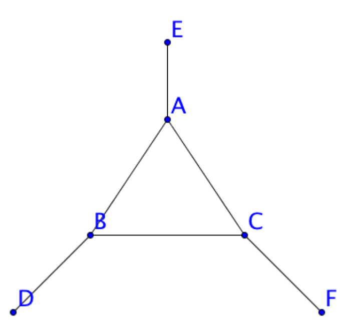


Figure 2.1 Example graph

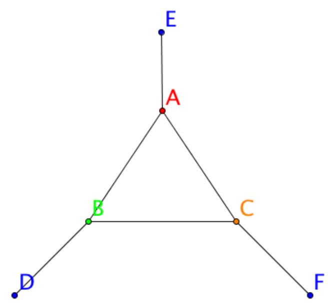


Figure 2.2 The 4-coloring described in Example 8

graph is and given an example, let us consider some definitions concerning different properties of graphs.

Definition 5. *Two vertices in a graph are said to be adjacent if they form an edge.*

Definition 6. *Given a graph* $\Gamma = (V, E)$ *and some* $v_i \in V$*, the degree of* v_i *denoted* $deg(v_i)$ *is the number of vertices to which* v_i *is adjacent.*

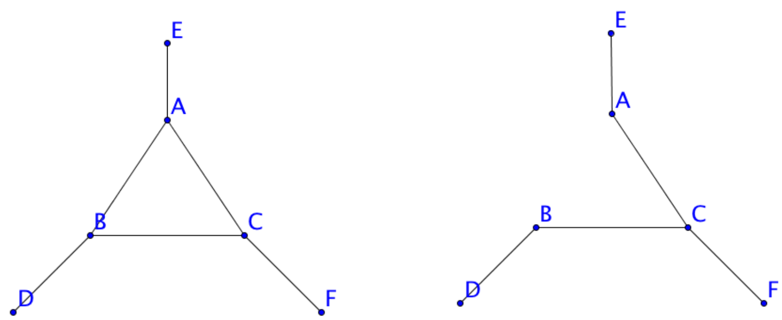
Our paper will concern itself with proper graph colorings:

Definition 7. *A proper k-coloring of a graph* Γ*, is a function* $c : V \to \{1, ..., k\}$ *such that* $c(u) \neq c(v)$ *for any edge* $\{u, v\}$*.*

Example 8. *We shall construct one of the* 4*-colorings for the graph from Figure* 2.1*. The* 4 *colors which we will use are blue, green, orange and red. We shall begin by coloring vertex A red. We know that no two adjacent vertices can be colored with the same color so when we go to color vertex B we know that it cannot be red so we shall color it green. We then see that vertex C is adjacent to A and B meaning that it cannot be colored red or green so we shall color it orange. Let us now consider vertex D, we see that we shall color it blue since its only adjacent to vertex A which is red. By this same logic we shall color vertices E and F both blue as well. We see that the coloring we have described can be seen in Figure* 2.2*.*

Now that the concept of a proper k-$coloring$ has been introduced we can define the chromatic polynomial.

Definition 9. (*Birkhoff*, 1912) *For a graph* Γ *and a positive integer* k*, let* $P_\Gamma(k)$ *be the number of proper k-colorings of the graph* Γ*. We shall refer to this as the chromatic polynomial.*



a. Γ **b.** $\Gamma \setminus \{A, B\}$

Figure 2.3 A graph Γ and $\Gamma \setminus \{A, B\}$

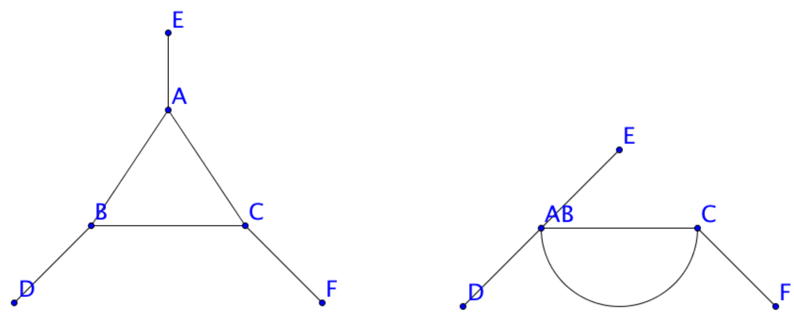
We have defined $P_\Gamma(k)$ to be the number of k-colorings of a graph Γ. Let us now compute $P_\Gamma(k)$ for a graph Γ.

Example 10. *Let us compute the chromatic polynomial for the graph in Figure* 2.1*. We shall begin by noting that the vertices A, B, and C are all adjacent to one another. Without loss of generality we shall begin by counting the number of ways to color vertex A first. Since vertex A is the first to be colored, there are k different colors which vertex A could be colored. We shall now consider the number of ways to color vertex B. We see that vertex B is adjacent to vertex A which we just colored so there are $k-1$ color options independent of the choice of color A for vertex B. Let us now consider vertex C, we see vertex C is adjacent to both vertex A and vertex B which have already been colored, so there is $k-2$ color choices independent of the color choices of vertices A and B for this vertex. So, there is $k(k-1)(k-2)$ ways to color these three vertices. Now let us consider the number of ways to color vertices D, E and F, we see these vertices are each adjacent to one previously colored vertex so there are $k-1$ possible color choices for each of these vertices. We can conclude that there is a total of $k(k-1)^4(k-2)$ k-colorings for the given graph. In other words the chromatic polynomial of the given graph Γ, is $P_\Gamma(k) = k(k-1)^4(k-2)$.*

As mentioned previously $P_\Gamma(k)$ satisfies a deletion-contraction formula. We shall now define deletion and contraction of edges for graphs.

Definition 11. *For a given graph Γ and an edge e from its edge set, $\Gamma \setminus e$ called Γ delete e, is the graph which is created by removing the edge e from the graph Γ.*

Figure 2.3 gives an example of a graph Γ and $\Gamma \setminus e$.



a. Γ **b.** $\Gamma/\{A, B\}$

Figure 2.4 A graph Γ and $\Gamma/\{A, B\}$

Definition 12. *For a given graph Γ and an edge e from its edge set, Γ/e called Γ contract e, is the graph which is created by removing the edge e from Γ and replacing the two vertices which were adjacent through e with a single vertex. All vertices which were adjacent to either of the two vertices will now be adjacent to this new vertex.*

Figure 2.4 gives an example of a graph Γ and Γ/e.

Intuitively it would seem as if there should be some relationship between $P_\Gamma(x)$, $P_{\Gamma\setminus e}(x)$ and $P_{\Gamma/e}(x)$. We see that there is a close relationship between these chromatic polynomials.

Theorem 13. *For a given graph $\Gamma = (V, E)$ and $\{u, v\} \in E$,*

$$P_\Gamma(k) = P_{\Gamma\setminus\{u,v\}}(k) - P_{\Gamma/\{u,v\}}(k).$$

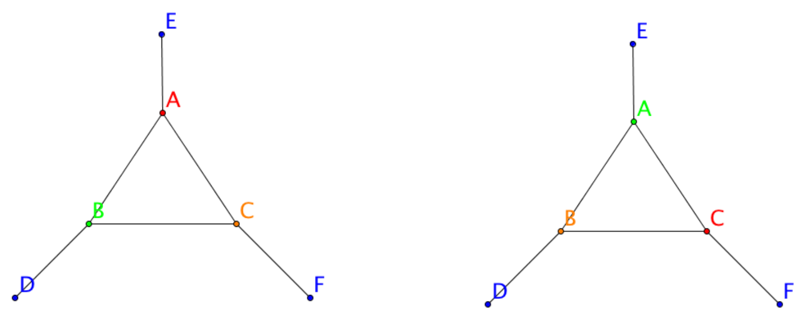
Proof : Let us begin by considering $P_{\Gamma\setminus\{u,v\}}(k)$ for a specific k. We know that $P_{\Gamma\setminus\{u,v\}}(k)$ will contain all of the colorings that were possible upon Γ because $\Gamma \setminus \{u, v\}$ is simply Γ with the edge $\{u, v\}$ deleted. Also note that $P_{\Gamma\setminus\{u,v\}}(k)$ will contain colorings that $P_\Gamma(k)$ does not. The extra colorings that are included in $\Gamma \setminus \{u, v\}$ which are not proper colorings of Γ are all colorings that can be attained since there is no edge between u and v. These are exactly the set of proper colorings of $\Gamma \setminus \{u, v\}$ where u and v are the same color. The number of proper colorings of $\Gamma \setminus \{u, v\}$ where u and v are the same color is the number of proper colorings of $\Gamma/\{u, v\}$. Recall that $\Gamma/\{u, v\}$ is the same as Γ except instead of u and v being represented as two separate vertices there is a single vertex which was adjacent to all of the vertices that are adjacent to u and v. We see that when we are considering $\Gamma/\{u,v\}$ that when we color the vertex which represents the contracted vertices u and v in Γ, this would be the same as coloring the vertices u and v the same color in $\Gamma \setminus \{u,v\}$. So we see that the number of colorings of $\Gamma \setminus \{u,v\}$ which are not proper k-colorings of Γ is exactly the number of k-colorings of $\Gamma/\{u,v\}$. In conclusion we see that for a specific k if we subtract the number k-colorings of $\Gamma/\{u,v\}$ from the number of proper $-k$colorings of $\Gamma \setminus \{u,v\}$ this will give us the number of proper k-colorings of Γ. Thus proving $P_\Gamma(k) = P_{\Gamma\setminus\{u,v\}}(k) - P_{\Gamma/\{u,v\}}(k)$. ■

Theorem 13 allows us to show that $P_\Gamma(k)$ is indeed a polynomial in k.

Theorem 14. *The chromatic polynomial is a polynomial.*

Proof : We shall prove the chromatic polynomial is a polynomial using induction on the number of edges. Let us begin by stating our base case. For our base case, we shall use the case where there are 0 edges, meaning that we simply have a collection of vertices. Suppose there are m vertices, where m is a non-negative integer. Since there are no edges each vertex can be colored any of the k colors so the chromatic polynomial is k^m, which is a polynomial. We see that when $m = 0$ that $k^0 = 1$ which is still a polynomial, so we see that the chromatic polynomial is still a polynomial on an empty graph. We can now conclude that the chromatic polynomial is in fact a polynomial for our base case. We shall now make the our inductive hypothesis,we shall say that for any graph with $n-1$ edges that the chromatic polynomial is a polynomial. For our inductive step let us now consider the chromatic polynomial of a graph which contains n edges. We know from Theorem 13 that the chromatic polynomial of the graph can be written as the difference of the chromatic polynomial of the graph with some edge e delete minus the chromatic polynomial of the graph with that same edge e contracted. We see that both the graph with the edge deleted and the graph with the edge contracted have $n-1$ edges, which from the inductive hypothesis means that thier chromatic polynomials are polynomials. Since the chromatic polynomial of our graph with n edges can be written as the difference of two polynomials we conclude that it is also a polynomial. We have thus proved that the chromatic polynomial of a graph is a polynomial using induction. ■

From our calculation of the chromatic polynomial we see there are many colorings which are very similar due to the symmetry of the graph we are



a. A graph

b. A graph with the colorings permuted by 120°

Figure 2.5 2 different colorings of the same graph under the chromatic polynomial

considering. Consider the colorings of the graph in Figure 2.5. These two colorings are different even though one coloring can be attained from the other by a rotation of the graph by an automorphism. We see that the orbital chromatic polynomial is a way to count the number of k-colorings of a graph which accounts for the symmetry of the graph. In order to understand the orbital chromatic polynomial we must first consider some concepts from group theory.

Let us begin by introducing the idea of a group action.

Definition 15. (*Dummit and Foote*, 2004) *For a group G and a set S, a group action denoted ϕ_g, is a bijection from S to itself,*

$$\phi_g : S \to S,$$

with the property that,

$$\phi_g \cdot (\phi_h \cdot s) = \phi_{gh} \cdot s \ \forall g, h \in G, \forall s \in S.$$

To help clarify this concept let us consider an example.

Example 16. *Let us begin by noting that graphs are abstract mathematical objects made up of two sets, as defined in Definition 4. For example, let G be the group generated by, $g = (AC)(B)$, which is a subgroup of the symmetric group on the*

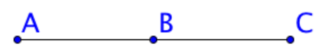


Figure 2.6 Path graph on 3 vertices

symbols $\{A, B, C\}$*. Throughout this paper we shall be talking about group actions as being rotations of certain physical representations of graphs, since it provides a simple way to give good visual examples. Let us now consider the group G composed of rotations by* $180°$*. We see that* $G = \{0°, 180°\}$*. Let S be the set of k-colorings of the path graph on three vertices depicted in Figure* 2.6*. Let us now consider the elements of G acting upon our graph. We see that when the identity element of the group* $0°$*, acts upon the graph that each k-coloring will remain the same. When the element of the group which represents rotation by* $180°$ *acts upon the graph we see that each coloring will be mapped to another coloring where the colors given to A and C switch while the color of vertex B stays the same. We see that we have now completely described the the group actions of our given group G on the given set.*

We see that for a given set that there are different ways to describe the effects of a group action on that set, one way is by grouping together different elements of the set by how they behave under the group action.

Definition 17. (*Dummit and Foote*, 2004) *For a group G and a set S on which G acts, the orbit of an element* $s \in S$*, denoted Gs, is all of the elements in S to which s can be sent to under different elements of the group G acting on S. That is,*

$$Gs = \{g \cdot s | g \in G\}.$$

To help clarify the concept of orbits let us consider the orbits of a specific set when acted upon by a specific group.

Example 18. *Let us consider the orbits of the set of k-colorings of the path graph on* 3 *vertices acted upon by the group G generated from* $180°$ *rotation. In Example* 16 *we considered all of the group actions of the given group on the given set. For the group element which is rotation by* $0°$*, we see this group action will send any k-coloring to itself. The group element of rotation by* $180°$ *will send a k-coloring to another k-coloring which has the same color for vertex B but where the colors of vertices A and C are flipped. We can then conclude if we are considering a k-coloring where vertices A and C are the same color that their orbit will include only* *that k-coloring because, rotation by* 180° *will return the same k-coloring. However the orbit of a k-coloring where vertices A and C are not the same color will contain itself and a k-coloring which has the same color for vertex B but where the colors of vertices A and C are flipped. We see that we have now described the orbits of all possible types of elements of our given set.*

Let us now consider another definition from group theory.

Definition 19. (*Dummit and Foote*, 2004) *For a group G and a set S on which G is acting on, for any $s \in S$ the stabilizer of s in G denoted G_s is,*

$$G_s = \{g \in G | g \cdot s = s\}.$$

In other words the stabilizer of s in G, for some fixed s in S is the set of elements of G such that when those elements act upon the element s, the element s gets sent to itself.

To help clarify the concept of stabilizers let us consider the stabilizers related to a given group and set.

Example 20. *We shall consider the stabilizers of the k-colorings of the path graph on* 3 *vertices depicted in Figure* 2.6 *and the group G of* 180° *rotations, $G = \{0°, 180°\}$. In Example* 18 *, we showed that for any k-coloring where vertices A and C were the same color that both rotation by* 0° *and* 180° *would return the same coloring so we see that the stabilizer of a k-coloring where vertices A and C are the same color contains both rotation by* 0° *and* 180°*. We also found in Example* 18*, that for k-colorings where vertices A and C were not the same color that while rotation by* 0° *returned the same coloring that rotation by* 180° *did not. We see that the stabilizer of a k-coloring where vertices A and C are not the same color contains only rotation by* 0°*. We have now described the stabilizer of all possible types of elements of our given set.*

Burnside's Lemma describes an interesting relationship between the number of orbits and stabilizers of a given group acting upon a given set. But in order to consider Burnside's Lemma we must begin with a definition.

Definition 21. (*Dummit and Foote*, 2004) *Let G be a finite group of permutations on a set S. For $g \in G$,*

$$\mathit{fix}(g) = \{s \in S | g \cdot s = s\}.$$

Lemma 22. (*Burnside's Lemma*) *If G is a finite group of permutations on a set S, then the number of orbits of G on S is*

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)|.$$

Proof : Let us begin by noting that if we consider the sum of $|fix(g)|$ over all $g \in G$ that this sum will be equal to the sum of the stabilizers s over G for all $s \in S$, since a stabilizer of an element s is an element from the group g such that $g * s = s$. We see,

$$\sum_{g\in G}|fix(g)| = |\{gs = s|\forall g \in G, s \in S\}| = \sum_{s\in S}|G_s|$$

where G_s is the set of all stabilizers of s. From this we can conclude that,

$$\sum_{g\in G}|fix(g)| = \sum_{s\in S}|G_s|.$$

We know that for a given $s \in S$ that $|Gs||G_s| = |G|$ by the Orbit-Stabilizer theorem. So if we sum all elements of G one orbit at a time we find,

$$\sum_{g\in G}|fix(g)| = \sum_{s\in S}|G_s| = |G|*(\textit{number of orbits}).$$

So we find the number of orbits of G on S is,

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)|.$$

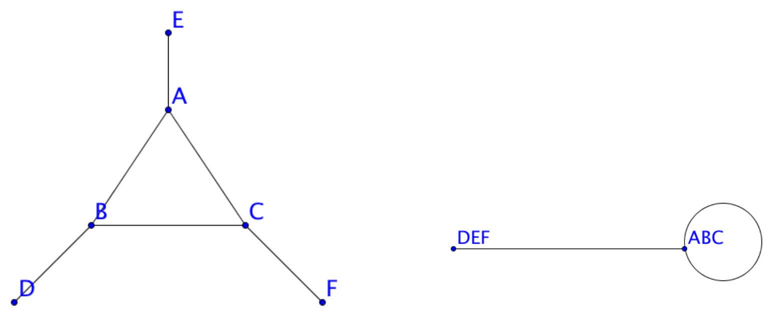
■

Let us now consider the contraction of graphs due to elements of a group of automorphisms of that graph, we will use this later on, along with other concepts from group theory to give a formula for the orbital chromatic polynomial.

Definition 23. *Given a graph Γ and a group of automorphisms G of Γ, for $g \in G$, Γ/g is defined in the following way: find all vertex cycles created by g acting on Γ, where a vertex cycle is all vertices that a vertex gets mapped to under repeated actions of g on Γ. Contract all vertices in a vertex cycle to a single vertex. If at least two elements in the vertex cycle are adjacent we shall instead of contracting the vertices of that cycle to a vertex instead contract this cycle to a loop.*

To help clarify this concept let us now consider an example.

Example 24. *Let us consider the vertex cycles of the given Γ in Figure 2.7 created by rotations by $120°$. We see that there are two cycles that form from these rotations, (ABC) and (DEF). Notice that D, E and F are not adjacent to one another so they will be represented as a vertex. Also notice that A, B and C are adjacent to one*



a. Γ **b.** $\Gamma/120°$

Figure 2.7 A graph Γ and $\Gamma/120°$

another so they will be represented as a loop. Since there are vertices in these two cycles which are adjacent to one another the loop and vertex are adjacent to one another. We can see Γ *and* $\Gamma/120°$ *in Figure* 2.7.

We see that we can make conclusions about k-colorings of the graph Γ which is acted upon by a group G from Γ/g where $g \in G$.

Theorem 25. *For a graph* Γ *and a group* G *of automorphisms of* Γ*, there exists a bijection between the number k-colorings of a graph* Γ *fixed by* $g \in G$ *and the number of colorings of* Γ/g.

Proof : Let us first consider the number of k-colorings of a graph Γ fixed by $g \in G$. We see that this is the number of colorings that are the same when the element g acts upon them. We see that each vertex within a vertex cycle created by g must be colored the same in order for it to be fixed by g. We see that we can also count the number of colorings of this type in the following way. First we shall contract each element in a vertex cylce to a single point since they all must be colored the same color. We will then notice that if two vertices in a vertex cycle are adjacent that there will be no possible k-colorings fixed by this g because all vertices in a vertex cycle must colored the same color in order to be fixed. In order to represent a vertex cycle which contains adjacent vertices and the inability to have any k-colorings fixed by this g, if a vertex cycle of this type is present, we shall contract the vertices in this vertex cycle to a vertex which has an edge from that vertex to itself. Let it be noted that there will be no possible proper colorings for a graph which has a vertex with an edge from that vertex to itself since there will be no way to color this vertex without coloring two adjacent vertices the same color, since this vertex is self-adjacent. Let us now count the number of k-colorings on the graph we just constructed this will be the same as the number of k-colorings of the graph fixed by g. We have now found another way to count the number of k-colorings fixed by g, let it be noted that the new graph we constructed is Γ/g. Thus proving that there is a bijection between the number k-colorings of a graph Γ fixed $g \in G$ and the number of colorings of Γ/g since they count the same thing. ■

Now that we have discussed some of the mathematical concepts that are used to when describing orbital chromatic polynomials, let us now introduce the orbital chromatic polynomial. We shall begin this discussion by giving a definition of the polynomial that depends upon counting groups of chromatic polynomials of a given graph.

Definition 26. (*Cameron and Kayibi*, 2007) *For a graph Γ, a group G of automorphisms of Γ, and a positive integer k, the orbital chromatic polynomial $OP_{\Gamma,G}(k)$, is the number of unique k-colorings of Γ. Where two colorings are equivalent if one can be obtained from another by an automorphism in G.*

To help clarify the concept let us now compute the orbital chromatic polynomial for the path graph on 3 vertices depicted in Figure 2.6 under the group of rotations by $180°$.

Example 27. *Let us begin by noting that the group of automorphisms of the graph that we will be using is that composed of rotation by $0°$ and $180°$. We will find $OP_{\Gamma,G}(k)$ by dividing up the k-colorings of Γ into different categories which are similar under $180°$ rotation. We see that there are 2 categories, the first category will be when A and C are the same color. We see that there are $k(k-1)$ unique colorings of this type when $180°$ rotaion acts upon these colorings. The second category will be when A and C are not the same color. Let us now consider this category. If we were to calculate the number of colorings of this type for the chromatic polynomial, there would be $k(k\text{-}1)(k\text{-}2)$ colorings of this type but, we see that if we rotate by $180°$ that we shall obtain a different coloring under the chromatic polynomial. We then conclude that two colorings where one can be attained from another by rotation by $180°$ will not be unique so we see that there are $\frac{k(k-1)(k-2)}{2}$ unique colorings of this type for the given graph and group. So we see that the orbital chromatic polynomial for the given graph is, $OP_{\Gamma,G}(k) = \frac{k(k-1)(k-2)}{2} + k(k-1)$.*

Although we have given a definition of the orbital chromatic polynomial, we can also state a closed form equation for the orbital chromatic polynomial.

Theorem 28. *The orbital chromatic polynomial of a graph* Γ *and a group G is,*

$$OP_{\Gamma,G}(x) = \frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(x).$$

Proof : From Definition 26 we see that the orbital chromatic polynomial is the number of orbits of k-colorings of a graph Γ acted upon by a group G, of automorphisms of the graph. From Theorem 25 we see that the number of k-colorings of a graph fixed by some element of the group is equivalent to the chromatic polynomial of Γ/g. We can now apply Burnside's Lemma and we find that the number of orbits of k-colorings of a graph Γ acted upon by a group G is

$$\frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(x),$$

thus proving,

$$OP_{\Gamma,G}(x) = \frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(x).$$

■

We see that Theorem 28 is significant because it allows us to view the orbital chromatic polynomial as a sum of chromatic polynomials. This interpretation of the orbital chromatic polynomial allows us to make a conclusion about its mathematical structure.

Theorem 29. *The orbital chromatic polynomial* $OP_{\Gamma,G}(k)$ *for a graph* Γ*, a group G of automorphisms of* Γ*, is a polynomial in k.*

Proof : From Theorem 28 we see that the orbital chromatic polynomial can be expressed as a sum of chromatic polynomials multiplied by some constant. From Theorem 14 we know that chromatic polynomials are polynomials so we see that the orbital chromatic polynomial can be expressed as a sum polynomials times a constant. We can therefore conclude that the orbital chromatic polynomial is a polynomial. ■

Chapter 3

Orbital vs Chromatic Polynomial Roots

In this chapter we shall discuss previous work done concerning the roots of orbital chromatic polynomials and chromatic polynomials. One of the first things that was of interest to mathematicians who studied of roots of chromatic polynomials was the density of their roots. Carsten Thomassen, a mathematician who was interested in chromatic polynomials studied the density of the roots of the chromatic polynomial in $\mathbb{R}$. In 1997 Thomassen published an article where he proved the following theorem concerning the density of the real roots of the chromatic polynomial,

Theorem 30. (*Thomassen*, 1997) *If* $\lambda_\circ > \frac{32}{27}$, $\epsilon > 0$, *then there exists a graph* G *such that* $P(G,\lambda)$ *has a root in* $(\lambda_\circ - \epsilon, \lambda_\circ + \epsilon)$.

This means that between any two real numbers greater than $\frac{32}{27}$, there exists a real root of some chromatic polynomial. This result is important not only due to its inherent mathematical value, but also because it sparked an interest in the mathematical community to study the density of the roots of the orbital chromatic polynomial. Cameron and Kayibi were interested in the results found by Thomassen and wanted to consider the density of the roots of the orbital chromatic polynomial, they proved,

Theorem 31. (*Cameron and Kayibi*, 2007) *Orbital chromatic roots are dense in* $\mathbb{R}$.

At the end of their paper Cameron and Kayibi posed a Problem 2. The answer to this problem leads directly to the problem which we shall be investigating in this thesis.

Kim, Mun, and Omar published a paper (2014), in which they showed a family of counterexamples to the problem stated above. They proved that the roots of the chromatic polynomial did not provide an upper bound for the roots of the orbital chromatic polynomial. Let us now consider the graph which Kim, Mun and Omar constructed and used as a counterexample to the problem posed by Cameron and Kayibi. First Kim, Mun, and Omar considered the following graphs,

Definition 32. (*Kim et al.*, 2014) *Given positive integers n,s, define the graph K_n to be the complete graph on n vertices, and N_s to be the graph consisting of s isolated vertices. Define the graph $H_{n,s}$, the join of K_n and N_s, to be the graph obtained by taking the union of K_n and N_s, and adding an edge between every vertex in K_n and every vertex in N_s.*

After $H_{n,s}$ was defined, it was then used along with a graph Γ to construct the graph $\Gamma^{(n,s)}$. The family of graphs $\Gamma^{(n,s)}$ formed a counter-example to Problem 2. In order for Γ to be used to construct $\Gamma^{(n,s)}$, it had to have the following properties,

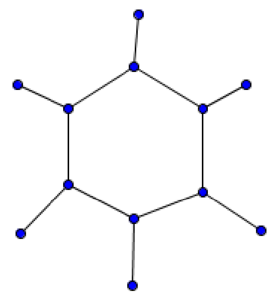
Theorem 33. (*Kim et al.*, 2014) *Let Γ be a graph and G be a group of automorphisms of Γ. Suppose that the following hold*:

1. *There is some $g \in G$ for which Γ/g contains fewer vertices than any of the graphs $\{\Gamma/h : h \in G, h \neq g\}$.*
2. *For the g in part* (1), *there is some $x_0 \in \mathbb{Z}$ greater than the largest real root of $P_\Gamma(x)$ such that $P_{\Gamma/g}(x) < 0$.*

After a graph with the apropppriate properties was selected $\Gamma^{(n,s)}$ was constructed,

Definition 34. (*Kim et al.*, 2014) *Let Γ be a graph with vertex set $\{v_1, v_2, ..., v_k\}$, and n, s be positive integers. Let $H^{(1)}_{n,s}, H^{(2)}_{n,s}, ..., H^{(k)}_{n,s}$ be k copies of the graph $H_{n,s}$ and choose vertices $u_i \in V(H^{(i)}_{n,s})$ so that there is an isomorphism from $H^{(i)}_{n,s}$ to $H^{(j)}_{n,s}$ sending u_i to u_j. We construct the graph $\Gamma^{(n,s)}$ by starting with Γ, and appending the k copies of $H_{n,s}$ to Γ by identifying the vertices u_i and v_i for $i \in \{1, 2, ..., k\}$.*

It was then shown for a graph Γ of this type, and the construction described above that with the correct choice of s that the orbital chromatic polynomial



a. $\Gamma^{(1,1)}$

Figure 3.1 A counterexample found by Kim, Mun and Omar

of $\Gamma^{(n,s)}$ would have a root larger than the largest real root of the chromatic polynomial, proving that the roots of the chromatic polynomial did not provide an upper bound for the roots of the orbital chromatic polynomial. Figure 3.1 shows the graph $\Gamma^{(1,1)}$ which is one in the family of counter examples given in Kim, Mun, and Omar (2014). At the end of this paper Conjecture 3 was stated, one of the main focuses of this thesis is to attempt to prove this conjecture.

Chapter 4

Results

We shall start by outlining a theorem that describes characteristics of graphs which would be sufficient for refuting Conjecture 3. The work throughout this chapter relies heavily on the methods that were used by Kim, Mun and Omar to create a family of counterexamples to Problem 2.

Theorem 35. *Let Γ be a graph and G be a group of automorphisms. Suppose the following hold*:

1. *The largest real root of $P_\Gamma(x)$ is m.*
2. *There is some $g \in G$ for which Γ/g has less vertices than any of the graphs $\{\Gamma/h : h \in G, h \neq g\}$.*
3. *For the g in* (2) *Γ/g is a complete graph on j vertices where $m < j$.*
4. *There exists some $x_0 > m$ such that $P_{\Gamma/g}(x_0) < 0$.*

Then one can construct from Γ and G, a graph Γ' and a group of automorphims of Γ', called G', such that the $OP_{\Gamma',G'}(x)$ has a real root that is at least $j-1-m$ larger than the largest real root of $P_\Gamma(x)$.

To prove this theorem we shall rely on preliminaries that were created by Kim, Mun and Omar. Let us begin by recalling the definition of the family graphs $H_{n,s}$.

Definition 32 (Kim et al., 2014) *Let Γ be a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$, and n, s be positive integers. Let $H^{(1)}_{n,s}, H^{(2)}_{n,s}, \ldots, H^{(k)}_{n,s}$ be k copies of the graph $H_{n,s}$ and choose vertices $u_i \in V(H^{(i)}_{n,s})$ so that there is an isomorphism from $H^{(i)}_{n,s}$ to $H^{(j)}_{n,s}$*

sending u_i to u_j. We construct the graph $\Gamma^{(n,s)}$ by starting with Γ, and appending the k copies of $H_{n,s}$ to Γ by identifying the vertices u_i and v_i for $i \in \{1, 2, ..., k\}$.

Next we will use the graph construction created by Kim, Mun, and Omar (2014) in Definition 34, to construct $\Gamma^{(n,s)}$.

Definition 34 (Kim et al., 2014) *Let Γ be a graph with vertex set $\{v_1, v_2, ..., v_k\}$, and n, s be positive integers. Let $H_{n,s}^{(1)}, H_{n,s}^{(2)}, ..., H_{n,s}^{(k)}$ be k copies of the graph $H_{n,s}$ and choose vertices $u_i \in V(H_{n,s}^{(i)})$ so that there is an isomorphism from $H_{n,s}^{(i)}$ to $H_{n,s}^{(j)}$ sending u_i to u_j. We construct the graph $\Gamma^{(n,s)}$ by starting with Γ, and appending the k copies of $H_{n,s}$ to Γ by identifying the vertices u_i and v_i for $i \in \{1, 2, ..., k\}$.*

Now let use the following fact found by Kim, Mun and Omar.

Proposition 36. (*Kim et al.,* 2014) *For any graph Γ, and positive integers n,s,*

$$\begin{aligned} P_{\Gamma^{(n,s)}}(x) &= ((x-1)\cdots(x-n+1)(x-n)^s)^{|V(\Gamma)|} P_\Gamma(x) \\ &= \left(\frac{P_{H_{n,s}}(x)}{x}\right)^{|V(\Gamma)|} P_\Gamma(x). \end{aligned}$$

Proof (of Prop. 36) Let us begin by calculating the chromatic polynomial of $H_{n,s}$. We shall begin by coloring the subgraph K_n. We see that the chromatic polynomial of the subgraph is,

$$x(x-1)(x-2)\cdots(x-n+1).$$

Now consider the subgraph N_s in $H_{n,s}$. We see that each one of these vertices is adjacent to all vertices in K_n so we see that the chromatic polynomial of the subgraph is,

$$(x-n)^s.$$

So we see that the chromatic polynomail of $H_{n,s}$ is,

$$x(x-1)(x-2)\cdots(x-n+1)(x-n)^s.$$

Now that we have found the chromatic polynomial of $H_{n,s}$ let us compute the chromatic polynomial of $\Gamma^{(n,s)}$. Let us begin by considering the subgraph Γ of $\Gamma^{(n,s)}$. We shall begin by counting the number of colorings of this graph, we see that the number of colorings of this graph is given by $P_\Gamma(x)$. Now consider coloring $H_{n,s}^{(i)}$ which we know is appended to v_i in the subgraph Γ of $\Gamma^{(n,s)}$. We see that the number of ways to color one of the vertices in the complete subgraph K_n of $H_{n,s}^{(i)}$ has already been counted since we have already counted the number of ways to color Γ. We will now count the number of ways to color the other $n-1$ vertices in the subgraph K_n, we see that the chromatic polynomial of the subgraph is,

$$(x-1)(x-2)\cdots(x-n+1).$$

Now consider the subgraph N_s in $H_{n,s}^{(i)}$ we see that each one of these vertices is adjacent to all vertices in K_n so we see that the chromatic polynomial of the subgraph is,

$$(x-n)^s.$$

Now notice that there is a total of $|V(\Gamma)|$ copies of $H_{n,s}$ appended to Γ in $\Gamma^{(n,s)}$. So we see that the chromatic polynomial of $\Gamma^{(n,s)}$ is,

$$P_{\Gamma^{(n,s)}}(x) = ((x-1)\cdots(x-n+1)(x-n)^s)^{|V(\Gamma)|}P_\Gamma(x).$$

Now notice,

$$((x-1)\cdots(x-n+1)(x-n)^s) = \frac{x(x-1)(x-2)\cdots(x-n+1)(x-n)^s}{x} = \frac{P_{H_{n,s}(x)}}{x}.$$

We can therefore rewrite $P_{\Gamma^{(n,s)}}(x)$ as,

$$P_{\Gamma^{(n,s)}}(x) = ((x-1)\cdots(x-n+1)(x-n)^s)^{|V(\Gamma)|}P_\Gamma(x) = \left(\frac{P_{H_{n,s}(x)}}{x}\right)^{|V(\Gamma)|} P_\Gamma(x).$$

■

We now have all we need to construct the proof of Theorem 35.

Proof (of Thm. 35) Let n and s be postive integers which are arbitrary for the moment. Construct the graph $\Gamma^{(n,s)}$ from Γ and let $G^{(n,s)}$ be the group induced by the group G which permutes the vertices of the subgraph Γ of $\Gamma^{(n,s)}$ just as G does, so that if g $\in$ G sends v_i to v_j then $H_{n,s}^{(i)}$ gets sent to $H_{n,s}^{(j)}$ via the isomorphism sending v_i to v_j, where v_i and v_j are in the subgraph Γ of $\Gamma^{(n,s)}$. We see that there is a natural isomorphimsm between the elements in G and the elements in $G^{(n,s)}$, so for and h in G, we denote $h^{(n,s)}$ to be its corresponding element in $G^{(n,s)}$.

Now observe that for any h $\in$ G, $\Gamma^{(n,s)}/h^{(n,s)} = (\Gamma/h)^{(n,s)}$, and so by Proposition 36,

$$P_{\Gamma^{(n,s)}/h^{(n,s)}}(x) = P_{(\Gamma/h)^{(n,s)}} = \left(\frac{P_{H_{n,s}}(x)}{x}\right)^{|V(\Gamma/h)|} P_{\Gamma/h}(x)$$

We can now write $OP_{\Gamma^{(n,s)},G^{(n,s)}}(x)$ as,

$$OP_{\Gamma^{(n,s)},G^{(n,s)}}(x) = \frac{1}{|G|}\sum_{h\in G}\left(\frac{P_{H_{n,s}}(x)}{x}\right)^{|V(\Gamma/h)|}P_{\Gamma/h}(x).$$

Which can then be rewritten as,

$$\frac{1}{|G|}\left(\frac{P_{H_{n,s}}(x)}{x}\right)^{|V(\Gamma/g)|}\left(P_{\Gamma/g}(x) + \sum_{h\in G,h\neq g}\left(\frac{P_{H_{n,s}}(x)}{x}\right)^{|V(\Gamma/h)|-|V(\Gamma/g)|}P_{\Gamma/h}(x)\right).$$

We can now choose appropriate values of n and s to control the roots of $OP_{\Gamma^{(n,s)},G^{(n,s)}}(x)$. First recall our assumptions, that there is some x_0 not in the integers for which $P_{\Gamma/g}(x_0) < 0$ and that Γ/g is a complete graph on j vertices. Let us now consider values of x_0 that will make our assumption concerning $P_{\Gamma/g}(x_0)$ true. First notice from our assumptions that we know Γ/g is a complete graph on j vertices, therefore the chromatic polynomial of Γ/g will be,

$$P_{\Gamma/g}(x) = (x(x-1)(x-2)\cdots(x-j+1)(x-j)).$$

Now notice that if we choose x_0 to be some value where $j-1 < x_0 < j$ then all terms in $P_{\Gamma/g}$ will be positive except for the last term in the product meaning that $P_{\Gamma/g}$ will be negative, we can conclude that for $j-1 < x_0 < j$, x_0 will satisfy our initial assumption. Now that we have found values for x_0 that satisfy our initial assumption, let us consider appropriate choices for n. For the purposes of our proof we want the term $\frac{P_{H_{n,s}(x)}}{x}$, in the expression for $OP_{\Gamma^{(n,s)},G^{(n,s)}}(x)$ to positive when evaluated at x_0, for this reason we shall choose n to be $j-1$. We see that this choice of n will make the term $\frac{P_{H_{n,s}(x_0)}}{x_0}$ positive from the expanded form of $\frac{P_{H_{n,s}(x)}}{x}$ which was given in Proposition 36 and since we are considering values of x_0 where $j-1 < x_0 < j$. Let us also notice from the expandeded form of $\frac{P_{H_{n,s}(x)}}{x}$ given in Proposition 36 that as we increase the value of s, $\frac{P_{H_{j,s}(x_0)}}{x_0}$ will remain positive and approach 0. Now consider the fact that $P_{\Gamma/g}(x_0) < 0$, and that $|V(\Gamma/h)| - |V(\Gamma/g)| > 0$ for all h $\neq$ g, this implies that we can choose a sufficiently large value of s that $OP_{\Gamma^{(n,s)},G^{(n,s)}}(x) < 0$. But since $lim_{x\to\infty}OP_{\Gamma^{(n,s)},G^{(n,s)}}(x) = \infty$ by the Intermediate Value Theorem we can conclude that $OP_{\Gamma^{(n,s)},G^{(n,s)}}(x)$ has a root larger than x_0. We can make this conclusion because we know $j-1 < x_0 < j$ and the largest real root of $P_{\Gamma^{(n,s)}}$ is m, therefore $OP_{\Gamma^{(n,s)},G^{(n,s)}}(x)$ must have at least one real root that is at least $j-1-m$ larger than the largest real root of $P_{\Gamma^{(n,s)}}$, thus proving Theorem 35. ■

In order to help find a family of graphs which have the properties described in Theorem 35, two lists of families of graphs was made. One list contains families of graphs where the real roots of the chromatic polynomial are large and the other list contains families of graphs where the real roots of the chromatic polynomial are small. If a family of graphs with small roots no matter number of vertices can be found which also has some automorphism group which contains an element that causes the graph to contract to a member of some family of graphs with arbitrarily large roots this would suffice to prove Conjecture 3.

Families of graphs with small roots.

1. The n-book graph is compoosed of $2n+2$ vertices and is composed of 2 star graphs each with n leaves, edges are then added between vertices in the two star graphs if they map to one another (Gallian, 1997). An example of a 3-book graph is shown in Figure 4.1. The chromatic polynomial of the n-book graph is given by,

$$P_{n-book}(x) = (x-1)x(x^2-3x+3)^n$$

(Gallian, 1997). The real roots of the chromatic polynomial of the n-book graph are 0 and 1. The imaginary roots of the chromatic polynomial of the n-book graph are not included since Theorem 35 is only concerned with the real roots.

2. The cycle graph denoted C_n is a graph on n vertices where every vertex is adjacent to exactly 2 other vertices. An example of a cycle graph C_5 is shown in Figure 4.2. The chromatic polynomial of a cycle graph on n vertices is given by,

$$P_{C_n}(x) = (x-1)^n + (-1)^n(x-1).$$

The roots of the cycle graph on n vertices are 0 and 1 if n is even and 0,1 and 2 if n is odd.

3. The n-ladder graph is a graph composed of 2 path graphs on n vertices where the ith vertex in one path graph is adjacent to the ith vertex in the other path graph (Rouse Ball and Coxeter, 2010). An example of a 4-ladder graph is shown in Figure 4.3. The chromatic polynomial of the n-ladder graph is given by,

$$P_{n-ladder}(x) = (x-1)x(x^2-3x+3)^{n-1}$$

(Rouse Ball and Coxeter, 2010).

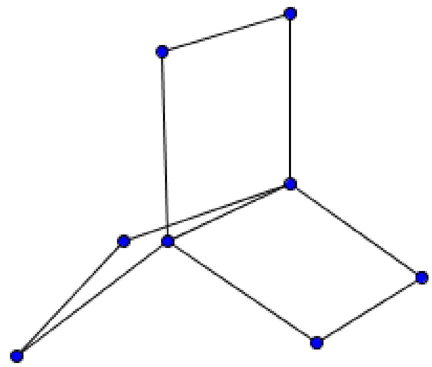


Figure 4.1 A 3-book graph

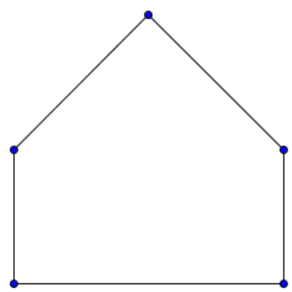


Figure 4.2 A cycle graph on 5 vertices

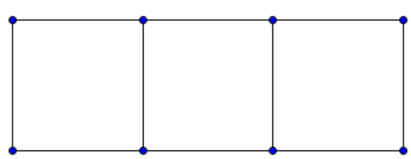


Figure 4.3 A 4-ladder graph

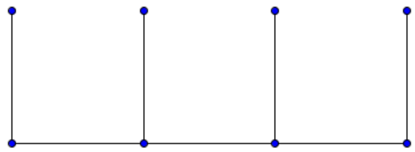


Figure 4.4 A 4-centipede graph

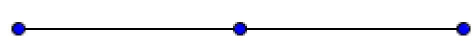


Figure 4.5 P_3
9

4. The tree graph, is an connected acyclic graph. We see that the chromatic polynomial of a tree graph on n vertices is,

$$P_{tree}(x) = x(x-1)^n.$$

The real roots of the chromatic polynomial of the tree graph are 0 and 1. We see that many well-known families of graphs fall within tree graphs. The following is a list of families of graphs which are all Trees.

(a) The n-centipede graph is a tree, composed of $2n$ vertices where n of the vertices form a path graph on n vertices and each of the other n vertices is adjacent to exactly 1 vertex in the path graph (Levit and Mandrescu, 2005). An example of a 4-centipede graph is shown in Figure 4.4.

(b) The path graph on n vertices is a tree, where all but two vertices are adjacent to exactly 2 other vertices and the other 2 vertices are adjacent to exactly 1 vertex. An example of a path graph on 3 vertices is given in Figure 4.5.

(c) The star graph on $n+1$ vertices denoted S_n is a tree with 1 central vertex and n leaves which are all adjacent to the central vertex. An example of S_4 is given in Figure 4.6.

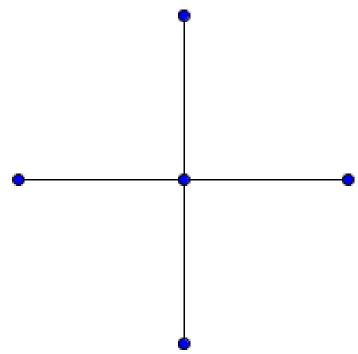


Figure 4.6 S_4

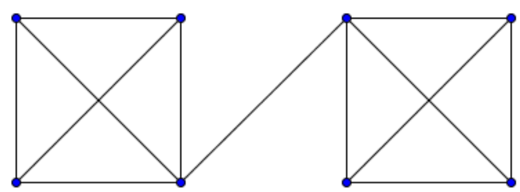


Figure 4.7 A 4-barbell graph

Families of graphs with large roots.

1. The n-barbell graph is a graph on $2n$ vertices. The n-barbell graph is composed two complete graphs on n vertices and then there is an edge added between some vertex in one of the complete graphs and some vertex in the other complete graph (Wilf, 1989). An example of a 4-barbell graph is given in Figure 4.7. The chromatic polynomial of the n-barbell graph is given by,

$$P_{n-barbell}(x) = \frac{[x(x-1)\cdots(x-n+2)(x-n+1)]^2(x-1)}{x}$$

(Wilf, 1989). The roots of the n-barbell graph are, 0,1,2,...,$n-1$

2. The complete graph on n vertices denoted K_n, is a graph where each vertex is adjacent to every other vertex in the graph. An example of a

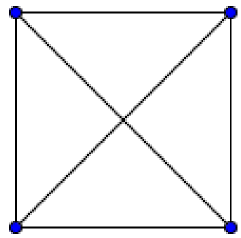


Figure 4.8 K_4

complete graph on 4 vertices, K_4 is given in Figure 4.8. The chromatic polynomial of K_n is given by,

$$P_{K_n}(x) = x(x-1)\cdots(x-n+2)(x-n+1).$$

The roots K_n are 0,1,2,..,$n-1$.

3. The n-sun graph is a graph composed of $2n$ vertices. n vertices form a complete graph and then each vertex in the complete graph is adjacent to 2 of the other n vertices, each of these n vertices is adjacent to exactly 2 vertices within the complete graph (Anitha and Lekshmi, 2008). An example of 4-sun graph is given in Figure 4.9. The chromatic polynomial of n-sun graph is given by,

$$P_{n-sun}(x) = [x(x-1)\cdots(x-n+2)(x-n+1)](x-2)^2$$

(Anitha and Lekshmi, 2008). The roots the n-sun graph are 0,1,2,..,$n-1$.

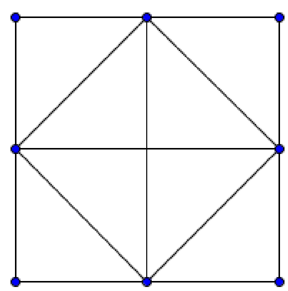


Figure 4.9 The 4-sun graph

Chapter 5

Conclusion

The purpose of this thesis was to gain a deeper understanding of chromatic and orbital chromatic polynomials along with attempting to prove Conjecture 3. After gaining a deeper and more thorough understanding of chromatic and orbital chromatic polynomials, attention was turned to attempting to prove Conjecture 3. The attempts made in this thesis to prove Conjecture were based strongly upon the methods that were used by Kim, Mun, and Omar (2014). The development in this thesis that would lead to a proof of Conjecture 3 was Theorem 35. Theorem 35 described a family of graphs and a group of automorphims whose properties were sufficient to provide an example of graphs that would prove Conjecture 3. Thus far, a search for the graphs that satisfy the conditions of Thereom 35 has come up inconclusive. The focus, then turned to trying to find families of graphs that were similar but did not fit the description of graphs in Theorem 35, because through this it may be possible to create a less strict set of characteristics than those outlined in Theorem 35, that would still be sufficient to prove Conjecture 3. However the search for graphs of these types also proved inconclusive.

If further attempts were to made to either prove or disprove this Conjecture, one of the first things that would be considered is expanding Theorem 35. Theorem 35 would be expanded so that the characteristics needed to prove Conjecture 3 were more broad and would allow for more graphs to be found. Along with trying to expand Theorem 35, the method for searching for graphs would be changed. Throughout the time spent searching for graphs and groups of automorphism which had the characteristics outlined in Theorem 35, a great amount of emphasis was put on first finding graphs and then evaluating their automorphism. Instead, we propose shifting the focus to first trying to find the main attributes of the group of automorphisms that are indicative of whether the graph had the properties outlined in Theorem 35 and then trying to construct graphs whose automorphism groups had these attributes. Although this thesis did not succeed in either proving or disproving Conjecture 3, headway was made and tools in the form of Theorem 35 were created that will give traction to any future attempts to prove Conjecture 3.

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