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# The Global Optimization of Incoherent-Phase Signals

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# THE GLOBAL OPTIMIZATION OF INCOHERENT PHASE SIGNALS*

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**Abstract.** The problem of global optimization of $M$ incoherent phase signals in $N$ complex dimensions is formulated. Then, by using the geometric approach of Landau and Slepian, conditions for optimality are established for $N = 2$, and the optimal signal sets are determined for $M = 2, 3, 4, 6$ and 12.

The method is the following: The signals are assumed to be equally probable and to have equal energy, and thus are represented by points $\mathbf{s}_j, j = 1, 2, \cdots, M$, on the unit sphere $S_1$ in $C^N$. If $W_{jk}$ is the half-space determined by $\mathbf{s}_j$ and $\mathbf{s}_k$ and containing $\mathbf{s}_j$, i.e., $W_{jk} = \{\mathbf{r} \in C^N : |\langle \mathbf{r}, \mathbf{s}_j \rangle| \geqq |\langle \mathbf{r}, \mathbf{s}_k \rangle|\}$, then $\{\mathscr{R}_j = \bigcap_{k \neq j} W_{jk} : j = 1, 2, \cdots, M\}$, the maximum likelihood decision regions, partition $S_1$. For additive complex Gaussian noise $\mathbf{n}$ and a received signal $\mathbf{r} = \mathbf{s}_j e^{i\theta} + \mathbf{n}$, where $\theta$ is uniformly distributed over $[0, 2\pi]$, the probability of correct decoding for the signal-to-noise ratio $A^2$ is

$$P_C = \frac{1}{\pi^N} \int_0^\infty r^{2N-1} e^{-(r^2 + A^2)} U(r)\, dr,$$

where

$$U(r) = \frac{1}{M} \sum_{j=1}^{M} \int_{R_j} I_0(2Ar|\langle \mathbf{s}, \mathbf{s}_j \rangle|)\, d\sigma(\mathbf{s}),$$

$R_j = \mathscr{R}_j \cap S_1$. For $N = 2$, it is proved that

$$U(r) \leqq \int_{C_\alpha} I_0(2Ar|\langle \mathbf{s}, \mathbf{s}_j \rangle|)\, d\sigma(s) - \frac{2K}{M} \cdot h\left(\frac{1}{2K}[M\sigma(C_\alpha) - \sigma(S_1)]\right),$$

where

$$C_\alpha = \{\mathbf{s} \in S_1 : |\langle \mathbf{s}, \mathbf{s}_j \rangle| \geqq \alpha\},$$

$2K$ is the total number of half-spaces that actually determine the decision regions, and $h$ is the strictly increasing, strictly convex function of $\sigma(C_\alpha \cap W)$ (where $W$ is a half-space not containing $\mathbf{s}_j$), given by

$$h = \int_{C_\alpha^*} I_0(2Ar|\langle \mathbf{s}, \mathbf{s}_j \rangle|)\, d\sigma(\mathbf{s}),$$

$C_\alpha^* = C_\alpha \cap W$. Conditions for equality are established and these give rise to the globally optimal signal sets for $M = 2, 3, 4, 6$ and 12.

**1. Introduction.** The problem of optimal (minimizing the probability of error) signal selection for transmission of messages over coherent phase and incoherent phase channels has been a subject of many investigations. Under the assumption of additive white Gaussian noise, equal energy, and equiprobable signal sets, Balakrishnan [1] showed in 1961 that with no bandwidth constraint the regular simplex is globally optimal for small and large signal-to-noise ratios for the coherent phase channel. In 1966, Landau and Slepian [6] established a condition for globally optimal signals for the coherent phase channel, independent of the signal-to-noise ratio, and claimed this condition was satisfied by the simplex code. In 1968, Farber [4] showed that in fact for more than three dimensions the simplex did not meet this condition.

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Also in 1966, using the approach of Balakrishnan [1], Scholtz and Weber [10] proved that the orthogonal signal set is locally optimal for the incoherent phase channel under no bandwidth constraint. For $M$ incoherent phase signals in $M - 1$ dimensions, i.e., a bandwidth constraint, the signals with $|\langle \mathbf{s}_j, \mathbf{s}_k \rangle| = 1/(M - 1)$ were established as locally optimal by Weber [13] in 1968. Stone and Weber [11] proved in 1969 that the orthogonal signal set is globally optimum for high signal-to-noise ratios under no bandwidth constraint. They also showed [12] in 1970 that this set is locally optimum with a bandwidth constraint.

Using the geometric approach of Landau and Slepian, we formulate a condition for global optimality of $M$ equiprobable incoherent phase signals in $N$ complex dimensions. In this approach, the square of the length of the signal vectors is proportional to energy, and the dimensionality of the space is analogous to bandwidth [5]. For a set of probability densities including the Gaussian, we prove the validity of this condition for $N = 2$ along with some related necessary conditions. We then perform a transformation that maps the unit sphere in $C^2$ onto the unit sphere in three-dimensional Euclidean space. With this transformation, we are able to use Euler's formula to show that there are global solutions obtainable by this method for $M = 2, 3, 4, 6$ and 12; and these have respectively 1, 2, 3, 4 and 5 half-spaces intersecting to form the decision regions. We then obtain the globally optimal signal sets for these values of $M$.

In particular, we demonstrate that the signal sets that are *globally* optimal in two complex dimensions do include the abovementioned signal sets for $M = 2$ and $M = 3$ (i.e., the orthogonal signal set $|\langle \mathbf{s}_j, \mathbf{s}_k \rangle| = 0$ for two signals and $|\langle \mathbf{s}_j, \mathbf{s}_k \rangle| = \frac{1}{2}$ for three signals). Furthermore, for four signals, the globally optimal signal set has $|\langle \mathbf{s}_j, \mathbf{s}_k \rangle| = 1/\sqrt{3}$. Finally, for six signals and for twelve signals, the inner products between one signal vector and the others determining the decision region have absolute values $1/\sqrt{2}$ and $\sqrt{2/(5 - \sqrt{5})}$ respectively.

**2. Formulation of the problem.** Suppose we have a communication system that transmits $M$ real signals, $x_1(t), x_2(t), \cdots, x_M(t)$, according to a given a priori probability distribution. The signals are transmitted over a phase-incoherent channel and are corrupted with additive Gaussian noise. For every choice of the set of transmitted signals and every decision rule, there is a probability of correct decoding. The general problem is to find the maximum value for this probability and to determine the corresponding decision rule and set of transmitted signals. One method of attacking this problem is the complex geometric approach that we shall now describe.

If $x_j(t)$ is the $j$th transmitted signal, with energy

$$\frac{E_j}{2} = \int_{-\infty}^{\infty} [x_j(t)]^2 \, dt,$$

let $z_j(t) = x_j(t) + i\hat{x}_j(t)$ be the analytical signal, where $\hat{x}_j$ is the Hilbert transform of $x_j$. If $\Omega > 0$ is the carrier frequency, let $s_j(t) = z_j(t)\, e^{-i\Omega t}$ be the complex envelope. Then the transmitted real signal $x_j(t)$ is equivalently described by the complex signal $s_j(t)$, with $E_j = \int_{-\infty}^{\infty} |s_j(t)|^2 \, dt$, since $x_j(t) = \operatorname{Re} z_j(t) = \operatorname{Re} [s_j(t)\, e^{i\Omega t}]$. Note that, if $x_j(t) = a_j(t) \cos \Omega t$, where $a_j$ is strictly band-limited to the interval $(-\Omega, \Omega)$, then $\hat{x}_j(t) = a_j(t) \sin \Omega t$, so that $z_j(t) = a_j(t)\, e^{i\Omega t}$. Hence, $s_j(t) = a_j(t)$ in this case.

If $w(t)$ is the real noise, which we assume to be a stationary Gaussian process with mean 0, let $W(t) = w(t) + i\hat{w}(t)$ be the corresponding complex noise. Next, let $n(t) = W(t)e^{-i\Omega t}$, so that $n(t)$ is also a complex stationary Gaussian process. Observe that if the spectral density of $n(t)$ is an even function, then the real and imaginary parts of $n(t)$ are independent processes, and this is equivalent to having the spectral density of $W(t)$ symmetric with respect to the carrier frequency $\Omega$.

If $u_j(t)$ is the signal that is received when $x_j(t)$ is transmitted over the phase-incoherent channel, then $u_j(t) = \mathrm{Re}\,[z_j(t)e^{i\theta}] + w(t) = \mathrm{Re}\,[z_j(t)e^{i\theta} + W(t)]$, where $\theta$ is uniformly distributed on $[0, 2\pi]$, $\theta$ is independent of the transmitted signal, and $\theta$ is independent of $w(t)$ and hence of $W(t)$. Let $r_j(t) = [z_j(t)e^{i\theta} + W(t)]e^{-i\Omega t} = s_j(t)e^{i\theta} + n(t)$. Then the received real signal $u_j(t)$ is equivalently described by the complex signal $r_j(t)$, since $u_j(t) = \mathrm{Re}\,[z_j(t)e^{i\theta} + W(t)] = \mathrm{Re}\,[r_j(t)e^{i\Omega t}]$. Note that in the case $x_j(t) = a_j(t)\cos\Omega t$, where $a_j$ is strictly band-limited to the interval $(-\Omega, \Omega)$, we have $u_j(t) = a_j(t)\cos(\Omega t + \theta) + w(t)$ and $r_j(t) = [a_j(t)\cos\theta + \mathrm{Re}\,n(t)] + i[a_j(t)\sin\theta + \mathrm{Im}\,n(t)]$ (see [14, pp. 511–527] for details).

Now let $\phi_1, \phi_2, \cdots, \phi_N$ be orthonormal functions that form a basis for $L^2(s)$, the subspace of $L^2$ spanned by $s_1, s_2, \cdots, s_M$. Assume that the spectral density of $n(t)$ is flat, with value $\eta_0$, over the significant frequencies of $L^2(s)$. Then, if $s(t) \in L^2(s)$ and $r(t) = s(t)e^{i\theta} + n(t)$, we have $r(t) = r_s(t) + n(t) - \underline{n_s(t)}$, where $r_s(t) = s(t)e^{i\theta} + n_s(t)$ and $n_s(t) = \sum_{j=1}^{n} n_j\phi_j(t)$, with the $n_j = \int_{-\infty}^{\infty} n(t)\overline{\phi_j(t)}\,dt$ independent complex normal random variables having independent real and imaginary parts of variance $\eta_0/2$. Also, $n(t) - n_s(t)$ is independent of $n_s(t)$, $s(t)$, and $\theta$, and hence is independent of $r_s(t)$. Consequently, since the optimal decision rule selects the signal $s_j(t)$ as having been transmitted if $s_j$ maximizes $p(s_k)\,p(r_s|s_k)\,p(n - n_s|r_s, s_k) = p(s_k)\,p(r_s|s_k)\,p(n - n_s)$, this rule can be based upon $p(s_k)\,p(r_s|s_k)$. That is, no loss of optimality results if the decision is determined by $r_s(t) = s(t)e^{i\theta} + n_s(t)$. Since these functions are complex linear combinations of $\phi_1, \phi_2, \phi_3, \cdots, \phi_N$, they can be represented by complex $N$-dimensional vectors. Thus, $\mathbf{r} = A\mathbf{s}\,e^{i\theta} + \mathbf{n}$, where $\|\mathbf{s}\| = 1$, $A^2 = (1/\eta_0)\int_{-\infty}^{\infty}|s(t)|^2\,dt = E/\eta_0$ is the signal-to-noise ratio, and $\mathbf{n}$ is a complex $N$-dimensional normal random vector with a density function $p_{\mathbf{n}}(\mathbf{r}) = (1/\pi^N)e^{-r^2}$, $r = \|\mathbf{r}\|$.

Assuming that the transmitted signals have equal energy, we find that the probability density of receiving a vector $\mathbf{r}$ given that $\mathbf{s}_j$ was transmitted is

$$
\begin{aligned}
p(\mathbf{r}|\mathbf{s}_j) &= \frac{1}{2\pi}\int_0^{2\pi} p(\mathbf{r}|\mathbf{s}_j, \theta)\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} p_{\mathbf{n}}(\mathbf{r} - A\mathbf{s}_j e^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\pi^N}\exp\{-\|\mathbf{r} - A\mathbf{s}_j e^{i\theta}\|^2\}\,d\theta \qquad (1) \\
&= \frac{1}{\pi^N}e^{-(r^2+A^2)}\frac{1}{2\pi}\int_0^{2\pi}\exp\{2A|\langle\mathbf{r}, \mathbf{s}_j\rangle|\cos\theta\}\,d\theta \\
&= \frac{1}{\pi^N}e^{-(r^2+A^2)}I_0\left(2Ar\left|\left\langle\frac{\mathbf{r}}{r}, \mathbf{s}_j\right\rangle\right|\right).
\end{aligned}
$$

Thus we can write

$$p(\mathbf{r}|\mathbf{s}_j) = \frac{1}{\pi^N} - e^{-(r^2+A^2)} P_r\left(\left|\left\langle \frac{\mathbf{r}}{r}, \mathbf{s}_j \right\rangle\right|\right), \tag{2}$$

where for each fixed $r > 0$, $P_r$ is a strictly increasing function on $[0, 1]$.

If we partition $C^N$ into decision regions $\mathscr{R}_j$ containing $\mathbf{s}_j e^{i\theta}$ for all $0 \leqq \theta \leqq 2\pi$, $j = 1, 2, \cdots, M$, and assume that our signals are equally probable, then the probability of no decoding error is given by

$$P_C = \frac{1}{M}\sum_{j=1}^{M} \int_{\mathscr{R}_j} p(\mathbf{r}|\mathbf{s}_j)\, dm(\mathbf{r}) = \int_0^\infty \frac{1}{\pi^N} e^{-(r^2+A^2)} U(r) r^{2N-1}\, dr, \tag{3}$$

where

$$U(r) = \frac{1}{M}\sum_{j=1}^{M} \int_{R_j^*} P_r(|\langle \mathbf{s}, \mathbf{s}_j\rangle|)\, d\sigma(\mathbf{s}), \tag{4}$$

$R_j^* = \mathscr{R}_j \cap S_r/r$ and

$$S_q = \{\mathbf{r} \in C^N : \|\mathbf{r}\| = q\}. \tag{5}$$

Clearly, $P_C$ is maximized if $U(r)$ is maximized for each $r > 0$. For given values of $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_M$, each $U(r)$ is certainly maximized by letting

$$\mathscr{R}_j = \{\mathbf{r} \in C^N : |\langle \mathbf{r}, \mathbf{s}_j\rangle| \geqq |\langle \mathbf{r}, \mathbf{s}_k\rangle| \text{ for all } k \neq j\}, \tag{6}$$

in which case $\mathscr{R}_j \cap S_r/r = \mathscr{R}_j \cap S_1 = R_j$ for all $r > 0$. Note also that if $W_{jk}$ is the *half-space* determined by $\mathbf{s}_j$ and $\mathbf{s}_k$ that contains $\mathbf{s}_j$, i.e., $W_{jk} = \{\mathbf{r} \in C^N : |\langle \mathbf{r}, \mathbf{s}_j\rangle| \geqq |\langle \mathbf{r}, \mathbf{s}_k\rangle|\}$, then

$$\mathscr{R}_j = \bigcap_{k \neq j} W_{jk}. \tag{7}$$

Consequently, our problem is to find a condition on the location of points $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_M$, on the unit sphere of $C^N$, such that

$$P_C = \int_0^\infty \frac{1}{\pi^N} e^{-(r^2+A^2)} U(r) r^{2N-1}\, dr$$

is maximized, where

$$U(r) = \frac{1}{M}\sum_{j=1}^{M} \int_{R_j} P_r(|\langle \mathbf{s}, \mathbf{s}_j\rangle|)\, d\sigma(\mathbf{s}),$$

$P_r$ is an increasing function on $[0, 1]$, $R_j = \mathscr{R}_j \cap S_1$, and the decision regions $\mathscr{R}_j$ are the intersections of finite numbers of half-spaces of $C^N$ determined by points on $S_1$.

**3. The method of Landau and Slepian.** For $0 < \alpha < 1$, we define the *cap* of $S_1$, centered at $\mathbf{s}_j$ and of size $\alpha$, to be

$$C_{j,\alpha} = \{\mathbf{s} \in S_1 : |\langle \mathbf{s}, \mathbf{s}_j\rangle| \geqq \alpha\}. \tag{8}$$

We let $\sigma(C_\alpha)$ denote the common value of $\sigma(C_{j,\alpha})$, $j = 1, \cdots, M$, and suppose that $(1/M)\sigma(S_1) \leqq \sigma(C_\alpha) \leqq \frac{1}{2}\sigma(S_1)$. If $W$ is a half-space that does not contain $\mathbf{s}_j$, let

$$h = \int_{C^*} P_r(|\langle \mathbf{s}, \mathbf{s}_m \rangle|)\, d\sigma(\mathbf{s}), \tag{9}$$

where $C^* = C_{j,\alpha} \cap W$. Then the method of Landau and Slepian is based on the following properties of $h$, which we shall prove later for $N = 2$:

(a) $h$ is a function of only $\sigma(C_{j,\alpha} \cap W)$ for fixed $\alpha$ and in fact is a strictly increasing and strictly convex function for $\sigma(C_{j,\alpha} \cap W) > 0$.

(b) If $V$ is the intersection of a finite number of half-spaces at least one of which does not contain $\mathbf{s}_j$, then

$$\int_{C^*} P_r(|\langle \mathbf{s}, \mathbf{s}_j \rangle|)\, d\sigma(\mathbf{s}) \geqq h(\sigma(C_{j,\alpha} \cap V)), \tag{10}$$

where $C^* = C_{j,\alpha} \cap V$, with equality for $\sigma(C_{j,\alpha} \cap V) > 0$ if and only if $V$ is a single half-space.

Assuming $h$ has properties (a) and (b), we proceed as follows. For $j = 1, 2, \cdots, M$, let $k_j$ be the smallest integer such that $\mathscr{R}_j$ is the intersection of distinct half-spaces $W_{j1}, W_{j2}, \cdots, W_{jk_j}$. Then $\mathscr{R}_j^c$ can be partitioned into regions $T_{j1}, T_{j2}, \cdots, T_{jk_j}$, where each $T_{jk}$ is the intersection of $W_{kj}$ and a finite number of other half-spaces. Hence, if we let $E_{j,\alpha} = R_j \cap C_{j,\alpha}^c$ and $T_{jk,\alpha} = C_{j,\alpha} \cap T_{jk}$, we have the identity

$$\sum_{j=1}^{M} \int_{R_j} f_j\, d\sigma = \sum_{j=1}^{M} \left[ \int_{C_{j,\alpha}} f_j\, d\sigma + \int_{E_{j,\alpha}} f_j\, d\sigma - \sum_{k=1}^{k_j} \int_{T_{jk,\alpha}} f_j\, d\sigma \right]. \tag{11}$$

We first let $f_j \equiv 1$ for each $j$, then let $f_j(\mathbf{s}) = P_r(|\langle \mathbf{s}, \mathbf{s}_j \rangle|)$, and use properties (a) and (b) along with the monotonicity of $P_r$ (see [6] for details).

The conclusion is that for any cap size $\sigma(C_\alpha)$ with

$$\frac{1}{M}\sigma(S_1) \leqq \sigma(C_\alpha) \leqq \frac{1}{2}\sigma(S_1), \tag{12}$$

we have

$$U(r) \leqq \int_{C_{1,\alpha}} P_r(|\langle \mathbf{s}, \mathbf{s}_1 \rangle|)\, d\sigma(\mathbf{s}) - \frac{2K}{M} h\left( \frac{1}{2K}[M\sigma(C_\alpha) - \sigma(S_1)] \right), \tag{13}$$

where

$$2K = \sum_{j=1}^{M} k_j.$$

Furthermore, there is equality if and only if such a cap size exists with the additional properties:

(i) $T_{jk,\alpha} = C_{j,\alpha} \cap W_{kj}$, where $W_{kj}$ is a half-space, for all $j$ and $k$.

(ii) $\sigma(T_{jk,\alpha}) = (1/(2K))[M\sigma(C_\alpha) - \sigma(S_1)]$ for all $j$ and $k$.

(iii) $\sigma(E_{j,\alpha}) = 0$ for all $j$.

**4. The case $N = 2 \leqq M$.** Consider the transformation that sends $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$ into $(r, \rho, \theta, \phi)$, where

$$z_1 = r\rho\, e^{i\theta}, \qquad z_2 = r\sqrt{1 - \rho^2}\, e^{i\phi},$$

$0 \leqq \rho \leqq 1$, $-\pi < \theta \leqq \pi$, $-\pi < \phi \leqq \pi$. The Jacobian of this transformation is $r^3\rho$ and so $dm = r^3\, dr\, d\sigma$, where $d\sigma = \rho\, d\rho\, d\theta\, d\phi$, and thus the unit sphere in $C^2$ has measure $\sigma(S_1) = 2\pi^2$.

If

$$\mathbf{s}_0 = e^{i\gamma_0}(1, 0) \quad \text{and} \quad \boldsymbol{\rho} = (\rho\, e^{i\theta}, \sqrt{1 - \rho^2}\, e^{i\phi}),$$

then for $0 < \alpha < 1$ the cap

$$C_\alpha = \{\boldsymbol{\rho} : |\langle \boldsymbol{\rho}, \mathbf{s}_0 \rangle| \geqq \alpha\} = \{\boldsymbol{\rho} : \rho \geqq \alpha\}. \tag{14}$$

For later convenience, we introduce the notation $v = \alpha^2 - \frac{1}{2}$ and $\beta = 1 - 2/M$. Then

$$\sigma(C_\alpha) = 2\pi^2(1 - \alpha^2) = 2\pi^2(\tfrac{1}{2} - v) \tag{15}$$

and the requirement $(1/M)\sigma(S_1) \leqq \sigma(C_\alpha) \leqq \frac{1}{2}\sigma(S_1)$ becomes

$$1/2 \leqq \alpha^2 \leqq 1 - 1/M \tag{16}$$

which is equivalent to $0 \leqq v \leqq \frac{1}{2}\beta$.

Now suppose that $\mathbf{s}_1$ and $\mathbf{s}_2$ are linearly independent points on $S_1$, i.e., $\mathbf{s}_1 \neq e^{i\eta}\mathbf{s}_2$, for any real $\eta$,

$$\mathbf{s}_1 = e^{i\gamma_1}(s_1\, e^{i\delta_1}, \sqrt{1 - s_1^2}) \quad \text{and} \quad \mathbf{s}_2 = e^{i\gamma_2}(s_2\, e^{i\delta_2}, \sqrt{1 - s_2^2}).$$

Let

$$\xi_1 = s_1\sqrt{1 - s_1^2}\, e^{i\delta_1}, \qquad \xi_2 = s_2\sqrt{1 - s_2^2}\, e^{i\delta_2}$$

and

$$\xi = \xi_1 - \xi_2 = |\xi|\, e^{i\delta}.$$

Then the half-space inequality $|\langle \boldsymbol{\rho}, \mathbf{s}_1 \rangle| \geqq |\langle \boldsymbol{\rho}, \mathbf{s}_2 \rangle|$ becomes

$$\sqrt{(\tfrac{1}{2})^2 - (\rho^2 - \tfrac{1}{2})^2}\,|\xi| \cos(\theta - \phi - \delta) \geqq (\rho^2 - \tfrac{1}{2})(s_2^2 - s_1^2). \tag{17}$$

If

$$t = |\xi|/(s_2^2 - s_1^2)$$

and we assume $s_2 \geqq s_1$, then $t$ is well-defined, $0 \leqq t \leqq \infty$, and we have the following cases for inequality (17). If $t = 0$, the inequality is $\rho^2 \leqq \frac{1}{2}$; if $t = \infty$, we have $\cos(\theta - \phi - \delta) \geqq 0$; and if $0 < t < \infty$, the result is

$$\cos(\theta - \phi - \delta) \geqq g_t(u) = \frac{u}{t\sqrt{(\frac{1}{2})^2 - u^2}}, \tag{18}$$

where $u = \rho^2 - \frac{1}{2}$ and $g_t$ is defined for

$$|u| \leqq \tau = \frac{t}{2\sqrt{1 + t^2}}.$$

*Proof of* (a). Let $W_t$ be a half-space, determined by $\mathbf{s}_1$ and $\mathbf{s}_2$ and not containing $\mathbf{s}_0$, which intersects $C_\alpha$ in a set of positive measure. That is, $s_1 < s_2$ and

$$W_t = \{\boldsymbol{\rho} : |\langle \boldsymbol{\rho}, \mathbf{s}_1 \rangle| \geqq |\langle \boldsymbol{\rho}, \mathbf{s}_2 \rangle|\} = \{\boldsymbol{\rho} : \cos(\theta - \phi - \delta) \geqq g_t(u)\}, \tag{19}$$

where $v < \tau < \frac{1}{2}$.

Define

$$\omega_v(t) = \sigma(W_t \cap C_\alpha) = \int_{W^*} d\sigma(\boldsymbol{\rho}) = \int_v^\tau k_t(u)\,du, \tag{20}$$

where $W^* = W_t \cap C_\alpha$ and

$$k_t(u) = 2\pi \arccos g_t(u). \tag{21}$$

Differentiating, we find

$$\frac{\partial \omega_v(t)}{\partial t} = \frac{4\pi\tau}{t^2}\sqrt{\tau^2 - v^2} > 0, \tag{22}$$

so that for fixed $v$, $\omega_v$ is a strictly increasing function of $t$. For $0 \leqq \omega < \pi^2(\frac{1}{2} - v)$, which is the range of $\omega_v$, we let $t_v$ be in the *inverse* function of $\omega_v$.

Next let

$$H_v(t) = \int_v^\tau k_t(u) P_r(\sqrt{u + \tfrac{1}{2}})\,du, \tag{23}$$

and let $h_v(\omega) = H_v(t_v(\omega))$, so that $H_v(t) = h_v(\omega_v(t))$. Then

$$\frac{\partial H_v(t)}{\partial t} = \frac{\partial h_v(\omega_v(t))}{\partial \omega} \cdot \frac{\partial \omega_v(t)}{\partial t},$$

so that we can calculate

$$\frac{\partial h_v(\omega_v(t))}{\partial \omega} = P_r(\alpha) + \int_v^\tau \sqrt{\frac{\tau^2 - u^2}{\tau^2 - v^2}}\, dP_r(\sqrt{u + \tfrac{1}{2}}), \tag{24}$$

which is positive and is a strictly increasing function of $t$ and hence of $\omega$. Therefore, we have proved that, for each fixed $v$, $h_v$ is a strictly increasing, strictly convex function.

*Proof of* (b). First observe that

$$\frac{\partial k_t(u)}{\partial u} = -\frac{\pi}{2} \frac{1}{[(\frac{1}{2})^2 - u^2]\sqrt{(t/2)^2 - (t^2 + 1)u^2}} \tag{25}$$

is a strictly increasing function of $t$ for fixed $u$ and is always negative. Next suppose that $W_{t_1}, W_{t_2}, \cdots, W_{t_n}$ are half-spaces such that

$$V = \bigcap_{i=1}^n W_{t_i}$$

intersects $C_\alpha$ in a set of positive measure and $\mathbf{s}_0 \notin W_{t_i}$ for $i = 1, 2, \cdots, m$, $m \leqq n$. For $\tau < u \leqq \frac{1}{2}$, define $g_t(u) = 1$ and $k_t(u) = 2\pi \arccos g_t(u) = 0$.

Then if $\mathbf{s}_0 \notin W_t$, we have

$$\sigma(W_t \cap C_\alpha) = \int_v^{1/2} k_t(u)\,du,$$

whereas for $\mathbf{s}_0 \in W_t$, we have

$$\sigma(W_t \cap C_\alpha) = \int_v^{1/2} [2\pi^2 - k_t(u)]\,du.$$

Therefore,

$$\sigma(V \cap C_\alpha) = \int_v^{1/2} k(u)\,du,$$

where $k(u)$ can be described as follows.

Let

$$d(i) = \begin{cases} +1, & i = 1, 2, \cdots, m, \\ -1, & i = m + 1, \cdots, n. \end{cases} \tag{26}$$

Then there is a partition $u_0 < u_1 < \cdots < u_k$ of $[v, \frac{1}{2}]$ such that for $u \in [u_{j-1}, u_j]$,

$$k(u) = \lambda + \tfrac{1}{2} \sum d(i_l) k_{t_{i_l}}(u),$$

where $\lambda$ is a constant, and $(i_l)$ is a collection of not necessarily distinct elements of $\{1, 2, \cdots, n\}$ such that at most two of them belong to $\{1, 2, \cdots, m\}$.

In particular, this description shows that $k$ is continuous on $[v, \frac{1}{2}]$, differentiable in $(u_{j-1}, u_j)$, and has right- and left-hand derivatives at the left and right endpoints, respectively. In fact, these derivatives are given by

$$\frac{dk(u)}{du} = \tfrac{1}{2} \sum d(i_l) \frac{\partial k_{t_{i_l}}(u)}{\partial u}.$$

Now let $W_t$ be a half-space such that $\mathbf{s}_0 \notin W_t$ and $\sigma(W_t \cap C_\alpha) = \sigma(V \cap C_\alpha)$. Then $\omega_v(t) = \sigma(W_t \cap C_\alpha) = \sigma(V \cap C_\alpha) \leqq \sigma(W_{t_i} \cap C_\alpha) = \omega_v(t_i)$ for $i = 1, 2, \cdots, m$, which implies $t_i \geqq t$ for $i = 1, 2, \cdots, m$. Consequently,

$$\frac{dk(u)}{du} \geqq \frac{\partial k_t(u)}{\partial u}$$

for $v \leqq u \leqq \tau$, with equality if and only if $V = W_{t_i}$ for some $i$ with $t_i = t$ and $1 \leqq i \leqq m$. But

$$\int_v^{1/2} k(u)\,du = \int_v^{1/2} k_t(u)\,du$$

and

$$\frac{dk(u)}{du} \geqq \frac{\partial k_t(u)}{\partial u}$$

for $v \leqq u < \tau$, which implies there is a point $u_0 \in [v, \frac{1}{2}]$ such that $k_t(u) \geqq k(u)$ for

$u \leqq u_0$ and $k_t(u) \leqq k(u)$ for $u \geqq u_0$. Thus

$$\int_{V^*} P_r(|\langle \boldsymbol{\rho}, \mathbf{s}_0 \rangle|)\, d\sigma(\boldsymbol{\rho}) - h_v(\omega_v(t)) = \int_v^{1/2} [k(u) - k_t(u)] P_r(\sqrt{u + \tfrac{1}{2}})\, du \geqq 0, \tag{27}$$

where $V^* = V \cap C_\alpha$, with equality if and only if $k(u) = k_t(u)$ for all $u \in [v, \frac{1}{2}]$, since $P_r$ is strictly increasing. Hence

$$\int_{V^*} P_r(|\langle \boldsymbol{\rho}, \mathbf{s}_0 \rangle|)\, d\sigma(\boldsymbol{\rho}) \geqq h_v(\sigma(V \cap C_\alpha)),$$

where $V^* = V \cap C_\alpha$, with equality if and only if $V$ is a single half-space.

**5. Some necessary conditions.** We consider the case in which $k_j \geqq 2$ for $j = 1, 2, \cdots, M$ which implies $2K \geqq 2M$ or equivalently $x = M\pi/(2K) \geqq \pi/2$. For a given allowable cap size $\sigma(C_\alpha)$, i.e., $0 \leqq v \leqq \frac{1}{2}\beta$, one necessary condition for an optimal signal set is the existence of a half-space $W_t$ such that $\sigma(W_t \cap C_\alpha) = (1/(2K))[M\sigma(C_\alpha) - \sigma(S_1)]$.

Now

$$\frac{1}{2K}[M\sigma(C_\alpha) - \sigma(S_1)] = 2\pi x(\tfrac{1}{2}\beta - v) = W_x(v) \tag{28}$$

belongs to the domain of $t_v$. Hence, if we define

$$T_x(v) = t_v(W_x(v)), \tag{29}$$

then

$$\omega_v(T_x(v)) = W_x(v). \tag{30}$$

Thus the half-space determined by the parameter $t = T_x(v)$ satisfies this necessary condition.

Since

$$\frac{\partial \omega_v(T_x(v))}{\partial t}\frac{\partial T_x(v)}{\partial v} + \frac{\partial \omega_v(T_x(v))}{\partial v} = \frac{\partial W_x(v)}{\partial v}, \tag{31}$$

we calculate

$$\frac{\partial T_x(v)}{\partial v} = \frac{T_x^2(v)[\arccos g_{T_x(v)}(v) - x]}{2\tau_x(v)\sqrt{\tau_x^2(v) - v^2}}, \tag{32}$$

where

$$\tau_x(v) = \frac{T_x(v)}{2\sqrt{1 + T_x^2(v)}}. \tag{33}$$

Also, if we let

$$U_x(v) = \int_{C_\alpha} P_r(|\langle \mathbf{\rho}, \mathbf{s}_0\rangle|)\, d\sigma(\mathbf{\rho}) - \frac{2K}{M} h_v\left(\frac{1}{2K}[M\sigma(C_\alpha) - \sigma(S_1)]\right)$$

$$(34) \qquad = 2\pi^2 \int_v^{1/2} P_r(\sqrt{u + \tfrac{1}{2}})\, du - \frac{\pi}{x} h_v(W_x(v))$$

$$= 2\pi^2 \int_v^{1/2} P_r(\sqrt{u + \tfrac{1}{2}})\, du - \frac{\pi}{x} H_v(T_x(v)),$$

then

$$(35) \qquad \frac{\partial U_x(v)}{\partial v} = \frac{2\pi^2}{x}[x - \arccos g_{T_x(v)}(v)] \int_v^{\tau_x(v)} \sqrt{\frac{\tau_x^2(v) - u^2}{\tau_x^2(v) - v^2}}\, dP_r(\sqrt{u + \tfrac{1}{2}}).$$

Furthermore,

$$(36) \qquad \frac{\partial U_x(v)}{\partial x} = \frac{\pi}{x^2}\left[ h_v(W_x(v)) - \frac{\partial h_v(W_x(v))}{\partial \omega} \cdot W_x(v) \right] \leqq 0$$

by the convexity of $h_v$ for each fixed $v$.

If we consider the requirement $\sigma(E_{j,\alpha}) = \sigma(R_j \cap C_{j,\alpha}^c) = 0$, we see a further necessary condition is the existence of a $v$, $0 \leqq v \leqq \frac{1}{2}\beta$, such that

$$(37) \qquad 2 \arccos g_{T_x(v)}(v) = \frac{2\pi}{2K/M}, \quad \text{i.e.,} \quad \arccos g_{T_x(v)}(v) = x.$$

We shall show, in fact, that there is exactly one such $v$, call it $V(x)$, and $V(x)$ is the unique point at which the maximum of $T_x$ and the minimum of $U_x$ occur in the interval $[0, \frac{1}{2}\beta]$.

First of all, for $x = \pi/2$, $g_{T_x(v)}(v) = \cos x$ is equivalent to $v = 0$, and so we let $V(\pi/2) = 0$. For $x < \pi/2$, $g_{T_x(v)}(v) = \cos x$ is equivalent to

$$(38) \qquad T_x(v) = \frac{1}{\cos x} \frac{v}{\sqrt{(\frac{1}{2})^2 - v^2}}.$$

On the other hand,

$$(39) \qquad \omega_v(t) = \int_v^\tau k_t(u)\, du = 2\pi[\tfrac{1}{2} \arccos j_t(v) - v \arccos g_t(v)],$$

where

$$(40) \qquad j_t(v) = \frac{1}{2\sqrt{1 + t^2}} \frac{1}{\sqrt{(\frac{1}{2})^2 - v^2}}.$$

Hence the defining equation for $T_x(v)$, $\omega_v(T_x(v)) = W_x(v)$, becomes

$$(41) \qquad \tfrac{1}{2} \arccos j_{T_x(v)}(v) - v \arccos g_{T_x(v)}(v) = x(\tfrac{1}{2}\beta - v).$$

Thus, we are looking for values of $v$ that satisfy the system

$$(42) \qquad \frac{v}{T_x(v)\sqrt{(\frac{1}{2})^2 - v^2}} = \cos x, \qquad \frac{1}{2\sqrt{1 + T_x^2(v)}\sqrt{(\frac{1}{2})^2 - v^2}} = \cos \beta x.$$

The unique solution is found to be

$$V(x) = \frac{1}{2}\frac{\tan \beta x}{\tan x} \quad \text{and} \quad T_x(V(x)) = \frac{\sin \beta x}{\sqrt{\cos^2 \beta x - \cos^2 x}}. \tag{43}$$

To show $V(x) \leqq \frac{1}{2}\beta$ for $0 < x < \pi/2$, we consider $2 \tan x(\frac{1}{2}\beta - V(x)) = \beta \tan x - \tan \beta x$. This function is 0 for $x = 0$ and has a derivative $\beta(\sec^2 x - \sec^2 \beta x) \geqq 0$, since $0 \leqq \beta < 1$.

Finally, if $U_0(x) = U_x(V(x))$ is the value of $U_x$ that will be attained if an optimal signal set occurs for a given value of $x$, we have $U_0'(x) = \partial U_x(V(x))/\partial x \leqq 0$. Thus, for fixed $M$, $U_0$ is a decreasing function of $x$, and hence the maximum possible value of $U_0$ is obtained when $K$ is as large as possible.

**6. The allowable values for** $M$. For every $r > 0$ and integer $M \geqq 2$, formula (13) gives an upper bound for $U(r)$ in terms of $\alpha$, the cap size, and $2K$, the total number of half-spaces that determine the decision regions. The values of $\alpha$ are restricted by $1/2 \leqq \alpha^2 \leqq 1 - 1/M$. Since we are interested only in half-spaces determined by the signal vectors, i.e., the maximum-likelihood decision regions given by (6), we must have $2K \leqq M(M - 1)$ and, for $M \geqq 3$, $2K \geqq 2M$. While keeping $2K$ fixed, the minimum of these upper bounds for $U(r)$, taken over the possible values of $\alpha$, is still an upper bound, which we denote by $U_0$. Since $U_0$ increases with $K$, we must find the maximum value of $K$—say $K_M$. Then a globally optimal set of $M$ signal vectors would be one for which equality is attained in (13) with $K = K_M$ and $\alpha$ the associated minimizing value; i.e., $U = U_0(x_M)$, where $x_M = \pi M/(2K_M)$. However, for equality in (13), each decision region must be formed by the intersection of the same number of half-spaces. Thus, a globally optimal signal set can be obtained by this method only if $2K_M = MI_M$, where $I_M$ is an integer.

In order to evaluate $K_M$, and thus determine the allowable values for $M$, we must consider the geometry more carefully. According to inequality (18), a half-space in $C^2$ may be expressed in terms of the two variables $u$ and $\chi$, where $u = \rho^2 - \frac{1}{2}$ and $\chi = \theta - \phi$. Hence, we can consider a transformation of $C^2$ onto $R^3$ given by

$$\begin{aligned} (z_1, z_2) &= r(\rho\, e^{i\theta}, \sqrt{1 - \rho^2}\, e^{i\theta}) \rightarrow \\ & (x, y, z) = r(2u, \sqrt{1 - 4u^2} \cos \chi, \sqrt{1 - 4u^2} \sin \chi). \end{aligned} \tag{44}$$

This type of mapping is mentioned briefly by Manning [7] on pp. 197–198 of his book on four-dimensional geometry and is discussed in detail in the paper by Blachman [2]. The main properties of this transformation are the following. The unit sphere in $C^2$ is mapped onto the unit sphere in $R^3$. The inverse image of each point in $R^3$ is a great circle, and a half-space in $C^2$ either contains every point or no point of this circle. Finally, the boundary of a half-space in $C^2$, which Manning [7] calls a conical hypersurface of double revolution (see pp. 197, 206, 207, 220), is mapped onto a plane through the origin in $R^3$. Therefore, the image of the decision region $\mathscr{R}_j$ is the intersection of $k_j$ half-spaces, each bounded by a plane through the origin, and thus is a convex cone with $k_j$ faces. Hence, the intersection of the image of $\mathscr{R}_j$ with the unit sphere in $R^3$, i.e., the image of $R_j$, is a spherical polygon bounded by $k_j$ arcs of great circles.

For $M = 2$, the only possible value for $K$ is $K = K_M = 1$, and so $I_M = 1$. For $M \geqq 3$, we note that $K$ is the number of edges of the network, on the unit sphere in $R^3$, determined by the images of the decision regions. Letting $V$ be the number of vertices, we see from Euler's formula that $V - K + M = 2$. But the convexity of the image cones means that a vertex is formed by at least three edges. Hence, $3V \leqq 2K$, which gives the inequality $2K \leqq 6(M-2)$. Since $6(M-2) \leqq M(M-1)$, we have $K_M = 3(M-2)$ for $M \geqq 3$. Thus, $2K_M = MI_M$, where $I_M$ is an integer, means

$$I_M = 6(M-2)/M \tag{45}$$

must be an integer. But this is true only for $M = 3$, 4, 6 and 12, for which the corresponding values of $I_M$ are 2, 3, 4 and 5. In connection with the geometry involved here, see Coxeter [3].

## 7. Solutions for $N = 2$.

*Two signals.* If $M = 2$, then $\beta = 0$, and the requirement $0 \leqq v \leqq \frac{1}{2}\beta$ implies $v = 0$ and $\alpha = 1/\sqrt{2}$. Also $1 \leqq 2K/M \leqq M - 1$ is equivalent to $K = 1$. Thus the decision region for $\mathbf{s}_1$ must be

$$\{\boldsymbol{\rho}: |\langle \boldsymbol{\rho}, \mathbf{s}_1\rangle| \geqq |\langle \boldsymbol{\rho}, \mathbf{s}_0\rangle|\} = \{\boldsymbol{\rho}: \rho^2 \leqq \tfrac{1}{2}\}$$

or

$$t = s_1/\sqrt{1 - s_1^2} = 0 \quad \text{or} \quad s_1 = 0.$$

That is,

$$\mathbf{s}_0 = e^{i\gamma_0}(1, 0) \quad \text{and} \quad \mathbf{s}_1 = e^{i\gamma_1}(0, 1)$$

and, hence, $\langle \mathbf{s}_0, \mathbf{s}_1\rangle = 0$.

*Three signals.* $M = 3$, $K = 3$ is the same as $x = \pi/2$, $\beta = 1/3$. Then

$$V(x) = 0, \qquad \alpha = 1/\sqrt{2},$$

and

$$t = T_x(V(x)) = \frac{\sin \beta x}{\sqrt{\cos^2 \beta x - \cos^2 x}} = \tan(\pi/6) = 1/\sqrt{3}.$$

Hence,

$$\frac{s_1}{\sqrt{1 - s_1^2}} = \frac{s_2}{\sqrt{1 - s_2^2}} = t \quad \text{or} \quad s_1 = s_2 = \frac{\sin \beta x}{\sin x} = \frac{1}{2}.$$

Therefore,

$$\mathbf{s}_0 = e^{i\gamma_0}(1, 0) \quad \text{and} \quad \mathbf{s}_j = e^{i\gamma_j}\left(\frac{1}{2}\exp\left\{i\left(\delta + \frac{2j-1}{2}\pi\right)\right\}, \frac{1}{2}\sqrt{3}\right)$$

for $j = 1, 2$, resulting in

$$|\langle \mathbf{s}_j, \mathbf{s}_k\rangle| = \tfrac{1}{2} \quad \text{for } j \neq k.$$

*Four signals.* $M = 4$, $K = 6$ is equivalent to $x = \pi/3$, $\beta = 1/2$. Then,

$$V(x) = \frac{1}{2}\frac{\tan(\pi/6)}{\tan(\pi/3)} = \frac{1}{6}, \qquad \alpha = \sqrt{\frac{2}{3}},$$

$$t = \frac{\sin(\pi/6)}{\sqrt{\cos^2(\pi/6) - \cos^2(\pi/3)}} = 1/\sqrt{2}, \qquad s_0 = 1,$$

and

$$s_j = \frac{\sin(\pi/6)}{\sin(\pi/3)} = \frac{1}{\sqrt{3}} \quad \text{for } j = 1, 2, 3.$$

Therefore, we have

$$\mathbf{s}_0 = e^{i\gamma_0}(1, 0) \quad \text{and} \quad \mathbf{s}_j = e^{i\gamma_j}\left(\frac{1}{\sqrt{3}}\exp\left\{i\left(\delta + \frac{2j-2}{3}\pi\right)\right\}, \sqrt{\frac{2}{3}}\right)$$

for $j = 1, 2, 3$, resulting in

$$|\langle \mathbf{s}_j, \mathbf{s}_k\rangle| = 1/\sqrt{3} \quad \text{for } j \neq k.$$

*Six signals.* $M = 6$, $K = 12$ is the same as $x = \pi/4$, $\beta = 2/3$. Thus,

$$V(x) = \frac{1}{2}\frac{\tan(\pi/6)}{\tan(\pi/4)} = \frac{1}{2\sqrt{3}}, \quad \alpha = \sqrt{\frac{1+\sqrt{3}}{2\sqrt{3}}},$$

$$t = \frac{\sin(\pi/6)}{\sqrt{\cos^2(\pi/6) - \cos^2(\pi/4)}} = 1, \qquad s_0 = 1,$$

$$s_j = \frac{\sin(\pi/6)}{\sin(\pi/4)} = \frac{1}{\sqrt{2}} \quad \text{for } j = 1, 2, 3, 4 \quad \text{and} \quad s_5 = 0.$$

Therefore we have the following set of signals:

$$\mathbf{s}_0 = e^{i\gamma_0}(1, 0), \qquad \mathbf{s}_j = e^{i\gamma_j}\left(\frac{1}{\sqrt{2}}\exp\left\{i\left(\delta + \frac{2j-1}{4}\pi\right)\right\}, \frac{1}{\sqrt{2}}\right)$$

for $j = 1, 2, 3, 4$, and

$$\mathbf{s}_5 = e^{i\gamma_5}(0, 1),$$

resulting in

$$|\langle \mathbf{s}_0, \mathbf{s}_j\rangle| = |\langle \mathbf{s}_5, \mathbf{s}_j\rangle| = |\langle \mathbf{s}_k, \mathbf{s}_j\rangle| = 1/\sqrt{2}$$

for $j = 1, 2, 3, 4$, $k = 1, 2, 3, 4$, with $k \not\equiv j \bmod 2$, and $\langle \mathbf{s}_0, \mathbf{s}_5\rangle = \langle \mathbf{s}_j, \mathbf{s}_k\rangle = 0$ for $j = 1, 2, 3, 4$, $k = 1, 2, 3, 4$, with $k \equiv j \bmod 2$, $k \neq j$.

*Twelve signals.* $M = 12$, $K = 30$ is equivalent to $x = \pi/5$, $\beta = 5/6$. Thus,

$$V(x) = \frac{1}{2}\frac{\tan(\pi/6)}{\tan(\pi/5)} = \frac{1}{2}\sqrt{\frac{3+\sqrt{5}}{3(5-\sqrt{5})}} \approx 0.39733,$$

$$\alpha = \sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{3+\sqrt{5}}{3(5-\sqrt{5})}}\right)} \approx 0.94727$$

and

$$t = \frac{\sin(\pi/6)}{\sqrt{\cos^2(\pi/6) - \cos^2(\pi/5)}} = \sqrt{\frac{2}{3 - \sqrt{5}}} \approx 1.6180.$$

Then $s_0 = 1$,

$$s_j = \frac{\sin(\pi/6)}{\sin(\pi/5)} = \sqrt{\frac{2}{5 - \sqrt{5}}} \approx 0.85065 \quad \text{for } j = 1, 2, 3, 4, 5,$$

$$s_k = \frac{\sqrt{\sin^2 x - \sin^2 \beta x}}{\sin x} = \frac{\sqrt{\sin^2(\pi/5) - \sin^2(\pi/6)}}{\sin(\pi/5)}$$

$$= \sqrt{\frac{3 - \sqrt{5}}{5 - \sqrt{5}}} \approx 0.52573 \quad \text{for } k = 6, 7, 8, 9, 10, \quad \text{and} \quad s_{11} = 0.$$

The optimum signal set is therefore

$$\mathbf{s}_0 = e^{i\gamma_0}(1, 0),$$

$$\mathbf{s}_j = e^{i\gamma_j}\left(\sqrt{\frac{2}{5 - \sqrt{5}}} \exp\left\{i\left(\delta + \frac{2j - 1}{5}\pi\right)\right\}, \sqrt{\frac{3 - \sqrt{5}}{5 - \sqrt{5}}}\right) \quad \text{for } j = 1, 2, 3, 4, 5,$$

$$\mathbf{s}_k = e^{i\gamma_k}\left(\sqrt{\frac{3 - \sqrt{5}}{5 - \sqrt{5}}} \exp\left\{i\left(\delta + \frac{2k - 12}{5}\pi\right)\right\}, \sqrt{\frac{2}{5 - \sqrt{5}}}\right) \quad \text{for } k = 6, 7, 8, 9, 10,$$

and

$$\mathbf{s}_{11} = e^{i\gamma_{11}}(0, 1).$$

Consequently, for $j, j' = 1, 2, 3, 4, 5$ and $k, k' = 6, 7, 8, 9, 10$, we have

$$|\langle \mathbf{s}_0, \mathbf{s}_j \rangle| = |\langle \mathbf{s}_{11}, \mathbf{s}_k \rangle| = |\langle \mathbf{s}_j, \mathbf{s}_k \rangle| = |\langle \mathbf{s}_j, \mathbf{s}_{j'} \rangle|$$
$$= |\langle \mathbf{s}_k, \mathbf{s}_{k'} \rangle| = \sqrt{2/(5 - \sqrt{5})}$$

whenever

$$k - j \equiv 0, 1 \bmod 5, \quad k' - k \equiv 1 \bmod 5, \quad j' - j \equiv 1 \bmod 5;$$

also,

$$|\langle \mathbf{s}_0, \mathbf{s}_k \rangle| = |\langle \mathbf{s}_{11}, \mathbf{s}_j \rangle| = |\langle \mathbf{s}_j, \mathbf{s}_k \rangle| = |\langle \mathbf{s}_j, \mathbf{s}_{j'} \rangle|$$

$$= |\langle \mathbf{s}_k, \mathbf{s}_{k'} \rangle| = \sqrt{(3 - \sqrt{5})/(5 - \sqrt{5})}$$

for

$$k - j \equiv 2, 4 \bmod 5, \quad k' - k \equiv 2 \bmod 5, \quad j' - j \equiv 2 \bmod 5;$$

and finally,

$$|\langle \mathbf{s}_0, \mathbf{s}_{11} \rangle| = |\langle \mathbf{s}_j, \mathbf{s}_k \rangle| = 0 \quad \text{when } k - j \equiv 3 \bmod 5.$$

In order to illustrate the globally optimal signal vector configurations, Fig. 1 presents the results for the case $M = 12$. The decision regions and caps are first projected onto a rectangular $(u, \chi)$ coordinate system and then onto the unit sphere in $R^3$.

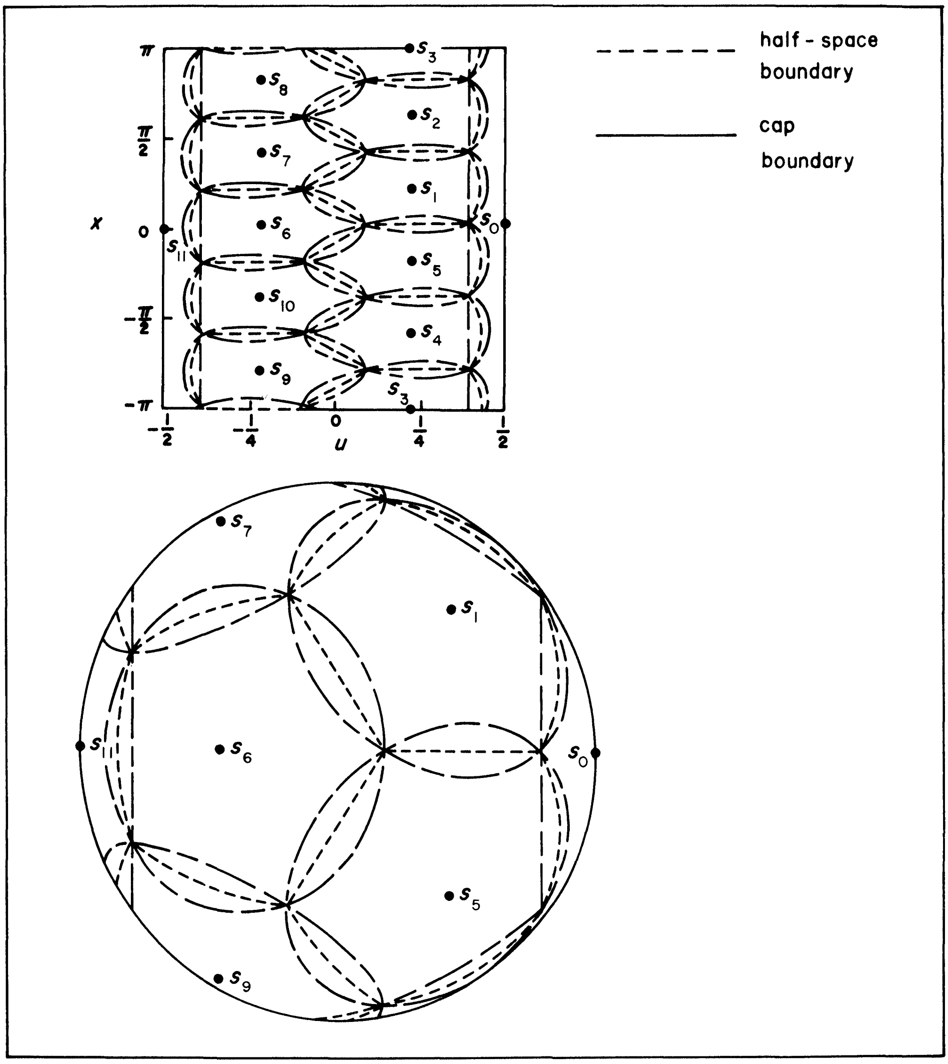


FIG. 1. *The optimum signal set for* $M = 12$, $K = 30$

**8. Calculation of the probability of error.** All the arguments above are independent of the signal-to-noise ratio, since they are based on only the monotone properties of the density. The probability of a correct decision for a globally optimal configuration with signal-to-noise ratio $A^2$ is

$$P_C = 2\int_0^\infty r^3 e^{-(r^2+A^2)}\left\{\int_v^{1/2} I_0(2Ar\sqrt{u+\tfrac{1}{2}})\,du - \frac{1}{2\pi x}\int_v^\tau k_t(u) I_0(2Ar\sqrt{u+\tfrac{1}{2}})\,du\right\} dr, \tag{46}$$

where $x$, $v$, $t$ and $\tau$ have the values associated with that particular configuration.

Using (46), we can compare the probability of error, $P_{\bar{C}} = 1 - P_C$, for the optimal signal sets with the probability of error for $M$ orthogonal signals in $M$

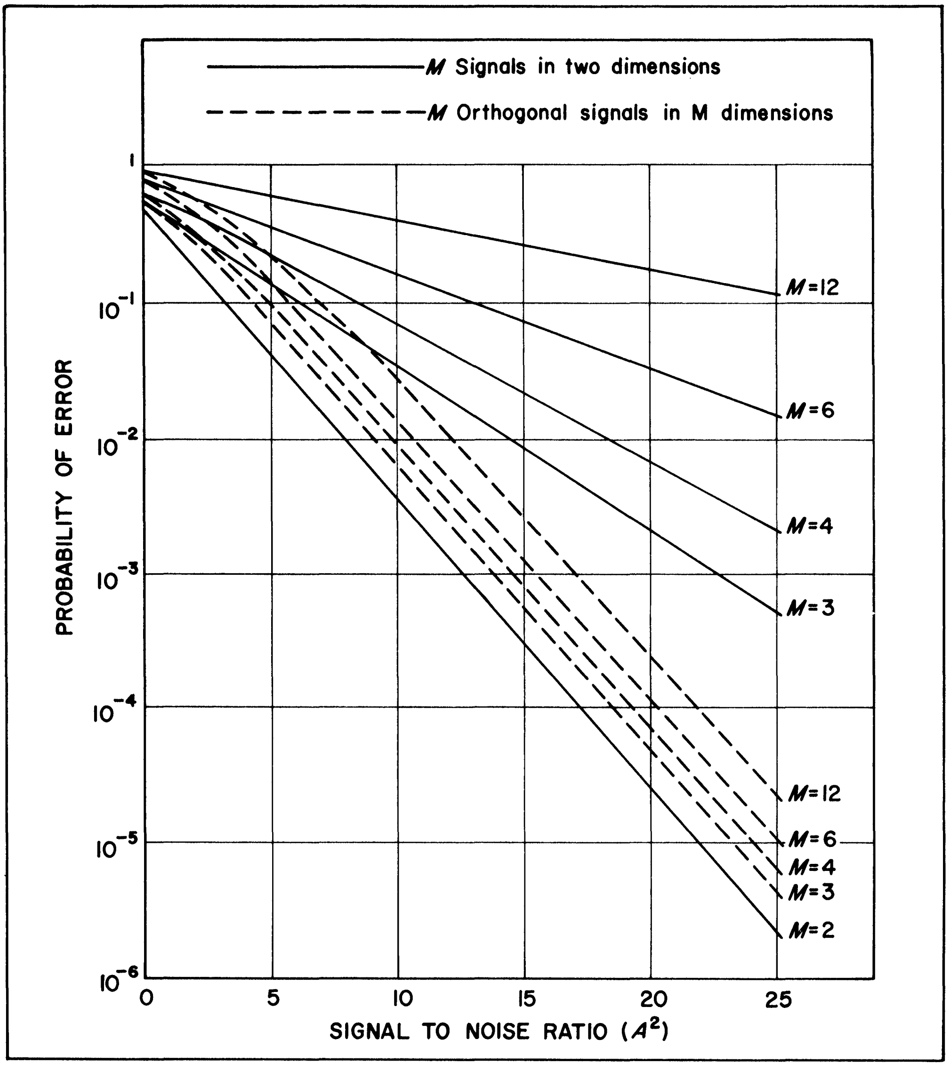


FIG. 2. *Comparison of the probability of error for orthogonal signals and globally optimal signals in $C^2$*

dimensions, which is given by

$$P_C = \frac{1}{M} e^{-A^2} \sum_{k=2}^{M} (-1)^k \binom{M}{k} e^{A^2/k}. \tag{47}$$

The results are presented in Fig. 2.

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