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Symmetry: A Link Between Mathematics and Life

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INTRODUCTION

In mathematics, certain basic concepts, such as symmetry and infinity, are so pervasive and adaptable that they can become elusive to the student. Understanding these concepts and the tools for studying them is often a long process that extends over many years in a student's career. Students first see infinity appearing as the potential infinite inherent in the positional number system, then implicit in plane geometry, and eventually underlying all of calculus and analysis. Students begin to use symmetry with commutativity and associativity in arithmetic, making more use of it in Euclidean geometry and plane geometry, and may eventually see it in terms of transformation groups. Nevertheless, it is natural to want to teach these concepts in their full value from the very beginning. This paper will describe how I have been introducing students in a general education geometry course to the concept of symmetry in a way that I feel gives them a comprehensive understanding of the mathematical approach to symmetry.

WHY TEACH SYMMETRY?

Symmetry is found everywhere in nature and is also one of the most prevalent themes in art, architecture, and design in cultures all over the world and throughout human history. Symmetry is certainly one of the most powerful and pervasive concepts in mathematics. In the *Elements*, Euclid exploited symmetry from the very first proposition to make his proofs clear and straightforward. Recognizing the symmetry that exists among the roots of an equation, Galois was able to solve a centuries-old problem. The tool that he developed to understand symmetry, namely group theory, has been used by mathematicians ever since to define, study, and even create symmetry.

Students are fascinated by concrete examples of symmetry in nature and in art. The study of symmetry can be as elementary or as advanced as one wishes; for example, one can simply locate the symmetries of designs and patterns, or one use symmetry groups as a comprehensible way to introduce students to the abstract approach of modern mathematics. Furthermore, the ideas used by mathematicians in studying symmetry are not unique to mathematics and can be found in other areas of human thought. By looking at symmetry in a broader context, students can see the interconnectedness of mathematics with other branches of knowledge.

For these reasons, many mathematicians today feel that the mathematical study of symmetry is worthwhile for general education students to explore.

A LINK BETWEEN SYMMETRY AND LIFE

The central idea in the mathematical study of symmetry is a symmetry transformation, which we can view as an isomorphism that has some invariants. For example, a symmetry transformation of a design in the plane is an isometry that leaves a certain set of points fixed as a set. I would like students to realize that this concept of symmetry transformation, as abstract as it may appear, can be connected to ideas that may seem more central to a view of life as a whole; for this, I introduce a verse from the Bhagavad-Gita.

In the Bhagavad-Gita, Lord Krishna lays out the complete knowledge of life to his pupil Arjuna, just as a great battle is about to begin. This work has long been appreciated for the great wisdom that is expounded in just a few short chapters. A verse that seems to me to capture the essence of the mathematical study of symmetry is part of Krisna's explanation of the field of action (chapter 4, verse 18, [1]):

> *He who in action sees inaction*
> *and in inaction sees action is*
> *wise among men. He is united, he*
> *has accomplished all action.*

How is this related to symmetry? A geometric figure that we wish to study is usually given as a set of points existing in some ambient space. For example, a tiling pattern may be given as a collection of line segments in the plane. A symmetry transformation can be regarded as "action" and invariants can be regarded as "inaction." We begin with a non-dynamic situation (the set of points of the tiling pattern sitting in the plane) and then find some dynamism (the symmetry transformation). Thus, in inaction, we see action. But a symmetry transformation is not just any action; it must leave the pattern (as a set of points) invariant. Thus, what is important to us is that in this action (the transformation), we are able to see inaction (the invariance of the set of points making up the pattern).

This is the seed of all that I want students to know about symmetry: action and inaction, a transformation and its invariants, what changes and what stays the same.

With this, the students gain a unifying perspective on the concept of symmetry that can help them understand it initially and that can later help them simplify and unify all the occurrences of this concept as they are met and eventually understand symmetry groups, invariants, and so on. This theme can also help students connect all instances of symmetry that they have already seen to this one unifying perspective. For example, in the commutative and associative properties of arithmetic, the positions of the numbers or parentheses change, but the answer does not change. For a tiling, the Euclidean plane can be rotated, reflected or translated in certain ways, but the pattern remains the same. A knot can be moved and redrawn, but its Conway polynomial is invariant, and so on.

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This verse from the Bhagavad-Gita not only captures the essence of symmetry, it also helps students understand the importance of invariants wherever they might see them. In his commentary on this verse, Maharishi Mahesh Yogi [1] explains that "in action sees inaction" means that one sees the nonchanging unmanifest absolute silent level of pure consciousness underlying the normal activity of thinking, perceiving, and acting. This silent level of life is the source of the active levels of life; it is subtler and more abstract than the active levels, but more powerful and more important. Elsewhere, Maharishi explains this using an analogy of the ocean. The ocean is silent at its depths and the dynamism of the waves is just the natural expression of the silent levels; the silent, nonchanging level is more fundamental. Thus, it is the invariants of a transformation that will be useful to us, even though at first they may seem difficult to grasp because of their subtlety or abstraction. With this perspective, whenever we see a transformation, our first question is, "What are the invariants? What doesn't change?"

For students at Maharishi University of Management, this understanding takes on a very personal meaning in terms of their practice of the Transcendental Meditation technique, which allows the active thinking mind to settle down to the silent, nonactive state of consciousness at its source. In their own experience, they see that their consciousness has two aspects, active and silent, and that the silent level is more fundamental and more powerful than the dynamic level. In a very concrete way, they are able to connect the ideas of symmetry transformation and invariants to their own personal experience.

TEACHING SYMMETRY

Students come to mathematics with rather limited ideas of symmetry; frequently the word symmetry is interpreted to mean "bilateral symmetry" and nothing more. Nevertheless, they will have seen symmetry in many forms already: nature, manufactured objects, art and architecture, and even in mathematics (commutativity, circles and squares, odd and even functions, and so on). It is good for students to have an understanding of symmetry that includes all the examples that they have seen and lays a foundation for further study. I want to introduce them to the idea of symmetry transformation, even though they may not know what a function is, so that they will remember it, feel that it is important, and be able to make some use of it. Students should realize that symmetry locates some underlying property that may be more abstract and less obvious but is more unifying and more discriminating. I also want students to have some insight into why symmetry is attractive and aesthetically appealing to us.

Using the verse from the Bhagavad-Gita as a guide, symmetry can be learned in a unifying way that students seem to enjoy.

We begin with a discussion of what symmetry is, recording some of the students' points on the board. Then we examine some finite designs from the artwork of different cultures and revise our notions of symmetry based on the fact that these designs should come under our definition of symmetry. To motivate this discussion, I bring up the idea that mathematics needs a precise definition that can allow us to definitively say whether something is symmetric or not and that a good definition will also help us to study objects in terms of the property.

Here, the idea of symmetry transformation is introduced. We look at some of the designs and find rotations and reflections and see that a rotation followed by a rotation is another rotation, a reflection followed by a reflection is a rotation, and the composition of a reflection and a rotation, in either order, is a reflection. Further investigation reveals the fundamental properties of finite designs: (1) A pattern can have only rotations, but not only reflections and (2) If the identity is treated as a rotation and there are reflections, then there are just as many rotations as reflections.

At this point, I introduce the Bhagavad-Gita verse and we spend quite a bit of time understanding the verse and how it can be interpreted in terms of symmetry transformations. The questions "What changes?" and "What stays the same?" start to become part of the students' way of thinking.

The first application of this way of thinking comes when we start working out the group table for the symmetry group of an equilateral triangle. After two symmetries are performed, one needs to determine what one symmetry is equivalent to the composition. We look at what stays the same. If one vertex of the triangle is left fixed, the composition is a reflection. If no vertices are left fixed (so that only the center is fixed), then the composition is a rotation. If we want to determine the type of a given transformation, look at what is fixed: if only a point is fixed, it is a rotation about that point; if a line is fixed, it is a reflection across that line; if everything is fixed, it is the identity.

As we move on to frieze ornaments and wallpaper patterns, to identify all possible transformations becomes more challenging. Now, we can think of "inaction" in terms of sameness, lack of change. Locate a motif or small design that is repeated throughout the whole pattern; then see if there is a way to transform that motif or design to as many of its repetitions as possible. If any of these transformations are symmetries of the pattern as a whole, then we have located a symmetry transformation. And the best way to describe the transformation is to say what stays the same: the center of rotation, the axis of reflection, the direction vector for a translation and the direction vector for a glide reflection (the lines determined by these vectors are fixed).

THE BEAUTY OF SYMMETRY

When students begin to design their own patterns, they start thinking in terms of aesthetics, what patterns they like and want to work on themselves.

Symmetry is beautiful and fascinating. From the charm of a snowflake to the deep spirituality of Leonardo's last supper, symmetry has an essential role in nature and art. Can the understanding of symmetry that we have gained here help us in any way to understand this role? We have seen that a symmetrical pattern gives rise to symmetries or transformations of the pattern which leave it essentially unchanged. From the Bhagavad-Gita, we see that life has two aspects, active and inactive. According to Maharishi [1], the silent level of life is pure consciousness, the source of thought, and it is subjectively experienced as bliss; whenever the active level of the mind begins to move in the direction of the silent level of the mind, there is increasing bliss. An artistic pattern or structure of nature expresses the diversity of relative existence, yet in the repetition of aspects of the design or structure, that is, in the symmetry, an underlying sameness or unifying value is indicated. The mind is spontaneously led to experience activity and silence simultaneously. This is in the direction of the nature of the experience described in the verse of the Bhagavad-Gita that we have examined. Thus, our analysis can help shed light on the charming nature of symmetry.

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CONCLUSION

Mathematics is part of life; mathematicians doing mathematics are subject to the same natural laws that govern all of life. A deep understanding of the whole of life should give us the kind of insight that will help us understand the parts of life, including some very specific aspect of mathematics. This paper presents how one expression of knowledge about the nature of life from the Bhagavad-Gita can be used to go deeply into the mathematical study of symmetry and, hopefully, acts as a suggestion that this bringing together of mathematics and life as a whole can be done in other ways.

REFERENCES

[1] Yogi, Maharishi Mahesh. *Bhagavad-Gita: A New Translation and Commentary with Sanskrit Text.* Penguin Press, Baltimore, 1969.

Life Math

Kathy Hayes

Don't take this 2 personal.
It can be +, not −.
Our thoughts might be ||.
The possibilities are relative.

Keep the R perspective.
Choosing division over subtraction,
It's surely the right theory.
But whole numbers are better than $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$.

Don't try to predetermine.
The addition is complete.
The $(x = a + b)$ is on the board.
The eraser is obsolete.

Will there be progression?
Will the ?'s be multiplied?
Problem solving can be fun!
Are the totals on your side?

Don't choose perfection.
Is that not possible?
Are you $<$ or $>$?
Is the answer plausible?

Study the basic principle.
Does quality = value?
A story problem? T or F?
Calculate the % and review,
Review, REVIEW!