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Teaching Differential Equations with Modeling and Visualization

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Abstract: In this article, I explain the history of using Interdisciplinary Lively Applications Projects (ILAPs) in an ordinary differential equations course. Students want to learn methods to "solve real world problems," and incorporating ILAPs into the syllabus has been an effective way to apply solution methods to situations that students may encounter in other disciplines. Feedback has been positive and will be shared. Examples of ILAPs currently used will be referenced. For more information about how to develop ILAPs, see Huber and Myers (in *Innovative Approaches to Undergraduate Mathematics Courses Beyond Calculus*, 2005).

1 Confident and Competent Problem Solvers

One of my goals each semester is to develop students into confident and competent problem solvers. The competence comes from practice, solving problem after problem. The confidence also comes with experience, from knowing what to do when facing a new problem. Many textbooks give students an equation and ask them to solve it. In the past, if the textbook gave students the harmonic oscillator equation, complete with values for the mass, damping coefficient and spring constant, students could put in the values and then use the appropriate technique to solve for the displacement or velocity of the mass. However, when confronted with a word problem and asked to develop the model's differential equation, students struggled. Students were comfortable with their "Plug and Chug" method of solving, as long as they knew what and where to plug. That got me thinking: What if we just gave them the forces acting on the problem, to include any non-homogeneous driving function, and asked them to predict long-term behavior?

A few years ago, as an experiment suggested by my colleague Don Small at West Point, I gave a group of students a differential equations problem in the form of a long word problem and asked them to set up the model (define variables, state what is given, make a few valid assumptions, write down what they were asked to determine, and explain the technique they would use to solve it). Suppose that the growth of an alligator population in the Okefenokee Swamp is proportional to its population at a rate of 0.05 per year, and that the initial population is 7200 gators. Alligator populations have risen to such high levels that the U.S. Fish and Wildlife Service "harvests" gators by allowing tightly controlled hunting of 650 to be hunted in one year, followed by no harvesting in each of the next two years of these animals, famous for their durable hide and excellent meat. This harvesting strategy is repeated every three years. Model the IVP and predict how many gators will there be after 6, 7, and 8 years. Further, what would the harvesting need to be to wipe out the gator population? I told them NOT to solve it, unless they had time. We had already covered using Laplace transforms to solve these types of problems, but I wanted to see if they could establish the problem to be solved. Afterwards, I asked the students to explain their problem-solving process, to include any assumptions they had to make. The real value of this exercise came from the explanations. Some students wrote about their anxieties about having a "vague" problem, while others wrote about their happiness in being able to set up a "real-world" problem.

The harvesting scheme involves a Heaviside function which is shown in Figure 1 below. The horizontal axis shows time in years and the vertical axis shows the number of gators hunted. The solution to the problem should show a series of population increases and decreases, due to the cyclic harvesting. A plot of the particular solution is shown in Figure 2.

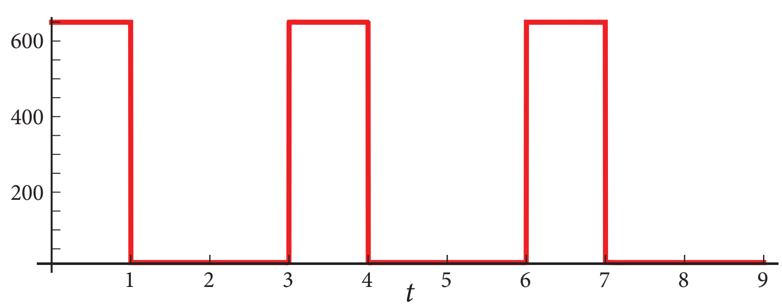

Figure 1: Harvesting Gators in the Okefenokee Swamp

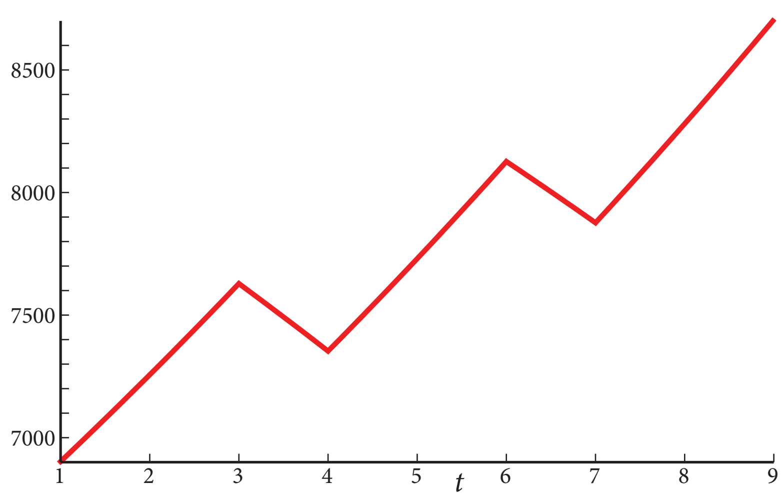

Figure 2: Gator Population in the Okefenokee Swamp

2 The Problem-Solving / Modeling Process

In Figure 3, students must transform a problem from a real world setting into a math model, solve that model, and then interpret the results in the real world scenario. With computer algebra systems, the Solution Process leg of the process has become less important than the other two legs. Instead of giving the students the model or algorithm and asking them to simply substitute numbers into the equation to find an answer, I began to ask students to develop the equation or the heuristics of the model. I also found that students were more receptive to learning solution methods when confronted with applications-oriented scenarios. Model creation depends heavily on defining variables with appropriate units, stating what is given, making valid assumptions, and figuring out what it is we are trying to find. The interpretation leg requires students to discuss the solution back in the context of the situation. The answer to the gator problem is not 50. Students are expected to discuss the effects of harvesting and how the population changes when harvesting rates are adjusted. That takes the solution back into the "real world."

Creating the model and interpreting the solution are just as important—maybe more important—than simply finding a solution. Today's computer algebra systems and numerical solvers (Mathematica, Maple, ODEToolkit, etc.) can solve most traditional differential equations, take derivatives, determine integrals, plot slope fields. Knowing what to enter as the equation to be solved becomes the critical task. Describing a situation where the rate of change of a body's temperature is proportional to the difference of the temperature of the body and the surrounding environment requires students to understand each term in the ODE; simply telling them to use Newton's Law of Cooling does not. Describing interaction terms in a competing species problem forces students to determine which might be a prey and which might be a predator, based on the situation. Making valid assumptions to simplify the model is stressed in the classroom. Asking, "So what?" or "Is my answer reasonable?" leads students into the interpretation stage. Writing is incorporated into the process by requiring students to explain their solutions in the context of the problem, with appropriate units. Does the solution pass the common sense test? The vertical side of the triangle shown below (Solution Process) becomes the (relatively) least important one. Further, the visualization aspect of a computer algebra system becomes a force multiplier in student learning. Plot the solution of the ODE. What happens as

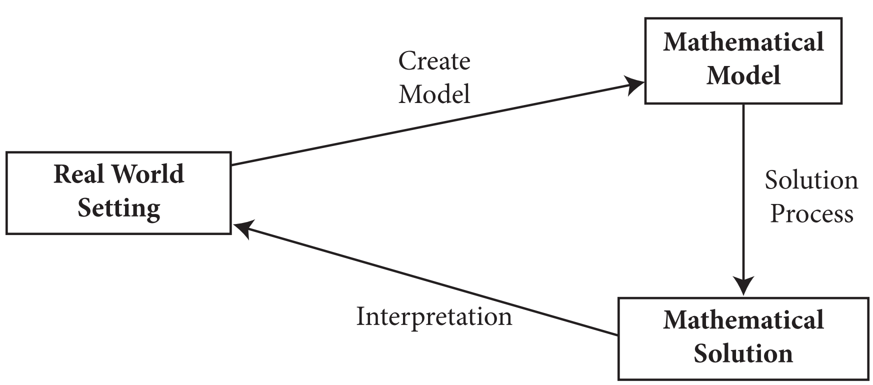

Figure 3: A Mathematical Modeling Approach

time increases? Does it make sense? Does the phase portrait or direction field of the system match our intuition (complex eigenvalues give rise to a spiral)? Can we predict the behavior based upon the direction field (without finding the particular solution)? Can students hand-draw a portion of the slope field near an equilibrium point? Far away from an equilibrium point? Then can they compare their drawing to that from the computer and claim success?

Figure 4: The Leaning Tower of Pisa (photo by author)

3 Using ILAPs

My lessons are now geared to solving groups of applied problems. The methods needed to approach applications are still very important, but I try to sequence theory with solving problems. Eigenvalue/eigenvector and matrix algebra skills are introduced as students try to solve systems of first-order ODEs. Visualization is incorporated into every solution, either by plotting a solution, direction field, or forcing function. Does the solution satisfy the ODE's forcing function? Plot them both and compare. Each block of material culminates in a word problem assessment (see the ILAPs below). A possible drawback is that not every type of problem can be assessed in a one-hour exam. However, this drawback existed with the previous mid-terms, before I began this modeling approach. In addition, students are now writing about their mathematics. The discussion section of the problem is not simply an answer that is double-underlined. It is the student's attempt at answering an applied problem and explaining the results.

As an example, I found an old exam from several years ago in our department, in which we asked students to find the general solution to the second-order equation

$$y'' + 7y' + 10y = 0. \tag{3.1}$$

We may have asked to determine the position at some time t, but students were given the model. They expected to be given the equation to be solved, and they really didn't question how it was derived. Now, we have placed a little more emphasis on how to put together the model. How do we sum the change of position in a first-order equation? What happens when we sum the forces acting on a body? Now, for instance, I like to give the following type of problem.

> A ball is dropped (no initial velocity) from the side of the Leaning Tower of Pisa (approximately 35 meters above the ground—see Figure 4) and bounces repeatedly; each time, the rebound is a bit lower. The following forces apply:
>
> 1. the ball has a mass of 3 kilograms and acceleration is due to the force of gravity alone;
> 2. air exerts a resistive force that is equal to two-and-one-half times the ball's velocity, and reduces the speed of the ball;
> 3. the ball exhibits a Hooke's Law-type force that can be modeled with a spring constant of 12 kilograms per second squared;
> 4. there is no external forcing function applied.
>
> Model the motion of the ball as a system of equations. Write the problem in matrix form. What are the eigenvalues and eigenvectors of the coefficient matrix? What is the expected behavior of the ball? Plot the particular solution. How many times does the ball bounce above 2.5 meters during a rebound?

Several objectives are assessed with this problem. Students show that they can transform a higher-order ODE into a system of first-order ODEs. Sometimes I don't explicitly ask for the eigenvalues, but by determining them, students can predict motion based upon the signs of the eigenvalues' real components. Most students can get the model of a damped harmonic oscillator and solve for the particular solution. However, not all will correctly plot the motion of a ball, as the solution crosses the t-axis (going from positive to negative). The more astute students realize they need to plot the absolute value of the ball's position, in order to get a bouncing effect (see Figure 5). In evaluating student writing about the solution, I look to see if students question the fact that the motion of their ball goes through the earth, as in the left side of Figure 5, or if it bounces, as in the right side.

This is an effective problem used to evaluate understanding, but it can be viewed as contrived for a classroom setting. For instance, notice that the time between bounces does not change, when in fact, it should. Don Small suggested that we develop connections with other departments at West Point, in order to expose students to problems that they may have to solve in their engineering or science or humanities classes.

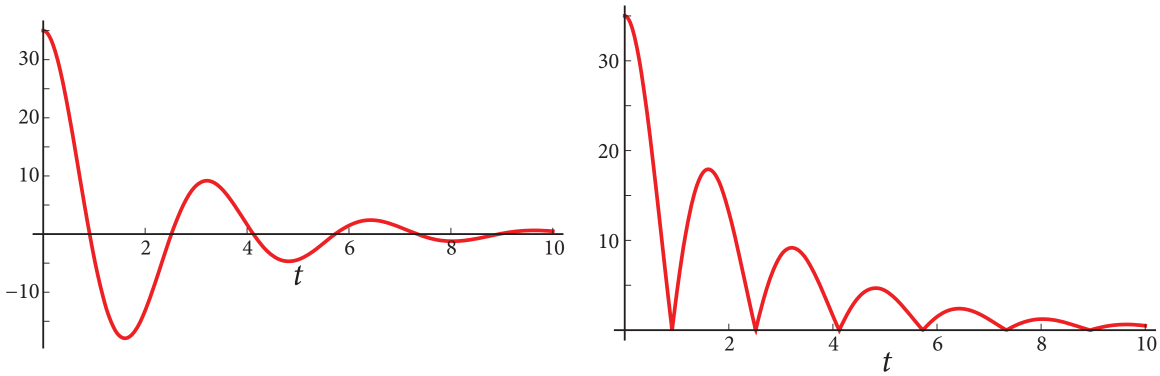

Figure 5: On the left, solution to damped harmonic oscillator; on the right, absolute value of solution, simulating a bounce.

When I taught at the United States Military Academy, we created several Interdisciplinary Lively Applications Projects (ILAPs). In the 1990s, our department at West Point led a consortium of 12 schools located throughout the country in an NSF-sponsored program hoping to improve the educational culture through the enhancement of interdisciplinary cooperation and coordination. The program was entitled Project INTERMATH, and its primary activity was to develop ILAPs. In 1997, the Mathematics Association of America published a book of ILAPs, edited by Dr. David C. Arney, department head at West Point at the time. In the preface, now-retired General Arney wrote, "From the student's perspective, ILAPs provide applications which motivate the need to develop mathematical concepts and skills, provide interest in future subjects that become accessible through further study and mastery of mathematics, and enable a broader, more interdisciplinary outlook at an earlier stage of development."

I was fortunate to have helped in some ILAP development, and I still try to develop them today. There is a cost. I have to find a colleague in another department who is willing to devote some time to assist with the context of the scenario. I have recently used ILAPs with ties to mechanical engineering, environmental engineering, physical education, and even mythology. I have submitted some of these ILAPs to the CODEE journal.

Some ILAPs are more sophisticated than others. For example, "The Phenomena of Mechanical Resonance" is an applied introduction to resonance with a linear second-order equation, while "Aircraft Flight Strategies" requires a basic understanding of lift, altitude, velocity, etc., and has nonlinear equations. Recently, I have been researching the Labors of Hercules and have developed ILAPs with connections to Greek Mythology. The Fifth Labor of Hercules required the hero to clean the stables of Augeas. This problem requires using Torricelli's law. The Eleventh Labor of Hercules involved wrestling a giant who gained energy from contact with the earth. The forcing function is a Heaviside term (a series of step functions), and we use Laplace transforms in developing the solutions. There are ILAPs for mixing problems. There was an episode on the popular television show The Simpsons ("Who Shot Mr. Burns?") that I have used involving the connection of mathematics and pop culture. Students have to solve a Newton's Law of Cooling problem to find the culprit. I have used ILAPs centered around bungee jumping and modeling oxygen in the bloodstream after exercise.

There is also a wealth of possibilities in COMAP journal issues. I am working to write up a few of them for use in my course. Many of the COMAP articles are very detailed, so I will condense them a bit, hoping to keep the interdisciplinary flavor for the students while extracting necessary information to develop the models.

4 Students and Feedback

What prerequisites have my students had? At Muhlenberg College, the student population in my ODEs class is usually mixed when considering the instruments in their mathematical toolkit. I usually have 15–24 students each semester (the course used to be only offered once each year, but enrollments over the last two years have allowed us to go every semester). Often I have sophomore mathematics majors who have just completed Integral Calculus (Calculus II is the only prerequisite for the course), and this is their first mathematics elective. In the same class I have juniors and seniors who take ODEs as their last elective to fulfill a major or minor in mathematics. Their abilities are much more advanced, but in other areas of mathematics (many have taken our proofs course, linear algebra, abstract algebra, and other electives). Many of the upper-classmen are physics majors, minoring in mathematics. Usually, the differential equations course is the first (or maybe second, after linear algebra) course which is heavily applications-oriented. However, as it is an elective, some mathematics majors will graduate from Muhlenberg without having had a course in differential equations.

When I assign ILAP projects, the students work in 2-, 3-, or 4-person teams. I usually allow up to two weeks for students to work the problems, and I require a formal write-up with all aspects of the modeling process. The format for the write-ups is provided in my syllabus. About a week into the projects, I hold informal 10-minute in-progress reviews (IPRs) with the student teams. This requires them to have done some work (instead of waiting until the last minute), and the teams brief me on their progress, addressing any questions they might have. The IPRs keep the groups focused and I find that the groups are trying to accomplish as much as they can before the IPRs. When the projects are turned in for a grade, the student groups are selected at random to brief a portion of the requirements and solutions. This allows for more groups to brief, reduces unnecessary repetition, and gets more students involved. The discussions we have are very valuable, to both the groups and to me. I find that they try to explain their thinking to me, before I have graded the write-ups), and the groups benefit from each other. I also have a portion of my final examination devoted to the ILAP, to ensure students were participating and understood the learning objectives.

Student feedback has been very positive. My experience has shown that students are more apt to really learn the mathematics when they can apply that math to solve problems that they might see again in a science or engineering course. Course-end surveys have revealed that once students sense that the lessons and assessments focus on problems which they might encounter in the real world, a metamorphosis occurs. The students feel that classes are more interesting. They also believe that their confidence levels rise when using mathematics to solve problems which surround them in life, because they have a feel as to whether the answer makes sense. They are beginning to understand that their reasoning skills are as important as their analytic skills. If they are confident in their model, and they obtain an answer that just does not make sense, they can receive partial credit for explaining what their intuition tells them should be the answer. For instance, “When I solved for the position of the mass in the damped harmonic oscillator problem, the solution to my equation had a positive exponent, which indicated exponential growth. Since there is damping, the displacement should decrease with time. I must have an algebra mistake somewhere, but I ran out of time.” When our department chairman conducts his exit interviews with graduating mathematics majors, most mention how valuable the applications-oriented courses have been to them, even naming the differential equations course.

Nationally, I see more sessions at the annual mathematics meetings on teaching/assessment using modeling or problem-solving every year. Several other professors across the country are getting their students to visualize and model word problems, and others are having great success in getting students to write about their mathematics. Textbooks are popping up with “modeling” in the title. The technology keeps getting better and more user-friendly; visualization of solutions is critical in learning to predict long-term behavior. Modeling population growth or decay, interaction between competing species, the spread of disease in a community, or even solving problems with piecewise continuous forcing functions are now problems that students think, “Oh, I’ve heard of that,” and “I think I can solve it.” Competence and confidence are achievable goals.