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The Philosophy of Mathematics: A Study of Indispensability and Inconsistency

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Abstract

This thesis examines possible philosophies to account for the practice of mathematics, exploring the metaphysical, ontological, and epistemological outcomes of each possible theory. Through a study of the two most probable ideas, mathematical platonism and fictionalism, I focus on the compelling argument for platonism given by an appeal to the sciences. The Indispensability Argument establishes the power of explanation seen in the relationship between mathematics and empirical science. Cases of this explanatory power illustrate how we might have reason to believe in the existence of mathematical entities present within our best scientific theories. The second half of this discussion surveys Newtonian Cosmology and other inconsistent theories as they pose issues that have received insignificant attention within the philosophy of mathematics. The application of these inconsistent theories raises questions about the effectiveness of mathematics to model physical systems.

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1 The Division of Platonism and Fictionalism

1.1 Introduction to the Philosophy of Mathematics

The history of mathematics is full of mathematical problems with philosophical significance. This is no surprise when we trace the roots of mathematics and philosophy. Both of these fields came into interest because of their ability make useful models of the world. The origins of mathematics are up for debate, but a rough sketch can be made by seeing mathematics as social field. As our ancestors began asking questions about the planets and stars they started forming models and theories accounting for what the saw. Over centuries these elementary models became the modern notion of orbits. These measuring tools gave rise to modern planetary science and astronomy. This human thirst to quantify is unquenchable, because in mathematics the more questions we answer the more questions we ask. The field of mathematics has grown exponentially, however, the same philosophical questions still remain. The notion of infinity or the idea of an empty set might be troubling for mathematicians and philosophers alike. A fundamental understanding of the nature of these problems is found by studying what mathematical objects and theorems consist of. The implications of accepting one philosophical theory over another spreads to problems in broader ontology and metaphysics. I wish to focus on The Indispensability Argument for platonism and the possible issues it faces with the inconsistencies in the field of mathematics. Here we focus on two of the most successful philosophical theories of mathematical objects.

The question about the existence of mathematical objects is more controversial than one may initially think. There are epistemological and metaphysical issues that are entangled in the acceptance of either theory. Philosophers initially thought Plato's theory of the Forms gave a comprehensive idea of the developing field of mathematics. The ideas from the broader metaphysical theory carried naturally to mathematical entities, mainly due to the formalization of the Forms. Plato scheemed an abstract heaven filled with perfect entities outside of our idea of space and time. To illustrate this idea it is useful to look at an example of a circle. We can draw a representation of a circle on a piece of paper but it is fundamentally different than the circle in the Forms. If we draw three circles in a row we can pick which one is the most perfect, which shows that we have some intuition of this perfect circle Plato suggests. However, the perfect circle is imperceivable because the circle in the Forms has no size, no color, and no physical properties. A reasonable question is if we can't perceive or have any causal relations with the Forms how are we able to have knowledge about them? Plato claimed that prior to our souls finding physical bodies on earth they had access to the Forms of abstract entities in the heaven. Furthermore, our experiences on earth reflect our efforts to remember the Forms to the best of our ability. Unsurprisingly the latter part of Plato's theory has been buried many times over. The idea of abstract mathematical entities, however, has remained as the foundation of mathematical platonism.

1.2 Mathematical Platonism

Similarly to Plato's theory, mathematical platonism is committed to objective mathematical entities. Plato's explanation requires accepting the soul to explain the possibility of connecting with The Forms. Although mathematical platonism has certain parallels with Plato's theories, it is best to see Plato's ideology as the starting point. Platonism (from here on describing modern mathematical platonism) is the metaphysical view that there are abstract mathematical objects that exist independent of the human practice, ambition, or language. If you asked a platonist what the Gaussian prime $(4+2i)$ refers to, they would say $(4+2i)$ is an abstract object that doesn't rely on our knowledge for it to exist. There is no physical representation of this prime. We cannot know the number three by experiencing three apples on a table based on our inability to have a causal connection with "3".

The number three is an abstract mathematical entity that is independent of our perception. Platonism is defined by the acceptance of three theses: existence, abstractness, and independence. Each of these theses weighs heavily on the ontological commitments of platonism, and without each of them being justified separately the rest will unravel.

1.2.1 Existence

I want to look at what abstract existence entails before looking at the specific cases of mathematical objects and their abstractness. The questionable existence of abstract objects creates a divide within different sectors of philosophy. Drawing the line between concrete and abstract objects has been controversial because of the epistemological consequences. David Lewis in his book "On The Plurality of Worlds", surveys the possible ways of distinguishing between the concrete and the abstract. The first way is called The Way of Abstraction which asks us to take a concrete object and remove all specificity from it. A blue shirt, blueberries, and the sky are concrete examples of blue things but "the color blue" is abstract. Class nominalists, who deny the existence of any abstracta, claim "the color blue" is nothing more than the set of all of the blue things but is not a distinct entity. Lewis defines to The Way of Example next. This urges a person to make a list of standard cases of abstract and concrete objects with the hope that the distinction will appear [17]. The Way of Negation points to the idea that concrete entities are in space and time and are causally efficacious, where abstract objects are not. Therefore, it suffices to label objects that do not fit these properties as abstract. Frege claimed an object could only be abstract if and only if it is non–physical, non–mental, and causally inefficacious. Further thoughts on epistemology and metaphysics depend on how a person categorizes abstract objects. For example, the empiricist takes issue with granting the existence of abstract entities based on their ideas of how we come to knowledge. The properties of causality and spatial location make it unclear how we are able to become aware of the existence of these abstracta. We will see an attempt to get around this issue in section 1.3.6 with James Robert Brown.

The argument for the existence of mathematical entities is the platonist's first responsibility. The second thesis of abstractness follows close behind, because clearly we cannot prove the existence of mathematical objects in the same way we would show the existence of a spider or a chair. Therefore, the platonist must show that abstract entities exist. The platonist wants to argue that mathematical objects are in existence similarly an electron. Electrons escape the verification through physical interaction and have properties that are unknown to us [18]. The independent lives of electrons are not governed by our understanding of them, however, we are able to know facts about electrons through experiments that capture the electrons salient properties. We have given the name “electron” to attempt to capture the abstract phenomenon of the electron. Mathematics is the same in practice, because we are trying to strategically tow mathematical truths out of our mathematical systems.

Possibly the most qualified speaker on the matter of the existence of mathematical objects was Gottlob Frege. Frege was a German mathematician, philosopher, and one of the founding fathers of modern logic. His ambition to argue for platonism worked alongside his argument that all mathematical truths could be boiled down to the general laws of logic. Frege made a separation between two kinds of expressions we use as linguistic models: saturated and unsaturated. He defined saturated expressions as singular terms such as proper nouns and unsaturated remarks as predicates or quantifiers. This distinction, according to Frege, accurately mirrors a metaphysical difference within our thoughts. Our thoughts are made from saturated “objects” and unsaturated “concepts” [8]. He uses this outlook to form his argument for arithmetic–object platonism. His argument begins with the claim that singular terms which refer to natural numbers can be seen within true simple statements. Simple statements with singular terms as components are only true if the referred objects exist. Therefore, the natural numbers must exist. If the natural numbers exist then they are mind–independent abstract objects because they are neither mental nor physical [8]. Frege's Existence Thesis relies on a number of controversial claims. He assumes the natural numbers exist because singular terms that refer to them appeal in true simple statements. Frege uses identity statements to illustrate how this is true. For example, $4 + 3 = 7$ is an identity claim using 4, 3, and 7. In order for this to be true, the numbers we refer to must exist. A similar argument for truth of mathematical sentences is seen within Putnam and Quine 's Indispensability Argument. However, in this case our sentences involve scientific statements that use mathematics indispensably.

Perhaps the most convincing argument for mathematical platonism is The Indispensability Argument given by Hilary Putnam and William Van Orman Quine. In the next chapter I take a closer look at the argument for platonism through the indispensability of mathematics in science, however, for now I simply want to give an overview of how the argument fits in with the rest of the outlook. The argument begins by making the claim that we ought to be ontologically committed to the things that are indispensable to our best scientific theories. We will see issues with this argument based on this notion of stagnant science later, however, it is important to recognize the evolution of scientific truths. Some argue that our scientific theories are not true but act as good approximations that help us predict things to a good degree. Through time these theories get abandoned and replaced by better theories that can adapt to our current picture of the world. Putnam and Quine state that mathematical entities are indispensable to our best scientific theories, therefore, we need to acknowledge the existence of mathematical entities. If both of the premises are true then mathematical sentences are true and we are justified in believing these truths according to Putnam and Quine's argument. Before granting these two theses as true, we need to have a closer look at the roots of the argument. In order to prove The Indispensability

Argument we must show there is an overlap between our mathematical theory and our empirical theory.

According to Putnam, platonism and the acceptance of science go hand in hand. To argue against platonism while accepting the scientific method would be like "maintaining that neither God nor angels exist but claiming that it was an objective fact that God put an angel in charge of each star." Putnam and Quine rely on the philosophical stances of naturalism and confirmation holism to take them from their observations of empirical evidence to their first premise. More specifically we should be ontologically committed to the things that are indispensable to our scientific theories. Naturalism practitioners safely abandon the search for a first philosophy by leaning exclusively on the scientific method to study and identify reality. Confirmation holism rejects sections of our theories in isolation, and demands we can only confirm or deny theories as a whole. Therefore, evidence for a set of scientific sentences that use mathematics indispensably is also support for the validity of the part of the theory tied to the mathematical entities. An entire theory can be confirmed if the theory's validity rides on empirical findings. In the case of this particular argument we see how mathematical entities are only confirmed by the confirmation of the larger scientific theory. Naturalism claims the mathematical part of the theory serves as a mirror of the reality we are modeling. The truth of the theory commits us to the existence of the entities involved, including the mathematical objects.

1.2.2 Abstractness

To add abstractness to the picture does not seem so controversial if we are already on board with the existence of mathematical entities. A well rounded notion of abstraction in this platonistic sense is easily understood when taking the characteristics of a physical object (size, weight, taste, color) and seeing how these descriptions can not possibly refer to mathematical objects. This is more explanatory when one thinks about what it would be like if these mathematical entities were physical. How much would the number twelve weigh? What is the empty set made of? These questions seem far-fetched and that is precisely the point. Linnebo speaks to this phenomenon by noticing that if mathematical entities were physical then mathematicians would concern themselves with the outright physical nature of mathematical objects [18]. This is the first time we have leaned on the working mathematician and their practice to help us solve philosophical problems. Later we will take a closer look at the intricate nature of the mathematician. A more hands-on argument for the abstractness of mathematical objects is given by the fictionalist later in the chapter, however, this is already fairly convincing.

1.2.3 Independence

Independence is the last premise for the platonist to verify, and seemingly the most difficult to overcome. The platonist must show that mathematical activity is independent from all other rational activity. This seems clear when thinking about what it would look like if the converse were true. Under the conditions that the platonist wants to subscribe, mathematics is independent of our thoughts, practices, and speech. The entity of pi would still govern much of our universe even if we didn't yet have the tools to uncover the pattern ourselves. The area of a circle involving pi was not a response to human recognition of the value. A dodecahedron would have the same features regardless of human's rational activity. If an alien from planet X saw a dodecahedron it would have the same properties as our dodecahedron on earth, even though our rational activities are very different. Mathematical properties are inherent to the object we refer to, which can help explain the unreasonable effectiveness of mathematics [28]. Mathematics simply would not be as accurate if the properties of mathematical entities relied on anything else but their innate properties. There is an objectivity of mathematics that is captured by this independence thesis. Leaning on our notion of the working mathematician, it would not make sense to have a dependent mathematical theory. What exactly would it be dependent on? There are some stimulating arguments for the stubborn philosopher who needs more reason to believe in this independence. Kurt Godel proved his Incompleteness Theorem and in doing so brought to surface two interesting notions within both mathematics and philosophy. I am going to reproduce a similar outline given by Rudy Rucker in “Infinity and the Mind” because he successfully captures how clever Godel's proof really is [25].

Let's pretend a woman approaches Godel and tells him that she had a machine that had captured all of the truth in the universe, let's call this Machine X. According to the creator, Machine X was capable of answering any question correctly. Godel asks the woman for the program and internal design of Machine X, P(X), and no matter how long this program is, it must be finite because it was programmed in a finite amount of time. Godel writes out the following sentence, sentence Y: “Machine X constructed on the basis of P(X) will never say that this sentence is true.” This sentence has the same logical setup as the sentence, X will never say Y is true. Now, Godel asks X if sentence Y is true or false. There are two possible answers that Machine X could give, X could say that Y is true or X could say that Y is false. If X says that the sentence is true then the sentence “X will never say Y is true” is false, therefore if X says Y is true then Y is in fact false. If “X will never say Y is true” is false, then Y is false. If X says Y is true then it must be the case that Y is false and Machine X made a false statement. Therefore, X can never say that sentence Y is true because it would be wrong. We can now conclude that Machine X is not universal because we found a truth that was unaccounted for.

This proof utterly demolishes the search for complete logical foundations for mathematics, much to Whitehead, Hilbert, Russell's dismay. The mathematical symbolism of this proof is that any system using arithmetic is incomplete. Given a set of axioms there exist some true mathematical sentences that simply can not be derived by the given set. Within a set where an infinite amount of truths can be developed there will similarly be some that can not be derived and therefore the set will always be incomplete. This points to the phenomenon that is happening objectively in mathematics, more specifically, that the ultimate truth can not be be attained by our rationality. Godel concludes mathematical entities are independent of our thought because our best efforts can never contain all truths in mathematics. Any attempt will be incomplete or inconsistent. The independence criterion is one of many consequences of Godel's Incompleteness Theorem. In Chapter Four we will see how the inconsistent aspect of Godel's proof shattered naive–set theory.

1.3 Transitioning to Fictionalism

In this account of mathematical platonism it is important to unpack the mathematical sentence. When looking at other alternatives to platonism we will see how this facet of the proof differs. So far mathematical sentences have been taken literally and at face value. Under this face–value notion of mathematical terms, the sentences "Three is an odd number" and "Theresa is an odd girl" are both literal. Both cases require the existence of the subject in question. In Putnam and Quine's argument, when we read our scientific theories at face value they are committed to existence of mathematical entities. We are mathematical realists when we read mathematical sentences at face value. In order to assign a truth value to these mathematical sentences we are ontologically committed to the objects they refer to. There are many ways to disagree with the platonist claim, however I want to focus on one regarding semantics. Mathematical fictionalism argues that the utterance of mathematical sentences should be taken at face value as if they were pointing to some object in the world. However, we should not accept these sentences as expressing any truth value.

Fictionalism provides the principal philosophical rebuttal to mathematical platonism. Fictionalism rejects the platonist account of metaphysics and creates an entirely new and convincing ontology along the way. Unlike platonism, fictionalism denies the existence of abstract objects. Referring back to the justification of abstract mathematical entities, the possibility of these objects being physical is seemingly out of the question. The fictionalist, like the platonist, approaches the metaphysical problem by noticing that our mathematical sentences refer to abstract entities. This can not be the case because there are no such things as abstract objects, therefore, our mathematical sentences are false. When engaged in mathematical discourse, we don't mean our sentences literally in regards to truth value. Instead, we are pretending that there are mathematical objects in order to get some point across. As we look closer at the consequences we may realize that from a purely philosophical standpoint the fictionalist may have it right. One way they could go about their attack is to argue for materialism. Instead, they argue against all other possibilities leaving fictionalism on the other side as the only choice.

1.3.1 Paraphrase Nominalism

The semantic claim of the fictionalist is especially troublesome to paraphrase nominalism. Referring to our previous example, the sentence "three is an odd number" refers directly to the number three. This is not controversial to the platonist, however, it is controversial within other schools of mathematical philosophy. The paraphrase nominalist holds that the sentence "three is odd" is really saying "if there were numbers, then three would be odd." The fictionalist responds to this by claiming the view of paraphrase nominalism involves an empirical claim about what we mean when we use mathematical language that seems unlikely. It is hard to believe that when mathematicians speak about their theories they really mean to say something like what the nominalist is pushing for. There is a lot of evidence that mathematical discourse should be read at face value, and very little for the converse. If mathematicians had the intention to be understood non literally it seems as though they would give some indication of this desire. However, there is an additional issue with this appeal to mathematicians that we will see when we look at the application of naive–set theory in Chapter Four.

The paraphrase nominalist might reply by saying that they do not need to appease the practice of the mathematicians. If they do reply with this, then their view will quickly collapse into fictionalism because they will be claiming that the discourse of actual mathematicians is false. If this is true, then it seems like they are simply fictionalists that think we should alter what we mean when we talk about mathematics. We can accept that mathematical sentences should be read at face value because paraphrase nominalism declines to offer a counter argument the fictionalist needs to take seriously.

1.3.2 Neo-meinongianism

The fictionalist must enforce their premise about the impossibility of the truth value in mathematical sentences. Fictionalism says that when we read sentences at face value and we claim that they are true, then there must be some mathematical object the sentence is referring to. Neo-meinongianism is the school of philosophy that claims the truth of mathematical sentences isn't contingent on the positive existence of abstract objects. The neo-meinongianism and the fictionalist are disagreeing about what must be the case in order for something to be true in ordinary conversations. The view is that there are non-existent entities that numerals refer to. According to the platonist, the sentence "17 is prime" says "there is something that is 17 and that thing is prime". Here if the first sentence is true then the second has to be true. On the other hand, to neo-meinongianism the sentence "17 is prime" is true without 17 being an abstract entity in existence.

They agree with the fictionalist that the mathematical sentences seem to be about irreferable abstract entities. If they had a standard acceptance of truth then the neo-meinongianist would collapse into fictionalism. To be distinct the neo-meinongianist must make a claim about the ordinary sense of the term true when the claim is about something that does not exist. The fictionalist can implore our intuition when we say a sentence is true. For example, when one says “it is true the grass is green”, our intuition is to think that there is such a thing as grass.

This semantic problem isn't the only issue with neo-meinongianism. The platonist offers another point of contrast. On the surface neo-meinongianism looks like we still have a way around platonism, especially if it collapses into fictionalism. Let's consider two notions of true, called X and Y. Truth X refers to the kind of truth that entails the existence of the subject of conversation, like in the grass example above. Truth Y is the kind of truth that the neo-meinongianist is trying to put into practice, in other words, Y is the kind of truth from a sentence where the sentence doesn't successfully refer to the existence of the object in question. A possible platonist response would be to slightly tweak their philosophy to show they have been arguing for X truth all along. When we use the word “true” in ordinary language we are unaffiliated whether we are talking about X or Y. This question only surfaces when philosophers get involved. The Indispensability Argument argues for X truth in mathematics. It seems as though the neo-meinongianist semantic thesis was irrelevant to this overall conversation about truth. We should really focus on the platonists argument for the X kind of truth. Therefore, it boils down to the two philosophies still in question, fictionalism and platonism.

1.3.3 Physicalism

The secured abstraction of mathematical entities is the next step in the fictionalist argument. We have seen a partial argument for the necessity of mathematical entities being abstract, however, the fictionalist gives us a different account. The fictionalist agrees with the platonist about the necessary abstraction of such entities as mathematical objects. Physicalism and psychologism give two opposite viewpoints that try to explain mathematical objects without abstraction. Physicalism claims our sentences refer to physical objects. This idea is less controversial when one thinks about elementary mathematics, however, when we consider mathematics in multidimensions it becomes less apparent. In knot theory, for example, we see the physical representation of the trefoil knot by simply making one out of string. Physicalism falls short when we are asked to find a physical representation of the knot with infinite crossings, or the knot made from an infinite number of trefoils together in a line. This issue of infinity is especially apparent in set theory, made more confusing with the idea of different sizes of infinity. Even if a physicalist could figure out a way to make the mathematical concept of infinity somehow physical, the notion of different sizes of infinities would surely stump them.

Georg Cantor, born in Russia in 1845, was the inventor of set theory. Set theory is of extreme importance in our conversation of philosophy of mathematics. Cantor allowed his mathematical achievements to shape his theological viewpoint. He claimed that a world without infinite numbers could not possible have a God with infinite power. His quest to prove the existence of the multitudes of infinite sets was perpetuated by his idea that God was the Absolute Infinite and all other infinities consequently were smaller than this one. The proof of this concept is relatively simple, and can be reproduced with an elementary notion of set theory. Cantor's diagonal argument was published in 1891. This proof shows that there are infinite sets that are unable to form a one-to-one correspondence with the infinite set of the natural numbers. His proof begins with a set that we are familiar with, the counting numbers. Let's define this as set S. $S = \{1, 2, 3, 4, 5, \dots\}$ The infinity property of the counting numbers is that we can always get another number by adding 1 to the last number in the set. Cantor defined a set as "countably infinite" if an infinite set has members that can be illustrated as a one–to–one correspondence with the counting numbers. These countable infinite sets have some interesting properties that provoked Cantor. For example, there are subsets of countably infinite sets that are also countable infinite. One might think there would be half as many even numbers as counting numbers, however, Cantor proved that you can make a one–to–one correspondence with these two sets. He paired the numbers up in the following set P.

$$P = \{(n, 2n) \qquad \text{where} \qquad n = 1, 2, 3, 4, \dots\}$$

This shows that there is still a countable infinity when we look at half of a countable infinity. He then showed that the set of real numbers is not a countable infinity by proving it is not possible to have a bijection between the reals and the natural numbers. He does this by using his diagonal method with creating a numbered list of the elements within an infinite set and flipping numbers to uncover a missing string. From here, Cantor concludes there are infinitely many sizes of infinite sets because one could complete this process with any infinite set at come to the same conclusion. For a complete translated proof refer to Keith Simmon's translation of Cantor's 1892 masterpiece [**?**].

Cantor shows through his proof that these infinities, although sometimes distinguishable (as seen with the real numbers vs. the listable integers), are mysterious in ways that are unimaginable precisely because of the finiteness of the human mind. It seems as though this proof, born from set theory, is one of the greatest challenges for the physicalist because of our inability to comprehend even the smallest infinity. The physicalist, if still serious about their philosophy, would have to be able to point to some physical entity for every mathematical entity because physicalism would unravel with the possibility of one aspect being unaccountable.

1.3.4 Psychologism

Psychologism is perhaps more plausible than physicalism, however, the issues with psychologism arise in a similar way. Psychologism, like physicalism, tries to argue against the idea that the existence of mathematical entities entails that these entities must be abstract. This view describes ordinary mathematical sentences as the true descriptions we make about the objects in our minds. Again, this seems plausible for elementary mathematical objects such as "3" however, the idea of infinity, let alone infinitely many infinities, seems unbreachable by this analysis. If psychologism is a possibility we would have to have infinitely many ideas in our minds, which simply isn't the case. Notice how this thesis is not saved by arguing that we have the idea of infinity in our minds. In order to save psychologism, there needs to be proof of infinitely many ideas in our minds because there are infinitely many mathematical entities that need to be accounted for. Therefore, when we utter mathematical sentences our theories are not descriptions of these mental entities. Another issue with this thesis is that it seems to miss out on the objectivity of mathematics. If we believe that mathematical entities are just figments of an individual's mind, then we better have an explanation for why mathematics is so universally agreed upon.

1.3.5 Humanism: The Social

We have seen the standard arguments for the mental and the physical existence of mathematical objects. In his book, "What is Mathematics, Really?", Rueben Hersh offers another possibility to the mix: the social. Hersh claims that mathematics is essentially a social phenomenon and a human activity. Mathematics is socially evolved and can only be understood within a social context [15]. Based on this definition of mathematics, he does not see the need to look for a definition of mathematics beyond this cultural meaning.

Hersh gives an example of a 4–dimensional cube as his initial "inquiry into mathematical existence" [15]. He notices by appealing to the characteristics of the 1–D, 2–D, and 3–D cube, that there is a pattern than extends to the cube outside of our perception. This experiment provokes Hersh to ask if the 4–D cube exists. If so, then what kind of existence is it? If not, then how can we infer so many details about it? Furthermore, does the 3–D cube exist in our space and time? Hersh claims that his humanist view of mathematics accounts for these questions more adequately than other philosophies. Humanism claims this cube exists within the shared consciousness of people as a thought or idea [15]. In a more simple case, Hersh explains how to think about the integers. The number "three" for example, is considered culturally as a noun and an adjective. It is noun insofar as it can refer to a real object but there is a social process that separates this distinction and creates a "shared concept in the minds/brains of people who know elementary arithmetic" [15]. On the other hand, the number "three" is an adjective because it amounts to the process of counting. Hersh claims the counting numbers are really finite because we do not see people counting to 9237837829^{39}. This idea is plausible with these specific cases, however, humanism falls short describing other mathematical truths– such as infinity. We have seen how psychologism and physicalism fumble with the idea of infinities, especially infinitely many sizes of infinity. Hersh agrees with psychologism that the brain is finite, but argues that "it is not the infinite that our brains generate, but the notions of the infinite" [15]. To claim that the "notions of infinity" can exist within the brain needs some back up that Hersh does not offer. This might not be enough of a reason to completely give up on Hersh's humanism, but the implications of this issue should push Hersh to address this more seriously.

1.3.6 Against Platonism

For the final premise of the fictionalist argument we see the direct jab at platonism, more specifically, there are no such things as abstract objects. This step is the most difficult because as we have seen the fictionalist has developed a standard argument to rule out most of the other philosophies in question. However, because the platonist agrees with the fictionalist under the aforementioned claims about semantics and the word "true", the fictionalist needs to come up with something else to overrule the platonist. In the attempt to cross platonism off of the list, the fictionalist appeals to their theory of human knowledge. This theory has been extremely popular within the history of philosophy and is rooted in the importance of sense perceptions to form belief. In order to form justifiable knowledge of an entity, one has to receive a sense perception from that entity in some way. According to the platonist picture, these mathematical objects exist outside of our reality. Therefore it is unreasonable to think we are able to sense them with our five senses let alone form any kind of substantial knowledge about them. In other words, causal contact is necessary to have a sense perception and according to platonism that this is not how mathematical entities exist.

James Robert Brown offers an explanation as to how platonism can respond to this problems of access [5]. He notes a disconnect in our understanding of everyday physical objects about how a sensation becomes a belief. The physiological process of how sense perceptions operate is understood, however, the belief formation process is still a grand mystery [5]. The nominalist claims the platonist must come up with an explanation as to how mathematical entities become mathematical beliefs. However, this issue is no worse than our inability to understand this phenomenon in the world outside of mathematics.

Brown gives a response to the additional issue of causality. The abstraction of these objects seems to disable possible causal connection between the realized object and the observer. Favored among naturalism, the causal theory of knowledge seems reasonable. For every thought there must be a causal chain between the object and our minds. I know there is a cup of coffee in front of me because photons from the coffee enter my eyes. I know events from the past because people who had direct causal contact recorded the events and their recordings were brought to me through this chain. If this model is correct, there would be no possibility of knowing abstract objects. Brown shows a flaw in the argument by providing an example in the physical world where this causal chain is unapparent.

There is a thought experiment in quantum mechanics posited by Einstein, Podolsky, and Rosen. In this experiment, there is a decay process that produces two photons going in opposite directions towards detectors on either side of the space. Each detector has polaroid filters that determine if the approaching photon has the spin–up or spin–down property. The two outcomes of the photons are always correlated. If one of the photons is spin–up then the other is necessarily spin–down. There is an aspect of randomness because before the experiment we do not know which one will end up at either side. The question is within this perfect correlation of the photons. One idea is that the measuring tool causes the outcome of the other. However, we can rule this explanation out by applying special relativity claiming that no causal power travel faster than light. Brown notes that the two simultaneous measurements are outside of each other's light cones so there is no possibility of them causally affecting the other. Another possibility is that there is something at the time of the creation of the photons that explains this property. This cannot be the case because of the Bell result showing that, "such a common cause predicts a different measurement outcome than either quantum mechanics predicts or experience determines" [5]. He concludes from this case that knowledge does not necessarily rely on a casual connection. If a person was on one side of the experiment and saw the result spin–up they could automatically infer the other side was spin–down without having any causal connection to the other wing. This example is thought–provoking, but rests on profound presumptions from the physical world. My issue with this example is that we are only able to infer any information about the photon because it exists in binary. If, for example, there were three possible states then we could not know anything about the photon. Our supposed causally unaffected knowledge rests on the knowledge that the photon must exist in one of two states which is previous knowledge.

The argument against abstracta is one of the most convincing arguments against platonism, however if someone is skeptical about this metaphysical issue there is an interesting mathematical argument against platonism as well. Originally by Benacerraf, the arithmetic multiple–reductions argument starts by postulating a set of numbers. Benacerraf claims that if there exists a set of abstract entities that are coherent with our theories in arithmetic then there are infinitely many of these sets. Furthermore, when we pick out the set of integers there is nothing metaphysically special about it in comparison to a different one of the infinite sets. However, Benacerraf notes that if we are platonists then we believe that there is a unique sequence of abstract entities that are the integers, and therefore platonism must be getting it wrong. He uses the structuralist view of the natural numbers to make the argument. He claims that numbers cannot be objects because when we give the properties of numbers we are simply characterizing the abstract structure. This view means that the numbers have no properties besides the properties that they have within their sequence relationally.

It is unclear how successful this argument is, however it is worth mentioning because it forces the deciding philosopher to ask questions about the nature of these abstract sets. If these arguments against all other possible philosophies are granted successful, then fictionalism is the only option left and we would be foolish not to believe it.

Now that I have explained the two propositions given to us by the platonist and the fictionalist it is time to decide on a stance as to who got it right. To reiterate, the question that has been up for debate between platonism and fictionalism is the question of mathematical objects and their questionable existence. A very simple recap of the two arguments is that the platonist is lacking on epistemological accounts, whereas the fictionalist is fumbling where metaphysics are involved. The problem here has seemingly reached a philosophical dead end because the only way for one argument to be refuted is for the other argument to restate one of their premises.

I want to move my conversation towards a focus on The Indispensability Argument given by Putnam and Quine. This argument is one of the more convincing arguments for platonism and therefore needs to be looked at closer to make sure we aren't missing anything before we admit the existence of abstract mathematical entities. The Indispensability Argument has high stakes because it illustrates accepting mathematical entities as a necessity to scientific inquiry. The Indispensability Argument might be precisely why followers of platonism were willing to take the metaphysical leap.

2 The Indispensability Argument and Scientific Inquiry

2.1 Another Look at The Indispensability Argument

While the first chapter merely glanced at The Indispensability Argument, we are now going to take a closer look into the strength of this argument. It garners its first premise from an observation embedded in the philosophy of science, particularly in the way that our theorems in mathematics act as an indispensable part of our best theories in science. The practice of science relies on the ability to use mathematical objects as explanatory entities. When scientists have evidence for a specific theory, the evidence corresponds to the theory as a whole as opposed to referring to the individual hypotheses. Given these two points, if we have evidence for a scientific theory, then this set of evidence is also evidence for the mathematical principles it presupposes. In addition to this claim, The Indispensability Argument says that science is the best vehicle for gathering evidence pointing to truth and existence. Therefore, the mathematical principles are true and the mathematical entities exist as much as the entities proposed by the scientific theories. This last point adds an interesting complication; by accepting our theories in science as true, we are forced to accept the mathematical entities involved as well. Vineberg put it bluntly when she claimed accepting the best results in science while denying abstract mathematical entities would be failing to accept the consequences of affirming the truth of our scientific theories [27].

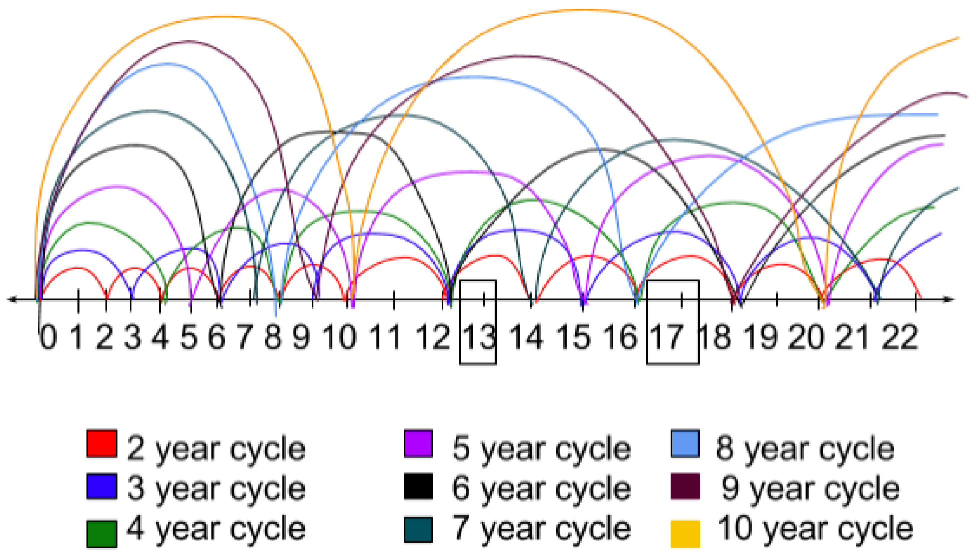
The acceptance or denial of this argument comes down to a study of the nature of mathematics, and in particular, how much explanatory power we warrant pure mathematics. All it takes is scanning through any science textbook to notice the prevalence of mathematics as support for all applications. Mathematical anti-realists such as Benacerraf have tried to come up with a science without mathematical algorithms or formulas. Even if this were somehow possible, the resulting "science" would still be lacking something crucial. As we shine this spotlight on the practice of science, we will see that the picture offered by Benacerraf only accounts for a small piece of what we consider the evolving field of science today. The complex relationships between mathematics and science are of utmost importance when studying The Indispensability Argument. I want to look at some specific cases to illustrate how this relationship should not be overlooked.

2.2 Mathematics Playing An Explanatory Role

First, let's explore how physical phenomena can be explained by science. The following examples are of mathematical truths, rather than explanations that merely utilize math. Alan Baker draws the distinction between representational mathematics and explanatory mathematics. Mathematical algorithms represent large sets of data, whereas mathematical truths explain physical phenomena. Baker quotes Mark Steiner noting that there are numerous instances where "when we remove the physics, we remain with a mathematical explanation of a mathematical truth!" [1]. The case of the North American cicada demonstrates exactly what Steiner had in mind.

2.2.1 The North American Cicada

The life cycle of the North American cicada highlights a mathematical truth about prime numbers. The cicada is a locust–like bug that buries its larvae in the ground. These larvae resurface simultaneously after a period of 13 to 17 years as mature bugs to mate, and die shortly afterwards. Although the life cycle varies with climate, within a single population the number of years is always synchronized. Concepts in number theory explain the ways this mass emergence after a prime number of years can be advantageous to the species. The periods of 13 and 17 years are notable, given that these numbers are not only prime but consecutive primes. The notion of a least–common–multiple (LCM) explains why cicada life cycles settled on these prime numbers and how this is beneficial to the cicadas' survival. The LCM of two numbers, x and y, is the smallest possible number z, where x divides z and y divides z. For example, the LCM of 3 and 4 is 12 because 3 divides 12 and 4 divides 12. When we have one prime number, the LCM of a prime and a composite will be the prime number multiplied by the composite number unless the composite number is divisible by the prime number. For cicadas, the LCM explains how the prime–numbered life cycle results in a diminished risk of being eaten by predators. The prime-numbered life cycle means that the life cycles of most other species do not overlap with that of the cicada. For example, the 13 year cycle cicada will be less likely to have intersections with predators whose life cycles are any number other than 1 or $0 mod 13$. On the other hand, if the cicada has a life cycle of 16 years we see that it intersects with predators of life cycles of 1, 2, 4, and 8 years. This hypothetical cicada life cycle would mean that a predator would have an increased chance of being able to eat the cicadas when they emerge for about five weeks to mate. The cicadas plant their larvae in the soil and then die after these brief weeks of life above ground. Below is a diagram I made to illustrate the first ten possible life cycles of predators and how they fail to overlap the primes in question. The only numbers that interact with 13 and 17 are 13 and 17 themselves. The other numbers on the number line are relatively prime to 13 and 17. Notice that the cicada could have had the same evolutionary advantage if their cycles were every 11 or 19 years. Biological factors, such as body size, put the cicadas within this specific 13 or 17 year window. If we were dealing with a smaller cicada perhaps we would see it shift down to 11 or 13.



The notion of an LCM also explains how the cicada finds an evolutionary advantage in avoiding other species by limiting the potential for hybridization. The theory that prime numbers have the lowest–common–multiple provides the best explanation for the evolution of prime–numbered life cycles. This theory comes from number theory, a realm of mathematics devoted to numbers. The Indispensability Argument holds that we must acknowledge the existence of any object that plays an indispensable explanatory role in our scientific theorems. In the case of the cicada, we see that number theory plays this role.

2.2.2 The Honeybee

Joe Morrison's talk, "The Ontological Extravagance of Honeybees," presents another example of mathematics playing an explanatory role in science. The

Honeycomb conjecture posits that bees solved a recondite problem, and have thus made their cells the proper shape to contain the most amount of honey using the least possible amount of wax in their construction. Bees solved this problem using the tools given to them through evolution. We are not claiming that these smart bees are mathematicians, only that they successfully solved this efficiency problem in order to reap the benefits for their species. Given one swarm of bees making a mathematically efficient honeycomb, and another making their honeycomb with excess wax, the more proficient swarm will be more likely to survive and pass on their wax-saving genes. Hypothetically, if there were a beehive using a more efficient shape than the hexagon, then the bees using the hexagon would eventually die out. Disregarding wax, the problem at hand is one of geometrical optimization. In other words, what shape has the largest surface area but the smallest possible perimeter?

This question dates back to 36 B.C., when Marcus Terentius Varro notes how the bee's chamber has six angles and claims, "the geometrician proves that this hexagon inscribed in the circular figure encloses the greatest amount of space" [2]. Although we have been observing bees for thousands of years, this honeycomb conjecture was not formalized until June 1999 by Thomas C. Hales in his 24 page proof,"The Honeycomb Conjecture". This may seem shockingly late, considering bees solved the problem originally. Hales had to use the notion of sphere packing in order to completely prove the conjecture without employing the convexity hypothesis. Before 1999, the hexagonal honeycomb conjecture was merely based on biological observation. Mathematics served the role of making this observation more rigorous by unifying the results without relying on our instinct about animal behavior and perhaps our religious affiliation. The explanatory power of mathematics becomes crucial for our understanding of the physical system of the bees, as well as providing an attractive argument for evolution. Hexagons are shown to be the most effective way to tile an area using the smallest perimeter by the mathematical truth highlighted in the two different approaches.

2.2.3 The Kirkwood Gaps

Kirkwood Gaps provide a final example of the power of explanatory mathematics. Daniel Kirkwood was the first person to notice the gaps in the asteroid belt between Jupiter and Mars. These gaps initially confused many scientists studying the orbits of these planets, however, by looking at the eigenanalysis of the orbits involved they derived the precise location of these gaps. The clearest gaps are dispersed at fractional orbit ratios of one-half, two-thirds, two-fifths, and three-sevenths. It was later discovered that gravitational resonances cause Jupiter's orbit to be unstable. If an asteroid is in these unstable orbits, it gets further and further off course until the asteroid is adopted by a stable orbit.

Colyvan explains the varying level of mathematical involvement within these Kirkwood gaps [10]. Starting with the less impressive case, we see eigenvalues of an operator identify the fraction of the unstable orbits compared to Jupiter's orbit. These discoveries are important, but they fail to explain why the Kirkwood gaps make it impossible for an asteroid to maintain a stable gap. The noteworthy explanatory involvement comes when looking at the relationship between the fractions through functional analysis. Through this analysis, scientists have found that the orbits are unoccupied because the vector operator squishes or stretches vectors. Colyvan notes that an asteroid circles the sun three times for every one rotation of Jupiter, will thereby be drawn into repeated interactions with Jupiter of a type to eventually pull it off course [10]. The similarities between the orbits are mathematical and not visibly physical, which means there is some mathematical truth hidden within these gaps. Here, the notion of eigenvalues in the study of linear algebra make it possible for us to understand a phenomenon happening in space. Without studying these orbits with mathematical rigor, these gaps in the asteroid belt would remain a mystery since there is no strictly physical explanation for their existence.

2.3 Scientific Discovery and Development

In all three of these cases, the explanatory connection is purely mathematical which makes it arguably impossible for scientists to explain the phenomena without mathematics. Thus, the practice of science goes beyond reading physical results, and in these cases the explanation requires an appeal to mathematical truth. When nominalists try to refute cases such as these, they miss out on the deep importance of mathematics in explaining the world. This lack of attention to explanatory power is the first mistake seen in the nominalists' attack. Alan Baker brings to light the second mistake, as he introduces the ideas of scientific development and scientific discovery.

We can clearly point to cases in science that suggest further mathematical innovations, such as Fourier analysis, however, more often than not, it is the formalization of mathematics that suggests the development of new physical theories [2]. The symbiotic relationship of calculus and mechanics provides a clear, somewhat controversial, example of this. Another example applies formal group theory to particle physics, which lets us predict the activity of entire families of unobservable subatomic particles [2]. Nominalism focuses their attention on a transection of a stagnant point in the history of science, which is not an adequate sample based on how science moves forwards hand in hand with mathematics. John Burgess goes so far as to say that if science goes nominalistic, that future science may never be discovered. As far as the nominalist successfully reformats specific examples, it is still a reformation that would not exist without the original platonistic account.

The nature of scientific discovery gives us further reason to discount nominalism. Baker underlines the distinction between proof verification and proof discovery, which helps us view mathematical practice through a new lens. Proof verification stems from the desire to verify something that is already known. In this kind of proof with a predetermined conclusion, the mathematician needs only to arrange the premises accordingly. Euler's totient function is an example of proof verification, and it follows from the fundamental theorem of arithmetic. This kind of proof creates lines connecting existing mathematics to help unify results in various fields. On the other hand, proof discovery is the type of proof that arrives at a previously unknown truth by combining previously known axioms and theorems. This distinction between mathematical proofs is not taken into consideration by nominalism.

2.3.1 Asymmetry In Cryptography

A great example of how this asymmetry can be exploited is seen in cryptography. One of the main coding systems used to transfer secret information is called the Diffie–Hellman key exchange. Baker and Colyvan focus on another public key system called RSA, however, Diffie–Hellman is slightly more user friendly. The Diffie–Hellman system is created to make encoding the message as easy as possible and makes decoding the message as hard as possible. Consider an interaction between two colleagues Adam and Beth, they are trying to share a secret without Eve knowing. They start with a public prime, let's use 67 with a public base of the primitive root 7. Adam chooses a secret integer a=6, and calculates $A \equiv 7^a \bmod 67$, which is equal to 64. Now Beth does the same calculation but with her own secret integer b=3, she gets B=8, then she send this to Adam. Now Adam computes $s \equiv B^a \bmod 67$ with his secret a, and gets the number 40. Beth also computes $s \equiv A^b \bmod 67$ with her secret b and also gets 40. This number, 40, is the shared secret between Adam and Beth that Eve does not know. In the NSA they are using much larger base prime numbers, however, even with this example the decoding is non–trivially difficult. This is thanks to the (hopefully forever) unsolvable discrete logarithm problem (I say hopefully forever because if it were to be solved our world would go to shambles.) In this example Eve could feasibly try all possible private keys, however, to decode the typical problem is impossible for the best modern supercomputers to do in a reasonable amount of time. For example, the typical time to factor a 100–digit product is around 60 years. Considering how simple it is to encode one of these crypto–systems, this is remarkable. Baker notes that in this example the encoding of the message is similar to verification and the decoding is analogous to discovery. The proof discovery relates to the mathematical truth about the difficultly to decode the discrete logarithm problem.

This mathematical disconnect shows that it is unwise to treat all mathematical problems with the same philosophical attitude. Baker claims "if the resources needed for proof discovery exceed those required for proof verification then mathematics might be dispensable for the latter task without being dispensable for the former one" [2] When examining The Indispensability Argument, it is possible that some rebuttals only capture a narrow segment of mathematical results. Even if proponents for nominalism could show the dispensability for mathematics in proof verification, it is not the case that this would translate to mathematics being dispensable to the discovery of new results in science [2]. For example, if nominalists such as Benacerraf can recreate scientific theorems without using numbers, it does not mean that the mathematics originally involved with discovery are dispensable to the theory. Mathematics can be responsible for the development and evolution of the theory in ways that cannot be replicated. Baker urges philosophers studying mathematics and science to stop focusing on the static aspect of theories and try to look at the larger picture of how science moves forward.

2.3.2 A History of Complex Numbers

At first glance, a static view of the complex number system looks like nonsense. What was once considered a mathematical trick of working with $\sqrt{-x}$, now complex calculus is applied to control theory, fluid dynamics, electromagnetism and electrical engineering. Complex numbers have a rather controversial history. The applicability of complex numbers in physical theories has contributed to the unification of previously distinct mathematical fields, and ultimately the unification of mathematics and science. For years mathematicians refused to believe in these numbers which caused tension within the community. However, after they were geometrically represented, people were somewhat less hesitant. Descartes famously coined these numbers as "imaginary," claiming "one can imagine as many as I said in each equation, but sometimes there exists no quantity that matches that which we imagine?" [12].

The unifying nature of complex numbers make them of utmost importance to the philosopher. Demonstrating the symmetrical nature of mathematics, complex numbers built the bridge between pure mathematics and applied mathematics. Colyvan claims that complex numbers had their finest hour when they were used to solve real integrals that could not be solved with the integration techniques available [9]. In calculus a student learns a handful of tricks when faced with an integral. There are certain cases where the only way to solve the integral by hand is through Euler's formula. It seems strange at first to complicate the integral by introducing complex variables, however, the identities of the trigonometric functions uncover aspects of the integral that were initially hidden.

Euler's formula is also used in second order differential equations acting as a middle step in the analysis of real (real as in not complex) functions. Here, complex numbers are a tool to unify exponential and trigonometric functions. Colyvan points to an example in differential equations that segues nicely into the physical application of the system. He defines the complex numbers as an extension of the typical operations in the reals with the addition of $i = \sqrt(-1)$ Complex exponentiation leans on Euler's formula, $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ where $\theta \in R$. This identity can be reworked to solve for sine and cosine in terms of the complex variable, z. These equations look like this,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

This rewriting of the sine and cosine functions shows that trigonometry in the reals is a specific case of a larger area. This coexistence of the exponential and trigonometric functions enables us to proceed gracefully with solving second-order differential equations. The applications of second-order differential equations in physics and engineering are impressive. A typical second-order linear homogeneous ordinary differential equation with constant coefficients takes the form, y"+ y'+y=0. I will proceed with a slightly more complicated example from the textbook "Differential Equations" by Paul Blanchard [4].

1. We want to solve

$$y'' + 4y' + 13y = 0 \tag{1}$$

2. We assume that $y = e^{rt}$ is a solution, so we have

$$y' = re^{rt} \qquad \text{and} \qquad y'' = r^2e^{rt} \tag{2}$$

3. Now, we substitute these back into the original equation

$$r^2e^{rt} - 4re^{rt} + 13e^{rt} = 0 \tag{3}$$

4. Now we divide by e^{rt}

$$r^2 - 4r + 13 = 0 \tag{4}$$

5. This quadratic doesn't factor, therefore we turn to to quadratic formula to find the roots

$$r = 2 + 3i \qquad \text{and} \qquad r = 2 - 3i \tag{5}$$

6. We conclude the general solution to our differential equation is

$$\begin{aligned} y &= a_1 e^{(2+3i)t} + a_2 e^{(2-3i)t} && \text{Substitution} && (6) \\ y &= a_1 e^{2t} e^{3it} + a_2 e^{2t} e^{-3it} && \text{Rule of Exponents} && (7) \\ y &= e^{2t}(a_1 e^{3it} + a_2 e^{-3it}) && \text{Factoring out the } e^{2t} && (8) \end{aligned}$$

7. This general solution involves complex exponents. We must use Euler's Formula to find the real answer. We find that

$$y = e^{2t}[a_1(\cos(3t) + i\sin(3t)) + a_2(\cos(-3t) + i\sin(-3t))] \tag{9}$$

$$y = e^{2t}[(a_1 + a_2)\cos(3t) + (a_1 - a_2)i\sin(3t)] \tag{10}$$

8. Now we let

$$c_1 = a_1 + a_2 \qquad \text{and} \qquad c_2 = i(a_1 - a_2) \tag{11}$$

9. Substitute these in to get the final answer

$$y = e^{2t}[c_1\cos(3t) + c_2\sin(3t)] \tag{12}$$

This example shows how we use complex numbers to get to a real answer. Euler's formula acts as the bridge between these two areas. This kind of differential problem is indispensable in order to explain physical phenomena in fluid mechanics, heat conduction, and to analyze population models. This elementary differential equations problem indicates the deeper nature of complex numbers. Imaginary numbers were thought to be entirely unreal, until they proved so useful in describing the structure of space. The introduction of complex variables help illuminate the underlying structure of the mathematical truths at play.

2.3.3 The Utility of Quaternions

Alongside the rigorization of the complex numbers, quaternions provide another great illustration of the complicated relationship between math and science. This case study given by Alan Baker and others illuminates the importance of utility when declaring a theorem as acceptable. In 1843 William Rowan Hamilton was on a quest for a three dimensional extension of complex numbers with the form $x + yi + zj$. He was particularly drawn to this form because he saw the possibility of it being used in physics for modeling of actual three dimensional space. He was unable to find such a framework, however, along the way he did discover a four dimensional extension of the complex numbers, quaternions. Quaternions are made up of three vector components, xi, yj, and zk, and one scalar element, w. These four components, $w+xi+yj+zk$, compose an associative but non commutative vectorial system.

This distinction between associativity and commutativity enabled quaternions to set the stage for algebra as a separate and rigorous field in mathematics. This system was the first to show there could exist a system that gives up commutativity without becoming trivial. This idea paved the road for division algebras, a whole scheme of algebraic structures which each have two operations. In addition to algebraic uses, quaternions were used in number theory to represent prime numbers as a sum of two squares. Quaternions helped introduce the contemporary system of vectors and scalars seen in most fields of mathematics and physics. When we consider a quaternion with a scalar part equal to zero, $xi+yj+zk$, we see how similar it is to the structure of a modern vector. Hamilton held a persistent geometrical lens throughout his mathematical infatuation with quaternions. Baker points out that he made the multiplication requirement for them to satisfy the 'law of moduli' which guarantees that every multiplication also has an inverse multiplication [2].

This geometrical side of quaternions made them a plausible tool for physical applications, but they never became the tool Hamilton had hoped. To his dismay, the specific geometric interpretation, particularly the rotation of the vector and the conical rotations, made quaternions an undesirable choice for physicists. Before quaternions were completely replaced by modern vectors, they were used for their transparency compared to Cartesian coordinates (Tait, 1875). The vector offered many benefits for scientists, and eventually pushed quaternions out of the picture. These benefits stem mostly from the intuitions of physicists being drawn to vector spaces. Specifically, the separation between the scalar and vector parts make it easier to formulate theories.

The case of quaternions is crucial to unpacking notions of indispensability. This study opens up a new sense of mathematics and how it relates to our progressive scientific theories. Strictly speaking, quaternions are not indispensable to any of our modern scientific theories – but they are indispensable to the history of physics and that is perhaps more powerful. The fault of quaternions in algebra was a gain for vectors in physics, and without this teeter totter of utility we would not be where we are today in physics. The road for quaternions, somewhat surprisingly, is not a dead end afterall. Quaternions were left in the dust due to their four dimensionality and their non-commutativity, allowing the modern vector to be the phoenix from their ashes. However, these faults were considered virtues when applied to Einstein's theories of special relativity and quantum mechanics. For the first time we begin to see the deductive indispensability of quaternions in physics.

This fuels Baker's conversation about distinguishing indispensability and deductive indispensability. Baker notes that this distinction has remain ignored by many philosophers, however, in light of the next section it is crucial to understand. He points to an example within the history of infinitesimals. Baker asks if infinitesimals were previously shown to be deductively dispensable before Cauchy, why were infinitesimals not dispensed by later 18th century mathematicians? [2]. This is a misunderstanding within the utility of infinitesimals. Although quaternions were not deductively indispensable to science they were completely indispensable for the discovery of unknown results [2].

The four dimensionality of quaternions was embraced by the development of special relativity. To model special relativity scientists needed something that would show the divide between the two spacetime points and the spatial separation minus the temporal separation. The norm of the real valued quaternion provided the perfect reformulation of special relativity [2]. Perhaps more impressive, the silver lining of the non-commutativity in quaternions is seen when they are applied to quantum mechanics. The application of quaternions to quantum mechanics has not only reformulated the theory but has unsurfaced other implications that go beyond the theory without quaternions. This is when this mathematics becomes indispensable to our scientific theory. Douglas Sweetser describes why quaternions are especially indispensable in quantum mechanics in his paper, "Doing Physics With Quaternions." The product of a quaternion transpose with another quaternion has the distinct property of a complete inner–product space. When applied to calculating the tensor product in quantum mechanics, the non–commutativity of quaternions ensure dependent results. This means that in any two systems of quaternionic quantum mechanics there will be a complementarity between some of the properties in the systems [2].

On the surface this might not seem like a groundbreaking discovery, but the fact that quaternions uncover a truth about the impossibility of independence of two systems is remarkable. This truth is brought to light by the properties of quaternions, specifically their non–commutativity. In this example, our choice of formalizing quantum mechanics with quaternions made it possible to further understand the structure of the physical reality at play. Quaternions prove to be deductively indispensable to our theory of quantum mechanics.

The hot and cold history of quaternions introduces an idea about scientific progress that is crucial to the study of indispensability. If quaternions were completely thrown out and replaced with vectors we would see quantum mechanics less clearly. With the exponential growth of scientific progress it is impossible to claim that a dispensable mathematical entity will always remain dispensable. The account for why quaternions were dispensable to early physics, specifically their non-commutativity and their four dimensionality, was precisely the reason why they became indispensable to quantum mechanics years later.

2.4 Questions About Indispensability: The Worries of Penelope Maddy

These examples give fruitful reasons to believe in The Indispensability Argument. However, there are important questions about the theory that have yet to be completely developed. Geometrical analysis, number theory, linear algebra, eigenanalysis, functional analysis, and complex analysis are some of the areas that look promising through an understanding of mathematics through science development and discovery. Penelope Maddy strategically rejects both of the premises of The Indispensability Argument but it is unclear if she is able to account for the explanatory power of mathematics we have discussed. Once we become familiarized with Maddy's objections we will investigate possible counterarguments. Maddy, once an eager mathematical realist, points to the practice of scientists and draws out a potential disconnect between naturalism and confirmational holism.

Her first objection is primarily given in her 1992 paper, “Indispensability and Practice,” and then made more concrete in her paper, “Taking Naturalism Seriously.” She appeals to the practice of scientists claiming that how they treat separate components of well–confirmed theories is rarely how confirmation holism would predict. The nature of science is to take the past discoveries and try to see it through a new lens until eventually we have a new system to better represent our world. Maddy provides an example about atomic theory, which was not accepted as true until the beginning of the 20th century through scientific justification. She claims that the majority of scientists remained skeptical of atomic theory although it was well-confirmed as early as 1860 [21]. Scientists remained comfortable with believing directly verifiable results from atomic theory as they weighed on their direct theory without believing in the atom's existence. In the case of a specific theory they granted the atom, however, they were not confirming the existence of the atom at large. Maddy sees this situation as disagreeing with Quine's confirmation holism because if he was correct then the scientist should have accepted the atom's existence the moment it became indispensable to their theories. In reality, scientists remained skeptical about the existence of atoms until as late as 1904 although atoms became indispensable to science around 1860 [9].

There is minimal leeway within this observation looking at Putnam and Quine's next premise, naturalism. Naturalism grants that we must lean on the working scientist and in this case the working scientist is giving us reason to reject the first premise. The inspection of the scientist in this situation indicates that they do not grant the existence of all entities proposed by our best theories. Maddy's next issue arises from the previous rejection of confirmation holism. If we no longer can regard a scientific theory as a homogenous element, there is a the possibility that the mathematical section of the theory is untrue. Maddy calls this into question claiming that it is up in the air whether the mathematical portions of the scientific theory correspond with the true parts of the confirmed theories [21]. She uses an example in fluid dynamics to show an incorrect use of indispensability in hypotheses that are purposely incorrect. Scientists provoke the premise a body of water is infinitely deep in order to analyse water waves in fluid dynamics. Maddy claims that these theories would be nonexistent without these unreasonable hypotheses, and are therefore indispensable to the theories they produce. It would be ill–advised to take this appearance in our best theory about water waves as sufficient reason to believe in infinity [21]. Maddy notices that scientists seem to use whatever mathematics they need to use to get the most accurate answer, even if the mathematics does not have ontological rights within the theory.

These rejections clearly do not sit well with The Indispensability Argument. Not only has Maddy pointed to errors in the thesis, she actually used one of the premises to refute the other premise. However, platonism should not give up quite so soon.

2.5 Colyvan Responds to Maddy

Mark Colyvan gives a coherent response to each of Maddy's issues with the argument. Starting with the first appeal to the practice of scientists, Colyvan tries to undercut Maddy's idea that scientists distinguish between fake and real entities in their theories. The first case is when the fictitious element is introduced to the theory as purposefully inconsistent. Colyvan gives a few examples, such as frictionless planes or inertial reference frames. He then claims that although these ideas are useful to the hypothetical situation, there must be a better way to theorize that is consistent with the other axioms at hand. The fact that scientists are hypothesizing with an inconsistent idea means that it is not the best possible practice to get at the truth of the system. In the case of the infinitely deep water example, we do not need to warrant the existence of the false hypothesis because we are not taking the literal truth of the entire theory.

The next case is illustrated by Maddy's previous observation of the atom. This issue occurs when scientists treat an object instrumentally that is in fact indispensable to the best theory. Colyvan retraces Maddy's steps and points out a misunderstanding of Quinean naturalism. Maddy wants to observe the Quinean notion that the philosopher overlooking science must second the ontological conclusions of science. However this is inconsistent with how she formalizes her ideas of naturalism, because she does not hold that naturalism prohibits all philosophical commentary of natural methodology. Colyvan claims that, "once this misconception is cleared up we see that the door is open for a critique of the sceptical scientists from a philosophical perspective located within the scientific enterprise" [10]. Colyvan selects Hilary Putnam to be the displeased philosopher and looks to Putnam's definition of "intellectual dishonesty". Putnam says to deny the existence of what one daily presupposes is intellectual dishonesty. In the case of the atom it is dishonest for the chemist to use these entities without granting that they exist. This doesn't mean that it is impossible for them to do so, however, it is an intellectual crime. It is impossible to know the motives behind scientists who suspended the existence of atoms, but from a Quinean point of view they were doing something wrong. Colyvan understands initial hesitation or skepticism to new objects, but after they become indispensable it seems worrisome to deny their existence. Colyvan then mentions another facet of Quine's thesis. This thesis is not a descriptive thesis about how science is or has been, it is a normative proposition about how we ought to decide our ontological commitments [10]. Quine is not committed to the history of science being unflawed, and in this case of the atom it seems like a human flaw that could be explain by various personal reasons and preferences. Consequently, this can not be seen as a real issue when surveying the practice of science.

Maddy argues that scientists use mathematics as a means to an end without considering the ontology of the mathematics in question. Colyvan notes that in examples of such negligence, the mathematics has already been widely used within the theory. Scientists do take caution when the they are working with mathematics that is being applied for the first time. This can be seen in the previous example of complex numbers. Mathematicians did not accept their existence until they were mentioned by Guass' fundamental theorem of algebra and then used successfully in modeling physical structures that were otherwise unapproachable. As well as in the early stages of the calculus, where the hesitation within applying the math was raised from the questionable use of the infinitesimal. Berkeley, as well as other mathematicians, were unconvinced that the calculus as proposed by Leibniz and Newton had consistent foundations. Colyvan gives an example Dirac's equation with the introduction of the delta function. Dirac knew this equation would cause a controversy based on its improper function. Along with his rigorization of the equation he wrote a sort of “users guide” to his equation claiming that there was a ontologically correct version of this equation, however, the solution becomes more salient with the given equation [10]. Here, the mathematician gives a philosophical warning in correlation to his theory. If Maddy is right in thinking mathematicians are not bothered by the philosophy they invoke, then we would not see Dirac's warning.

There is, however, a more concerning objection to account for in Colyvan's response. Maddy raises the idea that mathematical practice does not mirror mathematical realism as explained by The Indispensability Argument. Colyvan's rebuttal uses the account of holism given by Quine, and claims that Maddy is misunderstanding Quine's intentions. Maddy notices a disconnect between mathematical practice and scientific practice. Mathematicians do not commit themselves to theorems because of their utility; they believe the theorems based on the provability from given axioms. Colyvan draws a distinction between two types of belief to put at the background of Maddy's worry. When mathematicians assign truth to a theorem they are doing so because the theorem is in accordance to the other true axioms that contributed. Colyvan claims that there is no ontological grounds until the mathematical claim is rendered useful to empirical science [9]. The workings of pure mathematicians should not be concerned with the ontological commitments of their theories until they are applied as indispensable to empirical observations.

This seems right, however, Maddy claims this leaves a great deal of mathematics unaccounted for [21]. If we are only ontologically committed to the mathematics that makes the way into the empirical sciences it seems as though The Indispensability Argument isn't as strong as we once thought.

As we have seen in this survey of indispensability, there is not specific indication that a mathematical theorem will become indispensable. Therefore, it seems like we should allow the possibility for all mathematics to eventually be applicable to empirical findings. There are constantly branches of mathematics becoming linked by the discovery of hidden connections. However, Quine does raise the question of mathematical recreation, and claims some abstract branches of mathematics are considered to be without ontological rights according to The Indispensability Argument. This doesn't mean that mathematicians are misguided in continuing to pursue such abstract avenues. As we saw with complex numbers, mathematicians were not considering the possibility of them being mapped onto Euclidean space. The mathematicians in this example were not committed to the entities in their results, but eventually there became a reason to be committed to complex numbers. In this example mathematical recreation was shown to be increasingly important as complex analysis became more rigorous. Maddy misunderstands mathematical practice in this way and thinks we must endorse ontological rights to all mathematical entities in order for them to relevant. The practice of mathematicians illustrates their faith that the underlying current of mathematical patterns will eventually tie the theory together. Perhaps this does not happen in the majority of mathematical pursuits. The few times these miraculous connections are found gives enough reason to endorse mathematical recreation as a worthy endeavor.

3 Newtonian Cosmology and Inconsistent Infinite Sums

3.1 An Introduction To Newtonian Cosmology

Newtonian Cosmology possibly contributes an argument against indispensability that has not been touched on by Penelope Maddy. Thus far, each case study illustrates the power of mathematics to push scientific discovery and development further. In some cases the progress of science has demanded more rigorous mathematics to formalize the results. When we don't react to the demand we see examples of unsuitable mathematics that fails to stabilize scientific theory. Broadly speaking, this is what happened for two–hundred years with Newtonian Cosmology. The controversies embedded within Newtonian Cosmology are numerous from a physicist's standpoint. In John D. Norton's paper, "The Cosmological Woes of Newtonian Gravitation Theory", he points to a handful of issues within the theory that are mutually inconsistent. My focus will be how the theory deals with infinite sums, and more specifically, how some results require one outcome and other results require the opposite outcome. Our incomplete understanding of infinite sums lead us to overlook important results in the study of time evolution and the expansion of our universe.

Newtonian Cosmology describes the evolution of the universe exclusively using the language of Newtonian Dynamics. Newtonian Dynamics is the study of a particle as stated by Newton's laws of motion. In 1687, Newton devised an impressive picture of universe in his work "Principia" . He described an infinite universe with gravitational balance and uniformly distributed matter. This was remarkably advanced for his time and set the stage for many fields of science for years to come. The main question considered by Newtonian Cosmology aims to find the net gravitational force of a given test particle at any arbitrary place in the universe [26]. In order to attempt a solution scientists use a combination of Newton's law of gravitation, Poisson's equation, and considerations of symmetry and gravitational potential. Through this approach, inconsistencies arise.

Peter Vickers is particularly interested in understanding the nature of these inconsistencies. His book "Understanding Inconsistent Science" points to Newtonian Cosmology as an exemplification of a particular type of mathematical error leading science. He claims there are two main types of contradictions in this theory. Primarily, there is the kind of contradiction that arises when the force of a specific test particle is both X and Y where $X \neq Y$ The other notable contradiction surfaces when the force on the particle is both determinate and indeterminate. The bulk of the mathematical interest resides in the second kind of contradiction. Vickers calls the contradiction C5 in his book, where C5 is "it is the case that there is a unique gravitational force on a test mass and it is not the case that there is a unique gravitational force of a test mass." I will call the first part of C5, P, and the second part $\neq$ P. If the larger theory was consistent it would be impossible to derive both P and $\neq$ P.

These conclusions follow from applying Newton's three laws of motion, Newton's inverse square law of gravitation, and the fact that in an infinite Euclidean space matter is distributed isotropically and homogeneously. The issue can be seen through applying each of these aforementioned laws to arrive at the conclusion that the net force on a given test mass is undetermined. However, it is an axiom of Newton's three laws of motion that there is always a determinate force on a body. Simply by looking at Newton's inverse square law of gravitation and the fact about matter we get an indeterminate infinite sum. The mistake is to take this infinite sum as meaning there is no unique gravitational force on the test mass. It could simply mean that using the inverse square law and the fact about matter are not the right means to determine what the force is. Scientists can either take indeterminacy as meaning no solution *has been* reached or that there *is* no solution.

3.2 Infinite Sums

The possibility of arriving at a contradiction depends on how one handles the indeterminate sum. This fact was exploited by early scientists fond of Newtonian Cosmology. If we take this infinite sum to mean that there is some solution that has yet to be discovered, then we have a consistent theory. However, if the indefinite sum is inferred as no possible solution, then we have a contradiction to the laws of motion. It wasn't until two hundred years later that Cauchy showed that the contradiction-free application was the wrong way to deal with infinite sums.

Cauchy proved that when faced with an indeterminate sum there exists no solution. When faced with indeterminacy the only correct way to proceed is to conclude there is no solution. Overlooking this result in cosmology is nothing less than a metaphysical error. Scientists used incorrect mathematics to grasp a metaphysical understanding of forces in the universe. The mathematics was inconsistent, in turn causing inconsistencies in the metaphysical conclusions. Therefore, our conclusions about physical properties in the universe rely on a complete understanding of infinite sums.

This illustrates one of the issues that can arise when scientists use inconsistent mathematics as indispensable to their physical theories. The underlying structure of infinite sums points to a truth in cosmology that would have gone unnoticed using the incorrect framework. In other words, without understanding the actual structure of these infinite sums our cosmological truths are inconsistent. The indispensability of mathematical truths govern the consistency of our theories, and in this case we only got so far because we were not reading the infinite sums correctly. Perhaps without surprise, this is not the only time failing to understand infinity caused issues when applying mathematics. Infinity is prone to inconsistency, and before Cantor's proof regarding cardinality, mathematics concerning the size of infinity was inconsistent.

3.3 Applying Inconsistent Mathematics

The case of Newtonian Cosmology illustrates issues within the empirical sciences that arise based on mathematical error. This error was not trivial because there was not yet a clear understanding of how to approach infinite sums. Two hundred years later, this issue was ironed out with the help of Cauchy in his convergence test. There are cases where the inconsistent mathematics is impossible to iron out, leaving the physical problems at a stand still. Most of these situations stem from known paradoxes in mathematics that when applied to physical systems cause inconsistency.

Godel's Incompleteness Theorem, previously outlined in section 1.2.3, exemplifies that some mathematical statements are simply undecided. This proof caused many mathematicians, including Godel himself, to enter extreme existential crisis. Hilbert, and other mathematicians looking for a complete system of logic, were shocked to realize that any system they ever developed would rest on at least some assumptions that are unprovable. The example of the machine in 1.2.3 illustrates the theorem's obvious implications to artificial intelligence. Alan Turing was the first person to take Godel's work and apply it to the physical world. In an article from late last year Davide Castelvecchi reported on unanswerable physical problems based on this paradox given by Godel.

Condensed matter theory is one field where such unanswerable physical problems have surfaced. Scientists have tried to understand the gaps between the lowest energy levels of electrons in a given material. Using an idealized model of the atoms, scientists have found that it is impossible to calculate this property. Toby Cubitt, a quantum physicist, stumbled upon this result while studying spectral gaps between electrons. His research was focused on the gap between the two lowest energy levels that electrons occupy within a material. This is an especially important layer because in some materials, this gap is the determining factor of the material becoming a superconductor [7]. The researchers devised a theoretical model of an infinite two dimensional "crystal lattice of atoms" [7]. In this model, the "quantum states of the atoms in the lattice embody a Turing machine, containing the information for each step of a computation to find the material's spectral gap" [7] The issue occurred when Cubitt saw that for this kind of infinite lattice, it is impossible to see whether the calculation ends or the gap remains undecided. The team found that on a finite lattice the calculation was always finite, however, because of the possible undecidability with the infinite lattice, they cannot draw any general conclusions about the spectral gap. This question about infinity is problematic because it introduces the possibility of an abrupt change within the gap that cannot be accounted for.

Cubitt claims that this is the case with quite a few problems of matter theory. He points to a few examples that currently are getting massive mathematical attention and claims there is the great possibility that they are unanswerable. For example, the Yang–Mills mass-gap problem has been posed by the Clay Institute with a multi–million dollar prize attached to the solution. This type of problem stems from the inaccuracy of our current system when dealing with why force–carriers have mass while photons are massless. This case is of particular interest because the scientists in question would not know what to do with such a result, if Godel had not proven his Incompleteness Theorems. In this example, inconsistent mathematics proved to be indispensable to our theories about matter. "Undecided" as a possible scientific result came from Godel?s earth-shattering mathematical proof. Mathematics is typically appealed to for problem solving, however, in this case we see how a truth of mathematics actually creates undecidability for the sciences.

This understood undecidability in the sciences is based on the undecidability result illustrated by Godel. Godel's proof allows scientists to remain calm when an experiment yields "undecided". This ties into the conversation of inconsistency because we can see how science does not simply decide to be modeled by the best conclusive and consistent mathematics. This further reinforces the value of studying these typically forbidden areas within mathematics and science for further results.

4 The Worry of Inconsistency

Inconsistent mathematics is the branch of mathematics of non-classical logic that can tolerate the presence of a contradiction without turning every sentence into a theorem [23]. This definition provides a nice starting point, since clearly the mathematician is going to have to introduce a way to reform the existing logical structure. Francisco Miro Quesada introduced the logical basis called paraconsistent logic. Broadly speaking, this inconsistency–tolerant system allows certain mathematics to have some inconsistency without being completely disregarded. An inconsistency in number theory would entail a theorem claiming X and another theorem claiming $\sim$X simultaneously. As if this situation weren't enough of a quandary, we must also note the ontology of the objects at hand: more specifically, that the objects in question are consequently inconsistent due to governance by inconsistent theorems.

A perceived strength of mathematics is that the mathematical world is considered to be free from uncertainty. We like to hold math to this standard, as the subject where there is always a "right" and a "wrong"; unfortunately, the picture is slightly more muddled than early logicians may have hoped. The formalist goes so far to claim that in order for a mathematical structure to be considered as an object of study, it *must* be consistent. The platonist, however, might be in trouble if they are willing to take inconsistent mathematics seriously. According to The Indispensability Argument, they would have to also occasionally accept the existence of inconsistent objects, which is more philosophically troubling than the acceptance of abstract objects. Platonist must possess a firm understanding of where the indispensability thesis leaves them with inconsistent mathematics. A look at specific inconsistencies provides a better understanding of their place within the landscape of mathematical practice and the field of mathematics more broadly.

A glance at the history of mathematics is sufficient to understand the value of critically considering inconsistent mathematics. There are two cases of inconsistencies that need approaching. Primarily the inconsistencies that are apparent to the mathematicians. This kind of inconsistency is illustrated by Dirac providing mathematicians with a warning of the inconsistency issues within his equation. Another, more controversial example, can be seen in the early calculus when one proof uses two different notions of convergence. The second kind of inconsistency is seen when the mathematicians do not realize the inherent flaws in their theory. Seeing as there are times when we don't know when things are inconsistent we should have a way to deal with it philosophically before the inconsistency emerges. Pre–paradox naive–set theory is, of course, the notorious case of this strain of inconsistency. I want to focus on the overall application of inconsistent mathematics to see how inconsistencies weigh on the philosophy of mathematics as a whole.

4.1 Inconsistent Calculus

The Indispensability Argument is arguably the only reason to take mathematical platonism as a possibility. If this is the primary reason to believe in mathematical objects, then we believe in the existence of the objects indispensable to our best scientific theories. What if our best scientific theory involves inconsistent mathematics? If the reason is as stated above, then we have the same justification for inconsistent mathematics as we do for consistent mathematics. Colyvan illustrates this point nicely with the somewhat obvious example of the early calculus. The calculus, proposed by Leibniz and Newton, was inconsistent in a few ways. Primarily, the partial understanding of infinitesimals made it seem acceptable to write equations like this, $a = a + \delta$, where δ is a changing quantity [9]. It should be noted that this equation was actually the submitted revision for a more problematic mistake. Newton advised the above equation as a way to get rid of the issue seen when we divide by an infinitesimal, where sometimes the infinitesimal was treated like zero and other times as a fixed quantity. Early calculus required equations like the one above in order to capture the basic structure of changing quantities. However, clearly $a = a + \delta$ is an inconsistent function because when

δ is anything other than zero we get a false equation.

Eventually, calculus was put on a firm basis by Lagrange, Cauchy, Bolzano, and Weierstrass. Lagrange was the first person to realize that algebra would be an ideal base for calculus because it included infinite products and series without the necessary appeal to the derivative using ϵ and ρ notation. He used the coefficients of Taylor Series expansions to find derivatives. Cauchy then offered the revolutionary insight of the limit. He described how the rigorous foundations of the limit could account for the integral, infinite series, the derivative, and continuity. Weierstrass formally proved Cauchy's findings in the 1870's completing the improved calculus [13]. The main issue with this history is the fact that for over 150 years the inconsistent version of the calculus was being applied in all different areas. During this time, using calculus helped us understand the effects of changes in systems in economy, engineering, science, architecture, and more.

If we are taking The Indispensability Argument seriously, this poses quite a big problem, perhaps even larger than the issues raised by Penelope Maddy in section 2.4. To grant the existence of certain mathematical entities based on their apparent indispensability to science is an issue in the case of the early calculus. From the beginning of calculus up until 1873 we were committed to the existence of inconsistent objects. If one desires to argue against this commitment to inconsistency in this case, the reasons to believe in the indispensability of other consistent entities falls through simultaneously. An argument claiming that such inconsistent theories cannot be suitable for our best theories would go against naturalism and the appeal to the working scientist. People in favor of The Indispensability Argument need to understand the possible consequences of such a claim in mathematical ontology. It is subject beyond the scope of this paper but worthy of further inquiry.

This is not to say these inconsistent commitments force the platonist to immediately adopt inconsistent mathematics. We have seen cases where the mathematicians in question have not realized that they are committed to inconsistent objects. In the case of Newtonian Cosmology, scientists were indeed committed to two inconsistent theories which called for an adjustment of their overall theory. This case might initially look troublesome to the platonist, however, if there are two contradictory ideas in a theory we can still see a reason to believe in the existence of mathematical entities. Newtonian Cosmology was committed to there being either the entities on one side of the inconsistency or the entities on the other side of the inconsistency. Perhaps in the future it becomes clear that they are only justified in committing to the consistent side, and then dispel the necessity of the inconsistency. This illustrates the desire to iron out inconsistency eventually. The commitment to inconsistent objects might be uncalled for in certain cases because the platonist still is getting what they need to move forward.

4.2 Paraconsistent Logic

Colyvan makes controversial inferences from mathematical practice, however he considers inconsistencies in mathematics more than other philosophers in favor of platonism. It is suffice to say that neither nominalism nor platonism is comfortable committing to the existence of inconsistent objects. Colyvan argues that when faced with the question about the ontological commitments of an inconsistent theory we must embrace paraconsistent logic. Without paraconsistent logic we implore classical logic where from an inconsistency comes trivialism: where everything is true. Furthermore, this trivialism can be set within an indispensable theory which means that it can be shown using classical logic that anything exists.

1. $h = 0 \wedge h \neq 0$ $(Assumption)$
2. $h = 0$ $(1, \wedge - elimination)$
3. $h = 0 \vee \exists x T x$ $(2, \vee - introduction)$
4. $h \neq 0$ $(1, \wedge - elimination)$
5. $\exists x T x$ $(3, 4, Disjunctive\ \ Syllogism)$

Paraconsistent logic does not allow inference through disjunctive syllogism which means that we do not run the risk of lapsing into trivialism. Is this idea of using paraconsistent logic plausible?

If supporters of The Indispensability Argument decide to take paraconsistent logic seriously the consequences are not trivial. Paraconsistent logic acts as a foundation for many philosophical areas. Endorsing this kind of logic would not stop philosophers from applying this outlook to epistemology, metaethics, deontic logic, artificial intelligence, semantics, and electronics. The Indispensability Argument originally seemed to be a claim about entities in science and mathematics however the repercussions are vast. The inconsistent mathematics we have seen creates tension in the foundations of this argument. In order to implement indispensability further, we have to be weary of the weighty philosophical consequences.

4.2.1 Paraconsistent Logic in Science

Colyvan looks to scientific practice and observes it could be plausible for scientists to invoke paraconsistent logic when working with inconsistent theories [9]. In fact, it seems like there is no other option for scientists who knowingly involve inconsistent science or mathematics. We have seen a few inconsistencies within science based on mathematical inconsistencies but there are also inconsistencies that are purely scientific. An example that follows nicely from the conversation about Newtonian Cosmology as discussed in 3.1 is seen when scientists try to apply Newtonian gravitational theory to spiral galaxies and get a contradiction. In this this study we have titled an object as indispensable to the theory if there is no competing theory that is as good without the object being discussed. Inconsistency is not necessarily a reason for scientists to render the theory useless because there are many other factors to keep track of. Colyvan and Putnam agree on these factors, including empirical adequacy, simplicity, utility, and explanatory power of the theory in question. If consistency held more power than these other attributes, we would not see working scientists giving thought to inconsistent mathematics or science. It is worth pointing out that this is a clear application of Quine's naturalism because it is zooming in on the practice of scientists.

4.2.2 Paraconsistent Logic in Mathematics: Set Theory

Paraconsistent logic is more complicated when turning to mathematical practice. Applied paraconsistent logic in mathematics is best seen with the example of naive–naive–setset theory. Naive–set theory was a response to Hilbert's programme within formal logic. David Hilbert challenged mathematicians to find a consistent and complete set of axioms from which all mathematical theorems could be derived. Naive–set theory is based on two axioms, abstraction and extension. Abstraction says that for a given property there is a set of all of the objects that satisfy this property, for example, if the property is "tall" there is a set that is "objects that are tall". Extension claims that in order for two sets to be considered the same, it must be the case that their members are the same. With this theory, we can create any set as long as they satisfy these claims.

Godel's Incompleteness Theorem given in section 1.2.3 proves that there is no possible way for such a system to be complete. Russell's Paradox introduces a precise set that this set theory can not account for.

$$R = x|x \notin x$$

Here we see that there is no answer to Russell's question. In order to be a member of itself, x must not be a member of itself. Therefore if x is in R then x is not in R, and if x is not in R then x is in R. Naive–set theory can not rationalize this set, and therefore is incomplete just as Godel had anticipated. To formalize a theory to take set R into consideration, Ernst Zermelo and Abraham Fraenkel had to dispose the first axiom of abstraction and replace it with eight other much more complicated axioms. These axioms included the axiom of regularity, schema of specification, pairing, union, schema of replacement, infinity, power set, and the well–ordering theorem. In addition to these new axioms, they created a hierarchy of sets that are used to make new possible sets. It is safe to say set R cannot be constructed using this hierarchy.

This impressive reformation of naive–set theory given by Zermelo and Fraenkel comes with two main costs. First of all, it is unnecessarily confusing. The extra axioms seem to obviously be addressing a specific problem and not for any real systemic reason. To prove $1+1=2$ using this system, Russell and Whitehead needed 379 pages [22]. This cumbersome system does not make sense when mathematicians want to apply it to consistent mathematical sets. Does the issue of a possible inconsistency merit all of these arguably ad hoc specifications?

It turns out that most mathematicians do not seem to think a possible inconsistency labels the entire theorem as useless. Naive–set theory, or a similar brand, is still used with caution in fields such as analysis, topology, and algebra [11]. Maarten McKubre–Jordens, a modern mathematician practicing at the University of Canterbury, claims that although mathematicians might not be quick to admit it, they use naive–set theory in their informal arguments. Mathematicians rationalize this by claiming that it is reducible to a set in Zermelo–Fraenkel set theory and that is enough reason for them to proceed. We see mathematicians being tolerable of inconsistent theories remaining aware that if they are not careful they could run into difficulty. Perhaps the mathematicians are right and their theories can be put consistently, but the crucial point of interest is that they are not afraid of the possibility through using this method. Mathematicians use inconsistent theories to pursue the development of other branches that are applied across many fields [11]. As well as in set theory, mathematicians used the early calculus knowing it stood on inconsistent foundations. Despite the presence of contradiction, many useful conclusions were made here. It does not seem out of the question that mathematicians use paraconsistent logic when working because they are making use of theorems that are known to be inconsistent.

After this more consistent version of set theory was devised it was questionable whether it was as powerful as the original theory. Instead of classifying Russell's paradox as a problem that needs solving, paraconsistent logicians treat these instances as points of interest worthy of pursuing. This opens up an entirely new area of study because there are objects that are only accountable by inconsistent mathematics. McKubre–Jordens puts it nicely when he says, "Allowing inconsistencies without incoherence opens up many areas of mathematics previously closed to mathematicians, as well as being a stepping stone to making sense of some easily described but difficult to understand phenomena " [22]. Paraconsistent logic draws a line between contradiction and absurdity. This creates a system that can be inconsistent without being completely incoherent unlike in the troll example in 4.2. This application of paraconsistent logic is illustrated well with the inconsistent drawings on M.C. Escher and Oscar Reutersvrd. Reutersvard's drawing of the penrose triangle is unapproachable using consistent geometry. With this example we see a clearly inconsistent but coherent drawing of a shape that can only be accounted for through adopting paraconsistent logic.

A more relevant example can be seen in computer science with the halting problem. We touched on Alan Turing's application of Godel's theorems to computer programing in section 3.3. The halting problem is a consequence of this theorem. In computer analysis the halting problem arises when one cannot determine the fate of an arbitrary running program and an input. Alan Turing found that a general algorithm to solve this problem for all possible pairs of input and output cannot exist. This issue of finding whether or not an algorithm will halt in finite time is similar to many problems in mathematics and science. Through applying paraconsistent logic, this problem does not become a dead end worth avoiding at all costs. Instead, these kind of problems can be re-evaluated for further understanding.

4.2.3 The Ability To Model Reality Through Mathematical Models

As we saw with the case of complex numbers, the powerful world of mathematics is still a mystery to us. It never occurred to mathematicians that the controversial complex numbers could be used to adequately to model the structure of space. The refusal to move forward in studying these inconsistent systems is a scientific error because it does not encourage additional discovery. The successful application of inconsistent theories is also of interest to the philosopher. We have seen the inconsistent mathematical theories of calculus and pre–paradox set theory be used to model the real world in nontrivial ways. If we assume to be living in a consistent world then why is it the case inconsistent mathematics can so adequately model physical phenomena? Before tackling this problem, there is a question concerning the nature of mathematics in general that is worth looking at.

The applicability of mathematics, unlike inconsistent mathematics, has received a considerable amount of attention from philosophers. This problem relates to the further issue of how an *a priori* system like mathematics is an adequate tool for empirical science. There is a general consensus in philosophy that the success of a mathematical theory applied to a physical system is due to structural similarities between the structures of the two systems [9]. Colyvan points to physical space being modeled successfully by $\mathbb{R}_3$. This is a special case because the two systems are isomorphic, however, this does not accurately account for non-isomorphic structures. Colyvan claims that although extensive work has been done on this topic, no one has successfully explained how non-isomorphic structures can be explained by one another. These are huge topics worth further study, however, for my focus I want to how inconsistent mathematics adds a new facet to these problems.

The question revolves around the assumption that we live in a consistent world but on occasion it can be structurally assessed by inconsistent theories. For example, the world explained by calculus for 150 years before Cauchy set what Newton and Leibniz started on firm foundations. Colyvan argues that the early calculus was so successful in physical applications because it had the important features of the consistent calculus. It seems in this instance it matters only if the mathematical theorem captures the salient aspects of the empirical phenomenon at hand [9]. Colyvan then makes the observation that the proposed model can achieve this independent of the modelers knowledge [9]. The similarities between the two calculus'explains why the calculus presented by Leibniz and Newton was so useful. There are many incorrect theories that have been extremely useful for capturing the attributes in the contemporary questions. The fact that an inconsistent theory proves to be so useful could point to a consistent theorem in the distance. This is another reason to study and not abandon inconsistent mathematics. A critical study of the inconsistency will lead mathematicians to a thorough view of why the inconsistency is present. Narrowing down on a specific inconsistency is the crucial steps in finding the suitable consistent theory [9].

William Byers also speaks to contradictions in mathematics as a way to propel discovery. In his book "How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics", he touches on the creative process of mathematicians. He claims that mathematical progress is by way of present contradiction. The mathematicians job is to take at first sight unrelated perspectives on a mathematical structure and unify them through using known mathematics. Byers makes the argument that a comprehensive view of any mathematical system would not be feasible without the practicing mathematician using inconsistent ideas at some point in their discovery. Mathematics as a human activity is further testament to how the underlying mathematical truths are applicable to our physical systems.

4.3 Conclusion

The uncanny ability of mathematics to model physical systems speaks multitudes to the nature of mathematics. An evaluation of indispensability shows that a remarkable amount of mathematics has gained footing in the empirical sciences through explanatory power. When such mathematics is derived it is often complete with many unobservable corollaries. These abstractions, although not yet directly applicable to the physical sciences, further root our discoveries of mathematical truths. There is a trial and error within this discovery that is much more obvious within the evolution of science. Mathematicians are tapping away at the surface of mathematical truths trying to make our formulated notation consistent with other existing truths. This sheds light on mathematics as a fundamentally creative activity.

If it is important for working scientists to believe in the existence of the entities in their theories, then they must adopt paraconsistent logic as a consequence of the power of The Indispensability Argument. We have looked at scientific cases where the mathematics in question is inconsistent. In order to press forward in scientific development it seems reasonable to adapt the paraconsistent framework with caution. This being said, consistency remains a possible virtue of mathematical theorems. Nevertheless, consistency is not the only virtue a theorem can entertain. Mathematicians and scientists should strive for consistent models to apply to empirical occurrences. It is not, however, always counterintuitive to endorse inconsistencies along the way seeing as a better understanding of the particular inconsistency can point to a solution. The available inconsistent theorems might better capture the underlying mathematical structures at play. Without taking advantage of these available theorems, mathematicians and scientists leave the possibility of missing out on further understanding mathematical truths within their study.

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