2010

Arithmetic on Specializable Continued Fractions

Ross C. Merriam
Harvey Mudd College

Arithmetic on Specializable Continued Fractions

Ross Merriam

Nicholas Pippenger, Advisor

Christopher Towse, Reader

April, 2010

Department of Mathematics

Contents

Chapter 1

Introduction

Let a continued fraction be an object given by the sequence

$$\{a_i\} = [a_0, a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where $a_i \in \mathbb{Z}$ for all $i \geq 0$. We call the continued fraction simple if

$$a_i > 0 \text{ for } i > 0.$$

The elements of the sequence $\{a_i\}$ are referred to as the partial quotients or partial denominators of the continued fraction.

Well-known theory of simple continued fractions Hardy and Wright (1979) shows that if the sequence of partial quotients is finite, then the fraction corresponds to a rational number while infinite sequences correspond exactly to irrationals. Thus, we may view simple continued fractions as another representation of points on the real line. This notion is enhanced by the fact that the convergents

$$\{a_i\}_n = [a_0, a_1, a_2, \dots, a_n]$$

for an infinite continued fraction $A = \{a_i\}$ correspond to the best rational approximations of the irrational A. Thus it is clear that as partial quotients are added to the sequence, they make successively finer and finer adjustments to the approximation. From this result the study of such objects becomes quite interesting. Since they represent real numbers in a radically different way, it stands to reason that a proper theory of continued fractions will provide insight into the real numbers that conventional representations cannot. For example, sufficient conditions on the convergents of a continued fraction will show the number it represents is transcendental.

However, the task of finding a simple continued fraction representation for a given real number has proven to be difficult in the general case. Very few examples exist for an irrational with predictable patterns in its sequence of partial quotients. The most notable of these examples are quadratic irrationals, that is numbers of the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d represent integers, which by Euler and Lagrange are known to have eventually periodic simple continued fractions. To obtain more general results, it is natural to try to extend the partial quotients into the polynomial ring $\mathbb{Z}[X]$. Any result proved for such a continued fraction would provide insight into an infinite class of simple continued fractions, corresponding to specific, integer values of X. Formalizing this notion, define a specializable continued fraction van der Poorten and Shallit (1993) to be of the form

$$\{a_i(X)\} = [a_0(X), a_1(X), a_2(X), \ldots] = a_0(X) + \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \cfrac{1}{a_3(X) + \cdots}}}$$

where

$$a_i \in \mathbb{Z}[X] \forall i.$$

Also, we stipulate that

$$deg(a_i) > 0 \text{ for } i > 0.$$

Unless otherwise stated, we take all continued fractions in this paper as being specializable in this sense. Note that the convergents of these fractions are rational functions of X rather than rational numbers, so it will be prudent to consider specializable continued fractions as the limiting case for the sequence of functions defined by its convergents. Natural questions then arise about the nature of specializable continued fractions. Of these, the following are considered in this paper:

- When is the sum or product of specializable continued fractions itself specializable?
- Under what conditions will a formal power series have a specializable continued fraction expansion?

Chapter 2

Research in Specializable Continued Fractions

Recently, several interesting specializable continued fraction expansions have been found describing infinite series and products. Interestingly, the series given by

$$\sum_{n=0}^{\infty} \frac{1}{X^{F_n}},$$

where F_n denotes the n^{th} Fibonacci number with $F_0 = 0, F_1 = 1$, has a specializable continued fraction expansion. Using this as the primary result, it is shown in van der Poorten and Shallit (1993) as a corollary that the sum

$$2^{-1} + 2^{-2} + 2^{-3} + 2^{-5} + \cdots$$

converges to a transcendental number. These results are particularly interesting as they appear to generalize nicely to statements about any increasing sequence of nonnegative integers satisfying a sufficient recurrence relation and not simply the Fibonacci sequence. Van der Poorten and Shallit conjecture to this possibility and such generalizations which will be discussed later in the paper with other possibilities for continuing research.

Continuing, we consider Cohn (1996), in which Cohn notices that the following series

$$\sum_{n=0}^{\infty} \frac{1}{2^{2^n}} = [0, 1, 4, 2, 4, 4, 6, 4, \dots]$$

is merely a single specialization of the continued fraction investigated by

Shallit (1979, 1982) given by

$$\sum_{n=0}^{\infty} \frac{1}{x^{2^n}} = [0, x-1, x+2, x, x, x-2, x, x+2, \dots]$$

Further generalizing this result, Cohn classifies all f(x) for which the series

$$\sum_{n=0}^{\infty} \frac{1}{f^n(x)},$$

where f^n denotes the n^{th} iterate of f, has a specializable continued fraction expansion. It is also found in the literature that there exist many infinite products with easily understood specializable continued fraction expansions. Mc Laughlin (2007) shows that the infinite product

$$\prod_{n=0}^{\infty}(1 + \frac{1}{f^n(x)})$$

has a specializable continued fraction expansion for various families of polynomials f. He is also able to show transcendence of the irrationals gained by specializing a given expansion under some simple conditions.

These results illustrate the power of specializable continued fractions, allowing one to prove useful results on larger families of numbers than working with simple continued fractions. This increased efficiency helps mitigate the difficulty of finding specific continued fraction expansions for real numbers but does not eliminate it. We are still left with the dilemma of finding specializable continued fractions, for which the computations are much more grueling. It would thus be useful to obtain other specializable continued fractions from known ones, essentially discovering these new continued fractions for free. Cohn leaves as an open question in Cohn (1996) the conditions under which a sum of specializable continued fractions is itself specializable and it is my purpose to seek an answer to this question and generalizations thereof. It is my hope to then extend the properties for arithmetic on specializable continued fractions to obtain similar results to those discussed in this section.

Chapter 3

Preliminary Results

Recall that the *nth* convergent to an infinite continued fraction is formed by truncating the sequence after $n+1$ terms, including the $0th$. This finite sequence corresponds as a continued fraction to a rational number, denoted p_n/q_n. With this notation, we take the following well-known properties of continued fractions to be true:

Theorem 1. *(Fundamental Results) Let* $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$*. Then*

$$
\begin{gathered}
p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 1) \quad p_{-1} = 1, p_0 = a_0 \\
q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 1) \quad q_{-1} = 0, q_0 = 1 \\
p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \\
[a_n, \ldots, a_2, a_1] = \frac{q_n}{q_{n-1}} \\
[\ldots, a, 0, b, \ldots] = [\ldots, a+b, \ldots] \\
[\ldots, a, -b, \ldots] = [\ldots, a, 0, -1, 1, -1, 0, b] = [\ldots, a-1, 1, b-1].
\end{gathered}
$$

Proofs of these results can be found in most number theory texts, such as Hardy and Wright (1979). These results easily generalize to specializable continued fractions, as seen in van der Poorten (1998). Note that the first three of these results follow from the matrix identity

$$
\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.
$$

Lastly we take without proof a result known as the Folding Lemma. A proof and generalization of the idea can be found in Cohn (1996). Further discussion of the lemma can be found in van der Poorten and Shallit (1992).

Lemma 1 (Folding Lemma)**.** *Define the word* $\overrightarrow{w_n}$ *by* $\overrightarrow{w_n} = [a_1, a_2, a_3, \ldots, a_n]$. *Similarly, let* $\overleftarrow{w_n} = [a_n, a_{n-1}, \ldots, a_1]$ *and* $-\overrightarrow{w_n} = [-a_1, -a_2, \ldots, -a_n]$. *Then it follows that*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2} = [a_0, \overrightarrow{w_n}, x, -\overleftarrow{w_n}].$$

This lemma takes advantage of a symmetry present in many continued fractions and allows for the application of induction principles in proving a supposed continued fraction expansion is correct for all convergents. We can then conclude the validity of a continued fraction expansion for the infinite case, making this lemma invaluable to the study of specializable continued fractions.

Chapter 4

Gosper's Algorithm

We first examine the question of arithmetic on simple continued fractions, with the desire to generalize any results to the realm of specializable continued fractions. However, due to their unique structure, there exists no natural way of performing arithmetic on these objects. Despite this, it has been shown in Gosper (1972) that continued fractions are amenable to the basic arithmetic operations. As the algorithm developed in Gosper (1972) is invaluable to the purpose of this thesis, an analysis of how it operates is provided. Those wishing a more detailed discussion of the algorithm may consult the original source. Consider two simple continued fractions

$$A = [a_0, a_1, a_2, \dots]$$

and

$$B = [b_0, b_1, b_2, \dots].$$

To calculate the ratio

$$X = \frac{aAB + bA + cB + d}{eAB + fA + gB + h},$$

where a, b, c, d, e, f, g, h are constants, note that for all $A, B \in \mathbb{R}$, X varies between between the ratios $\frac{a}{e}, \frac{b}{f}, \frac{c}{g}, \frac{d}{h}$. Expressions such as X are called bihomographic functions in the language of Gosper (1972). If the integral parts of these ratios are equal, we may conclude that value is the integral part of X. To achieve this condition, the algorithm uses the following identity:

$$[a_0, a_1, a_2, \dots] = a_0 + \frac{1}{[a_1, a_2, \dots]}.$$

Applying this iteratively to either A or B will manipulate the four quotients until they have equal integral parts. For example, once a value for a_0 is input into the algorithm, $a_0 + 1/A$ is substituted back into the expression for X, where it is understood that the placeholder A is now given by $A = [a_1, a_2, \ldots]$. Notably, this substitution yields

$$\frac{a(a_0 + \frac{1}{A})B + b(a_0 + \frac{1}{A}) + cB + d}{e(a_0 + \frac{1}{A})B + f(a_0 + \frac{1}{A}) + gB + h} = \frac{(a_0 a + c)AB + (a_0 b + d)A + aB + b}{(a_0 e + g)AB + (a_0 f + h)A + eB + f},$$

which is another bihomographic function. Substituting for B yields similar results, reversing the roles of b and f with those of c and g, respectively. Once the ratios agree in their integral parts, applying the same identity to $X = [x_0, x_1, x_2, \ldots]$ we see that the next output term, x_1, is the integral part of $\frac{1}{[x_0, x_1, x_2, \ldots] - x_0}]$ which is equivalent to

$$\frac{1}{\frac{aAB+bA+cB+d}{eAB+fA+gB+h} - x_0} = \frac{eAB + fA + gB + h}{(a - x_0 e)AB + (b - x_0 f)A + (c - x_0 g)B + (d - x_0 h)}.$$

The algorithm may then be applied again to find the next partial quotient for X. Since this method may be applied for any bihomographic function X, we see that surprisingly, addition, subtraction, multiplication, and division can be performed using the same algorithm, varying only the values of the constants $a, b, c, \ldots, h$. To obtain the basic arithmetic operations, let $C = (a, b, c, d, e, f, g, h)$ take on the following values:

$$\begin{aligned} A + B &: C = (0, 1, 1, 0, 0, 0, 0, 1) \\ A - B &: C = (0, 1, -1, 0, 0, 0, 0, 1) \\ A * B &: C = (1, 0, 0, 0, 0, 0, 0, 1) \\ A/B &: C = (0, 1, 0, 0, 0, 0, 1, 0). \end{aligned}$$

Using this algorithm, I have created a program for determining X from its constants. A discussion of and link to the source code (in Mathematica) for this program is provided in Appendix A.

To better understand the given algorithm, consider the known simple continued fractions for $B = \sqrt{6} = [2, 2, 4, 2, 4, 2, 4, \ldots]$ and $A = \coth 1 = [1, 3, 5, 7, 9, \ldots]$. To calculate $F(A, B) = \frac{2AB+A}{AB+B}$, we begin with the following two-dimensional array:

$$
\begin{array}{ccccccccc}
\cdots & 7 & 5 & 3 & 1 & & & & \\
 & & & & 1 & & 0 & & \\
 & & & & & 0 & & 0 & \\
 & & & & 2 & & 0 & & 2 \\
 & & & & & 1 & & 1 & \\
 & & & & & & & & \\
 & & & & & & & & 2 \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & 4 \\
 & & & & & & & & \vdots
\end{array}
$$

In this array the initial matrix in the upper-right corner represents the values of the constants $a, b, c, \ldots, h$ for the bihomographic function F. Using this array it is trivial to notice that substituting $a_0 + \frac{1}{A}$ for A is equivalent to first multiplying the four coefficients $\begin{smallmatrix} b & f \\ a & e \end{smallmatrix} = \begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}$ by $a_0 = 1$ and adding the result to the matrix $\begin{smallmatrix} d & h \\ c & g \end{smallmatrix} = \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}$. The resultant matrix is then placed to the left of the previous matrix for $\begin{smallmatrix} b & f \\ a & e \end{smallmatrix}$. The new matrix of coefficients is obtained by ignoring the previous values for $\begin{smallmatrix} d & h \\ c & g \end{smallmatrix}$, and considering only the left-most eight digits in the array.

Substitutions for B can be handled similarly, this time working down the array by first multiplying $\begin{smallmatrix} a & c \\ e & g \end{smallmatrix}$ by b_0, adding the result to $\begin{smallmatrix} b & d \\ f & h \end{smallmatrix}$, and placing the new matrix below the previous array. In this case we then ignore all but the bottom eight digits of the array.

For the given example, note that $f = h = 0$, so the quotients $\frac{b}{f}, \frac{d}{h}$ must be shifted out of the array by substituting for B. This yields the new array:

$$
\begin{array}{ccccccccc}
\cdots & 7 & 5 & 3 & 1 & & & & \\
 & & & & 1 & & 0 & & \\
 & & & & & 0 & & 0 & \\
 & & & & 2 & & 0 & & 2 \\
 & & & & & 1 & & 1 & \\
 & & & & & & & & \\
 & & & & 5 & & 0 & & 2 \\
 & & & & & 2 & & 2 & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & 4 \\
 & & & & & & & & \vdots
\end{array}
$$

The constants of F now take on the values $\begin{smallmatrix} a & b & c & d \\ e & f & g & h \end{smallmatrix} = \begin{smallmatrix} 5 & 2 & 0 & 0 \\ 2 & 1 & 2 & 1 \end{smallmatrix}$. Since $\frac{a}{e}$ and $\frac{b}{f}$ have equal integer part, we next shift left by substituting for A. After two such substitutions the following array is generated:

$$
\begin{array}{ccccccccccc}
\cdots & 7 & 5 & & 3 & & 1 & & & & \\
 & & & & & & 1 & & 0 & & \\
 & & & & & & & 0 & & 0 & \\
 & & 8 & & 2 & & 2 & & 0 & & 2 \\
 & & & 7 & & 2 & & 1 & & 1 & \\
 & & & & 1 & & 0 & & & & \\
 & & & & & & & & & & \\
 & & 20 & & 5 & & 5 & & 0 & & 2 \\
 & & & 14 & & 4 & & 2 & & 2 & \\
 & & & & 6 & & 1 & & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & & 4 \\
 & & & & & & & & & & \vdots
\end{array}
$$

The resultant matrix gives immediately that $\lfloor \frac{a}{e} \rfloor = \lfloor \frac{b}{f} \rfloor = \lfloor \frac{c}{g} \rfloor = \lfloor \frac{d}{h} \rfloor = 1$, so $F(A, B)$ has 0^{th} partial quotient equal to 1. After substituting $1 + \frac{1}{F(A,B)}$ for $F(A, B)$, the new matrix of values for $\begin{smallmatrix} f & h \\ e & g \end{smallmatrix}$, is staggered as above with the resultant matrix to demonstrate the new constant values $\begin{smallmatrix} a & b & c & d \\ e & f & g & h \end{smallmatrix} = \begin{smallmatrix} 14 & 7 & 4 & 2 \\ 6 & 1 & 1 & 0 \end{smallmatrix}$. Continuing in this manner, we find that $F(A, B) = [1, 2, 1, 2, 1, 1, \ldots]$.

This algorithm will apply as long as the manipulations are carried out in a Euclidean Domain. Thus we can consider the case when A and B have partial quotients in $\mathbb{Q}(X)$, the ring of polynomials with rational coefficients. This domain encompasses all specializable continued fractions, since $\mathbb{Z}(X) \subset \mathbb{Q}(X)$. However, this shows that $F(A, B)$ may have partial quotients in $\mathbb{Q}(X)$ even if both A and B are specializable. Thus, while a generalized Gosper algorithm will determine the specializability of F for given A, B, it is ill-equipped to answer this question in the general case.

Chapter 5

Continued Fractions for Formal Power Series

As infinite specializable continued fractions correspond to the limit of rational functions, it is reasonable to attempt to represent specializable continued fractions as formal power series, that is objects of the form:

$$\Sigma_{n=0}^{\infty} c_n X^n, c_n \in \mathbb{R} \quad \forall n \geq 0.$$

Here, $\{c_n\}$ is called the sequence of coefficients, and For a general background on the ring of formal power series, consult Wilf (2006).

Within this ring, addition and multiplication are well-defined, so given some relationship between the partial quotients of $A = \{a_i(X)\}$ characterizing when sums and products of specializable continued fractions are still specializable is reduced to finding such sums and products that retain the desired relationship. However, for these formal power series to be well-defined, they should also be realizable as the limit of the power series which correspond to the convergents of A. Convergence in this sense means that the sequence of coefficients for the i^{th} convergent of A, denoted $A_i = [a_0(X), a_1(X), \ldots, a_i(X)]$, agrees to more and more terms with the series for A as i is incremented. Equating A_i to the power series $\Sigma_{n=0}^{\infty} c_n X^n$, we find that $c_0 = [a_0(0), a_1(0), \ldots, a_i(0)]$. This statement shows that A_i has as a constant term in the corresponding formal power series a finite simple continued fraction, that is, a rational number. However, as i increases, more and more terms are added to this fraction, so the sequence of constant terms for the A_i represent convergents of a simple continued fraction, which will not in general agree for any two indices. Thus the sequence of convergents for A does not converge as a formal power series.

Without convergence in this sense, it becomes necessary to examine a given series and attempt to extrapolate a specializable continued fraction. To this end, consider the ratio of power series given by

$$f(X) = \frac{c_{10} + c_{11}X + c_{12}X^2 + \cdots}{c_{00} + c_{01}X + c_{02}X^2 + \cdots}, c_{ij} \in \mathbb{R} \quad \forall i, j.$$

Following the method of Viskovatov (Jones et al., 2008), we find that

$$f(X) = \frac{c_{10} + c_{11}X + c_{12}X^2 + \cdots}{c_{00} + c_{01}X + c_{02}X^2 + \cdots} = \frac{1}{\frac{c_{00}}{c_{10}} + \frac{c_{10}+c_{11}X+c_{12}X^2+\cdots}{c_{00}+c_{01}X+c_{02}X^2+\cdots} - \frac{c_{00}}{c_{10}}} = \frac{c_{10}}{c_{00} + Xf_1(X)}$$

$$f_1(X) = \frac{c_{20} + c_{21}X + c_{22}X^2 + \cdots}{c_{10} + c_{11}X + c_{12}X^2 + \cdots}$$

$$c_{ij} = c_{i-1,0}c_{i-2,j+1} - c_{i-2,0}c_{i-1,j+1}, \quad i \geq 2 \quad j \geq 0.$$

Using the given relationship between the $c'_{ij}s$, one may iterate this process to find

$$f_n(X) = \frac{c_{n+1,0} + c_{n+1,1}X + c_{n+2,1}X^2 + \cdots}{c_{n0} + c_{n1}X + c_{n2}X^2 + \cdots}. = \frac{c_{n+1,0}}{c_{n0} + Xf_{n+1}(X)}, n \geq 1.$$

This sequence readily gives the result

$$f(X) = \cfrac{c_{10}}{c_{00} + \cfrac{c_{20}X}{c_{10} + \cfrac{c_{30}X}{c_{20} + \cdots}}}.$$

Setting $c_{00} = 1$, $c_{0j} = 0 \; \forall j \geq 1$ yields a continued fraction expansion for the power series $f(X) = c_{10} + c_{11}X + c_{12}X^2 + \ldots$:

$$f(X) = \cfrac{c_{10}}{1 + \cfrac{-c_{11}X}{c_{10} + \cfrac{(c_{10}c_{12} - c_{11}^2)X}{-c_{11} + \cdots}}}.$$

From this we see that in general, a formal power series has a continued fraction expansion that is not specializable, and in fact carries the indeterminate X in the partial numerators of the fraction, not the partial denominators.

Chapter 6

Further Research

While we have not explicitly answered the initial question of when sums and products of specializable continued fractions are themselves specializable, the results of the previous chapters suggest various avenues for further research into the problem. Bringing this question into the realm of formal power series appears to be a fruitful method for solving the problem, as there is a wealth of literature on finding continued fraction expansions of all forms for various series including Cohn (1996), Mc Laughlin (2007), Jones et al. (2008), van der Poorten (1998), and Wall (1948). Specifically, one may try to find an equivalence relation that takes a continued fractions of the form

$$f(X) = \cfrac{c_{10}}{1 + \cfrac{-\,c_{11}X}{c_{10} + \cfrac{(c_{10}c_{12} - c_{11}^2)X}{-c_{11} + \cdots}}}$$

and equates it with a specializable continued fraction. Are there certain values of the constants c_{ij} that admit such an equivalence relation?

Moreover, it is shown in Jones et al. (2008) that continued fractions of the form

$$\cfrac{\alpha_1}{\beta_1 + X + \cfrac{\alpha_2}{\beta_2 + X + \cfrac{\alpha 3}{\beta 3 + X + \cdots}}}$$

$$\alpha_i, \beta i \in \mathbb{C} \quad \forall i$$

correspond precisely to power series of the variable $\frac{1}{X}$, as long as the condition

$$det(\begin{bmatrix} c_0 & c_1 & \dots & c_k \\ c_1 & c_2 & \dots & c_{k+1} \\ & & \dots & \\ c_k & c_{k+1} & \dots & c_{2k} \end{bmatrix}) \neq 0$$

holds for all $k \geq 0$, where $\{c_i\}_0^\infty$ represents the sequence of coefficients for the series. Continued fractions of this form are known as J-Fractions and were historically used to solve the moment problem among other applications. Note that setting $\alpha_i = 1, \forall i$ yields a specializable J-Fraction, so there must be specializable continued fractions for some power series in $\frac{1}{X}$. With an easily verifiable condition for such a fraction to exist for a given series, sums and products of specializable J-Fractions should be easily characterized as specializable given sufficient conditions on the sequence of coefficients for the corresponding series. Further information on these series representations as well as many others can be found in Jones et al. (2008) as well as Wall (1948)

Throughout the history of mathematics, continued fractions have been used in countless applications from approximation theory to probability so a continued effort to understand these objects in their most general form may have far-reaching implications for mathematicians in nearly any field. The unique structure of these objects leaves them positioned to provide unique insight into the numbers and functions they represent.

Appendix A

Mathematica Source Code

Mathematica source code for some of the work in this thesis is available from

> `http://www.math.hmc.edu/seniorthesis/rmerriam/`
> `rmerriam-2010-thesis-source-code.nb`.

Note that the first two programs simply perform the transformation between bihomographic functions shown above. The main function is the third, titled CFArith. This function first checks if any of the denominators e, f, g, h are zero since it is a common problem one must work through in the algorithm, as evidenced by the input values given for the four basic arithmetic operations.

It is clear that the coefficients f and h are shifted out of the picture by a vertical move while h and g are shifted out through horizontal moves, and as such we check for the cases $e = 0$ and $f = 0$ first, performing a horizontal move in all other cases. The hardest case to work through is when $e = 0$, as one must perform two moves to remove the coefficient entirely.

Once we obtain nonzero denominators, CFArith checks if the integer parts of $\frac{a}{e}, \frac{b}{f}, \frac{c}{g}, \frac{d}{h}$ are equal, outputting a term and continuing the algorithm as described above. Otherwise, various pairs of ratios are examined. As $\frac{a}{e}$ and $\frac{b}{f}$ remain after a horizontal move, their equality forces a horizontal move from CFArith. Similarly, if $\frac{a}{e} = \frac{c}{g}$ a vertical move is performed to retain these two ratios. These checks in the algorithm disregard the ratio $\frac{d}{h}$ since it is shifted from the picture under either move. CFArith of course recursively calls at every possible exit from the series of nested if loops, to keep the algorithm running indefinitely if need be. The given code can be made to apply to specializable continued fractions by simply performing the same operations on the partial quotients, but in the more general class of polynomials over $\mathbb{Z}$.

Bibliography

Cohn, H. 1996. Symmetry and specializability in continued fractions. *Acta Arithmetica* 75(4).

Gosper, R.W. 1972. Continued fraction arithmetic. *HAKMEM Item 101B, MIT Artificial Intelligence Memo* 239.

Hardy, G.H., and E.M. Wright. 1979. *An Introduction to the Theory of Numbers*. Oxford.

Jones, W.B., V.B. Petersen, B. Verdonk, and H. Waadeland. 2008. *Handbook of Continued Fractions for Special Functions*. Springer.

Mc Laughlin, J. 2007. Symmetry and specializability in the continued fraction expansions of some infinite products. *J Number Theory* 127(2):184–219.

Shallit, J. 1979. Simple continued fractions for some irrational numbers. *J Number Theory* 11(2):209–217.

———. 1982. Simple continued fractions for some irrational numbers, II. *J Number Theory* 14(2):228–231.

van der Poorten, A.J. 1998. Formal power series and their continued fraction expansion. *Lecture Notes in Computer Science: Algorithmic Number Theory* 1423(2):358–371.

van der Poorten, A.J., and J. Shallit. 1992. Folded continued fractions. *J Number Theory* 40(2):237–250.

———. 1993. A specialised continued fraction. *Canad J Math* 45(5):1067–1079.

Wall, H.S. 1948. *Analytic Theory of Continued Fractions*. D. Van Nostrand.

Wilf, H.S. 2006. *generatingfunctionology*. A K Peters, 3rd ed.