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Tilings in Art and Science

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The title "Tilings in Art and Science" is a contraction of one that is longer and more descriptive: "Tilings of the Plane in Mathematics, Science, Nature, Art, and Design: A Personal View." Election to the Henderson Lectureship at Westminster College for 1989-90 was the occasion for the investigation of a topic that has long interested me and that has likewise long awaited the opportunity for deeper study. I was honored to be chosen by my colleagues and grateful to the generosity of Joseph and Elizabeth Henderson, whose endowment of the lectureship made the project possible.

One characteristic of our contemporary culture, viewed with distrust by many, is the increasing mathematization of more and more aspects of our lives. This mistrust or misunderstanding was underlined by the late C. P. Snow in his reference to "two cultures." He lamented the breakdown in communication between scientists and humanists. I attempted, through an illustrated presentation about tilings, to convince my audience that there is, in fact, a positive relationship between the abstract structure of mathematics and the sensory reality of the world in which we live and move.

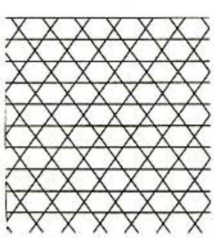
Human beings look for meaning and significance in the multitude of sensory stimuli with which they are bombarded by seeking pattern and order—organizing principles, schemes of classification, necessary relationships. Such abstractions enable them to understand, appreciate, evaluate, predict, and even shape and control certain portions of their surroundings. For example, the diagram below [Fig. 1] abstracts and idealizes the pattern of hexagons and triangles underlying the design on the ninth century Islamic bowl. The elaboration of relationships such as the one between these two pictures, in a wide variety of settings, formed the substance of the "show."

Two recurring themes in the history of thought are change and constancy. They are combined in patterns and tilings, where the theoretical urge to continue the design, and the practical need to curb and regularize it, are in dynamic tension. Sometimes a designer allows random variation to take a hand, as in the facing of a wall or building by irregular pieces of building stone [Fig. 2(a)] or in a random pattern of wood shingles. More often, regularity is imposed: a few basic shapes are chosen and used repeatedly, as in the many patterns that can be created with building bricks of the same shape [Fig. 2(b)]. We thus subject the flowing process of change to regulation, imitating the cyclic behaviors observed in nature.

A bridge to understanding these parallel regularities of pattern in design and nature is the mathematical theory of *tilings* or *tesselations*. Scientist and artist alike attempt to describe important features of the world about them—though the scientist's aim often goes beyond description to prediction and control. These can be as varied as a print by the graphic artist M. C. Escher or a structure diagram from organic chemistry [Fig. 3].

please see p.16

1(a)



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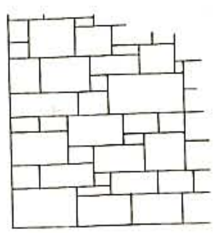

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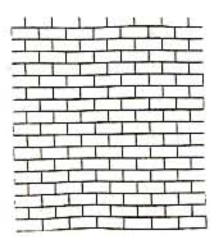
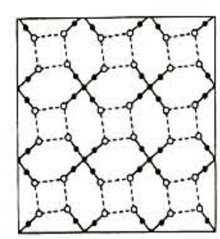


2(b)

please see p.16

3(a)

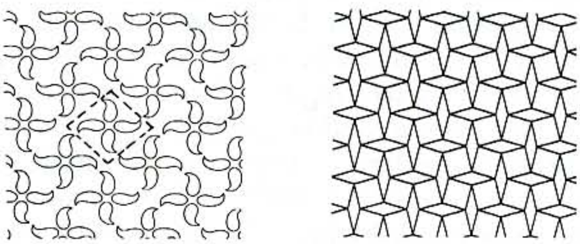


3(b)

Figure 1: Islamic bowl (a) and underlying tiling (b). Figure 2: Random stone (a) and regular brick (b).
1(a) Figure 3: Escher horsemen (a) and chemical diagram (b).

Subjecting our observations to the intellectual discipline of abstraction called mathematics deepens our appreciation of patterns and enables us to participate in creating new and better ones. Mathematical ideas can interact with all aspects of our experience.

Mathematical philosopher Philip J. Davis observes that "mathematics dreams of an order which does not exist. This is the source of its power; and in this dream it has exhibited a lasting quality that resists the crash of empire and the pettiness of small minds. Mathematical thought is one of the great human achievements. The study of its ideas, past and present, can [free] the individual . . . from the tyranny of time and place and circumstance. Is not this what liberal education is about?"



(a) (b)

Figure 4: Tiling (a) and pattern (b).

Let us distinguish between tilings and *patterns;* pattern is the more general word. When Grunbaum and Shephard were collecting materials for their definitive 1987 book on tilings and patterns, they found no rigorous technical definition of pattern in the mathematical literature; their book provides the first textbook treatment of the topic. That most of this material is new is, in their own words, "a surprising fact considering the immense amount of effort that artists and architects have expended in designing and analyzing patterns since time immemorial."

A pattern in the plane is simply a geometric design for which there is a small part whose repetitions create the whole [Fig. 4(a)]. The small prototypic part is called the *motif.* Patterns are studied and classified by considering various motions of the plane, such as rotations, and determining which of them leave the overall pattern indistinguishable from its original state.

Tilings represent a special case in which the plane is partitioned, without gaps or overlaps, into sets called *tiles* [Fig. 4(b)]. We will be mostly concerned with the case in which one or a few distinct shapes, the *prototiles,* are used to generate the entire configuration. Symmetries of these configurations are an important way of describing and classifying them.

There are several classical tiling problems that don't quite fit this description, yet are similar in nature. Though some of these have a "game" or "puzzle" character, they should not be taken lightly. Many mathematical puzzles are the key to understanding significant related applications; many have turned out in the long run to be more useful than their creators imagined. Such a classical puzzle is that of "tiling the crippled chessboard with dominoes." A standard 8 by 8 chessboard can easily be tiled with 32 dominoes, where the size of the domino is just that of two chessboard squares. If two opposite corners of the board are removed, however, it is no longer possible to tile it with the dominoes, even though its 62 square unit area would seem to accommodate precisely 31 of them.

A mathematical proof of this is illuminating and may suggest the appeal of this subject to those with a logical bent of mind. Any domino on the board must cover two adjacent squares, hence one square of each color. The 31 dominoes would thus cover 31 light and 31 dark squares. But the crippled chessboard, because the opposite comers which were removed had the same color, has 30 of one color and 32 of the other!

Like many mathematical demonstrations, this little argument has the virtue of settling the question without recourse to any tedious exploration of a large number of "nearly correct" solutions. For simply failing to find a solution by experiment isn't very convincing—we may just not have tried hard enough!

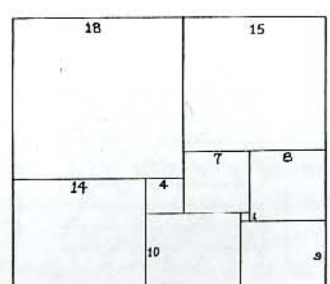


Figure 5: Stein decomposition.

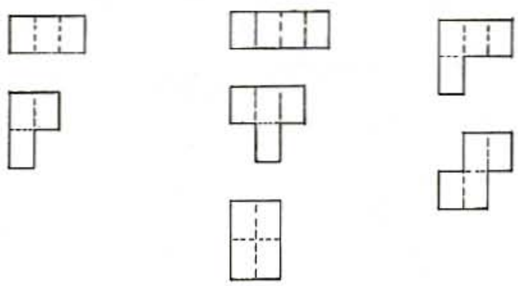


Figure 6: Trominoes (left) and tetrominoes (center and right).

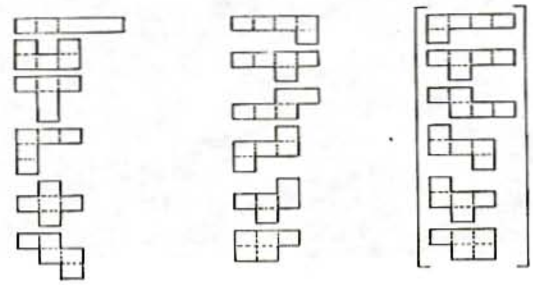


Figure 7: Pentominoes.

A related kind of question is that of tiling a given rectangle with squares that are all of different sizes. This is the antithesis of the notion of regularity mentioned earlier, but is another interpretation of the term tiling. The problem turns out to be difficult: not all rectangles can be tiled this way. There are restrictions on the dimensions of the rectangle as well as on the squares used to tile it. A minimal example is shown in Fig. 5 (from Stein's *Man-Made Universe)*. The rectangle is 33 by 32; the number in each square is the length of its side. (These tilings have interpretations as the equilibrium states of certain electrical circuits!)

Mathematical puzzles and recreations, especially those of a geometric nature, have played a significant role in the evolution of the subject. In addition to providing pleasure and diversion, mathematical puzzles and recreations have helped to develop the geometric intuition and insight of many a future geometer. They have stimulated creative and original contributions to the field, not only from mathematical professionals but from "amateurs" as well, that is, those with only modest training in formal mathematics.

For example, an interesting instance of a finite tiling is provided by an innovative jigsaw puzzle, called the "Shmuzzle," all of whose 168 tiles are alike. A typical piece is shaped like a lizard with six extremities: four legs, a head, and a tail. Since the outside border of this tiling is irregular, the puzzle maker provides a border into which to fit the pieces.

Some 25 years ago the mathematician Solomon Golomb generalized the familiar domino, made of two equal squares, to polygonal tiles called *polyominoes*. These can consist of three squares (triominoes or trominoes), four squares (tetrominoes), five squares (pentominoes), and so on. Fig. 6 shows the two trominoes and five tetrominoes; Fig. 7 illustrates the twelve pentominoes.

If we count mirror images as distinct—not allowing the figures to be flipped over—there are then seven "one-sided" or "oriented" tetrominoes and 18 "oriented" pentominoes. (The extra asymmetric figures are shown in brackets.) Higher order polyominoes have been studied, but to date there is no formula known that predicts the number of polyomino forms

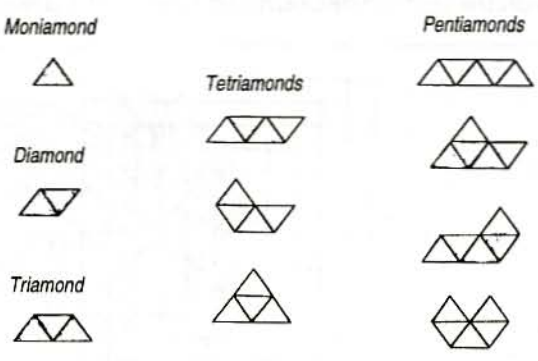


Figure 8: Moniamonds, diamonds, triamonds, tetriamonds, and pentiamonds.

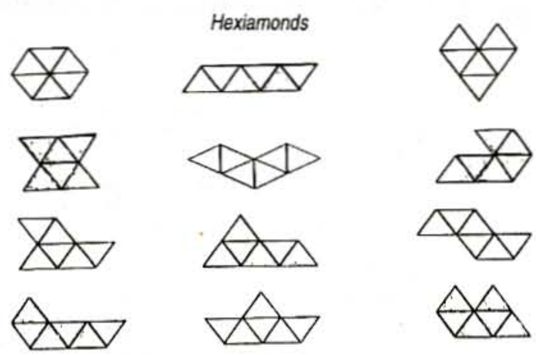


Figure 9: Hexiamonds.

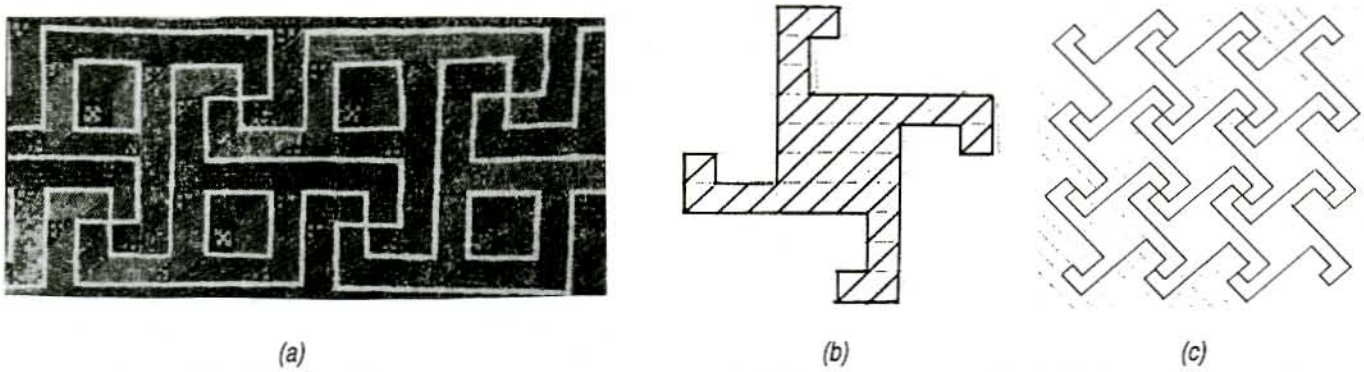


Figure 10: Ravenna mosaic (a), underlying figure of Ravenna mosaic (b), and tiling of the plane using Ravenna underlying figure (c).

of a given order. *Polyiamonds* are formed in a similar way using equilateral triangles instead of squares. A moniamond is one such triangle; a diamond consists of two. (The double-form is the source of the idea—as well as the terminology—for both polyominoes and polyiamonds!) There is only one triamond, but there are three tetriamonds, four pentiamonds, and a dozen hexiamonds (not counting the oriented or asymmetric forms) [Figs. 8 and 9].

Which polyominoes and polyiamonds are prototiles for plane tilings? Can copies of any one of these figures, laid out appropriately, be used to cover the plane periodically? It turns out that many of these figures do tile the plane, but some do not. There are many unsettled questions!

Mosaic is an art form closely related to tiling. A finite region is covered with small shapes, usually polygonal, but the requirement that there be no gaps between the tiles is not strictly observed. In addition, the tiles are colored. (This chromatic distinction can be made for strict tilings, too; the more colors allowed, the more complicated becomes the problem of classification.) Most mosaics are approximate tilings: the artists' creativity and realism have overruled strict structure.

A mosaic from Ravenna, the fretwork design from an alcove of the Galla Placidia Mausoleum [Fig. 10(a)], represents a very different meeting of creativity and structure. The shape outlined in white resembles a cross between a starfish and a swastika, but each example has only three arms instead of four.

The underlying regular figure [Fig. 10(b)], has fourfold rotational symmetry. Although each instance in the mosaic is missing an extremity in order that the design be confined to a linear border, the full design can easily be extended to a tiling of the plane [Fig. 10(c)]. Note how four tiles cluster together some restrictions commonly adopted when considering tilings of the plane.

The first of these is that the number of distinct prototiles be finite, ordinarily quite few in number. A second is that each tile itself be finite in extent. That these are not necessary can be seen in the examples in Fig. 11. The tiles on the left are unbounded strips. The tiling on the

Figure 11: Unbounded tiles (a) and infinitely many prototiles(b).

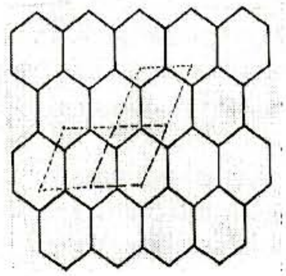


Figure 12: Hexagonal tiling with parallelogram motif.

right consists of infinitely many strips, of differing, ever-increasing size. While such tilings are not without interest, we won't pursue them further here.

As suggested earlier, the symmetries of a tiling and of its constituent prototiles are characteristics important in the description and classification of plane tilings. These are rigid motions of the plane after which the outline of the tiling appears identical to its original form.

One such motion is *translation:* slide the pattern in a fixed direction a specific distance, always parallel to its starting position. After a certain fixed distance, it coincides with its original configuration. Most of the tilings we'll look at are *periodic:* there are two distinct directions and distances in which translations will bring about coincidence. This means we can find a parallelogram whose contents form a motif for the overall pattern. The motif sometimes consists of fragmented copies of the prototiles [Fig. 12].

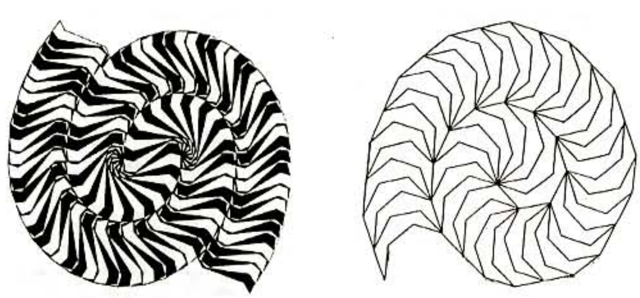
Many interesting tilings are *aperiodic*, however; artistic examples are provided by "spiral" tilings [Fig. 13]. The tiling on the left, constructed from enneagons, is due to Heinz Voderberg. Marjorie Senechal created the tiling on the right from concave heptagons.

Other symmetries are described in terms of *rotations, reflections*, and *glide-reflections*. These can be demonstrated most effectively using transparent models and other dynamic visual aids.

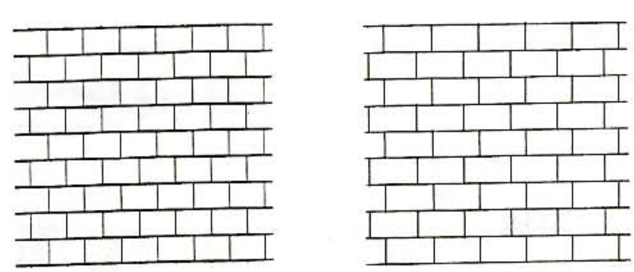
A further characteristic used in classifying tilings is that of being "edge-to-edge." This requirement limits the number of possibilities and makes the mathematical treatment easier. Yet many tiling patterns are not edge-to-edge. A familiar example is the usual pattern in which bricks are laid to face a building. Just by varying the offset, with no change in the shape of the prototile, an infinite number of distinct patterns is possible. In Fig. 14(a) the overlap is half a brick; in Fig. 14(b) it is one-third of a brick.

The examples mentioned so far may begin to convince you that tiling patterns can be found all around. Near-tilings appear in nature, while their imitations and idealizations abound in art and design. Once you begin to look for tilings, you will find them everywhere!

The pattern of patches on the hide of the giraffe provides a crude example of an irregular tiling; it is clearest on the reticulated giraffe. The cracks in drying mud often form similarly reticulated patterns, as do gelatinous preparations of tin oil, rock formations like the Devil's Postpile, packings of soap bubbles, and the pattern of cracks formed by the shrinking of plaque. The scales of fishes form tiling-like patterns. That reptiles provide illustrations of tiling patterns in their scales and skins is not surprising. We may be startled, though, to find similar patterns in the tail of the beaver and the paw of the mole. Additional examples of the



13(a) 13(b)



14(a) 14(b)

Figure 13: Vodeberg spiral (a) and Senechal spiral (b). Figure 14: Bricks with one-half brick offset (a) and one-third brick offset (b).

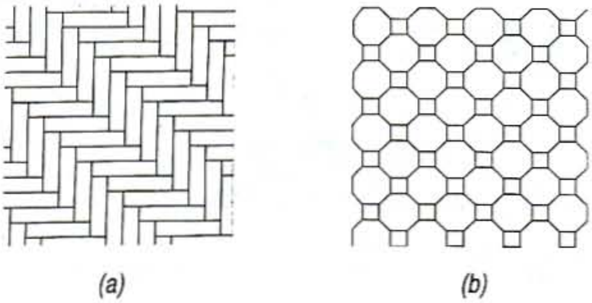


Figure 15: Herringbone bricks (a) and squares with octagons (b).

regular hexagonal tiling are found in cross-sections of the honeycomb and the nest of the paper wasp.

The study of crystallography has a close relationship with tiling. Although crystals are three-dimensional, suggesting a generalization of the tiling idea into space that is outside the scope of this discussion, their cross-sections and projections lead to configurations with tiling pattems.

Taking a cue from some of these natural phenomena, but adding creativity and the ability to make accurate copies of the motif to generate precise tiling pattems of any size, the human designer and artist have incorporated such elements into a wide variety of settings, using many different materials. To the underlying geometric pattern are often added the further dimensions of color and texture.

At a practical everyday level we see tilings in the roofs of the buildings in which we live and work, whether in asphalt shingles with an "anvil pattern," slate roofs arranged like diamonds, or in real old-fashioned ceramic roof tiles. The common postage stamp is another source of patterns for the student of tilings, since the sheets of which they are a part are tiled by the stamps [Fig. 27]. By far the most common pattern is that of simple squares or rectangles, but several countries have issued stamps shaped like triangles, trapezoids, parallelograms, and pentagons.

Earlier we mentioned pattems in brick facings. Bricks are used for walks and malls, too, as we see in Fig. 15, with a criss-cross pattern of rectangles on the left, a mixed tiling of octagons and squares on the right.

Glimpses of geodesic domes and some of nature's near-tilings may have inspired mathematician Doris Schattschneider, herself an expert on mathematical aspects of Escher's work, to use Escher designs to tile the surfaces of the five regular *or platonic* solids, three-dimensional analogs of regular polygons, having all edges, faces, and angles equal. These imaginative combinations of artistic creativity and mathematical regularity must be seen to be appreciated. Using tilings whose prototiles represent reptiles (is this a "rep-tiling"?), fishes, bats, lizards, shells, and starfish, Schattschneider and her colleague Wallace Walker present us with the tetrahedron, cube, octahedron, dodecahedron, and icosahedron resplendently clothed in their new "Escher prints."

This image of clothing leads naturally to the observation that many textiles are decorated with patterns and tilings. Somewhere between the sublime near-tessellations of a seventeenth century Persian carpet and the mundane (if not ridiculous) diamond tiling on a sweater in a fashion ad, are the colorful designs of quilts. Those of us living in the midst of Amish country, of course, find this no surprise. The designs vary from the relative simplicity of the Bow Tie pattern to the complexity of Pierced Star and Sunburst. The complexity results both from the intricacy of the underlying tiling and the further variations resulting

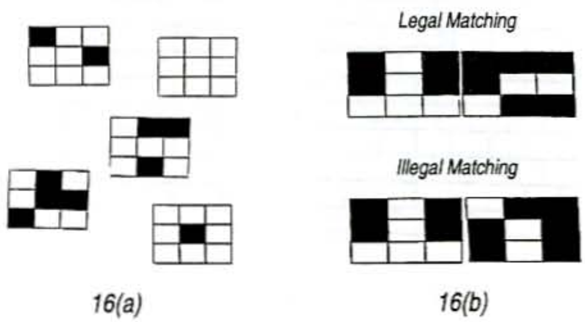


Figure 16: Novi tiles (a) and matching rules (b).

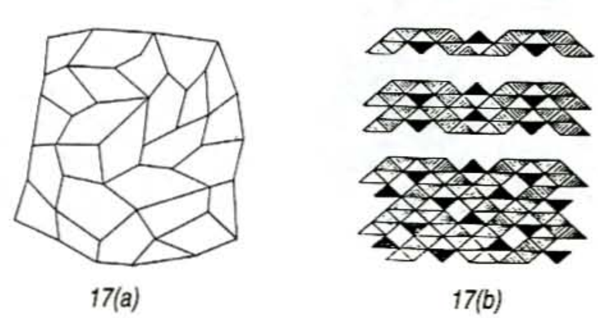
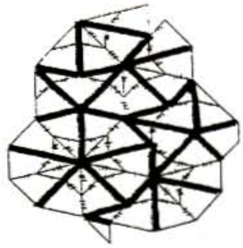


Figure 17: Irregular "glass" tilings (a) and crystal tilings (b).

18(a)

18(b)



19(a)

19(b)

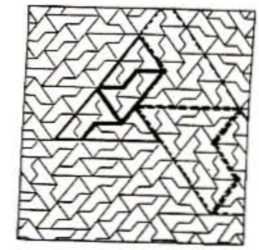


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Figure 18: Penrose kites and darts (a) and Penrose rhombi with matching (b).
Figure 19: Inflation (a) and deflation (b) of Penrose kite and dart tiling. Figure 20: Polyamond sphinxes.

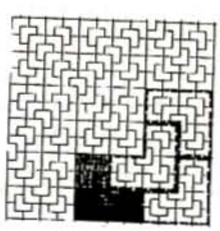
from the use of color or pattern within the tiles. Like the Escher-covered polyhedra, these must be seen in "living color" to be appreciated fully.

Recently a game has appeared called Novi, described by its creators as "a game of visual intelligence." Its 256 tiles, colored on both sides, represent the 512 possible ways of coloring either white or black the nine small squares of a square enneomino [Fig. 16(a)]. The tiles may be used to construct specified figures or played on a game board according to various sets of rules. A feature of all the games and pastimes associated with Novi is that tiles are to be placed next to one another only if their edge colorings match; Fig. 16(b) shows examples of a legal match and an illegal one.

This matching rule is similar to the matching rules imposed on *Penrose tiles*. In the 1970s, British mathematical physicist Roger Penrose was intrigued by tilings that were not quite periodic, not quite random. They exhibited the five-fold symmetry strictly forbidden by the crystallographers (apparently related to the fact that regular pentagons won't tile the plane). It wasn't until the discovery in 1982 of an icosahedral quasicrystalline phase of aluminum-manganese alloy that this pentagonal symmetry appeared convincingly in nature. Poised somewhere between the disorder of a glass, represented by the tiling by random polygons in Fig. 17(a), and the regimentation of a crystal in Fig. 17(b), these quasi-crystalline substances seem right in step with the recent emergence of the study of chaos as a scientific discipline: the remarkable discoveries, as David Eck puts it, of "unpredictability without randomness . . . [and] pattern without determinism."

Penrose's most interesting tiles, the kites and darts, must be matched according to rules enforced by corner labels or by matching colored arcs; these seem to reflect rules of chemical structure in the quasi-crystalline state of matter. Tiling by kites and darts [Fig. 18(a)], and its companion based on two differently shaped rhombuses [Fig. 18(b)], tantalize the physicist, intrigue the mathematician, and stimulate the artist to delight the eyes of all!

Penrose tilings are related to another topic of current interest, fractal geometry and sets of fractal or fractional dimension. Penrose tilings, like many fractals, exhibit *selfsimilarity*—identical structure at different scales of magnification. Tiles of a kite-and-dart configuration can be broken down into smaller versions or combined into larger ones [Fig. 19]. Other tilings have this property, too. The "sphinx-like" tile, repeated in Fig.



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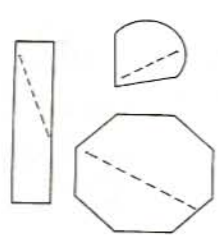
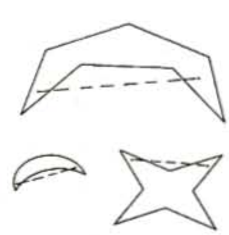
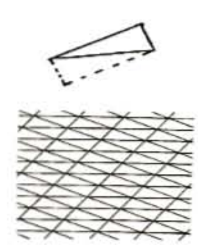


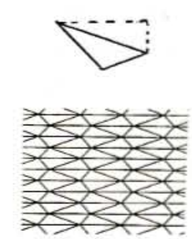
22(a)



22(b)

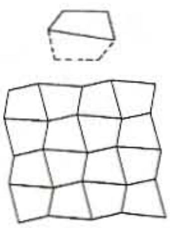


23(a)

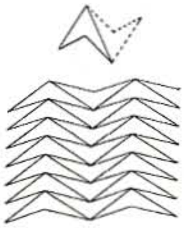


23(b)

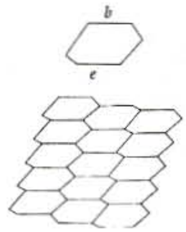
Figure 21: Triamond cells. Figure 22: Convexity (a) and non-convexity (b). Figure 23: Distinct tilings by the same triangle.



24(a)



24(b)



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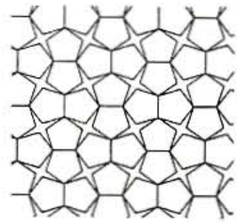
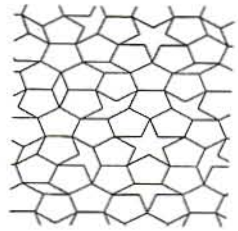


26(a)



26(b)

Figure 24: Quadrilateral tilings, convex (a) and non-convex (b).
Figure 25: Tiling by Type 1 hexagon. Figure 26: Tiling by pentagons (a) and stars (b).

20 at several scales, is just a hexiamond; a self-similar pattern is built from a tromino in Fig. 21.

A question about tilings that has intrigued both professionals and amateurs concerns the possible plane tilings with just one polygonal prototile—with the restriction that the tiles be *convex*. Convexity means that the boundary always bulges outward: there are no "dents" [Fig. 22]. In other words, the straight line path between any two points of the set remains entirely within the set.

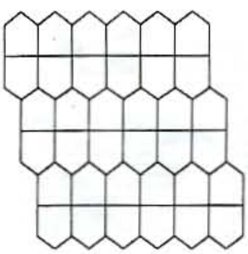
We know, of course, that we can tile with equilateral triangles, squares, and regular hexagons; these are the only regular edge-to-edge tilings. One can, in fact, tile with any triangle, and in more than one way [Fig. 23]. It's also possible to tile the plane with any quadrilateral [Fig. 24]. There are no restrictions on the relative sizes of the sides or on the angles: the quadrilaterals needn't even be convex!

For hexagons the situation is more complex. We've seen several examples of tilings by regular hexagons, but irregular ones work only under special conditions on sides and angles. It was shown by Reinhardt in 1918 that there are only three distinct types of hexagons which tile the plane. Each of these types is characterized by certain restrictions. The first, for example, has top side *b* and bottom side *e* equal in length; in addition, the three angles at the left end must add up to 360 degrees, as must the three at the right end. Fig. 25 illustrates these conditions and shows a tiling by a hexagon of this type.

Tiling by convex heptagons was believed for a long time to be impossible. No formal proof was written anywhere, and the fact was referred to as part of the "folklore" of the subject. An elementary, if tricky, proof was supplied by Niven in 1978. He actually proved the impossibility for 7 or more sides!

The alert reader will have noticed the omission of the case n = 5. Can we tile with pentagons? For regular pentagons the answer is no, though regular pentagons can be combined with other prototiles to produce interesting tilings like those in Fig. 26.

For many irregular pentagons, however, tiling is easy. The typical shape of a house drawn by a young child, for example, tiles as shown in Fig. 27(a), while the lopsided version of this pentagon [Fig. 27(b)] can also be used to tile. Possibly because its form suggests the



27(a)

please see p. 16

27(b)

28

29(a)

29(b)

Figure 27: Drawings of houses (a); stable-shaped stamps (b).
Figure 28: Tiling by pentagon species 10 (M. Rice). Figure 29: Escher tiling sketches from the Alhambra.

outline of a stable, it was chosen as the shape for a set of Maltese Christmas stamps in 1966.

Experience with hexagons suggests that there may be restrictions on pentagons as well. Indeed, the two examples above are special, since each contains two right angles and a pair of parallel and equal sides. The Reinhardt who classified hexagons also listed five types of pentagons that work and claimed that his list was complete, but in 1968, R. B. Kershner published three more types. He too claimed completeness for the now expanded list, but it was more "folklore"—not formally established. An engineer named Richard James produced a ninth one in 1975, stimulated to study the matter when the taxonomy of Reinhardt and Kershner was published by Martin Gardner in the Mathematical Games column of *Scientific American* earlier that year. This was not the end of the story, however.

Marjorie Rice, a San Diego housewife and mother of five, whose formal mathematics training ended with high school "general math" but whose informal training included years of reading Gardner's columns in *Scientific American*, devised her own scheme of classification and came up with type 10 in February of 1976, types 11 and 12 in December of 1976, and type 13 in December of 1977. The experts were amazed! Fig. 28 shows a tiling by her type 10. Its not quite parallel sides have a rather disconcerting effect, but it is a periodic tiling!

So the list stands at thirteen, but the question is still open. At this point not even the experts claim to know whether the list is complete. Perhaps some reader of this account will discover species number fourteen!

The artistic imagination leads one to seek more complex and interesting prototiles than just polygons.The Dutch graphic artist M. C. Escher was fascinated by tilings. In 1936 he made sketches of a number of Islamic designs from Moorish Spain [Fig. 29].

Ernest Ranucci and Joseph Teeters were so intrigued by Escher's drawings that they wrote a book entitled *Creating Escher-type Drawings*. While they produced some amusing efforts (for example, tilings by football players and by St. Bernard dogs), they didn't seriously challenge Escher, the master. From drawings of those Andalusian tiles in the Alhambra, Escher went on to lizards, angels and devils, fish and birds, and many other memorable designs. New York art publisher Harry Abrams has created a stunning collection of wrapping paper designs, colorfully continuing Escher's tilings into works of art too lovely to use on any package.

The art of Islam has been rich in geometric design because of Mohammed's prohibition against representing the human figure. The variety and inventiveness of these can be sampled by leafing through books on Islamic art. One will find such examples as a tiling by three species of octagon (two of them non-convex!) in a panel from a mosque at Isfahan and an intriguing design of interlocking arrows in a column from a tomb in Maragha. (A similar pattern appears in a nineteenth century French graphic by Cahier and Martin.)

The diversity and beauty with which these many tilings are executed in the world about us, both natural and man-made, together with the ingenuity of their neatly dovetailed designs, can provide both stimulation to the intellect and refreshment to the spirit. As you move about in your own world, be alert for tilings all around you—and enjoy!

POSTSCRIPT

In collaboration with my photographer/student assistant Mark Tanner, I assembled a library of approximately five hundred slides illustrating tilings in architecture, science, philately, Islamic art, and other areas. Eighty pairs of these formed the central visual vehicle for the Westminster Henderson Lecture and for subsequent versions of the lecture given in the US, New Zealand, and Australia. The narrative above conveys the structure of the presentation within the limitations of much simpler illustrations.

FOR FURTHER READING

COMAP, "Tilings," Chapter 22 of *For All Practical Purposes*, Third Edition (New York: Freeman, 1994), 693-722

Gardner, Martin, *Penrose Tiles to Trapdoor Ciphers* (New York: Freeman, 1989), 1-29

Grunbaum, Branko, and G. C. Shephard, *Tilings and Patterns: An Introduction* (New York: Freeman, 1989)

Locher, J. L., ed., *The World of M. C. Escher* (New York: Abrams, 1971)

Martin, George E., *Polyominoes: A Guide to Puzzles and Problems in Tiling* (DC: MAA, 1991)

Nelson, David R., "Quasicrystals;" *Scientific American, 255* (No.2, Aug. 1986), 42-51

Ranucci, E. R., and J. L. Teeters, *Creating Escher-type Drawings* (Palo Alto: Creative Publications, 1977)

Schattschneider, Doris, and Wallace Walker, *M. C. Escher Kaleidocycles* (Corte Madera, CA: Pomegranate Artbooks, 1977)

Senechal, Marjorie, and Jean Taylor, "Quasicrystals: the view from Les Houches"; *Mathematical Intelligencer, 12* (No. 2, 1990), 54-64

Seymour, Dale, and Jill Britton, *Introduction to Tesselations* (Palo Alto: Dale Seymour Publications, 1989)

Stein, Sherman K., "Tiling" (Chapter 5) and "Tiling and Electricity" (Chapter 6) in *Mathematics: the Man-made Universe*, Third Edition (San Francisco: Freeman, 1969), 77- 122

von Baeyer, Hans C., "Impossible Crystals"; *Discover, 11* (No. 2, Feb. 1990), 68-78

PICTURE CREDITS

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Fig. 1(a): Plate VII,B (Fig. 246), *Early Islamic Ceramics*, Helen Philon (Islamic Art Publishers, 1980; ISBN 0-85667-698-7).

Fig. 3(a): Plate 137, *World of M. C. Escher*, J. L. Locher, ed. (NY: Abrams, 1971; ISBN 0-81090107-2).

Fig. 3(b): adapted from Fig. 99, p. 223, *Organic Crystals and Molecules*, J. Monteath Robertson aca: Cornell Univ. Press, 1953).

Fig. 4(a): adapted from Fig. 4, p. 8, of *Contemporary Crystallography*, Martin J. Buerger (NY: McGraw-Hill, 1970).

Fig. 10(a): from "Fifteen Centuries Later, Ravenna's Mosaics Still Glow," Robert Warnick, in *Smithsonian*, Jan. 1990, p. 65.

Fig. 13(a): Fig. 37 p. 4, *Penrose Tiles to Trapdoor Ciphers*, Martin Gardner (NY: Freeman, 1989; ISBN 0-7167-1986-X).

Fig. 13(b): Fig. 16, p. 20, of "A Brief Introduction to Tilings," Marjorie Senechal, in *Introduction to the Mathematics of Quasicrystals*, Marko Jaric, ed. (San Diego: Academic, 1989; ISBN 0-12-040602-0).

Fig. 16: author; *Novi is* a registered trademark of the R/L Group, Cambridge, Massachusetts.

Fig. 17(b): Fig. 1, p. 80, of *Crystal Chemistry of Large-Cation Silicates*, A. N. Belov (NY: Consultants Bureau, 1963).

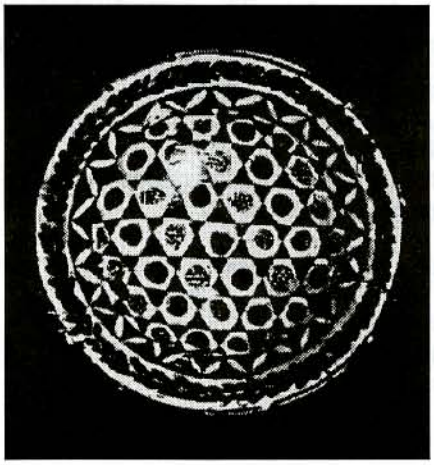
Fig. 18(b): adapted from Fig. 10.3.18, p. 543, of *Tilings and Patterns*, Grunbaum and Shephard (NY: Freeman, 1987; ISBN: 0-7167-1193-1).

Fig. 26(a): Fig. 2.5.4(q), p. 85, of *Tilings and Patterns*, Grunbaum and Shephard (NY: Freeman 1987; ISBN: 0-7167-1193-1).

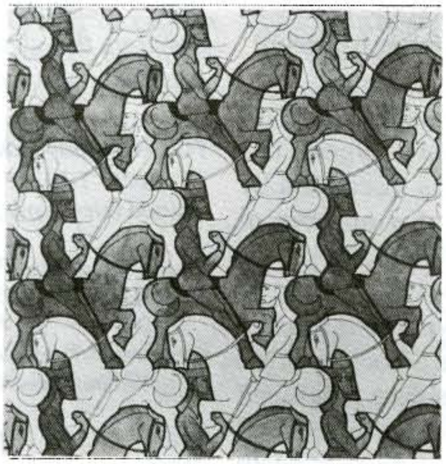
Fig. 26(b): adapted from Fig. 10.3.3, p. 532, of *Tilings and Patterns*, Grunbaum and Shephard (NY: Freeman, 1987; ISBN: 0-7167-1193-1).

Fig. 28: adapted from Fig. 6, p. 148, "In Praise of Amateurs," Doris Schattschneider, in *The Mathematical Gardner*, David A. Klarner, ed. (Boston: PWS, 1981; ISBN: 0-534-98015-5).

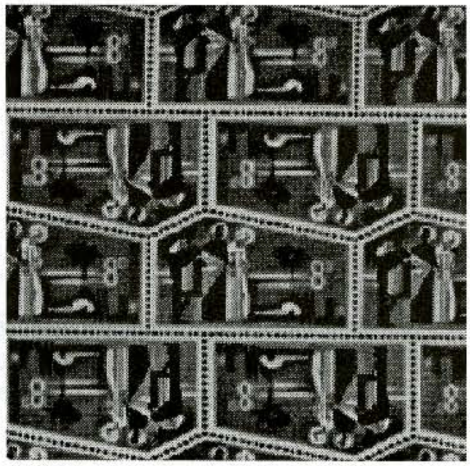
Fig. 29(a): tri-arrows from Plate 86 in *World of M. C. Escher*, J. L. Locher, ed. (NY: Abrams, 1971; ISBN 0-8109-0107-2).



1(a)



3(a)



27(b)

Figure 1(a): Islamic bowl. Figure 3(a): Escher horsemen. Figure 27(b): Stable-shaped stamps.