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1. Introduction

I have long been interested in the relationship between students' understanding of basic concepts in mathematics and their performance on material that is supposed to be built up out of these concepts. But such an interest immediately raises the question of what concepts underlie any particular body of mathematical material and how one might describe student understanding of these concepts as something separate from their mastery of the material.

To a mathematics teacher, the central concepts of a beginning calculus course are limit, derivative, and integral. But these concepts do not underlie the subject: they are to be developed in the course of studying it. Most teachers would agree that, while the course is in progress, students' understanding of these concepts could not stand much of a test. The concept that the subject is built out of, the one that lies behind such notions as limit, derivative, and integral is that of function. The question this paper addresses is: What do students understand of the concept of function while they are in the process of mastering the material of a beginning calculus course?

Boiled down to its simplest form, a function is a correspondence between two sets or between two variables. At first, one usually describes functions in terms of every day examples using tables, algebraic formulas, graphs, and various artificial rules. This idea is so simple that we have difficulty imagining that our students are not already familiar with it. And, indeed, if a function is given by a table, or we use its graph or formula as if it were only a table, reading particular numerical values of the independent variable as corresponding to particular numerical values of the dependent variable, then this *IS* a very simple concept—one that most of our students acquired in high school. I call such use or such a view of this concept a "Pointwise understanding" of functions. I have always found it interesting that when authors of calculus and precalculus texts give their obligatory introductions of the concept of function, it is a Pointwise view of the concept they are trying to get across. But this is not actually the way the concept of function is used in calculus.

A look at almost any page of a calculus book shows that the crucial question asked about functions is: How does change in one variable lead to change in others? How is the behavior of the output variables influenced by variation in the input variable? I call an ability to answer such questions an "Across-Time understanding" of the concept of function. The definition of the tangent line to a graph or of its slope would be utterly meaningless to someone who could only look at the graph or the function at a few specific points at a time. The essence of the definition is a tendency of the behavior of secant lines or difference quotients as the incremental change in the independent variable is decreased.

This paper is based on a study of student responses to two types of questions on final examinations in calculus classes, one of which requires only Pointwise understanding and the other of which requires Across-Time understanding. The results of this study show that these two kinds of understanding are clearly distinguishable. But, more crucially for teachers, they show that, at least in simple situations, students have a confident and secure Pointwise understanding of functions, but even at the end of the first or second quarter of calculus—when we tend to assume they have already acquired this ability—they are still struggling to see functions in an Across-Time manner.

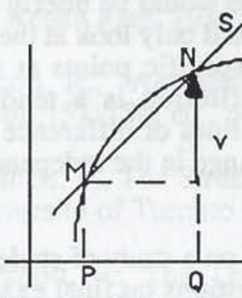
2. Description of the Study

This study is based on four multipart problems for beginning calculus students. Each problem has one or two questions that test for a Pointwise understanding of a function and one or two questions that test for an Across-Time understanding of the same function. A number of faculty members at the University of Washington each agreed to include one of these problems as a regular part of an examination in a calculus class with the results to be counted toward the exam grade. Each of the questions was used once on a final exam in a first-quarter calculus class (Math 124). One was also used on a late mid-quarter exam in the first quarter calculus class and two were used on final exams in second-quarter calculus classes (Math 125). In all, 628 students were involved in the study.

In order to facilitate analysis and discussion, these problems are presented below in a highly compressed form. They were expanded into a form more readily comprehensible to students for the exams. In most cases, they were written as multiple choice questions.

The Problems

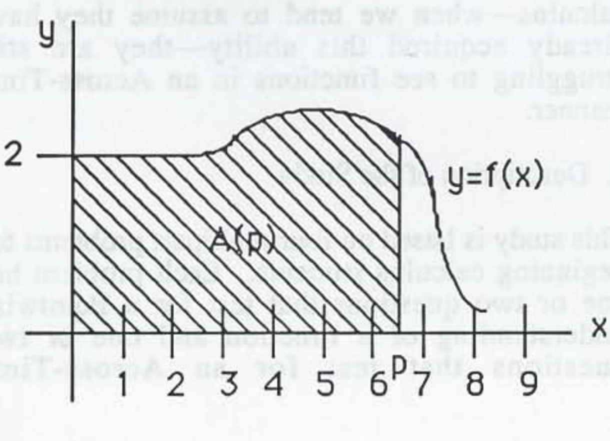
SLIDING SECANT



POINTWISE: Find the slope of the secant and value of v when M and N have coordinates (1.6) and (4.12).

ACROSS-TIME: The point Q moves toward P. Does the slope of S increase or decrease? Does the value of v increase or decrease?

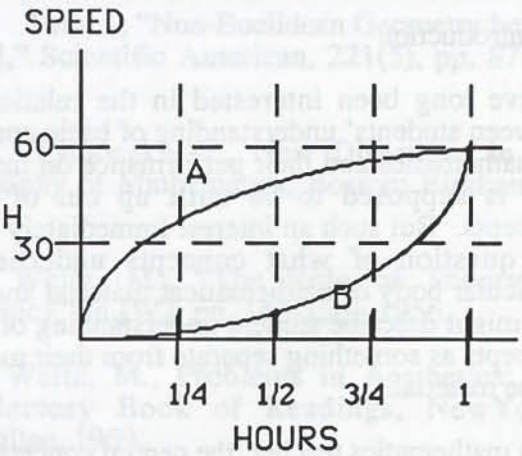
AREA UNDER THE GRAPH



POINTWISE: Determine the values of A(1) and A(3).

ACROSS-TIME: The point p moves from 4.5 to 6.0. Does the area A(p) increase or decrease?

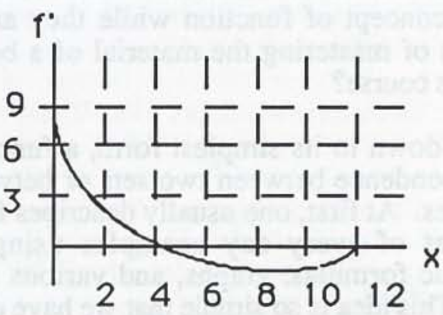
TWO SPEED GRAPHS



POINTWISE: Which car is going farther at time 3/4: Is Car A going faster at time 1/4 or 3/4?

ACROSS-TIME: Tell whether or not the cars are closest together at time t=1. In the time from 1/2 to 1 hour, do the two cars get further apart or closer together?

SECOND DERIVATIVE TEST



The graph shown is of the derivative, f' (x).

POINTWISE: Determine the slop of the tangent line to f(x) at x=2. Give a value of x for which the the tangent line to f(x) is horizontal.

ACROSS-TIME: Describe the shape of the graph of the function f(x) for the interval 1 to 4.

Each student's answers to the various questions were then coded and recorded, so that combinations of rightness and wrongness, or combinations of incorrect answers could be studied. A summary of the statistical results of this study is given in Table 1. Each grid indicates the results of one class's response to the two kinds of questions on one problem. Thus, for instance in the first grid of "Sliding Secant", we see that of the 116 students who answered the question, 53% got both the Pointwise and Across-Time Questions correct, and 34% got the Pointwise Question correct while they got the across-Time Question incorrect. We also see that, in all, 87% got the Pointwise Question correct. There are two grids for each of the problems Sliding Secant, Area Under the Graph, and Two Speed Graphs because each of these problems was given in two different classes.

Sliding Secant

		Across-Time		
	N=166	Right	Wrong	T
P	Right	53%	34%	87%
T	Wrong	4%	9%	13%
W	T	57%	43%	

Area Under the Graph

		Across-Time		
	N= 89	Right	Wrong	T
P	Right	63%	19%	82%
T	Wrong	0%	18%	18%
W	T	63%	37%	

Two Speed Graphs

		Across-Time		
		Right	Wrong	T
P	Right	40%	50%	90%
T	Wrong	0%	10%	10%
W	T	40%	60%	

Second Derivative Test

		Across-Time		
		Right	Wrong	T
P	Right	44%	8%	52%
T	Wrong	22%	28%	50%
W	T	66%	36%	

Sliding Secant

		Across-Time		
		Right	Wrong	T
P	Right	42%	40%	82%
T	Wrong	10%	10%	20%
W	T	52%	50%	

Area Under the Graph

		Across-Time		
		Right	Wrong	T
P	Right	60%	24%	84%
T	Wrong	4%	11%	15%
W	T	64%	35%	

Two Speed Graphs

		Across-Time		
		Right	Wrong	T
P	Right	49%	35%	84%
T	Wrong	0%	16%	16%
W	T	49%	51%	

Table 1 —Summary of Test Results

3. Interpretation of the Test Results

The most striking observation to be made about the data shown in Table 1 is that the percentages of completely correct solutions to these problems are so low. Overall, only 50% of the solutions to these problems were completely correct. Yet most mathematics instructors would agree that these are extremely simple, if not naive, problems, which test the kinds of understanding we tend to assume our students have as they work on the early material in calculus. The problems are so easy that

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many argue against the results by saying that the students were tricked or that they were probably not paying complete attention to them. But these problems were on exams which counted toward the students' grades in very competitive classes. These students were strongly motivated to carefully read and think about these questions. Moreover, each problem had a very simply Pointwise question, placed there, at least in part, to settle the student into the problem and to check from his or her answer that the quantity being asked about was clearly distinguished. The fact that such a high percentage did get the Pointwise questions correct (79% on all problems) indicates that the students were paying sufficient attention for us to make inferences about their thought processes.

In relation to the overall percentages of correct answers, it is probably worth noting that of the students who were assigned final grades in these classes, 83% received a grade of 1.6 or higher. Since all but a handful of students who take a final or very late mid-quarter exam receive a grade, it seems safe to assume that these data do not include a large pool of students who are lost in the material, as, say, an early mid-quarter exam might. Presumably, most of the students who took these tests felt reasonably comfortable with their ability to perform in the course.

To draw more refined inferences from these data we must separate the last problem, "The Second Derivative Test" from the others. It presents a different picture of students' understanding which can only be described in relation to the picture indicated by the other data—the pairs of results from the first three problems.

3.1 The First Three Problems

Common sense would indicate that at some stage students learn to read a graph one point at a time—in a Pointwise manner—and once they have done so, it is only a small jump to reading a graph at many points. From that position, it is again only a small step to reading a graph at infinitely many points, or with a continuously changing variable, i.e., reading graphs in an Across-Time manner. Indeed a study of precalculus texts indicates that authors and many teachers assume that this is how things happen when students are taught functions. However, since, in the first three problems in this study, 85% of the students got the Pointwise questions correct while only 53% of the students got the Across-Time questions correct, it does NOT seem to be the case that an Across-Time understanding comes easily and automatically after a Pointwise understanding has been developed.

A study of the first six grids in Table 1 illustrates this point forcefully. It shows that reading graphs in a Pointwise manner is a necessary, but far from sufficient condition to reading graphs in an Across-Time manner. In particular this data says:

a) Students who get the Pointwise questions wrong are unlikely to get the Across-Time questions right. (Overall, 15% who got the Pointwise questions wrong got the Across-Time questions right.)

b) Students who get the Pointwise questions right have variable chances of getting the Across-Time questions right—depending on the question, the class, etc. (Overall, 59% of the students who got the Pointwise questions correct also got the Across-Time questions correct.)

c) Students who get the Across-Time questions correct are extremely likely to get the Pointwise questions correct. (Overall, 96% who got the Across-Time question right got the Pointwise question right.)

Another way of putting this is that Pointwise understanding of graphs is prerequisite to Across-Time understanding, but the jump from the one to the other is a considerable one for students.

That the difference between these two kinds of questions is a qualitative one can be seen by analyzing the answers given to these questions by the population of those students who got the Pointwise questions correct. For these students we can be reasonably confident that they can read the graph in a rudimentary fashion and that they have a basic comprehension of the set-up of the problem. What one sees in these answers is that, when pressed, the students do inconsistent things; in each wrong answer there is a self-contradiction.

Sliding secant: 43% of this population got the Across-Time question wrong. The most common error was one in which the secant line is regarded as moving, with its slope increasing, while the vertical distance v is regarded as fixed. In fact, if one views this diagram as made up of a system of inter-connected "moving parts", (as the course material requires) then all of the incorrect answers are self-contradictory. They either give one part as moving and another as fixed, or they give two parts moving, but in inconsistent ways. It must be that these students do not see this diagram as such a system, that such a dynamic, Across-Time use of a graph is quite alien to them.

Area Under the Graph: 25% of this population got the Across-Time question wrong. These students knew well enought how to compute area from the two dimensions, height and base, for the two particular values p=1 and p=3 given, but then, when they were asked about the behavior of A(p) as p goes from 4.5 to 6.0, they responded as if they thought that A(p) was to be found by simply looking at the height of the given graph. This seems to be a case of a student transforming a question into one that is easier when the given question cannot be answered.

Two Speed Graphs: 50% of this population got one of the Across-Time questions wrong. Most of these, (34% of this population) gave answers to both Across-Time questions as if the graphs shown were of position vs time—while answering the Pointwise questions as if the graphs were of speed vs time. The remaining students in this population were conflicted. They answered one Across-Time question as if the graphs were of speed vs time and the other as if the graphs were of position vs time. In both cases, we see the students shifting their interpretation of the quantity represented by the vertical axis when confronted with a need to use the information in the graphs in a novel way.

It would seem to be the case that students who give contradictory answers have lost their hold on the situation described in the problem—that they are, in some way, overloaded. To understand why these Across-Time questions have this effect would require a separate study that would at least include an analysis of the mental processes of students who do get the correct answers to these problems. My own speculation is that in order to do the Sliding Secant and Area Under the Graph problems correctly, the student would have to evoke in his or

It would seem to be the case that students who give contradictory answers have lost their hold on the situation described in the problem—that they are, in some way, overloaded.

her own mind a version of the given diagram that can be made to move and then draw conclusions from these mental experiments. In order to do the Two Speed Graphs problem correctly the student would have to evoke a mental model of two cars, read the information from the graphs that one car is going farther than the other for the entire period, and draw conclusions from this fact, while specifically disattending from the striking pictorial qualities of the graphs. There is very little in the experience of first-quarter calculus students that would prepare them to do these things, and so it does not surprise me that the students do so badly.

3.2 Interpretation of Results of "Second Derivative Test"

The statistical results of "Second Derivative Test" are very different from the results of all of the first three problems. On this problem students did better on the Across-Time question than on the two Pointwise questions (66% vs 52%). Furthermore, in contrast to the results we saw before:

a) 22% of the students who got the Across-Time question wrong got the Pointwise questions right.

b) 66% of the students who get the Across-Time question correct got the Pointwise questions correct.

c) 85% of the students who got the Pointwise question correct also got the Across-Time question correct.

Thus, extending the kind of analysis used to confirm that Pointwise understanding of graphs is prerequisite to Across-Time understanding on the first three problems, we arrive at the conclusion that an Across-Time understanding of this problem is prerequisite to a Pointwise understanding. But this makes no sense at all, because any understanding of how to draw inferences from the shape of the graph of a derivative, would have to include an ability to make pointwise readings of this graph.

But, in fact, it can be seen that of the 108 students who took the test, 24 were able to use the graph of $f'(x)$ to select the correct shape of $f(x)$ over the interval $[1,4]$, but at the same time, could not read from the graph the value of $f'(2)$ or the value of x for which $f'(x) = 0$. These students are getting correct answers, but not from a base of understanding. For this class, this problem is not a good test of understanding, since the students seem to be responding with fragments of partially digested course material. If anything, this problem underscores the distinction to be made between understanding and the ability to produce correct answers to selected questions.

4. Implications of this Study

The problems used in this study were specifically chosen for their proximity to the standard calculus curriculum, so that the results of each problem bear upon assumptions made by calculus instructors as they teach particular topics in the subject.

It is difficult to imagine how one could present the notion of a tangent line and its slope as approximated by slopes of secant lines without using some version of the diagram in the Sliding Secant problem. Likewise, it is hard to conceive of a discussion of the Fundamental Theorem of Calculus that is not illustrated by something like the diagram used in the Area Under the Graph problem. The results of this study indicate that these diagrams do NOT carry the meaning for our students that we assume they do. The students can read them in a Pointwise manner, but large numbers of them cannot read them in an Across-Time manner, as the subject demands they do. Of course, it could be argued that an outcome of 50% to 65% of the students getting these problems correct is not all that bad, but the rejoinder is to point out how naive these questions are in relation to those that arise when the slope of the tangent or the area function are actually used in a calculus class. For instance, in the Sliding Secant problem,

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one could ask whether or not the slope of the secant line increases without bound, and in the Area Under the Graph problem, one could ask about the behavior of the function ΔA (for a fixed increment of p) as the variable p increases. It seems clear that the results would have been so much worse that a legitimate issue would have arisen as to the fairness of such problems on a final exam.

One of the key issues in the first quarter of calculus is that the students come to understand how the behavior of the derivative $f'(x)$ of a function gives us information about the behavior of the function $f(x)$ itself. A student who is faced with graph sketching or optimization problems who has no such understanding is forced to memorize a series of arcane rules and procedures, which will only move him or her further from the possibility of comprehension of this subject. The Two Speed Graph and Second Derivative problems indicate that large numbers of students have little or no basis for arriving at such an understanding, because under the pressure of making Across-Time reading of f '(x) and then Across-Time inferences about f(x), they confound these two quantities, thinking part of the time that they are being given the graph of one and part of the time that they are being given the graph of the other.

The implications of this study are not restricted to the particular aspects of the calculus curriculum that the problems refer to. It would not be difficult to write problems related to other crucial topics in calculus that would show just as clearly that students can use functions in a Pointwise manner, but not in an Across-Time manner as the subject demands. For instance, we could begin with almost any related rate problem, show the students the corresponding diagram, and have them indicate specific corresponding numerical values of the variables. What they would not be able to do is tell how change in one of these variables causes change in another, either in the form of how constant increments in one lead to a pattern of increments in the other, or qualitatively, in terms of related rates of change.

Across-Time understanding of functions is critical to an understanding of calculus. This study indicates that most students come to a calculus course neither equipped with it nor on the verge of acquiring it. Specific instruction toward Across-Time understanding is clearly indicated, but this raises genuine questions as to the type of instruction that would be effective; what sorts of activities should the students carry out to be able to draw Across-Time conclusions about a function? I have written, and have used for several years, material that takes one approach to this problem. It is based on a variety of graphs and diagrams at the same level of complexity as the above problems. As with the Two Speed Graphs problem, much of the material requires that students translate between a graph and a concrete context. The material is in the form of instructions and questions that ask students to distinguish and interpret the various kinds of information contained in the graphs and to draw further conclusions from this information. Watching students struggle with this material has convinced me even more that the difficulties students have with Across-Time understanding are real and that one of our tasks as calculus and precalculus instructors is to directly address these difficulties and help our students overcome them.

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