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Applied Mathematics Should Be Taught Mixed

Gary I. Brown
College of Saint Benedict

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Gary I. Brown
College of Saint Benedict
Saint Joseph, Minnesota 56374

1. INTRODUCTION

Many would argue that "the power of mathematics" is derived from the great variety of problems that can be modeled and solved mathematically. According to *The Power of Mathematics, Applications to Management and Social Science*, by Whipkey, Whipkey and Conway, "the power of mathematics is derived from two sources. First, the same mathematical concept can be used to solve a multitude of problems from diverse academic fields. For example, the algebra of matrices can be applied to problems arising in mathematics, physics, chemistry, economics, sociology, psychology and statistics" (Introduction, p 3).

The hidden message coming from such a derivation seems to be that (pure) mathematics is this self-contained, autonomous abstract subject completely devoid of any human value until it is "applied" to a given problem or discipline. As a young student I first felt this hidden message when encountering "modeling" diagrams such as the one in figure 1*:

One was supposed to transform a "real world problem" into a "mathematical model" and solve the resulting problem mathematically. Then, one was supposed to interpret the mathematical solutions in terms of the real world problem. It always seemed that the "real world problem" was completely separated from the "mathematical problem". As an aspiring mathematician, I hated to deal with the "real world problem" and only wanted to deal with the "mathematical model."

The metaphor emanating from the term "applied" seems to reinforce this separation between (pure) mathematics and applied mathematics. Is is possible to change metaphors and reverse this hidden message? Historically, the term "applied" was not used in the literature until about 170 years ago ("applied" appeared in J. Gergonne's journal called Annales de mathématiques pures et appliquées" that was first published in 1810). Prior to this, mathematics was divided into "pure mathematics" and "*mixed* mathematics." The purpose of this paper is to outline historically the meaning of the term "mixed mathematics" and then to choose aspects of its meaning that can be modified to present a different vision of how to teach applied mathematics. I will first argue that there is a dichotomy today between "pure" and "applied" mathematics. Then I will provide some historical analysis of mixed mathematics in England in the early seventeenth century and in France in the eighteenth century. This will be followed by a brief examination of two nineteenth century practitioners of mixed mathematics, Gaspard Monge and William Whewell. Finally, I will provide some ideas for a possible new model for the teaching of applied mathematics that is partially based upon a recent article in the *Bulletin of the American Mathematical Society* by Arthur Jaffe and Frank Quinn. I an not arguing that mixed mathematics is historically "better" than applied mathematics, but that one can adapt some of the main ideas from mixed mathematics to our modern evolving view of applied mathematics. I hope that

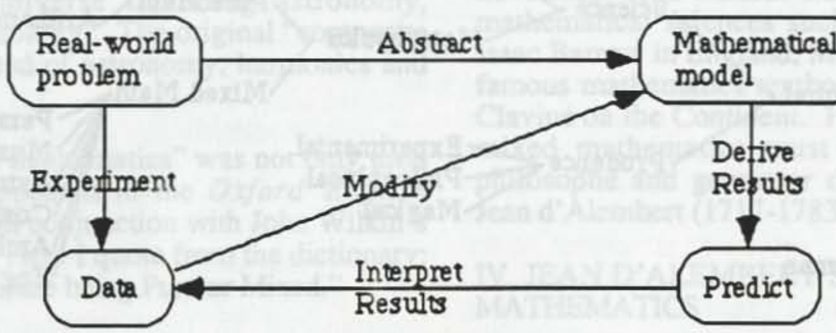

Figure 1

the changing of metaphors will result in reversing this hidden message of applied mathematics.

II. THE EXISTENCE OF A DICHOTOMY BETWEEN "PURE" AND "APPLIED"

Before discussing "mixed mathematics," it perhaps is necessary to demonstrate the existence of a dichotomy between "pure" and "applied" mathematics today. The first thing I did was to examine how the terms "pure" and "applied" were used in everyday English language. According to Funk and Wagnalls Dictionary,

> *Pure*— Free from mixture or contact with that which weakens or impairs or pollutes; considered apart from practical application, opposed to applied.
> *Applied*— To bring into contact with something; to devote or put to a particular use; as to apply steam to navigation or money to payment or debts.

Secondly, I looked at many general source books such as encyclopedias. According to the *World Book Encyclopedia*,

> The work of mathematicians may be divided into *pure mathematics* and *applied mathematics*. Pure mathematics seeks to advance mathematical knowledge for its own sake rather than for any immediate practical use...applied mathematics seeks to develop mathematical technique for use in science and other fields.

Also from the *Mathematics Dictionary* edited by James and James, I found the following:

> *applied mathematics*: A branch of mathematics concerned with the study of physical, biological and sociological worlds... In a restricted sense, the term refers to the use of mathematical principles as *tools* in the fields of physics, chemistry, engineering, biology, and social studies.

From Avner Friedman, current SIAM President, testifying to the House of Representatives on NSF funding, and discussing the *goals* of the mathematical sciences,

> to lead the development and transfer of applications of mathematics to problems in science, technology and industry.

All of these sources seem to reinforce the idea that to do "applied mathematics" one must apply theory A from "pure mathematics" to problem B from "the real world." This separation did not seem to exist prior to the French Revolution (It is difficult to find evidence of this separation in non-European cultures).

III. "MIXED MATHEMATICS IN EARLY SEVENTEENTH CENTURY ENGLAND

The term "mixed mathematics" occurred frequently in many so called "trees of knowledge" in the Seventeenth and Eighteenth centuries.** One of the first prominent "trees of knowledge" comes from Francis Bacon's *Of the Proficience and Advancement of Learnings* (1605). In Bacon's tree of knowledge (See Figure 2), philosophy was divided into three categories (Divine Theology, Natural, and Human). Natural Philosophy was

Bacon's Tree of Knowledge (Human Learning)

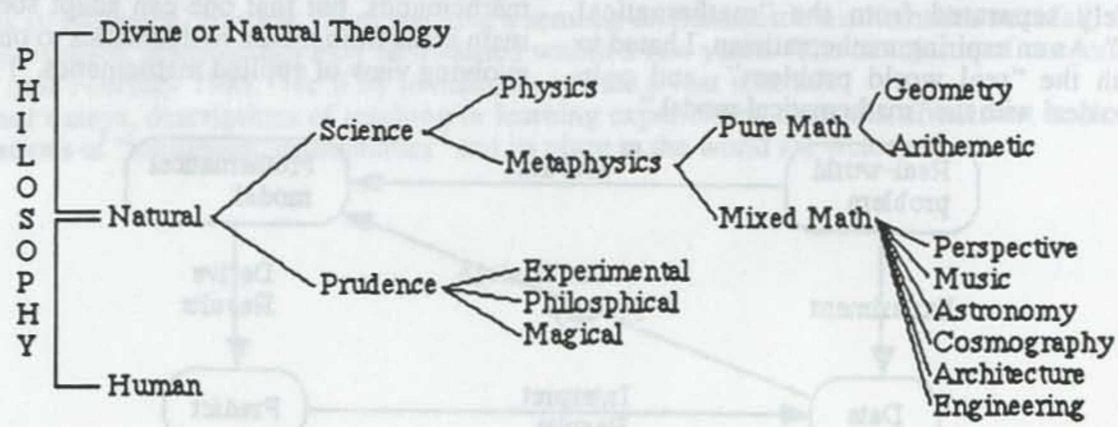

Figure 2

divided into Science and Prudence; Science was divided into Physics and Metaphysics. Finally mathematics was classified under Metaphysics (See Figure 2).

According to Bacon, mathematics belonged to metaphysics because its aim was to inquire *about fixed and constant causes not indefinite causes.* He said "all other forms are the most abstracted and separated from matter and therefore most proper to Metaphysics."

After placing mathematics in the category of metaphysics Bacon subdivided mathematics into "pure" and "mixed". He said "to pure mathematics belong those sciences which handle Quantity entirely severed from matter and from the Axioms of Natural Philosophy. These are Geometry and Arithmetic...Mixed Mathematics has for its subject some axioms and parts of natural philosophy, and consider quantity *determined as it is auxiliary and incident unto them. For many parts of nature can neither be inverted with sufficient subtlety nor demonstrated with sufficient perspicuity* without the aid and intervening of mathematics."

According to d'Alembert, "quantity, the object of mathematics, could be considered either alone and independent of real and abstract things from which one gained knowledge of it"

Bacon added to the Ancient Greek's list of the "composite sciences" the branches of *architecture* (important in the 16th Century Renaissance), *Engineering* and *cosmography* (science that describes the universe including astronomy, geography and geology). The original "composite sciences" consisted of astronomy, harmonics and optics.

The term "mixed mathematics" was not only used by Bacon but appeared in the *Oxford English Dictionary 1648* in conjunction with John Wilkin's "Math Magick". Here I quote from the dictionary: "The Mathematics are being Pure or Mixed."

The reference to John Wilkins pertains to the former Puritan clergyman and popularizer of Bacon's approach to science. Wilkins was one of the founders of the Royal Society (1660), the first scientific organization whose purpose was the promotion and advancement of science and its applications to the improvement of society. The book "Math Magick" refers to the longer title "Math Magick: The Wonders that may be Performed by Mechanical Geometry." Mechanical Geometry was described as "one of the most easy, useful and yet most neglected parts of mathematics. It is a liberal art like astronomy and music." Topics included a discussion of pulleys, cranes, bows and catapults used in levers and wedges; the possibilities of various kinds of machines, such as submarines and carriages propelled by sails; and "secret and speedier ways of attacking forts by approaches and galleries." (inventions in fortification)

The purpose of "Math Magick" was to familiarize the average person with the basic and long-accepted principles of mechanics. Wilkins begins with a defense of mechanics as a liberal art that was more like astronomy and music than the so called "illiberal sciences" which involved some physical activity such as manufacturing or trade. The basic subject of mechanics was the relationship between weight and power. Weight was no longer to be considered a "natural quality, whereby condensed bodies do of them selves tend downwards," but "an affection which might be measured." Quantity, the subject of seventeenth century mathematics, could therefore pertain to qualities of physical objects such as weight and power. If mechanics were to become a mathematical science then one could not separate the physical from the theoretical. The scientist must use the proper mixture of theoretical reasoning, direct observation and experimentation.

During the seventeenth century there were other advocates of a careful handling of the mixed mathematical sciences such as John Wallis and Isaac Barrow in England, Marin Mersenne and the famous mathematics textbook writer Christopher Clavius on the Continent. However, any study of mixed mathematics must include the famous philosophe and geometer of the Enlightenment, Jean d'Alembert (1717-1783).

IV. JEAN D'ALEMBERT'S VIEWS OF MIXED MATHEMATICS

D'Alembert was a key advocate of extending the kind of thinking involved in geometry to decision making in society. As a philosophe he considered himself as a literary man who exercised public responsibility and whose function was to criticize, analyze, and redefine intellectual norms and institutional practice. He became one of the leaders among a group of philosophes that called themselves rationalists.

D'Alembert thought that rational thinking involved the processes of *analysis* and *methodical doubt*. Analysis was the process of starting with an idea and breaking it into simpler ideas until one reaches simple ideas (atoms). Simple ideas are self-evident truths based upon sensory receptions. One would know from *experience* that so and so is a simple idea. Then, once the simple idea (first principles) was found by analysis, one tried to construct a deductive chain of ideas that built back to the original idea. This was the process of synthesis found in geometry.

Notice that this form of analysis is similar to that form used in algebra. Consider examples 1 and 2 below:

<u>Example I</u>: Solve $2x+3=2$

Assume there is a solution x.

analysis → $2x=-1$

$x=-1/2$

synthesis → $x=-1/2$

$2x=-1$

$2x+3=2$

<u>Example 2</u>: Solve $2x^2+3=2$

Assume there is a solution x.

analysis → $2x^2=-1$

Impossible

synthesis → Hence one of the elements in the chain is false

In Example 1, one assumes the equation is true and finds "simpler" statements that follow from the assumption. Eventually, one will arrive at a "simplest" statement that cannot be broken down any further. Then one tries to build a chain of statements from this "simplest" statement to the original statement. Such a procedure will constitute a proof that there is a solution to $2x+3=2$. In proofs involving mixed mathematics (which might include example 2 as a trivial example), one must factor in experience when analyzing each step in the chain. This is apparent in example 2 where an assumption of $2x^2+3=2$ having a real solution will lead to a positive real number equalling a negative real number. The chain has been severed and one must re-examine the problem.

D'Alembert was convinced that man had knowledge on only a few links that joined that gigantic chain of each discipline together. Hence, the philosophe had to look at arguments with a healthy dosage of *methodical doubt* for errors in any chain. The philosophe especially had to search for abuses of theory as might occur in the second example above. Furthermore, the philosophe should use methodical doubt then "indicating the paths which have deviated from the truth, so that it facilitates the search for the path which conducts to it." This chain always had to factor in common experience (the use of the sense perceptions), i.e. *one should never separate the pure from the applied.*

Now d'Alembert was involved in the 1750s with Diderot in the Encyclopedia project. They wanted to classify all of knowledge and d'Alembert was especially interested in classifying mathematics. After all, there were many advances in mathematics

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For Bacon "quantity" was "determined by fixed or constant causes" while for d'Alembert was applicable to objects that were independent of real things (pure) or could be applicable to objects involving physical things (mixed).

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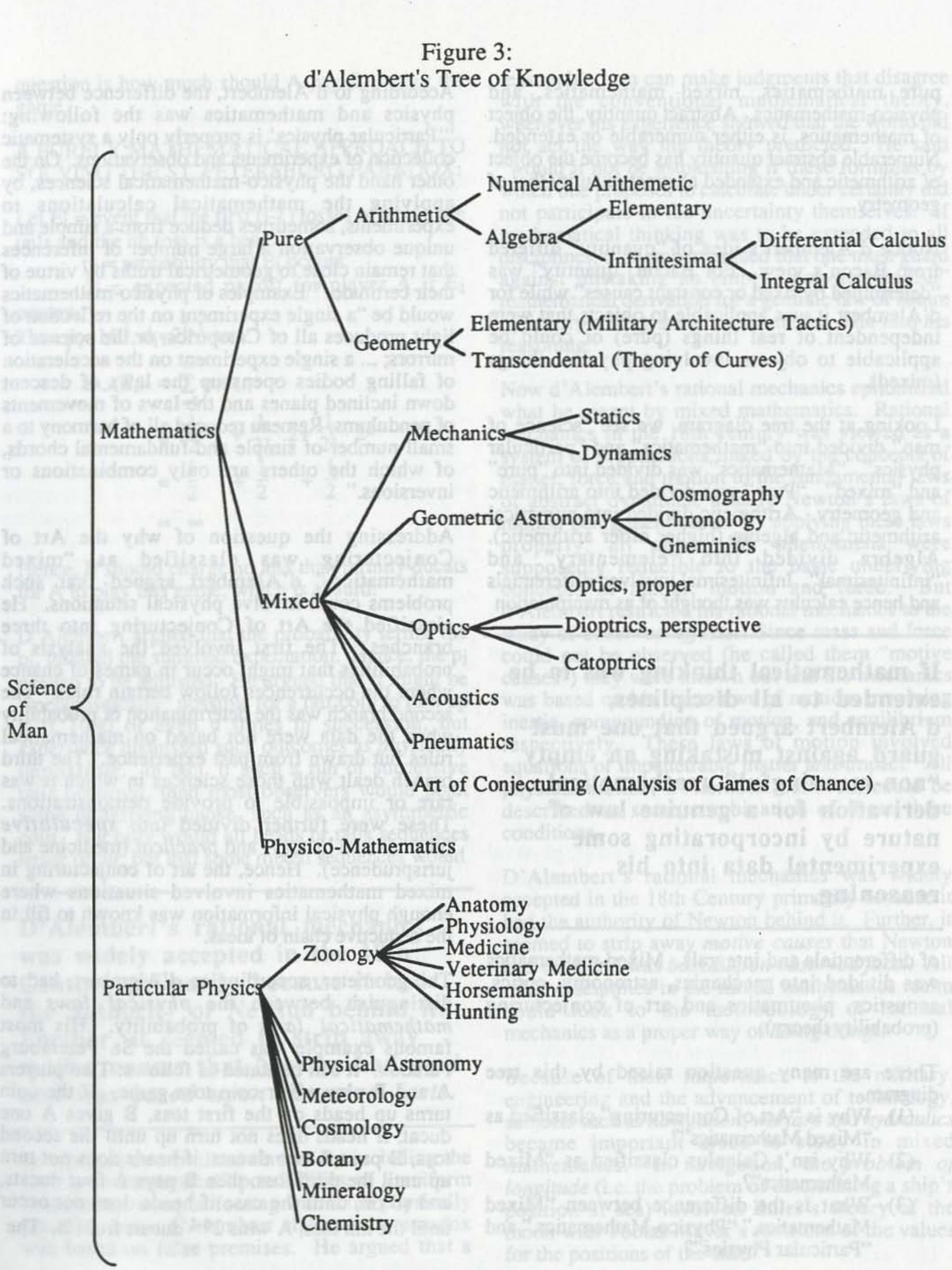
since the last time someone had tried to do such a project. We find d'Alembert's classification of mathematics in his famous *Discours Preliminaire*. (See d'Alembert's tree of knowledge).

According to d'Alembert, "quantity, the object of mathematics, could be considered either alone and independent of real and abstract things from which one gained knowledge of it, or it could be considered in their efforts and investigated according to real or supposed causes; this reflection leads to the division of mathematics into

Figure 3:
d'Alembert's Tree of Knowledge



pure mathematics, mixed mathematics, and physico-mathematics. Abstract quantity, the object of mathematics, is either numerable or extended. Numerable abstract quantity has become the object of arithmetic and extended abstract quantity that of geometry.

Notice d'Alembert's idea of "quantity" differed from Bacon's view. For Bacon "quantity" was "determined by fixed or constant causes" while for d'Alembert it was applicable to objects that were independent of real things (pure) or could be applicable to objects involving physical things (mixed).

Looking at the tree diagram, we see "science of man" divided into "mathematics" and "particular physics". "Mathematics" was divided into "pure" and "mixed". "Pure" was divided into arithmetic and geometry. Arithmetic divided into numerical arithmetic and algebra (higher order arithmetic). Algebra divided into "elementary" and "infinitesimal". Infinitesimal involved differentials and hence calculus was thought of as manipulation

If mathematical thinking was to be extended to all disciplines, d'Alembert argued that one must guard against mistaking an empty "non-meaningful" mathematical derivation for a genuine law of nature by incorporating some experimental data into his reasoning.

of differentials and integrals. Mixed mathematics was divided into mechanics, astronomy, optics, acoustics, pheumatics and art of conjecturing (probability theory).

There are many question raised by this tree diagram.

(1) Why is "Art of Conjecturing" classified as "Mixed Mathematics"?
(2) Why isn't Calculus classified as "Mixed Mathematics"?
(3) What is the difference between "Mixed Mathematics," "Physico-Mathematics," and "Particular Physics"?

According to d'Alembert, the difference between physics and mathematics was the following: "'Particular physics' is properly only a systematic collection of experiments and observations. On the other hand the physico-mathematical sciences, by applying the mathematical calculations to experiments, sometimes deduce from a simple and unique observation a large number of inferences that remain close to geometrical truths by virtue of their certitude." Examples of physico-mathematics would be "a single experiment on the reflection of light produces all of Catoptrics, or the science of mirrors; ... a single experiment on the acceleration of falling bodies opens up the laws of descent down inclined planes and the laws of movements of pendulums; Rameau reduced all of harmony to a small number of simple and fundamental chords, of which the others are only combinations or inversions."

Addressing the question of why the Art of Conjecturing was classified as "mixed mathematics," d'Alembert argued that such problems could involve physical situations. He classified the Art of Conjecturing into three branches. The first involved the analysis of probabilities that might occur in games of chance where the occurrences follow certain rules. The second branch was the determination of probability when the data were not based on mathematical rules but drawn from past experience. The third branch dealt with those sciences in which it was rare or impossible to provide demonstrations. These were further divided into *speculative* (physics and history) and *practical* (medicine and jurisprudence). Hence, the art of conjecturing in mixed mathematics involved situations where enough physical information was known to fill in the deductive chain of ideas.

The geometer, according to d'Alembert, had to distinguish between the *physical laws* and *mathematical laws* of probability. His most famous example was called the St. Petersburg Paradox. It can be stated as follows: Two players A and B play a fair coin toss game. If the coin turns up heads on the first toss, B gives A one ducat, if heads does not turn up until the second toss, B pays A two ducats, if heads does not turn up until the third toss, then B pays A four ducats, and so on, until the case if heads does not occur until the nth toss, A wins 2^{n-1} ducats from B. The question is how much should A pay B to play the game.

A PURELY MATHEMATICAL APPROACH TO SOLVING THE ST. PETERSBURG PARADOX:

Let E_i = event that the first (i-1) tosses of a coin are tails and the ith toss is a head.

p_i = probability that E_i occurs.

d_i = expected payoff for player A if E_i occurs.

Then $p_i=1/2^i$, $d_i=2^{i-1}$ and

$$E\cup(E_i) = \sum_{i=1}^{\infty} p_i d_i.$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2^2}\cdot 2 + \frac{1}{2^3}\cdot 2^2$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= \infty$$

Hence, B should charge no less than infinity ducats for A to play this game, which is absurd.

D'Alembert argued that the probability terms, the p_i, inflated the summed expectation. Hence, the p_i needed to be adjusted. He argued that it might be "metaphysically" possible for a fair coin to turn up tails 1000 or even n times in succession, but *experience* dismissed such outcomes as *physically impossible*. In fact, if such a run of successive tails occurred, then most observers would posit some underlying cause, such as an asymmetric coin. Not only would all heads or tails sequences never occur, but that some mixed sequences would be repeated two or three times. By including the mainly "metaphysical" possibilities of a uniform sequence on an equal footing with more physically plausible ones, d'Alembert claimed the paradox was based on false premises. He argued that a reasonable man can make judgments that disagree with the conventional mathematical theory, whenever his experience showed that the world did not act the way the theory predicted. He said "would it not be astonishing if these formulas by which one proposes to calculate under certainty did not participate in the uncertainty themselves. If mathematical thinking was to be extended to all disciplines, d'Alembert argued that one must guard against mistaking an empty "non-meaningful" mathematical derivation for a genuine law of nature by incorporating some experimental data into his reasoning.

D'Alembert's rational mechanics was widely accepted in the 18th Century primarily because it had the authority of Newton behind it. Further, it seemed to strip away *motive causes* that Newton had used and was *based upon observed facts*.

Now d'Alembert's rational mechanics epitomized what he meant by mixed mathematics. Rational mechanics in the 18th century was viewed as a system of propositions linked by the concepts of matter, force and motion to the fundamental laws of abstract dynamics, usually Newton's laws of motion being mentioned. By applying these laws properly, all observable phenomena were supposedly reducible to the basic underlying concepts of matter, motion and force. But d'Alembert restricted his rational mechanics to the study of *observed effects*. Since mass and force could not be observed (he called them "motive causes") they were thrown out and his mechanics was based on his three laws of motion, involving inertia, compounding of motion, and equilibrium respectively. These laws of motion involved equations of impenetrable matter and impact. All physical motion, d'Alembert believed, could be described as some combination of these three conditions.

D'Alembert's rational mechanics was widely accepted in the 18th Century primarily because it had the authority of Newton behind it. Further, it seemed to strip away *motive causes* that Newton had used and was *based upon observed facts*. All other disciplines in the mixed mathematics realm could look to the methodology of rational mechanics as a proper way of doing things.

Because of their importance to the military, engineering and the advancement of technology, subjects such as *navigation*, *warfare* and *hydraulics* became important areas to study in mixed mathematics. In navigation, the *problem of longitude* (i.e. the problem of determining a ship's position at sea) involved Euler's theory of the moon with Tobias Mayer's revisions of the values for the positions of the stars.

In *warfare*, the problem of projectile motion involved the so called quadratic theory of ballistics in combination with experimental data on the velocity of projectiles shot from a cannon.

In *hydraulics*, the problem of building waterworks and ships involved the theory of architecture and hydrodynamics with experimental results. In all of these cases, mixed mathematics was involved because all quantities used magnitudes subsisting in material bodies and interwoven everywhere with physical considerations. All subjects modeled their approach to mixed mathematics after that of rational mechanics.

V. SOME PROPONENTS OF THE TEACHING OF MIXED MATHEMATICS DURING THE NINETEENTH CENTURY

(a) Gaspard Monge

French geometers such as Laplace and Condorcet continued their advocation for mixed mathematics in the spirit of Jean d'Alembert. Laplace reiterated d'Alembert's idea that mixed mathematics involved the search for first principles and the building of a deductive chain when he said "natural phenomena are mathematical results of a small number of invariable laws." However, the major advocate for the teaching of mixed mathematics in the spirit of d'Alembert was Gaspard Monge, known today as the Father of Descriptive Geometry.

Monge was a major advocate of using physical drawings in doing geometry and an advocate of incorporating mathematics and science in an education intended to be broadly based. He was a major advocate of integrating science and mathematics into a humanistic education. Monge did not just teach mathematics, he molded future French citizens. His descriptive geometry lent itself beautifully to both the theoretical and practical sides of the curriculum because it consisted of a purely rational theory which could be translated into concrete graphic relations. This involved learning the essential skills of mechanical drawing by passing constantly from the abstract to the concrete and back again. Monge expressed this idea of mixing mathematics best in the preface of his famous *Gèomètrie descriptive*: "The second object of descriptive geometry is to deduce from the exact description of bodies all which necessarily follows from their forms and respective positions. In this sense it is a means of investigating truth; it perpetually offers examples of passing from the known to the unknown; and since it is always applied to objects with the most elementary shapes, it is necessary to introduce it into the plan of national education and make use of this geometry of the representation and determination of the elements of machines by which man, controlling the forces of nature, reserves for himself, so to speak, no other labor in his work but that of his intelligence."

In contrast to Monge's position of integrating descriptive geometry into a humanistic eduction, we have Cauchy's position that mathematics should be treated educationally as a separate, specialized discipline. According to Cauchy, "let us then admit that there are truths other than those in algebra, realities other than those of sensible objects. Let us ardently pursue mathematics without trying to extend it beyond its domain." Different fields are different, Cauchy asserted: they rest on different forms of evidence, use different kinds of arguments, and generate different kinds of knowledge. Cauchy's mathematics was an austere and rigidly circumscribed subject for which he offered no justification outside of its own integrity.

According to Cauchy, "let us then admit that there are truths other than those in algebra, realities other than those of sensible objects. Let us ardently pursue mathematics without trying to extend it beyond its domain."

As time advanced during the nineteenth century, Cauchy's view tended to win out over Monge's view. Public assertions that scientific knowledge is irrelevant to humanistic education can be found as early as 1802. The final years of Gaspard Monge epitomized this change from a holistic to separatist view of mathematics. Monge taught geometry at the Ecole Polytechnique until the restoration in 1816 at which time the school was closed for several months. What is more, he was summarily removed from his position in the Institute and another appointed to take his place. He died miserably soon thereafter.

If Cauchy's separatist view of mathematics began to prevail in French mathematics in the early nineteenth century, then one must also conclude that d'Alembert's view of mixed mathematics must be waning during this period. If mathematics is to be done separately from other disciplines, how could it be mixed with these disciplines? On the other hand, it makes perfectly good sense to talk about how a body of mathematics could be *applied* to some discipline. Since the mathematics had been already established, one could simply "lift" the needed mathematics from its total embodiment and "insert" it where it could be applied in that discipline.

(b) William Whewell

As Cauchy's approach to mathematics was winning out over Monge's approach on the European Continent, the changes in mathematics in England were far more gradual. The English had a more "exemplary" view of mathematics which tended to grow stronger, both institutionally and philosophically, well into the middle of the 19th Century. Their view was based upon Newton's approach to calculus using fluxions and to the unified Enlightenment view of science. By 1800, Newton's approach to the calculus was still being taught at institutions like Cambridge. This approach was more rigorous than the Ancient Greeks' perspective of geometry but more difficult to learn and apply to problems than the calculus of Cauchy. There was an early 19th century awakening of English interest in Continental Mathematics led by a short-lived organization of Cambridge students known as the Analytic Society (1812-1813). This group led by Charles Babbage, George Peacock and John Herschel advocated the more analytical methods of calculus using Lebnitz notation. Although this group died out, many of its aspirations had effects on how mathematics was taught at Cambridge. By the 1820s, the Cambridge mathematics exam, the Tripos was radically modified to include some of these analytic techniques. However, the most powerful theme at Cambridge still remained the *connectedness and universality of knowledge*. The person that epitomized this approach was William Whewell.

In 1800 the mathematics curriculum at Cambridge included arithmetic, algebra, trigonometry, geometry, fluxions, mechanics, hydrostatics, optics and astronomy. The focus was from Newton's *Principia*. When Whewell came along there was pressure from the Analytic Society to change the curriculum towards Continental mathematics. Whewell forged a compromise. He wanted the students to be grounded in *physical realities*—in pulley machines, and forces rather than mathematical symbols. In his textbook *Elementary Treatise on Mechanics* (1819) he placed a considerable portion of mechanics prior to any discussion of differential calculus. Today, we argue that calculus should be taught before and at worst concurrently with mechanics. Whewell thought the best way to achieve rigorous results was through *geometric-physical reasoning* and to confirm the results thus achieved by "more direct" reasoning. It was his mission to prevent the establishment of the study of abstract analysis as a discipline independent of, and as prestigious, as mixed mathematics. He accomplished this mission with his many textbooks that were primarily used during the first half of the 19th century and his influence in the writing of the Tripos exams. This extended even as late as 1848 when many changes were being considered with regard to the Tripos exam. By 1854 (James Clerk Maxwell's year) the Tripos exam consisted of a ratio of three applied mathematics problems to every two pure mathematical problems; problems calling for synthetic-geometric solutions, including Newtonian ones, made up more than 40% of the examination. Obviously the emphasis was still on

Whewell thought the best way to achieve rigorous results was through *geometric-physical reasoning* and to confirm the results thus achieved by "more direct" reasoning.

mixed mathematics, i.e. geometry over algebra, intuitive rather than abstract rigor, on detail more than generization, extensiveness more than intensiveness, on problem solving rather than mathematical processes, and on Newton rather than Lagrange.

Even in the 2nd half of the 19th century mixed mathematics did not go away. There were battles between Arthur Cayley supporting Continental Mathematics and Henry Airy supporting mixed mathematics. Only after another generation of mathematicians led by Bertrand Russell and G.H. Hardy do we clearly see a decline in mixed mathematics.

VI. THE DECLINE OF MIXED MATHEMATICS IN THE NINETEENTH CENTURY

(a) A Consequence of the French Revolution—Changing Views of Probability Theory

As mentioned earlier, the rise of military and engineering schools brought an increase in the study of mixed mathematics. Furthermore, the importance of mathematics as a method of searching for the truth was important to some Enlightenment philosophes. All of these trends collided with the tremendous upheavals of the French Revolution.

The French Revolution seemed to create a discontinuity and a rethinking of some of the trends of the 18th century. Philosophes of the 18th century stressed the importance of enlightening the individual. Society would ultimately prosper only if its citizens were enlightened individuals. This meant the individual had to master rational thinking which was based upon the reasoning of geometry (synthesis) and to a lesser extent algebra (analysis). Mathematically, this mean that psychological arguments could be mixed with mathematical technique in solving problems such as the St. Petersburg paradox and the Inoculation Problem.

The events of the French Revolution seemed to shatter this belief that "good sense" was monolithic and a constant for a selected few. Passions seemed to prevail over reason, and what was "good sense" and who practiced it was no longer so clear. The chaos and many political shifts from the Revolution in 1789 to the restoration of the Bourbons in 1814 shook the confidence out of the remaining philosophes, some of whom suggested that "good sense" is more intuitive than rational. Mixed mathematics involved the proper combination of theory with experience, but by the time of the restoration of the monarchy, these ideas seemed to some philosophers diametrically opposed.

While the philosophers of the Enlightenment thought the way to improve society was by concentrating on enlightening the individual (individual→society), the social scientists of the 19th century thought one can improve the lot of the individual by improving society (society→individual). This change of worldly views especially affected how probability theory was used. The philosophes used it by considering the rational individual while the social scientist used it in statistics to say something about the "average person with property A." For example, Quetelet used the frequentist interpretation of probability theory and the normal distribution to find the mean

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in the case of statistical regularities of data, and thus associated a number to some social trait of the average man. The average man could be assigned "a penchant for crime" equal to the number of criminal acts committed divided by the total population. In this way a set of discrete acts by distinct individuals was transformed into a continuous magnitude, "the penchant," which was an attribute of the average man. *It was the responsibility of the social scientist to apply the mathematical theory to the social chaos*; the normal distribution was *applied* to all kinds of social (and later biological) phenomena.

(b) Some Mathematical Discoveries of the 19th Century

If the rise of statistical methods affected how scholars viewed the role of mathematics in the "real world," then mathematical events occurring later in the 19th century had a further influence on this changing view of mathematics. Here I am referring to (1) the discovery of non-Euclidean geometry, (2) the discovery of non-commutative rings (i.e. Hamilton's quaternions), (3) the discovery of nowhere differentiable but everywhere continuous functions on the real line, (4) the changing relationship between geometry and mechanics.

The discovery of non-Euclidean geometry shifted the foundations of mathematics towards arithmetic and away from geometry. No longer was there an example of absolute, infallible knowledge. Furthermore, the sensory perceptions could no longer be trusted. This was especially re-iterated with the discovery of non-intuitive mathematical entities like the quaternions and nowhere differentiable, everywhere continuous functions. The truth of such propositions must rely on more formalistic arguments and less on experience.

With regard to mechanics, recall d'Alembert used his rational mechanics as a model for the other mixed mathematical sciences. His rational mechanics was kind of an extension of geometry with three axioms of motion added. However, with the discovery of non-Euclidean geometry and the rise of experimental physics in the 19th century some physicists thought that mechanics was less a branch of geometry and more one of analysis (calculus).

VII. THE CHANGING MEANING OF APPLIED MATHEMATICS TODAY

In a recent article appearing in the July, 1993 issue of the *Bulletin of the American Mathematical Society*, Arthur Jaffee and Frank Quinn discuss "Theoretical Mathematics: Toward a Cultural Synthesis of Mathematics and Theoretical Physics."

According to Jaffe and Quinn, mathematics should be divided into "theoretical" and "rigorous" mathematics. They argue that applied mathematics is generated in two stages: First intuitive insights are developed, conjectures are made, and speculative outlines of justifications are suggested. This they call "theoretical" mathematics. Then the conjectures and speculations are corrected and are made reliable by proving them. This they call "rigorous" mathematics. Later in their article, they say "pure" was used in the past instead of "theoretical" but the term "pure" is "no longer common."

Jaffe and Quinn argue that mathematicians have even better experimental access to mathematical reality than the laboratory sciences have to physical reality. They say, "this is the *point of modeling*: a physical phenomena is approximated by a mathematical model; then the model is studied precisely because it is more accessible." Today, the same person is likely to be doing "theoretical mathematics" and "rigorous mathematics" in solving a problem. This is especially true in the areas of *string theory*, *conformal field theory* and *topological quantum field theory*.

Finally, they argue that the bifurcation of mathematics into theoretical and rigorous communities has partially begun but has been inhibited by consequences of *improper speculation*. They argue that "speculative mathematics" ought to be publishable but with the stipulation that it can be acknowledged as speculative.

VIII. CONCLUSION

I wish to argue that applied mathematics should be taught "mixed" allowing for "speculative mathematics." Solving mathematical problems has always involved factoring in "experience" even if that experience could be non-intuitive. To say that some mathematics was applied to some physical problem is a distortion and misleading. This view tends to trivialize the modeling process.

"Mixing mathematics" involves breaking down a given problem into simpler parts until one arrives at "first principles." One is supposed to create a chain of truths starting from these first principles and logically arrive at a solution to the problem. It seems reasonable in today's changing mathematical world, that this chain could include speculative statements, conjectures, refutations, data accumulation, experimental strategies and even some incorrect conclusions as long as this chain is constructed in a logical progression. The student could be required in some sort of notebook to include any analytical or synthetic way of thinking involved in the solving of a problem. This may include the stripping down to "first principles" that have no obvious or apparent relevance to the original problem. It could include the use of "common experience" to make conjectures or

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refutations. It would be the opposite of the lean, economical and formalized kind of written mathematics emphasized today. Why should acceptable mathematics always be in some finished form free from dead ends and speculations? Why should acceptable mathematics be free from the motivations that lead to the theorems? Why does it have to appear from the finished product that mathematics has been applied to a problem when it was really mixed? Why do we continue to reinforce this distorted view of applied mathematics? Mathematics should be taught mixed and clearly advocated to students as being mixed.

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Footnotes

* This approach to modeling appears in Bittinger (1992), page 76. It would be an interesting study to examine to what extent the Bittinger diagram represents the presentation of modeling in elementary mathematics textbooks over the last fifty years. I am conjecturing that it would typlify other such diagrams.

** For a more thorough, scholarly explanation, including complete citations and footnotes, of the historical evolution of the term "mixed mathematics" see Brown (1991) and Brown (to appear).