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A Borsuk-Ulam Equivalent that Directly Implies Sperner's Lemma

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A Borsuk–Ulam Equivalent that Directly Implies Sperner's Lemma

Kathryn L. Nyman and Francis Edward Su

Abstract. We show that Fan's 1952 lemma on labelled triangulations of the n-sphere with $n+1$ labels is equivalent to the Borsuk–Ulam theorem. Moreover, unlike other Borsuk–Ulam equivalents, we show that this lemma directly implies Sperner's Lemma, so this proof may be regarded as a combinatorial version of the fact that the Borsuk–Ulam theorem implies the Brouwer fixed-point theorem, or that the Lusternik–Schnirelmann–Borsuk theorem implies the KKM lemma.

1. INTRODUCTION. The Brouwer fixed-point theorem, the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma, and Sperner's lemma are known to be equivalent. Equally powerful, they form a triumvirate of theorems whose interconnections have been exploited with great success in fixed point algorithms [**15, 17**] as well as in game theory [**1**]. Similarly, the Borsuk–Ulam theorem, the Lusternik–Schnirelmann–Borsuk (LSB) theorem, and Tucker's lemma are another triumvirate of equivalent results. In each of these triples, the first is a topological result, the second is a set-covering result, and the third is a combinatorial result.

Moreover, these triples are related to each other. Since the Borsuk–Ulam theorem implies the Brouwer fixed-point theorem, any theorem in the second triple must imply any theorem in the first. It is an interesting question to find *direct* proofs of each implication. For instance, a topological construction shows how a Brouwer fixed point follows from Borsuk–Ulam antipodes [**13**], and with set-coverings, the LSB theorem can be used to directly prove the KKM lemma [**11**]. But in the combinatorial domain, we are unaware of a direct proof that Tucker's lemma implies Sperner's lemma.

In this article, we show that another combinatorial lemma, Fan's $N+1$ Lemma, may be a more natural combinatorial analogue to the Borsuk–Ulam theorem, and therefore more worthy to sit in the Borsuk–Ulam triumvirate than Tucker's lemma. In particular, in Section 3 we show that Fan's $N+1$ Lemma is equivalent to the Borsuk–Ulam theorem, and in Section 4 we exhibit a direct proof that it implies Sperner's lemma (see Figure 1).

2. BACKGROUND. We first review these theorems. Let Σ^n be a polyhedral version of the n-sphere, the set of all points in $\mathbb{R}^{n+1}$ of distance 1 from the origin in the L_1 norm:

$$\Sigma^n = \{(x_1, \ldots, x_{n+1}) : \sum |x_i| = 1\}.$$

In $\mathbb{R}^3$, Σ^2 is just the boundary of the octahedron. As with the octahedron, note that Σ^n is naturally subdivided into orthants; we will study labelled triangulations of Σ^n that refine the orthant subdivision. A *triangulation* is a subdivision by simplices that either meet face-to-face or not at all. Each simplex is the affine hull of its *vertices*; these are the *vertices of the triangulation*. A triangulation of Σ^n is *symmetric* if, when σ is a simplex of the triangulation, then $-\sigma$ is a simplex as well.

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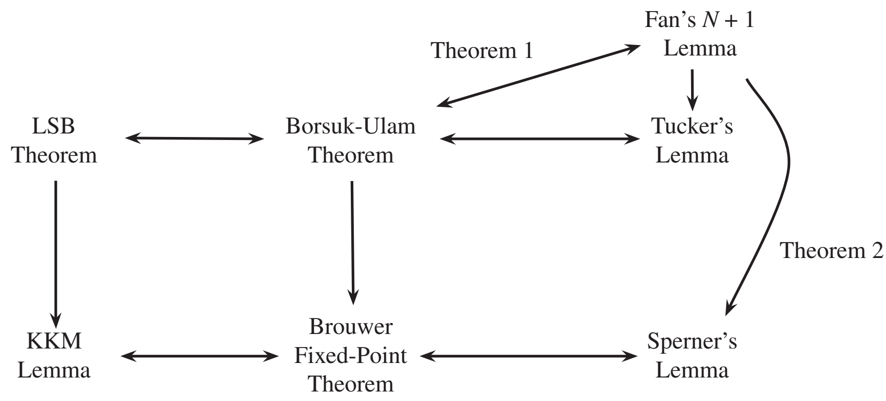

Figure 1. Connections between the topological, set-covering, and combinatorial results.

Define an *m-labelling* to be a function ℓ that assigns to each vertex v one of $2m$ possible integers: $\{\pm 1, \pm 2, \ldots, \pm m\}$. A symmetric triangulation of Σ^n has an *anti-symmetric* labelling if $\ell(-v) = -\ell(v)$ for all vertices v. A labelling has a *complementary edge* if some adjacent pair of vertices has labels that sum to zero, e.g., $\{+i, -i\}$.

Call a simplex *alternating* if its vertex labels are distinct in magnitude and alternate signs, when arranged in order of increasing value. So the labels have the form

$$\{k_1, -k_2, k_3, \ldots\} \quad \text{or} \quad \{-k_1, k_2, -k_3, \ldots\}$$

when $1 \leq k_1 < k_2 < k_3 < \cdots$. The first kind is called *positive alternating* and the second is *negative alternating*, based on the sign of k_1. For instance, a triangle labelled $\{-1, +3, -7\}$ would be negative alternating, and an edge labelled $\{+2, -3\}$ would be positive alternating.

Fan's $N + 1$ Lemma. *Let T be a symmetric triangulation of Σ^n with an $(n+1)$-labelling that is anti-symmetric and has no complementary edge. Then T has a positive alternating n-simplex.*

Thus, if the boundary of an octahedron (e.g., see Figure 7) has a triangulation anti-symmetrically labelled by $\{\pm 1, \pm 2, \pm 3\}$ and no complementary edges, then it must have a $\{+1, -2, +3\}$ triangle.

We call this Fan's $N + 1$ Lemma because Fan's original lemma [**4**] is more general; it says that for any m-labelling with the same hypotheses, there are an odd number of positive alternating n-simplices and an equal number of negative alternating n-simplices. And as [**9**] shows, the result holds for more general triangulations of S^n with a constructive proof. When $m = n + 1$, an m-labelling has only one kind of positive alternating simplex—namely, the simplex with labels of every magnitude: $\{1, -2, +3, \ldots, (-1)^n(n+1)\}$.

Note that if an anti-symmetric m-labelling has no complementary edge, then $m \geq n + 1$, because alternating simplices must have $n + 1$ different label values (apart from sign). Since an n-labelling is an $(n+1)$-labelling with one label missing, then as noted by Fan [**4**], the contrapositive of Fan's $N + 1$ Lemma yields Tucker's lemma as a corollary.

Tucker's Lemma. *Let T be a symmetric triangulation of Σ^n with an n-labelling that is anti-symmetric. Then T has a complementary edge. See Figure* 2.

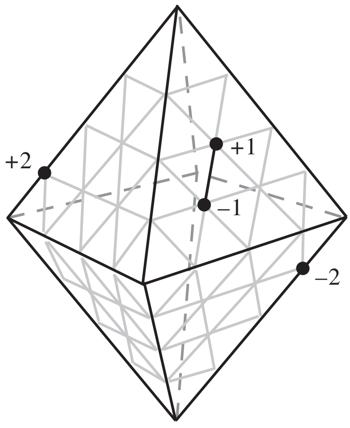

Figure 2. A complementary edge is guaranteed by Tucker's lemma when the polyhedral 2-sphere has a symmetric triangulation with an anti-symmetric 2-labelling.

Tucker's lemma [**6, 16**] was originally proposed as a combinatorial equivalent of the Borsuk–Ulam theorem [**2**], though it has found other applications as well (e.g., [**10**]).

Borsuk–Ulam Theorem. *Let $h : S^n \to \mathbb{R}^n$ be a continuous function such that $h(-x) = -h(x)$ for all $x \in S^n$. Then there exists $w \in S^n$ such that $h(w) = 0$.*

A set covering result due to Lusternik–Schnirelman–Borsuk [**2, 7**] is also equivalent to the Borsuk–Ulam theorem.

LSB Theorem. *Let $C_1, \ldots, C_{n+1}$ be a collection of closed sets that cover S^n. Then at least one of the sets must contain a pair of antipodal points.*

These theorems (Fan, Tucker, Borsuk–Ulam, LSB) concern topological or polyhedral n-spheres. The next three theorems concern topological and polyhedral n-balls.

Let B^n denote an n-ball, the set of all points within unit distance of the origin in $\mathbb{R}^n$. A polyhedral version of an n-ball is an n-simplex, which is more naturally described by its embedding in $\mathbb{R}^{n+1}$:

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) : x_i \geq 0, \sum x_i = 1\}.$$

It is homeomorphic to an n-ball. For any $v = (v_1, \ldots, v_{n+1}) \in \Delta^n$, let

$$Z(v) = \{i : v_i \neq 0\}$$

be the set of indices of coordinates of v that are nonzero. Thus in Δ^2, $Z((0, 1, 0)) = \{2\}$ and $Z((.3, 0, .7)) = \{1, 3\}$. Suppose T is a triangulation of Δ^n. A *Sperner-labelling* ℓ assigns to each vertex v a label from $\{1, \ldots, n+1\}$ such that

$$\ell(v) \in Z(v). \tag{1}$$

This forces each main vertex of Δ^n to have a different label (the index of its one nonzero coordinate), and any vertex on a face of Δ^n can only be labelled by one of the main vertices that span that face. Call an n-simplex in the triangulation *fully-labelled* if its vertices have distinct labels (and therefore all labels $\{1, \ldots, n+1\}$).

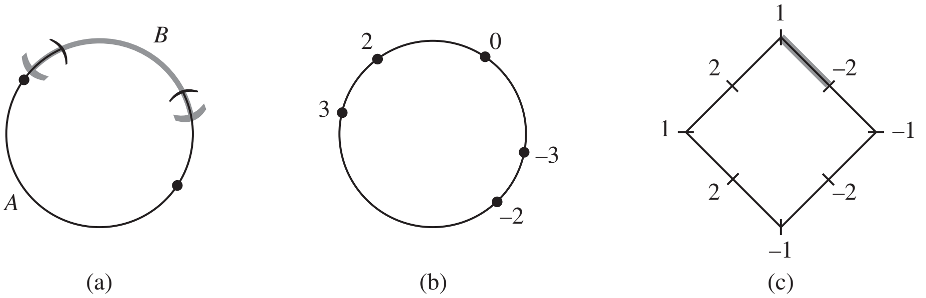

Figure 3. (a) LSB: a pair of antipodal points contained in one of two closed sets that cover S^1. (b) Borsuk–Ulam: Given a continuous, anti-symmetric function from S^1 to $\mathbb{R}$, there is a point mapped to 0. (c) Fan's $N+1$: an antisymmetric 2-labelling of Σ^1 with no complementary edge must have a positive alternating edge (shaded).

Sperner's Lemma. *Any Sperner-labelled triangulation of Δ^n must have a fully-labelled n-simplex.*

In fact, there are an odd number of such simplices [**12**]. An exposition and applications may be found in [**14**]. Sperner's lemma provides the simplest route to proving this famous theorem of Brouwer [**3**].

Brouwer Fixed-Point Theorem. *For any continuous function $f : B^n \to B^n$, there exists a point $x \in B^n$ such that $f(x) = x$.*

Knaster–Kuratowski–Mazurkiewicz [**5**] provided the original link between the Brouwer theorem and Sperner's lemma.

KKM Lemma. *Let $C_1, \ldots, C_n$ be a collection of closed sets that cover Δ^n such that for each $I \subseteq [n+1]$, the face spanned by the set $\{e_i | i \in I\}$ is covered by $\{C_i | i \in I\}$. Then $\cap_{i=1}^n C_i$ is nonempty.*

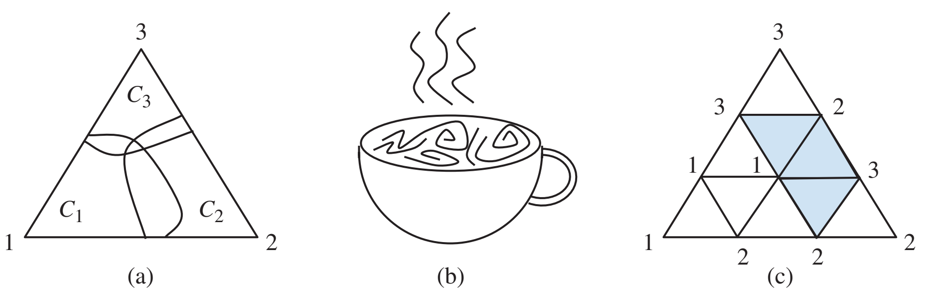

Figure 4. (a) KKM: these sets have a non-empty intersection. (b) Brouwer: the stirred coffee has a point that is in the same place as before the stirring. (c) Sperner: there's an odd number of 123-triangles.

3. EQUIVALENCE OF FAN'S $N+1$ LEMMA AND THE BORSUK–ULAM THEOREM.

As discussed earlier, Fan's general lemma with m-labellings [**4**] implies the Borsuk–Ulam Theorem through Tucker's lemma. Here we show that Fan's $N+1$ Lemma is equivalent to the Borsuk–Ulam theorem.

Theorem 1. *Fan's $N+1$ Lemma is equivalent to the Borsuk–Ulam Theorem.*

Proof. We first show that the Borsuk–Ulam Theorem implies Fan's $N+1$ Lemma. Let T be a symmetric triangulation of Σ^n with an anti-symmetric $(n+1)$-labelling L, in which there are no complementary edges. Let $w_i \in \mathbb{R}^{n+1}$ be the point with ith coordinate n and other coordinates -1:

$$w_i = (-1, \dots, -1, n, -1, \dots - 1).$$

Let $W_+ = \{w_1, \dots, w_{n+1}\}$ and $W_- = \{-w_1, \dots, -w_{n+1}\}$. The set $W = W_+ \cup W_-$ comprises $2n+2$ points that lie on the n-dimensional hyperplane: $H = \{(x_1, \dots, x_{n+1}) : \sum_{i=1}^{n+1} x_i = 0\}$.

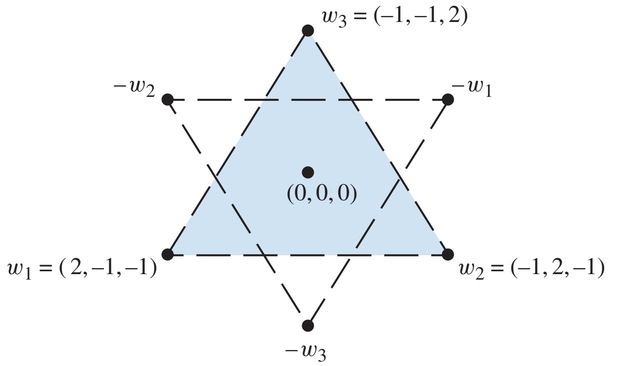

Figure 5. For $n=2$, the points w_1, w_2, w_3 and $-w_1, -w_2, -w_3$ in the hyperplane H. The shaded region indicates the image under h of a positive alternating 2-simplex, which maps to a simplex containing all the positive w_i (and the origin).

Define a continuous map $h : \Sigma^n \to H$ as follows. For each $v \in T$, let

$$h(v) = \begin{cases} w_{L(v)} & \text{if } L(v) \text{ is odd} \\ -w_{L(v)} & \text{if } L(v) \text{ is even,} \end{cases} \tag{2}$$

where $w_{-i} = -w_i$ in case $L(v) < 0$. Extend h linearly to each simplex of T. Since L is an anti-symmetric labelling, we see $h(-x) = -h(x)$ for all $x \in \Sigma^n$. Therefore, by Borsuk–Ulam there is a $z \in \Sigma^n$ such that $h(z) = 0$.

Thus z is in some n-simplex σ such that $h(\sigma)$ contains the origin. The images of the vertices of σ form a set $K = \{h(v) : v \in \sigma, v \in T\}$, a subset of W of size $n+1$ or smaller (if there are repeated labels). Since there are no complementary edges in T, the set K contains no pair $\{w_j, -w_j\}$. Then $K = \{w_j\}_{j \in B} \cup \{-w_j\}_{j \in B'}$, where B and B' are disjoint subsets of $\{1, \dots, n+1\}$.

Now consider the sum of vectors in K:

$$\hat{v} = \sum_{j \in B} w_j - \sum_{j \in B'} w_j.$$

Note that the dot products $w_i \cdot w_i = n(n+1)$ for all $i \in [n+1]$, and $w_i \cdot w_j = -(n+1)$ for all $j \neq i$. So, for $i \in B$, the dot product

$$\begin{aligned} w_i \cdot \hat{v} &= n(n+1) - (|B|-1)(n+1) + |B'|(n+1) \\ &= (n+1)(n+1-|B|+|B'|), \end{aligned}$$

which is positive unless $|B| = n+1$ and $|B'| = 0$, i.e., $K = W_+$. And for $i \in B'$,

$$-w_i \cdot \hat{v} = |B|(n+1) + n(n+1) - (|B'|-1)(n+1)$$
$$= (n+1)(|B| - |B'| + n + 1),$$

which is positive unless $|B'| = n+1$ and $|B| = 0$, i.e., $K = W_-$. Since the convex hull of K contains the origin, it cannot be the case that all vectors in K have a positive dot product with $\hat{v}$. So either $K = W_+$ or $K = W_-$ (and indeed, in these cases, K's convex hull contains the origin).

If $K = W_+$, then (2) shows the original simplex σ has labels $\{1, -2, \ldots, (-1)^n(n+1)\}$. If $K = W_-$, then (2) and anti-symmetry of L shows that $-\sigma$ has these labels. In either case we find a positive alternating simplex, as desired.

Now we show Fan's $N+1$ Lemma implies the Borsuk–Ulam Theorem. Let $h : \Sigma^n \to \mathbb{R}^n$ be a continuous function such that $h(-x) = -h(x)$ for all $x \in \Sigma^n$. Assume, by way of contradiction, that there is no point $z \in \Sigma^n$ such that $h(z) = 0$. If $h(x) = (x'_1, \ldots, x'_n)$, let $\hat{h} : \Sigma^n \to \mathbb{R}^{n+1}$ be the function defined by $\hat{h}(x) = (x'_1, \ldots, x'_n, -\sum_{i=1}^n x'_i)$. So $\hat{h}$ maps Σ^n to the hyperplane H and preserves continuity and anti-symmetry. Furthermore, there is no point z such that $\hat{h}(z) = 0$.

Let T be a symmetric triangulation of Σ^n, and let the set W be as above. We wish to construct a labelling L on the vertices of T that is anti-symmetric.

For $v \in T$, define $L(v)$ to be the index i such that w_i is closest to $\hat{h}(v)$ in $\mathbb{R}^{n+1}$. Note that $i \in \{\pm 1, \ldots, \pm(n+1)\}$. In the case of ties, choose the index with the smallest absolute value. This is well-defined because $\hat{h}(v)$ is never 0, and no nonzero point can be equidistant from w_i and $w_{-i} = -w_i$. That L is anti-symmetric follows from noting that $\hat{h}$ is anti-symmetric, so $\hat{h}(v)$ is closest to w_i if and only if $\hat{h}(-v)$ is closest to w_{-i}.

Therefore, by Fan's $N+1$ Lemma, there exists either a complementary edge $(+i, -i)$, for some i, or an alternating simplex with labels $\{1, -2, \ldots, (-1)^n(n+1)\}$. By taking finer and finer triangulations, and by the compactness of the Σ^n, there exists a convergent subsequence of shrinking positive alternating simplices or a convergent subsequence of shorter complementary edges involving the same index i. This gives a limit point which, by the continuity of $\hat{h}$, is either equidistant from both w_i and $-w_i$, or is equidistant from all points in $\{w_1, -w_2, w_3, \ldots, (-1)^n w_{n+1}\}$. But the only point with this property is 0. Thus, the limit point z must satisfy $\hat{h}(z) = 0$ and therefore, $h(z) = 0$. ■

4. FAN'S $N + 1$ LEMMA IMPLIES SPERNER'S LEMMA.

Now we establish how Fan's $N+1$ Lemma will indeed prove Sperner's lemma by a direct construction, so it is the "right" combinatorial result to sit in the Borsuk–Ulam triumvirate. Prescott [**8**] established this implication in dimension two by a different method.

Theorem 2. *Fan's $N+1$ Lemma implies Sperner's lemma.*

Proof. Consider a triangulation S of Δ^n with a Sperner-labelling ℓ. We first extend S to a triangulation T of Σ^n by reflecting copies of S to the other orthants of Σ^n. Let $G = \{\pm 1\}^{n+1}$ denote the group of symmetries of Σ^n generated by reflections that flip the sign of selected coordinates; then the action of $g = (g_1, \ldots, g_{n+1}) \in G$ on $v = (v_1, \ldots, v_{n+1}) \in \Sigma^n$ produces $gv = (g_1 v_1, \ldots, g_{n+1} v_{n+1}) \in \Sigma^n$. So g reflects v in all coordinates i for which $g_i = -1$. Note that $g = (1, 1, \ldots, 1)$ is the identity in G. The idea of this construction is illustrated in Figure 6.

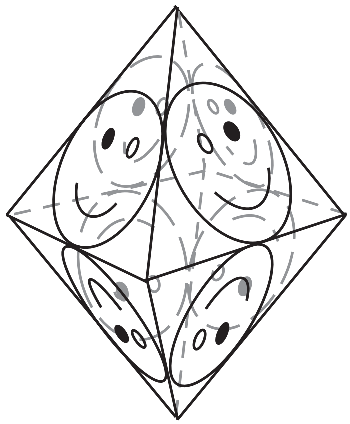

Figure 6. The actions of G on Σ^n, as shown by their effects on Mr. Smiley.

Similarly, if σ is a simplex in S spanned by a set of vertices V, we define $g\sigma$ to be the simplex spanned by the vertices in $gV = \{gv : v \in V\}$. Let T be the collection of simplices $\{g\sigma : \sigma \in S \text{ and } g \in G\}$. Then T is a triangulation of Σ^n, since the reflection method ensures that simplices of T meet face-to-face along reflected facets of S.

Now we extend the labelling ℓ on vertices of S to a labelling L on vertices of T by reflection but with possible sign modifications. Define

$$L(gv) = g_{\ell(v)} \cdot (-1)^{\ell(v)+1} \cdot \ell(v) \tag{3}$$

for each $v \in S$. Notice that $L(gv)$ and $\ell(v)$ have the same label value (but possibly different signs). When $g = (1, 1, \ldots, 1)$, this defines L on S and the factor $(-1)^{\ell(v)+1}$ turns fully-labelled simplices into positive alternating simplices. When g is non-trivial, L defines a labelling of vertices on reflected copies of S (see Figure 7).

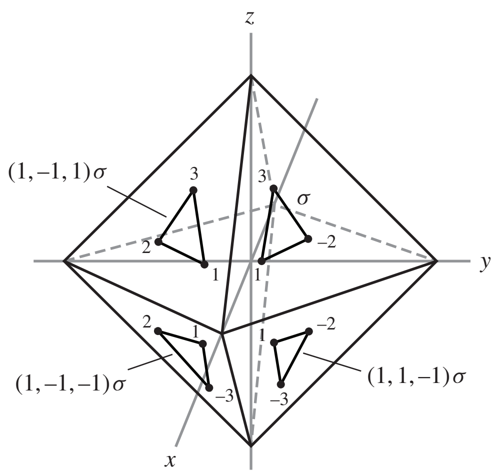

Figure 7. A positive alternating simplex σ in T arising from a fully-labelled simplex with labels $\{1, 2, 3\}$ in S, and reflected simplices $g\sigma$ for $g = (1, 1, -1)$, $(1, -1, 1)$, and $(1, -1, -1)$ with their L-labellings indicated.

We might worry that L is not well-defined where orthants meet. However, orthants meet where $gv = \hat{g}\hat{v}$, for some $g, \hat{g} \in G$ and some $v, \hat{v} \in S$. But then $g_i v_i = \hat{g}_i \hat{v}_i$

for each i, which implies $v_i = \hat{v}_i$ since $g_i, \hat{g}_i = \pm 1$. Then $g_i = \hat{g}_i$ when $v_i \neq 0$, i.e., when $i \in Z(v)$. But $\ell(v) \in Z(v)$ by (1), so that $g_{\ell(v)} = \hat{g}_{\ell(v)}$. It follows from (3) that $L(gv) = L(\hat{g}v)$, so L is well-defined.

Now we show that L satisfies the conditions of Fan's $N+1$ Lemma. Antipodal labels sum to zero by construction: The point antipodal to v is $-v = \bar{g}v$, where $\bar{g} = (-1, -1, \ldots, -1)$, so that (3) gives $L(-v) = -L(v)$. Also, we can show that L has no complementary edges. Every edge in T is a reflected copy of some edge in S via some $g \in G$, and the Sperner-labelling ℓ of S has no complementary edges (all labels are positive). Then the rule (3) shows that for any choice of g, two vertices $v, w \in S$ will have identical ℓ-labels ($\ell(v) = \ell(w)$) if and only if their g-reflections have identical L-labels as well ($L(gv) = L(gw)$). So L has no complementary edges, because ℓ did not.

Thus Fan's $N+1$ Lemma applies, so there exists a positive alternating n-simplex in T. Since Δ^n is the only facet of Σ^n that contains the labels $\{1, -2, 3, \ldots, (-1)^n(n+1)\}$, there must be a fully-labeled n-simplex in S. ■

In fact, as noted earlier, a stronger version of Fan's $N+1$ Lemma holds, whose conclusion is that there are in fact an odd number of positive alternating n-simplices. Then the above argument would demonstrate the stronger version of Sperner's lemma, which concludes that there are an odd number of fully-labelled n-simplices in S.

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An Easy Proof of the Divergence of the Harmonic Series Sum

The sum of the harmonic series, 1, 1/2, 1/3, diverges, even as the terms tend to zero. Many proofs of this significant fact are available, such as the well-known proof by N. Oreseme, and the more recent ones (see, for instance, [**1, 2**]). We give another.

Let $t_n = 1 + 1/2 + \cdots + 1/n$, $n = 1, 2, \ldots$ We may note that

$$t_{k+m} = t_k + 1/(k+1) + 1/(k+2) + \cdots + 1/(k+m) > t_k + m/(k+m),$$

for a finite, fixed $k, m/(k+m) \to 1$, as $m \to \infty$. Therefore, we may consider $\epsilon > 0$, sufficiently small, and get an m such that $p/(k+p) > 1-\epsilon$, $p = m, m+1, \ldots$ So, $t_{k+m} - t_k > 1 - \epsilon$. Fix ϵ. For such ϵ, we then consider t_{k+m}, and get a finite r, such that $t_{k+m+r} - t_{k+m} > 1 - \epsilon$. This may be continued so that terms, with finite indices, are obtained, each of which exceeds the previous term by at least $1 - \epsilon$. So there cannot exist any upper bound for the series $t_n, n = 1, 2, \ldots$. Hence, the sum of the harmonic series diverges and cannot have a limit.

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