Journal of Humanistic Mathematics

Volume 6 | Issue 2 July 2016

A Truly Beautiful Theorem: Demonstrating the Magnificence of the Fundamental Theorem of Calculus

Dan McQuillan
Norwich University

Darlene M. Olsen
Norwich University

Recommended Citation

McQuillan, D. and Olsen, D. M. "A Truly Beautiful Theorem: Demonstrating the Magnificence of the Fundamental Theorem of Calculus," *Journal of Humanistic Mathematics*, Volume 6 Issue 2 (July 2016), pages 148-160. DOI: 10.5642/jhummath.201602.11 . Available at: https://scholarship.claremont.edu/jhm/vol6/iss2/11

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | http://scholarship.claremont.edu/jhm/

A Truly Beautiful Theorem: Demonstrating the Magnificence of the Fundamental Theorem of Calculus

Dan McQuillan

Department of Mathematics, Norwich University, Vermont, USA
dmcquill@norwich.edu

Darlene M. Olsen

Department of Mathematics, Norwich University, Vermont, USA
dolsen1@norwich.edu

Synopsis

In standard treatments of calculus, the Fundamental Theorem of Calculus is often presented as a computational method to evaluate definite integrals, with such powerful utility that one is tempted to overlook its beauty. To improve students' appreciation for the first part of the Fundamental Theorem of Calculus, we suggest a few classroom examples focusing on the accumulation function, to be introduced early and often throughout an introductory calculus course. These examples are small enough that they would not necessarily result in changes to a typical course schedule; yet we believe their contribution to student understanding can be significant. Furthermore, such examples might allow students to share more of the excitement that the pioneers of the subject surely experienced along the way.

1. Introduction

What is the most difficult part of Calculus? Many answers abound. In this paper we focus on one specific idea to respond to one of the most typical answers.

Calculus students often say that they understood everything up until integration. Sadly our students see integrals only as the inverse operation of derivatives and fail to see why the definite integral of a non-negative continuous function yields the area under the curve. They do not seem to develop an intuition for the relationship between limits, derivatives, and integrals. In summary, students fail to see the beauty of the Fundamental Theorem of Calculus. There are more calculus books available than ever; however, they all approach the subject in essentially the same way. This standard approach does not work for everyone; we need alternative ways of teaching the topic to provide additional insight into the Fundamental Theorem of Calculus.

After a brief background section (Section 2), we present four short examples involving the accumulation function that can help students understand the first part of the Fundamental Theorem of Calculus long before it would typically be introduced in a calculus class. These geometric examples elucidate the beauty of the Fundamental Theorem of Calculus. The main added value of the examples is that students can visualize the idea of the rate of change of the accumulation function, which leads to an increased understanding of the Fundamental Theorem of Calculus. The last section (Section 7) summarizes student feedback from a calculus course where these examples were highlighted.

2. Background

The concept of the integral is central to the study of calculus. Research shows that most students see integral calculus as a sequence of steps to solve a problem and do not emerge from the course with a strong understanding of the Fundamental Theorem of Calculus [1, 2]. The lack of clarity has been attributed to students' poor understanding of the accumulation function

$$F(x) = \int_a^x f(t)dt$$

that appears in the first part of the Fundamental Theorem of Calculus [1, 2, 3]. While the accumulation function may entail a higher order of thinking, Thompson and Silverman [2] argue that a major source of this lack of understanding is that the idea of the accumulation function is rarely taught with the intent that students will actually understand it. Instead, the Fundamental Theorem of Calculus is typically taught as just a tool for evaluating definite integrals [2].

Thompson and Silverman [2] suggest that educators include accumulation functions in the calculus curriculum as a central idea. We offer the examples in the next four sections to provide opportunities to explore the concept of accumulation as early as the introduction of the limit definition of the derivative. We recommend that these problems be discussed in class; active dialogue will enhance student understanding. Furthermore, spacing these problems throughout the course will conceptualize and reinforce the idea of accumulation several times prior to presenting the Fundamental Theorem of Calculus.

These examples were developed and reworked over several years teaching Calculus I at a small private college. The intent was to give students additional insight into the main idea behind the subject, as opposed to just using the standard approach given in textbooks. A formal presentation of these examples was incorporated into a calculus course, taught to 24 students, who were primarily engineering and mathematics majors, during the spring semester of 2014. The student feedback we share in Section 7 corresponds to that semester's experience.

3. The Accumulation Function as the Area under a Curve

The function $y = x^2$ is almost always represented as a *parabola* in the xy-plane. However, functions can be represented in a multitude of ways. For example, the same function can be visualized as the area of a triangle with base x and height $2x$. As such, we can use the straight line $y = 2x$ instead of a parabola, provided that we imagine $y = x^2$ as the area under this line, and above the x-axis. This motivates our first example:

> **Example 1.** Graph the function $f(x) = cx$ for $x > 0$ and the line $x = z$ where $z > 0$ (Figure 1). Let $Acc(z)$ represent the area of the triangle bounded by the x-axis, the vertical line $x = z$, and the curve $y = f(x)$. Prove that $Acc'(z) = f(z)$ using the limit definition of the derivative.
>
> **Solution.** Students will use the formula for the area of a triangle to get $Acc(z) = \frac{1}{2}zf(z) = \frac{1}{2}cz^2$. From there, the problem fits perfectly as a typical problem involving the limit definition of the derivative,

$$Acc'(z) = \lim_{h\to 0} \frac{Acc(z+h) - Acc(z)}{h} = \lim_{h\to 0} c\Big(z + \frac{h}{2}\Big) = f(z).$$

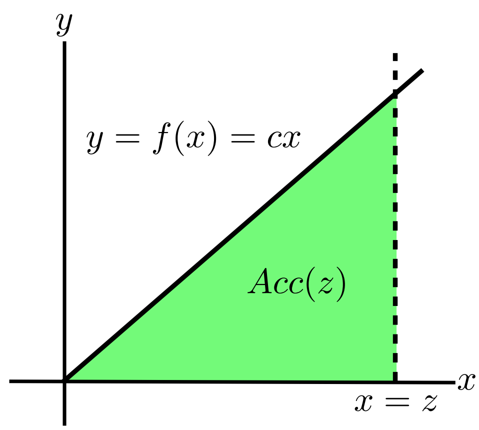

Figure 1: Illustration of the bounded region given in Example 1.

Students appreciate an *alternative* solution that provides geometric intuition without any algebraic manipulations. *The main idea is that the derivative of the accumulation function can be computed without an explicit formula for $Acc(z)$.* Indeed the difference $Acc(z+h) - Acc(z)$ represents the area of a trapezoid, with parallel sides of length cz and $c(z+h)$, and with height h, where the height of the trapezoid is lying on the x-axis (Figure 2).

The difference quotient

$$\frac{Acc(z+h) - A(z)}{h} = c\left(z + \frac{h}{2}\right)$$

is the average of the lengths of the two parallel sides. Taking the limit as $h \to 0$ gives the result.

With this solution, we *think* of the rate of change of the accumulated area under the curve *as equal to the value of f at the stopping point $x = z$*. This is the heart of the Fundamental Theorem of Calculus.

Students are so comfortable with this example that they can handle an intuitive discussion of the notation

$$Acc(z) = \int_0^z f(x)dx.$$

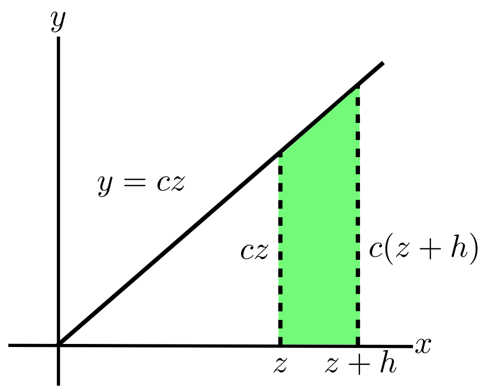

Figure 2: $Acc(z+h) - Acc(z)$ gives the area of a trapezoid, which is sufficient for proving $Acc'(z) = f(z)$.

In this context it means that, in order for our quadratic function to be represented by the straight line $f(x) = cx$, we need a starting point, $x = 0$, and a stopping point, $x = z$. While it is standard in calculus texts to switch from the variables (x, z) to (t, x), thus writing $\int_0^x f(t)dt$ instead of $\int_0^z f(x)dx$, we believe this to be a mistake, as it leads to unnecessary confusion for beginners. An additional advantage of using (x, z) is that it more closely aligns the notation for the accumulation function with the notation used in the second part of the Fundamental Theorem of Calculus.

Depending on the students and the amount of time the instructor has for discussing class exercises, there may be some interest in the following question:

Does this still work if a non-linear function $f(x)$ is used instead of $y = cx$?

After all, it would not be possible to replace

$$c\left(z + \frac{h}{2}\right)$$

with

$$f\left(z + \frac{h}{2}\right)$$

in the preceding solution. If this question is asked, it provides an excellent opportunity to motivate the essential role of existence theorems, such as the Intermediate Value Theorem.

4. The Accumulation Function as the Area of an Expanding Circle

A key point from the previous example has to do with *imagining* an expanding triangle. It is now intuitive to think that the rate of change of area depends on the value of f at the stopping point. As an additional exercise to build on this type of thinking, students can be challenged to explain the connection between the area of a circle and its circumference through the use of a suitably chosen derivative. The next example illustrates that the area of the circle can be thought of as the accumulation of the circumferences of all the circles of radius $r \leq z$.

Example 2. Graph the circle with radius z, $z > 0$. Let $Acc(z)$ represent the area of the circle. Prove that

$$Acc'(z) = 2\pi z$$

using the limit definition of the derivative.

Solution. Using the formula for the area of a circle, $Acc(z) = \pi z^2$, where the variable z represents the radius of the circle.

Again, from the limit definition of the derivative,

$$\begin{aligned} Acc'(z) &= \lim_{h \to 0} \frac{\pi(z+h)^2 - \pi z^2}{h} \\ &= \lim_{h \to 0} \frac{\pi}{h} \Big[\Big((z+h) - z \Big) \Big((z+h) + z \Big) \Big] \\ &= \lim_{h \to 0} \frac{\pi h}{h} [2z + h] \\ &= 2\pi z. \end{aligned}$$

It is interesting to note that $Acc(z+h) - Acc(z)$ gives the area of the annulus (Figure 3).

The main value of this example is that it encourages the kind of *thinking* required for finding new applications for calculus. A circle is a static object; yet by *imagining* an expanding circle, we see that the rate of change of the area is given by the circumference of the circle at the stopping point $x = z$. In the class trial, students were most impressed with this example; while all students had memorized formulas for the circumference and area of a circle, none were made aware of the connection between the two formulas.

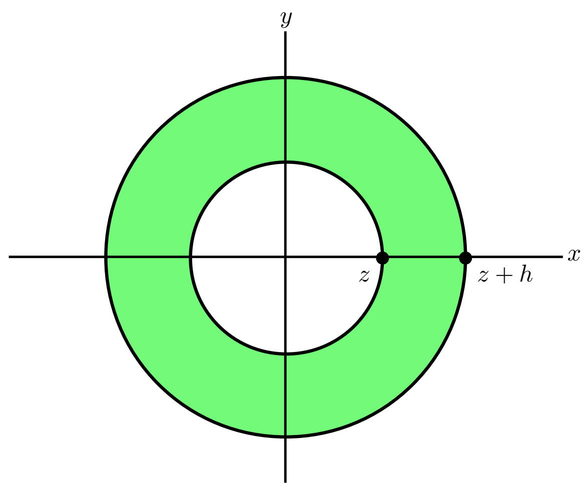

Figure 3: $Acc(z+h) - Acc(z)$ gives the area of the annulus, which is sufficient for proving $Acc'(z) = 2\pi z$.

In fact most calculus text books inadvertently prove that the derivative of the area of the circle is the circumference of the circle while introducing cylindrical shells. By contrast, we are explicitly mentioning and proving this connection at the time when derivatives are first introduced, since it is interesting! It is also a good time to preview the formula for finding the volume of a solid by rotating the region under the curve $y = f(x)$ from a to b about the y-axis [4]:

$$V = \int_a^b 2\pi x f(x) dx \text{ where } 0 \leq a < b.$$

Naturally, Example 2 can be reviewed prior to the proper treatment of cylindrical shells.

5. The Accumulation Function as the Volume of a Cone

This next example combines features of the previous two. Once again we see that the derivative of the accumulation function can be computed without an explicit formula for $Acc(z)$. The example also captures the essence of how calculus is used to find volumes, by treating it as an accumulation of areas.

As a result we get an elegant explanation of the poorly-understood formula for the volume of a cone.

Example 3. Graph the function $f(x) = cx$ for $x > 0$ and the line $x = z$ where $z > 0$, as presented in Example 1 (see Figure 1). Let $Acc(z)$ denote the volume of the cone obtained by revolving the triangle about the x-axis (Figure 4). Prove that $Acc'(z) = \pi f(z)^2$ using the limit definition of the derivative.

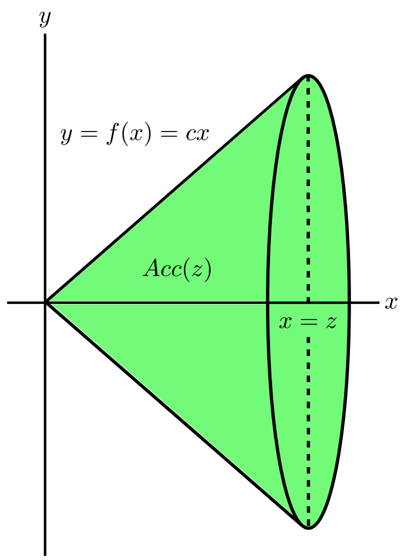

Figure 4: Illustration of the cone generated in Example 3.

Solution. We use the formula for the volume of a cone to get $Acc(z) = \frac{1}{3}\pi f(z)^2 z = \frac{1}{3}\pi c^2 z^3$ where the variable z represents the height of the cone. Using the limit definition of the derivative,

$$\begin{aligned} Acc'(z) &= \lim_{h\to 0} \frac{Acc(z+h) - Acc(z)}{h} \\ &= \lim_{h\to 0} \frac{1}{3}\pi c^2 \frac{(z+h)^3 - z^3}{h} \\ &= \pi(cz)^2 \\ &= \pi f(z)^2. \end{aligned}$$

Again, this example fits perfectly into the beginning part of the course. Fortunately, the following beautiful solution does not require the formula for the volume of a cone. Indeed, if $h > 0$, then $Acc(z + h) - Acc(z)$ is the volume of a frustum (Figure 5).

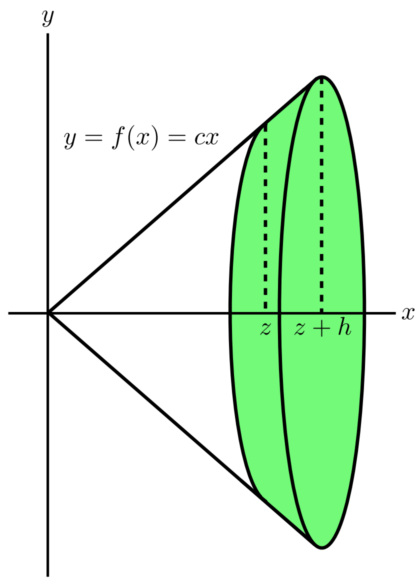

Figure 5: $Acc(z + h) - Acc(z)$ gives the volume of the frustum, which is sufficient for proving $Acc'(z) = \pi f(z)^2$.

This volume is greater than the volume of a cylinder with height h and radius $f(z)$, and it is smaller than the volume of a cylinder with height h and radius $f(z+h)$. It follows that

$$\pi c^2 z^2 \leq \frac{Acc(z+h) - Acc(z)}{h} \leq \pi c^2 (z+h)^2.$$

Using the Squeeze Theorem and taking the limit as $h \to 0$ gives the desired result.

Just as in Example 1, where we emphasize that the rate of change of an expanding triangle is given by the value of f, here we see that the rate of change of the volume of an expanding cone is given by the area of the circular base. We see once again that the derivative of the accumulation function is entirely dependent on the evaluation of f at the stopping point $x = z$, the heart of the Fundamental Theorem of Calculus.

While it is easy to memorize the formula for the volume of the cone, it is more inspiring to understand the formula as an antiderivative of the area of the circular base. Prior to formally introducing antiderivatives, it is straightforward enough for students to determine the antiderivative of the equation representing the area of the circular base as

$$\frac{1}{3}\pi c^2 z^3 = \frac{1}{3}\pi f(z)^2 z.$$

This is an excellent explanation of the volume of the cone formula.

6. Bonus: A Related Rates Example

The expanding cone is central to related rates problems where, typically, the height of the cone is assumed to be increasing. Example 3 allows students to think of the expanding cone problem as if it is the circular base of the cone that is expanding.

Consider the following ubiquitous related rates example:

> **Example 4.** Sand is pouring from a pipe at a constant rate of $\frac{dV}{dt}$ (where V is the volume of the sand). The sand forms a conical pile such that the ratio of the radius to height, r/h, is the constant c. How fast is the height increasing when the pile is at a fixed height?
>
> **Solution.** Let
> $$V = \frac{1}{3}\pi r^2 h$$
> be the volume of the cone. Write $r = ch$ to get
> $$V = \frac{1}{3}\pi c^2 h^3$$
> and differentiate with respect to t to get
> $$\frac{dV}{dt} = \pi c^2 h^2 \frac{dh}{dt}.$$
> Substitute the given values appropriately and the problem is solved.

This solution is certainly succinct. Students generally ask: "Where does the $\frac{dh}{dt}$ come from?" The standard answer is useful. Namely, if we differentiate the right hand side with respect to t, then we must do the same to the left hand side, and use the Chain Rule. Here we provide an additional answer, to provide extra meaning to the first one. Our proposed solution involves a lot of imagination.

> **Proposed Solution to Example 4**. Turning the expanding cone sideways, the altitude of the cone being measured is given by the x-coordinate. Thus, the cone would be the solid of revolution obtained by revolving $y = cx$, $x > 0$, about the x-axis and the cone would be continually expanding to the right as it grows. This suggested change provides an opportunity to illustrate that real-world details can be temporarily imagined as altered, to make the mathematics seem more familiar, provided that the variables are defined carefully, and therefore the mathematical solution stays essentially the same.
>
> Start from the proposed solution to Example 3, where students see that
> $$\frac{dAcc(z)}{dz} = \pi c^2 z^2.$$
> In other words, the rate of change of the volume of an expanding cone is given by the area of the circular base. Replacing $Acc(z)$ with V yields
> $$\frac{dV}{dz} = \pi c^2 z^2.$$
> Multiply both sides by $\frac{dz}{dt}$ and the left hand side can be replaced with $\frac{dV}{dt}$ by the chain rule. Substitute the given values appropriately and the problem is solved.

An important advantage of this solution is that the equation

$$\frac{dAcc(z)}{dz} = \pi c^2 z^2$$

suggests *thinking* about how the volume changes with respect to z. This indirectly, yet emphatically answers the student's commonly asked question.

As a bonus, another possible benefit of the proposed solution is that it encourages students to recognize a derivative *as having been calculated using the chain rule*. Indeed, the last step, replacing $\frac{dV}{dz}\frac{dz}{dt}$ with $\frac{dV}{dt}$ encourages recognizing patterns that should help students when they eventually integrate using the method of substitution.

7. Classroom Feedback and Final Thoughts

At our institution students have the option to complete a course evaluation at the end of the semester. This is anonymous feedback that is available to the course instructors and college administrators several weeks after course grades are submitted. We have observed that the inclusion of the examples we shared here has coincided with substantial improvements in student evaluations of the course. For our Spring 2014 course, student comments[1] included the following:

> *"I like the professor's way of teaching and how he connected the material ... helped me to understand the material better."*
>
> *"The professor's explanations of the mathematical concepts of Calculus are the best I have ever heard. Because of his teaching methods and thorough detail, I actually understand not only Calc 1, but also large chunks of Pre-Calc that I failed to grasp in multiple previous passes."*
>
> *"[The professor] explained topics in a more understandable way."*
>
> *"[The professor taught the] class in many different styles of teaching to makes sure all students were gaining an understanding."*
>
> *"I strongly believe in his teaching methods. To be specific [his presentation of the Fundamental Theorem of Calculus] was a far more effective strategy in my opinion."*

[1] We have completed Norwich University's IRB process regarding quoting student comments.

Although the accumulation function is fairly trivial when introduced in the context of these examples, it seems to come across as a very complex concept if its first appearance coincides with the presentation of the Fundamental Theorem of Calculus. These examples provide an informal introduction to part I of the Fundamental Theorem of Calculus before it is formally covered. They provide an ideal opportunity to communicate the main idea of this theorem, focusing on its simplicity, beauty, and applicability. We believe that greater connections between topics will result in greater enjoyment of the material as well as a deeper understanding of mathematics.

References

[1] P. W. Thompson, “Images of rate and operational understanding of the Fundamental Theorem of Calculus,” *Educational Studies in Mathematics*, Volume **26** (1994), pages 229-274.

[2] P. W. Thompson and J. Silverman, “The concept of accumulation in calculus,” in M. P. Carlson and C. Rasmussen (editors), *Making the Connection: Research and Teaching in Undergraduate Mathematics*, Mathematical Association of America, Washington DC, 2008, pages 43-52.

[3] A. Kouropatov and T. Dreyfus,“Constructing the integral concept on the basis of the idea of accumulation: suggestion for a high school curriculum,” *International Journal of Mathematical Education in Science and Technology*, Volume **44** Issue 5 (2013), pages 641-651.

[4] J. Stewart, *Calculus: Early Transcendentals*, 7th edition, Brooks/Cole, Cengage Learning, Belmont, CA, 2008.