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Understanding Voting for Committees Using Wreath Products

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May, 2010

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Abstract

In this thesis, we construct an algebraic framework for analyzing committee elections. In this framework, module homomorphisms are used to model positional voting procedures. Using the action of the wreath product group $S_2[S_n]$ on these modules, we obtain module decompositions which help us to gain an understanding of the module homomorphism. We use these decompositions to construct some interesting voting paradoxes.

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Acknowledgments

I would like to thank Maso, for leading me to this beautiful place called math. Endless thanks to Professor Orrison; I really see you as more of an Obi-Wan, honestly.

Chapter 1

Introduction

Extensive work has been done to expose and analyze the flaws in common single-candidate election procedures. Perhaps most pertinent to this thesis, significant work on voting theory has been done by Saari (1999), and an algebraic interpretation of Saari's work has been done by Daugherty et al. (2009). Saari's work has exposed disturbing flaws in common voting systems which potentially allow for unintended election results or manipulation of election results. The problem of electing committees in a fair and representative procedure is even more complex to develop and analyze. For example, voters may have preferences regarding the relationships between committee members, and it may be appropriate to develop a voting procedure that respects those preferences. Various committee election procedures have been proposed by Ratliff (2003a, b), Fishburn and Pekeč (2009), Brams et al. (2007), and others. In particular, an election procedure investigated by Ratliff (2003a) is the prime motivation for this thesis; other works on committee elections are summarized in order to present a sample of the ideas which surround this problem.

1.1 Ratliff and Wheaton College

Ratliff (2003a) discusses an intriguing example of a committee election that failed to respect the voter's preferences regarding the committee as a whole. In 1992, the faculty at Wheaton College in Massachusetts conducted an election to compose a Presidential search committee. To represent the college as best as possible, the search committee was to be comprised of three candidates, one from each of the three academic divisions of the college: Arts & Humanities, Natural Sciences, and Social Sciences. The voting procedure was simple and accessible. An initial ballot was distributed to first reduce the number of potential candidates to two from each division; next, the final ballot required voters to select their preferred candidate from each division. From each division, the candidate who received the most votes was then placed on the search committee.

The results produced by the election received a nearly unanimous disapproval from the college. Until four years before the election, Wheaton had been a women's college; despite this, their newly elected Presidential search committee consisted of three men. In 2003, another Presidential search was needed, and Wheaton realized the errors of the previous election and used a new voting procedure. After reducing the pool of initial candidates down to two members from each division, the ballot asked voters to rank from first to eighth all of the eight possible committees that could be constructed. The votes were then scored using a Borda count, in which for each vote, the first-place committee received seven points, the second-place committee received six points, and so on.

Ratliff analyzes the outcome this election produced (a committee with two women and one man), arguing that the election procedure had an excellent representation of voters' preferences. Ratliff then extends his analysis to ask questions about committee elections of larger size. This particular election at Wheaton worked smoothly because there were only eight possible committees to fully rank. However, for example, if a ballot for an election was produced to elect a committee from five departments with each department putting forth three candidates, the number of possible committees would be 243. Firstly, asking voters to fully rank all 243 possibilities is unreasonable. Secondly, even if these rankings were produced, the significance of the rankings is questionable; voters may have a good idea of their most and least preferred committees, but given 243 possibilities it may be difficult for a voter to list the middle-ranked committees with any confidence. The differences between the 1992 and 2003 elections at Wheaton College suggest that the method of fully ranking committees is better for representing voters' opinions. Unfortunately, when the number of possible committees increases, this strong voting procedure becomes impractical.

1.2 Two More Methods

To get a sense of the scope of the committee election problem, we briefly introduce two other voting methods that have been developed. These methods are known as *nonpositional* voting methods, as they do not require voters to position the committees in any kind of ranking.

One voting procedure is proposed by Fishburn and Pekeč, and their procedure, like Ratliff's, is based on the notion that it is insufficient to only obtain voter preferences on individual candidates. The procedure is designed to be simple for voters, avoiding the manageability problems that can arise in large committee elections; all that voters are required to do is identify a single subset of candidates that they would approve of having on the committee. These votes are then aggregated using a threshold function—for each voter, this function identifies a possible committee as approved by the voter if it contains a sufficiently large intersection with the voter's approved group of candidates (the size of this intersection is known as the threshold). For example, if the threshold is two and a voter approves of the candidate group {A,B}, then the threshold function will assume that the voter approves of the committee {A,B,C} but not of {A,C,D}. The committee approved by the most voters is selected as the winner.

This procedure guarantees that voters can express preferences on the relationship between committee members. However, the simplicity of the method may pose unwanted issues for voters. For example, suppose a voter approves of having candidates A and B on the committee as long as candidate C is not; approving of the candidate group {A,B} is no longer in the interest of the voter, even though she approves of many committees containing those candidates. In addition, Section 2 of Fishburn and Pekeč finds that the problem of applying the threshold function is NP-complete, so computation for large elections may be troublesome.

Another voting procedure proposed by Brams et al. (2007) is known as the minimax procedure. The minimax procedure asks all voters to declare their most preferred committee. The procedure then selects the committee such that the maximum Hamming distance from the voters' choices is minimized. The Hamming distance between two committees is simply the number of candidates by which they differ. Thus this method seeks to pick a committee so as not to antagonize any of the voters too much. By only asking voters for their favorite committee, the method obtains information about preferences on the committee as a whole in a simple manner. Section 5 of Brams et al. shows that the minimax method is manipulable, meaning that it is possible for a voter to obtain a preferred outcome by misrepresenting his preferences; however, the chance is small that a voter will have enough information to perform a desired manipulation.

1.3 The Problem At Hand

This thesis will attack the committee election problem from an algebraic perspective, similarly to the work in Daugherty et al. (2009). With such an enormous problem to tackle, we will begin by narrowing our attention. This paper begins by seeking to understand committee elections which share a structure similar to the Wheaton elections: elections which elect members from separate divisions or departments, each with the same number of candidates. With an understanding of these elections come two important results: firstly, a better idea of how to construct a practical and representative voting procedure for elections of any size, and secondly, an understanding of the potential voting paradoxes which may arise. A primary goal of this thesis is to construct a strong algebraic framework that will allow us to analyze different voting procedures while using the same algebraic perspective.

While both of the aforementioned nonpositional voting methods are appealing, in this thesis we choose to focus on positional voting methods such as the one used in the Wheaton elections. The selection of this particular focus is motivated by Daugherty et al. (2009), which shows that we can construct an algebraic framework with which we can analyze the behaviors of positional voting procedures. Before constructing a general algebraic framework for committee elections, we try to get a sense of the breadth of the task by looking at the various kinds of committee elections that exist. In general, committee elections vary based on a few key parameters. One parameter is the manner in which candidates are grouped. An election might seek to construct a committee from a large, uniform pool of candidates, or it might be selecting candidates from multiple departments or categories. For example, a little league kickball team might hold an election to vote their starting lineup by simply choosing nine players from the whole group. On the other hand, they could instead decide to elect their starting lineup by voting for their favorite pitcher, catcher, etc.

Other key parameters involve quantities of people. When an organization constructs a committee election they have to decide on some fixed or variable number of people they want on the committee. If the voters are voting for candidates from different departments, there may be a fixed or variable number of candidates the voters want to elect from each department. Another parameter is the number of people allowed to run for the committee. If there are multiple departments, there may be rules about how many candidates each department is allowed to put forth; there may even be rules regarding whether candidates are allowed to run from multiple departments or categories. Some elections may allow voters to even vote for candidates not on the ballot.

Perhaps the parameter which most heavily influences our choice to use an algebraic framework is the voting procedure. Different voting procedures include the two methods mentioned in Section 1.3, approval voting (pick anyone of whom you approve), and positional voting (rank all or some of the candidates, and then points are assigned based on the ranking). As demonstrated in Daugherty et al. (2009), an algebraic framework can be applied to positional and approval voting procedures. The existing literature demonstrates that an algebraic perspective is a powerful way to study elections, and so in this paper we restrict our focus to voting procedures realizable by this kind of framework. The framework constructed in Daugherty et al. (2009) applies to single candidate elections; in this thesis we construct a natural extension of the framework suited to handle committee elections.

As we can see, the committee election problem is a broad and complex one. Constructing a general algebraic framework for all of these parameters and scenarios is a daunting task, so we heavily narrow our focus. Primarily, we will seek to construct a framework to handle positional voting methods with the intent that there will be natural extensions to methods such as approval and plurality, such as in Daugherty et al. (2009). The focus of this paper will be on committee elections which share a similar structure to the Wheaton elections: elections which elect members from separate divisions or departments, each putting forth two candidates. With an understanding of these elections come two important results: firstly, a deeper understanding of how voters influence the outcome of an election, and secondly, an understanding of the potential voting paradoxes which may arise. Hopefully, this research will serve as a strong step towards understanding more general committee elections, perhaps even providing ideas on how to construct a practical and effective voting procedure for elections of any size.

We assume that the reader is familiar with linear algebra and abstract algebra, with a basic knowledge of module theory and character theory. For nice introductions to module theory and character theory, see James and Liebeck (2001).

Chapter 2

The Algebraic Perspective

In this chapter, we build our algebraic framework by modeling an election procedure as a linear transformation between two vector spaces. Using group actions, we then view the linear transformation as a module homomorphism between two modules, which allows us to apply strong results from representation theory. The framework allows us to analyze the ways in which voting procedures use the information that is input by voters. We discuss the choice of group with which to construct our modules and define the group action on the modules, and argue that the wreath product group is the optimal choice given the particular election structure we are studying.

2.1 The Profile Space and the Results Space

Based on the work of Saari (1999), Daugherty et al. (2009) present an algebraic approach to voting theory which serves as an illuminating interpretation of the inner workings of single-candidate elections. In particular, Saari and Daugherty et al. focus heavily on positional voting procedures in which points are assigned to alternatives based on how each voter ranks or positions them. For example, the commonly used plurality method is a positional procedure, as it assigns one point to each voter's top-ranked candidate; the Borda count is another example of a positional procedure for n candidates, awarding $n-1$ points to a voter's top-ranked candidate, $n-2$ points to the second-ranked, and so on. An understanding of the complexities of positional methods can help us to explain many paradoxes which occur in elections. More on voting paradoxes will be discussed in Chapter 4.

The algebraic perspective that is applied to positional voting methods by Daugherty et al. (2009) extends in a natural way to committee elections. Consider an election in which there are m possible committees. With these m committees, a voter can produce a list of all m committees in which committees higher on the list are preferred to those lower on the list. We call such a list a *full ranking* of the committees. There exist $m!$ full rankings of the m committees, and these full rankings form the basis of an $m!$-dimensional vector space (over $\mathbb{Q}$) which we call the *profile space*. We may think of a vector, or *profile*, in the profile space as a tally of votes for each possible full ranking. For example, suppose there were three possible committees A, B, and C; then a profile using the basis of full rankings might look like the profile:

$$\mathbf{p} = \begin{array}{c} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{array} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}.$$

This example profile $\mathbf{p}$ represents an election in which 3 voters preferred the full ranking ABC, none preferred ACB, 2 preferred BAC, etc.

The profile space contains all possible ways in which a group of voters may vote for rankings of committees. In an election, after we have collected these votes, we then score them in some way to obtain a winner. This scoring procedure is realized as a linear transformation from the profile space to the *results space*, an m-dimensional vector space in which the set of m possible committees forms the basis. Vectors in the results space simply indicate the "scores" each committee has received; in an election, the committee with the highest score is usually considered the winner.

The linear transformation from the profile space to the results space is encoded as a matrix using a *weighting vector* $\mathbf{w}$. Every positional voting procedure has a corresponding weighting vector; the weighting vector for a procedure reflects the manner in which full rankings are scored under that procedure. For our three committee example above, the weighting vector is of the form $\mathbf{w} = [w_1, w_2, w_3]^t \in \mathbb{Q}^3$. Given any full ranking, the first place committee receives w_1 points, second place receives w_2 points, and third place receives w_3 points. For example, if we use the plurality method to score the three committee election, the weighting vector is $\mathbf{w} = [1, 0, 0]^t$, indicating that the highest-ranked committee receives 1 point, and the two others receive none. Under the Borda count, the weighting vector is instead

$\mathbf{w} = [2,1,0]^t$. The linear transformation $T_{\mathbf{w}}$ from the profile space to the results space is constructed using $\mathbf{w}$: the columns of $T_{\mathbf{w}}$ permutations of $\mathbf{w}$ corresponding with the full rankings indexed in the profiles. For example, the transformation of $\mathbf{p}$ via $T_{\mathbf{w}}$ for the Borda count (where $\mathbf{w} = [2,1,0]^t$) is

$$T_{\mathbf{w}}\mathbf{p} = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 7 \end{bmatrix}.$$

The first row of $T_{\mathbf{w}}$ indicates how many points A receives for each full ranking; the second row corresponds to B, and the third row corresponds to C. So, for example, the third column $[1,2,0]^t$ scores the full ranking BAC. Notice that the results vector indicates that A receives 11 points, winning the election under the Borda count, since B only receives 9 points with C receiving 7.

By itself, this linear algebraic framework already gives us tools with which we can begin to analyze elections. In particular, we can study the kernel of $T_{\mathbf{w}}$ and the image of $T_{\mathbf{w}}$, or we could talk about the row space of $T_{\mathbf{w}}$. The kernel of $T_{\mathbf{w}}$ can be interpreted as the space of profiles which contribute nothing to the results of an election, while the row space of $T_{\mathbf{w}}$ can be thought of as the space of profiles that do contribute to the results of an election. The image of $T_{\mathbf{w}}$ can be seen as the space of possible results given a weighting vector $\mathbf{w}$.

With this perspective, we can study how elections behave differently under different voting procedures. In particular, note that for two different weighting vectors $\mathbf{w}$ and $\mathbf{v}$, the linear transformations $T_{\mathbf{w}}$ and $T_{\mathbf{v}}$ may differ in their kernels, row spaces, and images. Thus, once the votes are collected in an election, it may be possible to achieve vastly different results by choosing different voting procedures. For example, suppose we scored our three committee election with the plurality procedure instead of the Borda count. Using the weighting vector $\mathbf{v} = [1,0,0]^t$, we obtain the results vector

$$T_{\mathbf{v}}\mathbf{p} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

While under the Borda count, A wins the election, under plurality the three committees are all tied. This strong dependence of election results on the voting procedure is one of the paradoxes of voting theory, and we further explore voting paradoxes in Chapter 4.

2.2 Applying Representation Theory

It is helpful to understand positional voting procedures as linear transformations, but even more can be illuminated if we turn the profile and results spaces into modules and think of $T_{\mathbf{w}}$ as a *module homomorphism*. If we define the action of a group algebra $\mathbb{Q}G$ on the profile space and results space, the spaces become $\mathbb{Q}G$-modules. In Daugherty et al. (2009), the profile space and results space are viewed as $\mathbb{Q}S_n$-modules; the vector spaces become modules under the action of the symmetric group S_n. Expanding the framework to use modules grants us even more tools for analyzing the profile and results spaces.

Most importantly, we know that a $\mathbb{Q}G$-module P can be written as a direct sum of $\mathbb{Q}G$-*submodules*, which are subspaces of P that are invariant under the action of $\mathbb{Q}G$. In other words, we can write P as

$$P \cong P_1 \oplus \cdots \oplus P_k$$

such that for all $\alpha \in \mathbb{Q}G$ and $\mathbf{p_i} \in P_i$, $\alpha\mathbf{p_i} \in P_i$.

A module P is said to be *irreducible* if P is nonzero, and the only submodules of P are $\{\mathbf{0}\}$ and P itself. As a consequence of Maschke's Theorem (see Chapter 8 in James and Liebeck (2001)), every nonzero $\mathbb{Q}G$-module is a direct sum of irreducible $\mathbb{Q}G$-modules. Up to isomorphism, there exist only a finite number of distinct irreducible $\mathbb{Q}G$-modules, so knowing how a $\mathbb{Q}G$-module decomposes into irreducible submodules allows us to compare its structure with other $\mathbb{Q}G$-modules. As an example, consider the results space of the three committee election from the previous section. As a $\mathbb{Q}S_3$-module, the results space R can be decomposed as

$$R = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle,$$

where the subspaces on the right are distinct irreducible $\mathbb{Q}S_3$-modules, say R_1 and R_2 (so $R \cong R_1 \oplus R_2$). Irreducible submodules can be easier to grapple with than the entirety of a module.

Together with the theory of module homomorphisms, module decompositions help us understand the profile and results spaces. Viewing $T_{\mathbf{w}}$ as a module homomorphism is extremely useful for two reasons. Firstly, if $T_{\mathbf{w}} : P \to R$ is a module homomorphism, then the kernel of $T_{\mathbf{w}}$ is a submodule of P, and the image of T is a submodule of R. If we define the *effective space* $E(T_{\mathbf{w}})$ as the orthogonal complement (under the usual inner product) of the kernel of $T_{\mathbf{w}}$, then $E(T_{\mathbf{w}}) \cong T_{\mathbf{w}}(P)$ as modules. Given a weighting vector $\mathbf{w}$, the effective space contains the only information in the profile space that $T_{\mathbf{w}}$ will map to a nonzero vector in the results space.

Secondly, an important insight from representation theory, Schur's Lemma, provides an extremely useful tool to understand module homomorphisms. A reference for these results may be found in Chapter 9 in James and Liebeck (2001).

Lemma 2.1 (Schur's Lemma)**.** *Every module homomorphism between irreducible modules is either an isomorphism or the zero homomorphism.*

To illustrate how Schur's Lemma is useful, consider our three committee example. We found that as a $\mathbb{Q}S_3$-module, the results space has the decomposition $R \cong R_1 \oplus R_2$. It turns out that as a $\mathbb{Q}S_3$-module, the profile space has the decomposition $P \cong R_1 \oplus R_2 \oplus R_2 \oplus R_3$, where R_3 is another irreducible module. Letting $T_{\mathbf{w}}$ be the module homomorphism $T_{\mathbf{w}} : P \to R$, Schur's Lemma guarantees that R_3 must be in the kernel of $T_{\mathbf{w}}$. Thus Schur's Lemma helps us easily identify a subspace of the profile space that will have no effect on the results of the election.

These observations can be useful since the profile space is $m!$-dimensional, while the results space is only m-dimensional. Understanding the irreducible submodules that make up the small results space will greatly simplify our analysis of the effective space of any module homomorphism from the large profile space. Once we know the irreducible submodules that compose the results space, we know which irreducible submodules of the profile space must be in the kernel, so we can essentially ignore them. Decompositions of the profile and results spaces can help us see what kinds of profiles are being considered or completely ignored by any positional voting procedure. This kind of analysis will show up in Chapters 3 and 4 when we decompose the profile and results spaces.

2.3 Choosing the Group

To view the profile and results spaces as modules, we of course must select a group. We will have to define the group action on the profile and results spaces, so we want a group for which there will be a very natural action on the committees. Furthermore, we want a group that will reflect the symmetries that exist in our election structure. As a more technical consideration, in choosing our group G, we also want to consider how our profile and results spaces would decompose as $\mathbb{Q}G$-modules. For example, if all irreducible $\mathbb{Q}G$-modules are 1-dimensional, then our modules will decompose into 1-dimensional components, from which it may be difficult to extract information. On the other hand, if the irreducible $\mathbb{Q}G$-modules are very large, then our decompositions will consist of very large components which may be either unwieldy, uninformative, or both.

As demonstrated in Daughterty et al., in a single candidate election with n candidates, a very appropriate choice of group is the symmetric group S_n; the group action simply permutes the n candidates. For our committee elections, the symmetric group is also a plausible choice. With n departments we have 2^n possible committees, so we could use the group S_{2^n}; the group action would simply permute the 2^n committees. One benefit of using this group is that the theory of $\mathbb{Q}S_n$-modules applied to elections is already well-understood, so many of the results would likely carry over. However, there are a couple important reasons that suggest we should use a different group. Firstly, even for small n the size of S_{2^n} grows rapidly, as do the sizes of irreducible $\mathbb{Q}S_{2^n}$-modules. As these irreducible modules are the structures from which we hope to gain information, they would be difficult to handle and understand given their large sizes. Secondly and more importantly, using this group only reflects the symmetries of the committees but not of the candidates themselves. We are heavily motivated to pay more attention to the effect of the candidates in committee elections, and the symmetric group ignores which candidates make up which committees.

Instead of a group that acts only on the committees, we should look for groups that act directly on the candidates. Once we define a group action on the candidates, we can then extend the action to the committees. We search for groups that reflect the symmetries of our election structure. Such groups will somehow permute the candidates within their respective departments; we can consider groups that permute the departments as well. There exist plenty of such groups, but in particular three groups that act very naturally on the set of candidates are those in the chain

$$S_2 \times \ldots \times S_2 \leq S_2[S_n] \leq S_{2n},$$

where $S_2[S_n]$ is the wreath product group, which will be explained shortly. In an election with n departments, we have a total of $2n$ candidates, so the largest group in the chain is simply the symmetric group on $2n$ elements. While this certainly permutes the candidates, it ignores the departmental structure we have set, and so it is not an appropriate group to use here.

The smallest group in our chain is a direct product of n copies of S_2. This group permutes each pair of candidates, but ignores any symmetry between the departments. This is a very natural group action that fits our election structure, but its module theory reveals that this is not quite the group we want. Firstly, all irreducible $\mathbb{Q}(S_2 \times \ldots \times S_2)$-modules are 1-dimensional, suggesting that it may be difficult to extract information from our decompositions. Secondly, as a $\mathbb{Q}(S_2 \times \ldots \times S_2)$-module, the results space contains copies of every distinct irreducible $\mathbb{Q}(S_2 \times \ldots \times S_2)$-module. Recall our illustration of how to use Schur's Lemma to determine which submodules of the profile space must be in the kernel of $T_{\mathbf{w}}$. Because every distinct irreducible $\mathbb{Q}(S_2 \times \ldots \times S_2)$-module exists in the results space, we are unable to use Schur's Lemma to determine if any submodules of the profile space are always in the kernel. Thus in our discussions of the effective space of $T_{\mathbf{w}}$, we are forced to accommodate the entirety of the large profile space in our analysis.

Finally, we consider the group in the middle of the chain, $S_2[S_n]$, the wreath product group. The wreath product $S_2[S_n]$ is defined as a semidirect product $H \rtimes S_n$ where H is a direct sum of n copies of S_2. There is a more natural way to interpret this wreath product for our purposes, as it reflects the symmetries of our election structure. An element of $S_2[S_n]$ will permute the pair of candidates in each department (not necessarily the same permutation), and then it will permute the n departments. The reasons to use the wreath product extend beyond the simple reason that the election structure is highly appropriate for a wreath product action. A highly beneficial result of using the wreath product instead of the symmetric group is that we achieve a finer decomposition of the spaces; given n, irreducible $\mathbb{Q}(S_2[S_n])$-modules are generally smaller than irreducible $\mathbb{Q}(S_{2^n})$-modules. In other words, our modules will decompose into smaller, more manageable irreducible submodules. Demonstrations of this result will be discussed in later chapters.

Another wonderful advantage the wreath product has over the symmetric group is that for our problem, the wreath product group we will be

n	$S_2[S_n]$	S_{2^n}
2	8	24
3	48	$40,320$
4	384	2.09×10^{13}

Table 2.1: Comparing Wreath Product and Symmetric Group Sizes.

using is much smaller than the associated symmetric group. With 2^n possible committees, the symmetric group we would use, S_{2^n}, is of size $2^n!$. The wreath product group $S_2[S_n]$ has size of only $2^n n!$. Table 2.1 shows the size of the wreath product for small values of n. The size of the necessary symmetric group balloons magnificently compared to the size of the associated wreath product group. The wreath product seems to make the best compromise between the qualities we want from our group, and so we choose to use its group action to make our profile and results spaces into $S_2[S_n]$-modules.

2.4 Using the Group

To illustrate the action of this group on a profile space, we begin with the simple case using $n = 2$. The structure shown in Figure 2.1 is used to represent the election structure; A and B represent two departments, where a_1 and a_2 are the two candidates from department A, and b_1 and b_2 are the two candidates from B. An element in the wreath product $S_2[S_2]$ will either leave alone or swap the departments A and B (leaving a_1 and a_2 attached to A, and b_1 and b_2 attached to B), and it will either leave alone or swap the pairs of candidates a_1,a_2 and b_1,b_2.

In this situation, there are four possible committees, which we will define as W,X,Y, and Z as follows:

$$W = \{a_1, b_1\} \qquad X = \{a_1, b_2\} \qquad Y = \{a_2, b_1\} \qquad Z = \{a_2, b_2\}.$$

We define the group action as follows. Let all possible committees be indexed by lists of the form (c_1,c_2), where the committee consists of the c_i^{th} candidate from the i^{th} department for each i. Applying this notation to the committees we have just defined, we have

$$W = (1,1) \qquad X = (1,2) \qquad Y = (2,1) \qquad Z = (2,2).$$

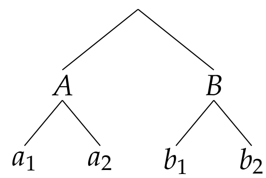

Figure 2.1: The election structure with two departments each putting forth two candidates.

We begin by defining the action of $S_2[S_n]$ on the candidates. This action then induces an action of $S_2[S_n]$ on the committees, which in turn induces an action on the full rankings of the committees.

An element σ of $S_2[S_n]$ will first act on the candidates within their respective departments, either swapping them or leaving them. Then the element will perform some permutation on the n departments. This action will produce a new list which can be identified as one of the possible committees. For example, in our above example, let σ be the element in $S_2[S_2]$ that swaps both pairs a_1, a_2 and b_1, b_2 but leaves A and B in place. Then the action of σ on $W = (1, 1)$ turns W into the list (2,2), which we identify as committee Z; thus $\sigma(W) = Z$. One way to interpret the action of $S_2[S_n]$ is to think of elements of $S_2[S_n]$ as permuting the committees. So in our example, σ takes the committee W and changes it into committee Z.

We can easily extend this group action to an action on full rankings in a natural way. Let $\mathcal{F}$ be some full ranking in the profile space, say $\mathcal{F} = (C_1, C_2, \ldots, C_{2^n})$, where the C_i's are the possible committees in some ranked order. Then the action of an element $\sigma \in S_2[S_n]$ on $\mathcal{F}$ is simply $\sigma\mathcal{F} = (\sigma(C_1), \sigma(C_2), \ldots, \sigma(C_{k^n}))$, using the action on committees defined above. For example, if σ is the wreath product element defined in the previous paragraph, then $\sigma(WXYZ) = ZYXW$. We can show that a wreath product group action will always permute a full ranking into another full ranking. This follows from the observation that due to the uniqueness of inverses in $S_2[S_n]$, if $\sigma(C) = D$ for committees C and D, then C is the unique committee such that $C = \sigma^{-1}(D)$. Therefore, if $\mathcal{F}$ is a full ranking, then $\sigma\mathcal{F}$ must be a list of all of the possible committees with none repeated—a full ranking.

Finally, we can extend the group action to an action on the profile space. Profiles are of the form $\sum_{\mathcal{F}_i} \alpha_i \mathcal{F}_i$, where $\alpha_i \in \mathbb{Q}$ and the sum is over the set of all possible full rankings $\mathcal{F}_i$. Thus the wreath product action naturally extends to an action on the profile space such that $\sigma(\sum_{\mathcal{F}_i} \alpha_i \mathcal{F}_i) = \sum_{\mathcal{F}_i} \alpha_i \sigma(\mathcal{F}_i)$ for $\sigma \in S_2[S_n]$.

The action of $S_2[S_n]$ on the results space R is fairly straightforward, since the basis elements of R are simply the possible committees. So if $\sum_{C_i} \alpha_i C_i \in R$ where $\alpha_i \in \mathbb{Q}$ and the sum is over all possible committees C_i, then the group action on R is defined as $\sigma(\sum_{C_i} \alpha_i C_i) = \sum_{C_i} \alpha_i \sigma(C_i)$ for $\sigma \in S_2[S_n]$.

Now that we have defined the group action on both the profile and results spaces, we may treat them as $\mathbb{Q}S_2[S_n]$-modules. We now argue that for any weighting vector, $T_{\mathbf{w}}$ is indeed a module homomorphism from the profile space to the results space.

As defined in Chapter 7 of James and Liebeck (2001), if P and R are modules, then $T : P \to R$ is an FG-module homomorphism if $T(g\mathbf{p}) = gT(\mathbf{p})$ for all $g \in G$ and $\mathbf{p} \in P$. In other words, T is a module homomorphism if the group action commutes with T. In our case, the action of any $\sigma \in S_2[S_n]$ on any profile $\mathbf{p}$ essentially effects a relabeling of the committees, and the same relabeling occurs for any results vectors. Thus, if committee A first receives x points and is then relabeled as committee B, we will have the same outcome if committee A is first relabeled as committee B and then receives x points. By this argument we see that the group action commutes with $T_{\mathbf{w}}$, therefore $T_{\mathbf{w}}$ is indeed a $\mathbb{Q}S_2[S_n]$-module homomorphism.

Now that we have a framework with a module homomorphism between two modules under the action of the wreath product group, we may begin to use our representation theoretic tools to uncover the inner workings of the profile space and the results space.

Chapter 3

Characters and the Results Space Decomposition

In this chapter, we explore the character theory of the wreath product group to enable us to decompose our modules. We proceed to find the characters of $S_2[S_2]$ and use them to decompose the results space for the $n = 2$ case. We continue on and work with the $n = 3$ case, decomposing the results space with $S_2[S_3]$. The decompositions we find tell an intriguing story about the way results vectors decompose into smaller parts which contain information about the way the candidates make up the committees.

3.1 The Irreducible Characters of $S_2[S_n]$

In order to harness the property that our modules will decompose into irreducible submodules, we first obtain the irreducible characters of the wreath product group. These characters will allow us to decompose the modules using tools from representation theory. The character tables for $S_2[S_n]$ seem difficult to find in literature, but Section 3.1 in Rockmore (1995) provides a method to produce the complete set of irreducible characters for $S_2[S_n]$. This method delves into representation theory and Clifford theory and would be difficult to explain completely here.

A visual that can help us navigate the irreducible characters of $S_2[S_n]$ is the branching diagram Γ in Figure 3.1. The n^{th} row of Γ, where the 0^{th} row is the lone $(\varnothing, \varnothing)$ node, corresponds with the irreducible characters of $S_2[S_n]$ such that each node in row n corresponds to a distinct irreducible character of $S_2[S_n]$. The irreducible characters of $S_2[S_n]$ are indexed by double partitions $\lambda = (\alpha, \beta)$ such that $|\alpha| + |\beta| = n$. Note that α and β are not

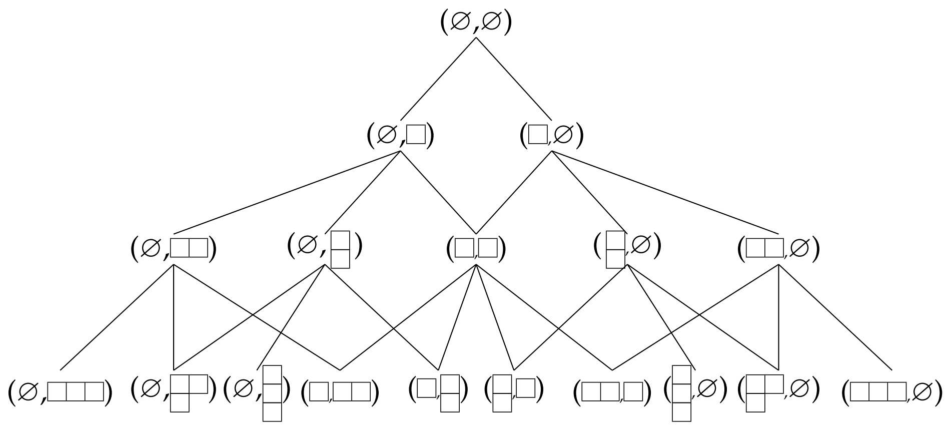

Figure 3.1: The branching diagram Γ, indexing the irreducible characters of $S_2[S_3] \geq S_2[S_2] \geq S_2[S_1] \geq 1$.

merely integers; they are *partitions* whose sizes add to n. The rows of Γ contain all of the possible double partitions for each $n \geq 0$. Partitions are often visually represented with Young diagrams, the boxes shown in Γ. For example, the only partitions of size 2 are (2) and (1,1), visually represented as $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, respectively.

The branching diagram Γ can help us find the dimensions of the irreducible characters as well. The restriction rules for these characters are such that if $V^{(\alpha,\beta)}$ is an irreducible $S_2[S_n]$-module, then it restricts to $S_2[S_{n-1}]$ as

$$V^{(\alpha,\beta)} \downarrow_{S_2[S_{n-1}]} \cong \oplus_{(\mu,\nu)\in(\alpha,\beta)^-} V^{(\mu,\nu)}$$

where the direct sum is over the set of double partitions (μ, ν) of size $n-1$ which are obtained by removing one box from (α, β), where a box can only be removed if there are no boxes bordering it from the right and from below. Note that in Γ, a path between nodes in rows n and $n-1$ indicates that a box can be legally removed from the node in row n to form the node in row $n-1$.

It follows that the dimension of the irreducible character indexed by (α, β) is equal to the number of paths from (α, β) to the $(\varnothing, \varnothing)$ node in Γ, where a path may only move towards a smaller double partition. In checking the dimensions of the irreducible characters for small values of n, we find that these dimensions are small but nontrivial, reinforcing our selection of $S_2[S_n]$ as our group of choice. The dimensions of the irreducible characters are precisely the dimensions of the irreducible $\mathbb{Q}S_2[S_n]$-modules,

	1	a	a^2	b	ab
$\chi_{(\square\square,\varnothing)}$	1	1	1	1	1
$\chi_{(\varnothing,\substack{\square\\\square})}$	1	1	1	-1	-1
$\chi_{(\substack{\square\\\square},\varnothing)}$	1	-1	1	1	-1
$\chi_{(\varnothing,\square\square)}$	1	-1	1	-1	1
$\chi_{(\square,\square)}$	2	0	-2	0	0

Table 3.1: Character table for $S_2[S_2]$. Note that $S_2[S_2] \cong D_8 = \langle a, b | a^4 = b^2 = 1, ab = ba^{-1} \rangle$.

and so using the wreath product group action should give us a manageable but nontrivial decomposition.

We actually already know the irreducible characters of $S_2[S_2]$ without having to use the method in Rockmore (1995). Conveniently, $S_2[S_2]$ is isomorphic to the dihedral group D_8, for which the character table may be found in James and Liebeck (2001). Although we already have the irreducible characters of $S_2[S_2]$, we would like to try and link these characters to the double partitions, and so we perform the method in Rockmore (1995) anyways, to see which irreducible characters correspond to which double partitions. Let $\chi_{(\alpha,\beta)}$ be the irreducible character corresponding to the double partition (α, β). Table 3.1 shows the character table of $S_2[S_2]$ indexed with its double partitions.

Checking Γ we may verify that the dimensions of the irreducible characters (the values of $\chi(1)$) indeed equal the number of paths to the $(\varnothing, \varnothing)$ node.

3.2 Decomposing the Results Space With $S_2[S_2]$

Once we have the irreducible characters of our group, we can use them with our defined group action to decompose the results space into its irreducible submodules. There is a well known technique used for module decomposition; one such explanation of the technique which we briefly summarize here can be found in Chapter 14 of James and Liebeck (2001). Suppose V is a $\mathbb{Q}G$-module. Let χ_i be an irreducible character of G, and let $V_i = (\sum_{g \in G} \chi_i(g^{-1})g)V$. Then V_i is a sum of all of the irreducible $\mathbb{Q}G$-submodules of V which have character χ_i. It follows that $V \cong \bigoplus_i V_i$;

in other words, V decomposes into submodules, each associated with a unique irreducible character of G.

As the results space is 2^n dimensional, this technique is computationally manageable for the small cases. We begin with the $n = 2$ case, using the characters for $S_2[S_2]$, shown in the character table in Table 3.1. For the purpose of our discussion of irreducible submodules, we make a quick definition. Let $S^{(\alpha,\beta)}$ be the irreducible $S_2[S_n]$-module associated with the double partition (α, β) of size n, where $\chi_{(\alpha,\beta)}$ is its character.

Recall the committees we defined in Chapter 2, shown again here for convenience:

$$W = \{a_1, b_1\} \qquad X = \{a_1, b_2\} \qquad Y = \{a_2, b_1\} \qquad Z = \{a_2, b_2\}.$$

Let the vectors in the results space be indexed lexicographically. For example, the results vector

$$\mathbf{r} = \begin{bmatrix} 7 \\ 4 \\ 2 \\ 5 \end{bmatrix}$$

indicates that committee W received 7 points, X received 4 points, Y received 2 points, and Z received 5 points.

Upon decomposing the results space using our decomposition technique, we find that $R \cong S^{(\square\square,\varnothing)} \oplus S^{(\square,\square)} \oplus S^{(\varnothing,\square\square)}$, where

$$S^{(\square\square,\varnothing)} = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \qquad S^{(\varnothing,\square\square)} = \left\langle \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle \qquad S^{(\square,\square)} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle.$$

Note that we performed the technique for all of the characters; applying the technique for the characters $\chi_{(\substack{\square\\\square},\varnothing)}$ and $\chi_{(\varnothing,\substack{\square\\\square})}$ produced the zero submodule in both cases.

We performed this decomposition with the hope of breaking the results space into submodules which would be informative in some way. Thus we make the interesting observation that these basis vectors actually tell a kind of story.

For $\alpha \in \mathbb{Q}$, the space $S^{(\square\square,\varnothing)}$ contains results vectors of the form

$$\mathbf{r} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \end{bmatrix}.$$

We can see that all results vectors from $S^{(\square\square,\varnothing)}$ are score tallies in which all committees receive the same number of votes. We refer to this space as the *trivial space*.

For $\alpha \in \mathbb{Q}$, the space $S^{(\varnothing,\square\square)}$ contains results vectors of the form

$$\mathbf{r} = \begin{bmatrix} \alpha \\ -\alpha \\ -\alpha \\ \alpha \end{bmatrix}.$$

Thus we find that the results vectors from $S^{(\varnothing,\square\square)}$ are score tallies which award α points to committees W and Z, and $-\alpha$ points to commitees X and Y. But notice that W and Z are *disjoint committees* - that is, they have no candidates in common; X and Y are disjoint committees as well. Thus we see that these score tallies are awarding points equally to two disjoint committees, while docking the same amount of points from the other pair of disjoint committees.

For $\alpha, \beta \in \mathbb{Q}$, the space $S^{(\square,\square)}$ contains results vectors of the form $\mathbf{r} + \mathbf{s}$, where

$$\mathbf{r} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ -\alpha \end{bmatrix}, \mathbf{s} = \begin{bmatrix} 0 \\ \beta \\ -\beta \\ 0 \end{bmatrix}.$$

Thus the results vectors from $S^{(\square,\square)}$ are linear combinations of score tallies that tell an interesting story. We find that $\mathbf{r}$ awards α points to W and $-\alpha$ points to Z, giving no points to X or Y. But note that W and Z are disjoint committees, while X and Y each share one candidate with W and Z, respectively. For some voters, this kind of tally may reflect a very natural way to score votes for committees: award the most points to a favorite committee, say W, award no points to committees who share one candidate with W, and deduct points from Z, the committee with no candidates in common with W. We can see that $\mathbf{s}$ behaves similarly to $\mathbf{r}$, so results vectors from $S^{(\square,\square)}$ contain linear combinations of these kinds of score tallies.

These basis vectors reveal that we have discovered a decomposition which reflects how the candidates make up the committees. Each irreducible submodule reflects a different kind of preference between the committees. The trivial space is much less interesting, as it only indicates ties between committees. The two submodules $S^{(\square,\square)}$ and $S^{(\varnothing,\square\square)}$, on the other hand, reflect completely different kinds of preferences with respect to candidate make-up of the committees.

n	DTP	$Total$
2	3	5
3	4	10
4	5	20
5	6	36
6	7	65

Table 3.2: Comparing Number of Double Trivial Partitions and Total Irreducible Characters.

3.3 Characters for Larger n

For the larger cases of n we could simply repeat this procedure, and we would produce similar decompositions. However, along the way we observe an interesting property of the results space decomposition that may make the application of the procedure easier and faster. From the decomposition we have obtained for the $n = 2$ case, along with the restriction rule on the irreducible characters of $S_2[S_n]$, we make the following conjecture on how the results space decomposes for all n.

Conjecture: For all $n \geq 2$, the results space R decomposes into a direct sum of the irreducible submodules which are indexed by double partitions (μ, ν), where both μ and ν are trivial partitions (in the graph Γ, these are the "flat" partitions). Each of these irreducible submodules only appears once in the decomposition. In other words, $R \cong \bigoplus_{(\mu,\nu)} S^{(\mu,\nu)}$, where the direct sum is over all such double partitions. Let us refer to these double partitions as *double trivial partitions*.

For example, we saw that for $n = 2$, $R \cong S^{(\square\square,\varnothing)} \oplus S^{(\square,\square)} \oplus S^{(\varnothing,\square\square)}$. Note that each irreducible submodule is indexed by a double trivial partition. If our conjecture is true, then we no longer need to find all of the irreducible characters of $S_2[S_n]$. Instead, we only need to find the characters of $S_2[S_n]$ for double trivial partitions! For each n, the number of double trivial partitions (μ, ν) is simply $n + 1$ (there are $n + 1$ choices for the size of μ, and the size of ν is then fixed). The total number of irreducible characters of $S_2[S_n]$ does not have a closed form, but the total number for small values of n is shown in Table 3.2.

We can support our conjecture by checking the dimensions of the irreducible characters with double trivial partitions. This is due to the fact that if our conjecture is true, the dimensions of the associated irreducible submodules must sum to the dimension of R. We can check the dimensions

	I	II	III	IV	V	VI	VII	VIII	IX	X
$\chi_{(\square\square\square,\varnothing)}$	1	1	1	1	1	1	1	1	1	1
$\chi_{(\varnothing,\square\square\square)}$	1	-1	-1	-1	-1	-1	1	1	1	1
$\chi_{(\square,\square\square)}$	3	3	1	0	-1	-1	-1	1	0	-1
$\chi_{(\square\square,\square)}$	3	-3	-1	0	1	1	-1	1	0	-1

Table 3.3: Irreducible characters of $S_2[S_3]$ with double trivial partitions.

using the graph Γ, counting the paths from (μ, ν) to $(\varnothing, \varnothing)$.

Recall that, as described with the restriction rules in Section 3.1, the number of paths is equal to the number of ways we can remove boxes one at a time from (μ, ν). Consider a double trivial partition (μ, ν) with $|\mu| + |\nu| = n$. To follow a path from (μ, ν) to $(\varnothing, \varnothing)$, at each node of the path we either remove a box from μ or from ν. For trivial partitions, we can only remove the rightmost box. Therefore any path can be uniquely determined with a list of $|\mu|$ L's and $|\nu|$ R's, indicating the order in which boxes were removed from either the left or right partition. For example, if $|\mu| = 2$ and $|\nu| = 1$, then the list (L,R,L) represents the path for which we first remove a box from μ, then remove the sole box from ν, and finally remove the last box of μ. Any such list will determine a path, so given $|\mu|$ and $|\nu|$ the number of paths equals the number of such lists. The number of such lists is equal to $\binom{n}{|\mu|}$, the number of ways to first place the L's in the list—the R's are then fixed.

Thus we find that the dimension of $S^{(\mu,\nu)}$ is $\binom{n}{|\mu|}$. It follows that to sum over all of the double trivial partitions, we simply take $\sum_{|\mu|=0}^{n} \binom{n}{|\mu|}$, which by the binomial theorem is equal to 2^n, the dimension of the results space. Thus we confirm that the dimensions of the submodules add up the way they should, supporting our conjecture.

Following this conjecture, we use the Rockmore method to find the irreducible characters of $S_2[S_3]$ that are associated with double trivial partitions, so that we may find the results space decomposition for the $n = 3$ case. The characters we find are shown below. The ten conjugacy classes of $S_2[S_3]$ are represented with roman numerals, where roman numeral I represents the identity element. For more on these conjugacy classes and on the conjugacy classes of $S_2[S_n]$ in general, see Appendix A.

We can verify that these are indeed the characters of the irreducible submodules of R. We know from representation theory that the character of a

	I	II	III	IV	V	VI	VII	VIII	IX	X
χ_R	8	0	0	0	0	0	0	4	2	0

Table 3.4: The character of R.

module is equal to the sum of the characters of its irreducible submodules (see Chapter 14 in James and Liebeck (2001)), thus we find the character of R, χ_R, and see if it equals the sum of the irreducible characters of $S_2[S_3]$ that we found. Since the action of $S_2[S_3]$ on R simply permutes the basis elements (the committees), R is a *permutation module*. In a permutation module, the value of χ_R for a conjugacy class is the number of basis elements left fixed by the action of any representative from that conjugacy class (see Chapter 13 in James and Liebeck (2001)). Thus, for each conjugacy class of $S_2[S_3]$, we take any element of that conjugacy class and act on every committee with that element; the number of committees that are left fixed by the action is equal to the value of χ_R for elements in that conjugacy class. Using this process, we find that the character of R is as shown in Table 3.4.

The character of R is precisely the sum of the four irreducible characters of $S_2[S_3]$ that we found earlier! Thus we conclude that $R \cong S^{(\square\square\square,\varnothing)} \oplus S^{(\varnothing,\square\square\square)} \oplus S^{(\square,\square\square)} \oplus S^{(\square\square,\square)}$. Not only do we obtain the decomposition of R, but we also confirm our conjecture for the $n = 3$ case: the irreducible submodules of R correspond to double trivial partitions! This thesis does not complete any computations for larger values of n, but hopefully this conjecture holds, simplifying the process for all such n.

3.4 Decomposing the Results Space With $S_2[S_3]$

In the $S_2[S_2]$ case, not only did we find all of the irreducible characters, but we also performed the decomposition method for all of the characters. Now that we have already determined which irreducible $\mathbb{Q}S_2[S_3]$-modules are found in R when $n = 3$, we only need to apply the decomposition method with respect to those characters.

For the purpose of discussing results vectors, we define new committees for the $n = 3$ case. Let the three departments be A, B, and C, where the candidates from A are a_1, a_2, the candidates from B are b_1, b_2, and the candidates from C are c_1, c_2. Define the committees as follows:

$$\begin{array}{llll} S = \{a_1, b_1, c_1\} & T = \{a_1, b_1, c_2\} & U = \{a_1, b_2, c_1\} & V = \{a_1, b_2, c_2\} \\ W = \{a_2, b_1, c_1\} & X = \{a_2, b_1, c_2\} & Y = \{a_2, b_2, c_1\} & Z = \{a_2, b_2, c_2\}. \end{array}$$

Again, vectors in the results space are indexed lexicographically, so the results vector $\mathbf{r} = [5, 3, 6, 2, 7, 5, 1, 0]^{\mathrm{T}}$ represents a score tally in which S receives 5 points, T receives 3 points, and so on.

Applying the decomposition method, we find that the irreducible submodules of R have the following basis vectors:

$$S^{(\square\square\square,\varnothing)} = \left\langle \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix} \right\rangle \qquad S^{(\square\square,\square)} = \left\langle \begin{bmatrix} 3\\1\\1\\-1\\1\\-1\\-1\\-3 \end{bmatrix}, \begin{bmatrix} 1\\3\\-1\\1\\-1\\1\\-3\\-1 \end{bmatrix}, \begin{bmatrix} 1\\-1\\3\\1\\-1\\-3\\1\\-1 \end{bmatrix} \right\rangle$$

$$S^{(\varnothing,\square\square\square)} = \left\langle \begin{bmatrix} 1\\-1\\-1\\1\\-1\\1\\1\\-1 \end{bmatrix} \right\rangle \qquad S^{(\square,\square\square)} = \left\langle \begin{bmatrix} 3\\-1\\-1\\-1\\-1\\-1\\-1\\3 \end{bmatrix}, \begin{bmatrix} -1\\3\\-1\\-1\\-1\\-1\\3\\-1 \end{bmatrix}, \begin{bmatrix} -1\\-1\\3\\-1\\-1\\3\\-1\\-1 \end{bmatrix} \right\rangle.$$

Again, these basis vectors also tell an interesting story! Three of these submodules even resemble the ones we found in the $n = 2$ case. Here the irreducible submodule $S^{(\square\square\square,\varnothing)}$ is the trivial space, containing results vectors which award the same number of points to all committees.

The results vectors found in $S^{(\square\square,\square)}$ resemble those found in $S^{(\square,\square)}$ in the $n = 2$ case. This time, the story is definitely similar, but it also extends to accommodate the third member of the committee! For example, in the first basis vector listed above, $[3, 1, 1, -1, 1, -1, -1, -3]^T$, the committee S receives three points. The committees T, U, and W each receive one point; these are the committees with *two* candidates in common with S. The committees V, X, and Y each lose one point; these are the committees with exactly *one* candidate in common with S. And of course, Z loses three points, having no candidates in common with S. Essentially, results vectors of this form seem to suggest a scoring in which a favorite committee receives the most points, and the rest of the committees are awarded fewer points based on how many candidates they have in common with the favorite. Notice that there are four pairs of disjoint committees but only three basis vectors; the vector $[-1,1,1,3,-3,-1,-1,1]^T$ which awards 3 points to committee V is not listed as a basis vector, but is still a linear combination of the three basis vectors. Thus this irreducible submodule contains results vectors of this form for all pairs of disjoint committees, including all linear combinations of such results vectors.

The results vectors found in $S^{(\square,\square\square)}$ resemble those found in $S^{(\varnothing,\square\square)}$ in the $n = 2$ case. The story told here is a similar one. For example, in the first basis vector shown above, $[3,-1,-1,-1,-1,-1,-1,3]^T$, the committees S and Z—which are disjoint committees—each receive three points, while the rest of the committees lose one point. The other basis vectors demonstrate the same scoring, with two disjoint committees receiving a large positive number of points while the rest of the committees lose a small number. Again, there are four pairs of disjoint committees but only three basis vectors; just as in $S^{(\square\square,\square)}$, this issue is no concern, as the vector $[-1,-1,-1,3,3,-1,-1,-1]^T$ is a linear combination of the three basis vectors shown. Thus this irreducible submodule contains results vectors of this form for all pairs of disjoint committees, as well as all linear combinations of such results vectors.

The irreducible submodule that does not have a cousin in the $S_2[S_2]$ case is $S^{(\varnothing,\square\square\square)}$, a peculiar one-dimensional submodule. For $\alpha \in \mathbb{Q}$, results vectors from this submodule are of the form

$$\mathbf{r} = \begin{bmatrix} \alpha \\ -\alpha \\ -\alpha \\ \alpha \\ -\alpha \\ \alpha \\ \alpha \\ -\alpha \end{bmatrix}.$$

These vectors represent a score tally in which the committees S, V, X, and Y each receive α points, while the committees T, U, W, Z each receive $-\alpha$ points. The relationship between these committees is an interesting one. Any pair of committees in the set {S,V,X,Y} shares exactly one candidate;

the same holds for the set {T,U,W,Z}. Notice that in $S^{(\square,\square\square)}$, the most points are awarded to committees which have no candidates in common; in $S^{(\square\square,\square)}$, the most points are awarded to a favorite committee and to the committees which have two candidates in common with the favorite. Thus it seems natural that we have a submodule in which points are awarded to some committee along with the committees which share exactly one candidate.

The fact that the decompositions still tell a story in the $n = 3$ case is both interesting and encouraging. Since the set of all the basis vectors for the irreducible submodules forms a basis for R, any results vector can be expressed uniquely as a linear combination of the basis vectors. In other words, we can interpret any results vector as the composition of various pieces which carry different kinds of information. For example, if an election procedure returns a results vector with a winner S that is mostly comprised of vectors from $S^{(\square\square,\square)}$, we may interpret this in two ways. Firstly, we may see it as a suggestion that voters somehow more heavily preferred committees which had two candidates in common with S. Secondly, this may suggest that the effective space for the voting procedure that was used contains copies of $S^{(\square\square,\square)}$ in the profile space as opposed to the other submodules of the profile space.

These decompositions enhance the strength of the algebraic framework, allowing us to make conclusions about the profile space and about $T_{\mathbf{w}}$. We are also encouraged by the idea that even in cases with larger values of n, the decomposition of the results space will continue to tell a similar kind of story.

Chapter 4

Decomposing the Profile Space

In this chapter we seek to decompose the profile space so that we may apply our findings on the decomposition of the results space. We discuss the decomposition of the profile space into smaller, more manageable orbits and the properties of these orbits. We then show that the module homomorphism $T_{\mathbf{w}}$ also breaks down into smaller components. Finally, we investigate the implications of using these smaller components, including a discussion on voting paradoxes.

4.1 Orbits in the Profile Space

Since we have defined the group action of the wreath product $S_2[S_n]$ on the profile space P and have found the group's irreducible characters we want, we have the ability to decompose P into irreducible submodules using the same process with which we decomposed the results space. However, given the massive size of P (2^n!-dimensional, as 2^n is the number of possible committees), this process quickly becomes computationally difficult. Furthermore, any decomposition we obtain will be expressed as a set of 2^n!-dimensional basis vectors for each irreducible submodule, which may still be unwieldy. Fortunately, $S_2[S_n]$ acts on P in such a way that there exists a much more manageable and preferable means of decomposing P.

For any full ranking $\mathcal{F}$ in P, consider the orbit of $\mathcal{F}$ under the action of $S_2[S_n]$; as we saw in Chapter 2, every element of $S_2[S_n]$ turns $\mathcal{F}$ into another full ranking. Since different elements of $S_2[S_n]$ effect different relabelings of the committees, no two full rankings in the orbit of $\mathcal{F}$ are identical. As an example, here is the orbit of the full ranking WXYZ under the action of

$S_2[S_n]$, where the committees are defined as they were in Chapter 2:

$$\begin{array}{cccc} WXYZ & XZWY & ZYXW & YWZX \\ XWZY & WYXZ & YZWX & ZXYW. \end{array}$$

Given any orbit of full ranking $\mathcal{F}$, consider the subspace $P_{\mathcal{F}}$ of P whose basis elements are the elements of the orbit. By definition, vectors in this subspace are closed under the action of $S_2[S_n]$. Because of this property, $P_{\mathcal{F}}$ becomes a submodule of P.

Since the action of $S_2[S_n]$ permutes the full rankings that form the basis of $P_{\mathcal{F}}$, $P_{\mathcal{F}}$ becomes a permutation module. We then note that any non-identity element of $S_2[S_n]$ leaves no full rankings fixed under the group action. This is due to the fact that aside from the identity element of $S_2[S_n]$, no element of $S_2[S_n]$ leaves all committees fixed; thus every non-identity element of $S_2[S_n]$ must permute all of the full rankings. It follows that the character of $P_{\mathcal{F}}$ has the values $\chi_{P_{\mathcal{F}}}(1) = |S_2[S_n]|$ where $1 \in S_2[S_n]$ is the identity element, and $\chi_{P_{\mathcal{F}}}(\sigma) = 0$ for all non-identity $\sigma \in S_2[S_n]$. From these character values we conclude that under the action of $S_2[S_n]$, $P_{\mathcal{F}}$ is isomorphic to a regular $\mathbb{Q}S_2[S_n]$-module (a proof may be found in Chapter 13 of James and Liebeck (2001)).

Any full ranking belongs to a unique orbit under the $S_2[S_n]$ action. Thus, the set of all full rankings can be viewed as the collection of orbits of full rankings under the $S_2[S_n]$ action. It follows that as a module the profile space is isomorphic to a direct sum of regular $\mathbb{Q}S_2[S_n]$-modules. A regular $\mathbb{Q}S_2[S_n]$-module has dimension equal to the order of $S_2[S_n]$. The wreath product has order $|S_2[S_n]| = 2^n n!$, and the profile space is $2^n!$-dimensional, so the profile space is the direct sum of exactly $\frac{(2^n-1)!}{n!}$ regular $\mathbb{Q}S_2[S_n]$-modules.

This decomposition of the profile space is important for three main reasons. Firstly, the decomposition of regular modules is well understood. If $U_1, U_2, \ldots, U_k$ forms a complete set of irreducible $\mathbb{Q}S_2[S_n]$-modules, then $\mathbb{Q}S_2[S_n] \cong d_1U_1 \oplus d_2U_2 \oplus \ldots \oplus d_kU_k$, where U_i has dimension d_i. Secondly, this decomposition allows us to narrow our analysis of the profile space to these smaller orbits which are much more manageable. Furthermore, there exists an interesting difference between these orbits. Consider the following orbit in the profile space P for the $n = 2$ case, and compare it with the orbit shown above:

$$\begin{array}{cccc} WXZY & XZYW & ZYWX & YWXZ \\ XWYZ & WYZX & YZXW & ZXWY. \end{array}$$

Recall that W and Z are disjoint committees, having no candidates in common. Notice that in all of the full rankings in the first orbit shown, the committees W and Z either occupy first and fourth places, or second and third places. On the other hand, in the full rankings of the second orbit shown, the committees W and Z either occupy first and third places, or second and fourth places. In this manner, for any full ranking we can partition the ranks into pairs based on where the disjoint committees are ranked. We refer to such a partition as a *disjoint pair partition*. For example, in the first orbit shown we can notate the disjoint pair partition as $D = \{(1,4),(2,3)\}$, indicating that in full rankings with the disjoint pair partition D, disjoint pairs of committees are either placed first and fourth, or second and third.

In short, a disjoint pair partition tells us how all the pairs of disjoint committees are ranked. A disjoint pair partition is an indication of a kind of structure that exists in every full ranking; note that this is a structure based on how the candidates play a role in the ranking of committees. An interesting property of the disjoint pair partitions is demonstrated with the following lemma.

Lemma 4.1. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are full rankings of committees, and $\sigma \in S_2[S_n]$ such that $\sigma(\mathcal{F}) = \mathcal{G}$ under the defined group action. Then $\mathcal{F}$ and $\mathcal{G}$ have the same disjoint pair partition.*

Proof. Suppose committees W and Z are disjoint committees such that W is ranked mth and Z is ranked nth in $\mathcal{F}$. By the definition of the group action, $W' = \sigma(W)$ is ranked mth and $Z' = \sigma(Z)$ is ranked nth in $\mathcal{G}$. We show that W' and Z' are disjoint. Let x_1 and x_2 be the candidates from department X. Suppose $x_1 \in W$, so $x_1 \notin Z$. If $x_1 \in W'$, then the element σ does not swap x_1 and x_2, and so $x_1 \notin Z'$. If $x_1 \notin W'$, then σ does swap x_1 and x_2, and so $x_1 \in Z'$. The same argument follows if we suppose candidate $x_1 \notin W$. Thus W' and Z' are disjoint committees. It follows that the disjoint pair partition is preserved under the group action. $\square$

This lemma implies that every full ranking within one orbit has the same disjoint pair partition. In addition, for every possible disjoint pair partition there must exist an orbit with that partition; thus, different orbits may differ in their disjoint pair partitions. Each kind of orbit has something different to say about the candidates' roles in the rankings.

4.2 Pockets in the Transformation

When we view the profile space as a direct sum of smaller subspaces, we can then study how this concept of orbits applies to the module homomorphism $T_{\mathbf{w}}$. In particular, we will be able to study $T_{\mathbf{w}}$ as a collection of smaller pieces as well. In the profile space P, let us index the full rankings so that all of the rankings from an orbit are grouped together. For example, in the $n = 2$ scenario we can view a profile $\mathbf{p}$ as $\mathbf{p} = [\mathbf{p_1}|\mathbf{p_2}|\mathbf{p_3}]^{\mathbf{t}}$, where the $\mathbf{p_i}$'s are each 8-dimensional column vectors corresponding to different orbits of $S_2[S_2]$ in P.

It follows that we can divide up the $T_{\mathbf{w}}$ matrix in corresponding fashion. We can view the 4x24 $T_{\mathbf{w}}$ matrix as $T_{\mathbf{w}} = [T_{\mathbf{w_1}}|T_{\mathbf{w_2}}|T_{\mathbf{w_3}}]$, where the $T_{\mathbf{w_i}}$ are 4x8 matrices corresponding to different orbits. In other words, the columns of $T_{\mathbf{w_i}}$ are the appropriate permutations of $\mathbf{w}$ used to score the full rankings found in the ith orbit. Following from this, we can then write $T_{\mathbf{w}}$ as a sum

$$T_{\mathbf{w}} = [T_{\mathbf{w_1}}|\mathbf{0}|\mathbf{0}] + [\mathbf{0}|T_{\mathbf{w_2}}|\mathbf{0}] + [\mathbf{0}|\mathbf{0}|T_{\mathbf{w_3}}].$$

Notice that each of these transformations which sum to $T_{\mathbf{w}}$ is still a module homomorphism, and so we can still use the results we obtain from using module homomorphisms, such as Schur's Lemma. For convenience, we will refer to these smaller transformations as "pockets" of $T_{\mathbf{w}}$.

Another interesting door we open by splitting up $T_{\mathbf{w}}$ into pockets is the ability to talk about the effective spaces $E(T_{\mathbf{w_i}})$ of each of the pockets. We define the effective space $E(T_{\mathbf{w_i}})$ of a pocket of $T_{\mathbf{w}}$ as the orthogonal complement to the kernel of the mapping $T_{\mathbf{w}} \circ \iota$, where $\iota : P_i \to P$ is the inclusion map from the ith orbit to the profile space. We find that $\ker(T_{\mathbf{w}} \circ \iota) = (P_i^{\perp} + \ker(T_{\mathbf{w}})^{\perp})^{\perp}$. Thus the orthogonal complement to the kernel is $P_i \cap (\ker(T_{\mathbf{w}} \circ \iota))^{\perp} = P_i \cap (P_i^{\perp} + E(T_{\mathbf{w}}))$. Thus we can think of the effective space of a single pocket as the space of profiles from a single orbit that contributes to the results of the election.

We can then ask if and how these pockets' effective spaces differ given a particular weighting vector $\mathbf{w}$. To investigate this question, we recall the results space decomposition we obtained earlier and apply Schur's Lemma. Recall that in the $n = 2$ case, we found that the results space has decomposition $R \cong S^{(\square\square,\varnothing)} \oplus S^{(\square,\square)} \oplus S^{(\varnothing,\square\square)}$. It follows that for any pocket $T_{\mathbf{w_i}}$, its effective space $E(T_{\mathbf{w_i}})$ must be isomorphic to the direct sum of irreducible $S_2[S_2]$-modules from the set $\{S^{(\square\square,\varnothing)}, S^{(\square,\square)}, S^{(\varnothing,\square\square)}\}$.

To find the effective spaces of a given pocket $T_{\mathbf{w_i}}$, it is actually sufficient to find how its image decomposes in the results space R. For example, if the image of $T_{\mathbf{w_1}}$ is isomorphic to $S^{(\square,\square)}$ in the results space, then by the First Isomorphism Theorem, $E(T_{\mathbf{w_1}})$ must also be isomorphic to $S^{(\square,\square)}$. The process of finding this image decomposition is simple, given that we have already found decompositions of R. Given a weighting vector $\mathbf{w}$, suppose we wish to find the image of $T_{\mathbf{w_i}}$. The image of $T_{\mathbf{w_i}}$ is simply its column space, but the columns of $T_{\mathbf{w_i}}$ are simply permutations of $\mathbf{w}$ that correspond to the full rankings indexed in this pocket. Thus, let $\mathbf{w_i}$ be any one of these permutations of $\mathbf{w}$. The set of basis vectors for all the irreducible submodules of R forms a linearly independent set of vectors; thus we are able to uniquely decompose $\mathbf{w_i}$ as a linear combination of these basis vectors. If $\mathbf{w_i}$ contains a component that is a basis vector for $S^{(\square,\square)}$, for example, then we can conclude that $S^{(\square,\square)} \subseteq E(T_{\mathbf{w}})$. We can make this conclusion because $\mathbf{w_i}$ is in the column space of $T_{\mathbf{w_i}}$, and so if a component of $S^{(\square,\square)}$ is in the column space of $T_{\mathbf{w_i}}$, then all of $S^{(\square,\square)}$ is in the column space, or image, by Schur's Lemma. It follows that some copy of $S^{(\square,\square)}$ in the corresponding orbit of the profile space must be in the effective space of $T_{\mathbf{w_i}}$.

We only need to perform this process on a single permutation $\mathbf{w_i}$. To see why, we view $\mathbf{w_i}$ not just as a weighting vector, but also as a vector in the results space. The column space of $T_{\mathbf{w_i}}$ is the span of all columns of $T_{\mathbf{w_i}}$, but this set of columns is simply the set $S_2[S_2](\mathbf{w_i})$, the wreath product group acting on $\mathbf{w_i}$ as a results vector. We harness the fact that $S_2[S_2]$-modules are invariant under the group action, and that the irreducible submodules of R all have multiplicity 1. Thus, if for example $\mathbf{w_i}$ is the sum of components from $S^{(\square,\square)}$ and $S^{(\square\square,\varnothing)}$, then under the group action from any element of $S_2[S_2]$, these components remain in their respective submodules. It follows that all permutations of the components of $\mathbf{w_i}$ under the group action belong to the same irreducible submodules, and thus we only need one of those permutations to identify which irreducible submodules contribute to $\mathbf{w_i}$.

The interesting result we find in the $n = 2$ case is that the effective spaces of different pockets of $T_{\mathbf{w}}$ may be isomorphic to different subsets of the irreducible $S_2[S_2]$-modules we found in the results space. Performing this process for the 2 department case using the Borda count weighting vector $\mathbf{w} = [3, 2, 1, 0]$ reveals that the three pockets have the following effective spaces:

$$E(T_{\mathbf{w_1}}) \cong S^{(\square\square,\varnothing)} \oplus S^{(\square,\square)}, E(T_{\mathbf{w_2}}) \cong E(T_{\mathbf{w_3}}) \cong S^{(\square\square,\varnothing)} \oplus S^{(\square,\square)} \oplus S^{(\varnothing,\square\square)}.$$

We have found that under the Borda count, the effective spaces of different pockets decompose into different direct sums of irreducible $\mathbb{Q}S_2[S_n]$-

modules. This can be fairly intriguing if we recall that different irreducibles in the results space tell different stories. Since irreducible submodules in P are isomorphic to their counterparts in R, it would seem that the irreducible submodules of P have as much a story to tell as do the submodules of R. This seems to imply that different pockets can tell different stories; in particular, different kinds of full rankings—different in their disjoint pair partitions—produce different kinds of results vectors in R. For example, voters who vote in the orbit corresponding to $T_{\mathbf{w_1}}$ only contribute information that is mapped to results vectors which reflect the stories told by $S^{(\square\square,\varnothing)}$ and $S^{(\square,\square)}$, whereas the other voters contribute information that is mapped to results vectors in $S^{(\varnothing,\square\square)}$ as well.

In the $n = 2$ case it is possible for the effective spaces of two different pockets to be completely disjoint. Consider the weighting vector $\mathbf{w} = [1, 1, -1, -1]$. This weighting vector represents a voting procedure in which voters assign one point to their two most preferred committees and deduct one point from their two least preferred committees. Finding the effective spaces of the pockets shows that

$$E(T_{\mathbf{w_1}}) \cong E(T_{\mathbf{w_2}}) \cong S^{(\square,\square)}, E(T_{\mathbf{w_3}}) \cong S^{(\varnothing,\square\square)}.$$

If we know an irreducible submodule U is in the effective space, then it must be the case that the effective space of at least one pocket must contain U (otherwise U would be in the kernel of $T_{\mathbf{w}}$ and not in the effective space). Thus the question of how the pockets' effective spaces differ can perhaps be rephrased as: which submodules $U \subseteq E(T_{\mathbf{w}})$ does each pocket lack, if any? If $E(T_{\mathbf{w_i}})$ contains U while $E(T_{\mathbf{w_j}})$ does not, then the two pockets contribute differently to the results space.

Our conjecture is that the effective space of a given pocket is closely related to the disjoint pair partition of the full rankings in that pocket, but more research will have to be completed to support this hypothesis. For example, in our $n = 2$ Borda count example, the pocket $T_{\mathbf{w_1}}$ differs from the other two pockets in that its effective space contains only the trivial irreducible submodule and $S^{(\square,\square)}$. The corresponding orbit has disjoint pair partition $\{(1, 4), (2, 3)\}$, meaning disjoint committees are either ranked first and fourth, or second and third. Notice that this kind of full ranking reflects the story told by results vectors in $S^{(\square,\square)}$.

4.3 Voting Paradoxes

A common paradox found in single-candidate voting theory occurs when a fixed profile yields different results under different scoring methods. Despite the voters' data remaining the same, the fate of the election rests in the hands of whoever determines the scoring method. In single-candidate elections, paradoxes can be constructed by taking advantage of the different results yielded by profile data when scored with different weighting vectors (see Section 3 in Daugherty et al. (2009)). The transformations for different weighting vectors have different effective spaces, so it is possible to carefully construct a profile that behaves very differently under those weighting vectors. Unsurprisingly, committee elections are prone to these paradoxes as well, and the profile space decompositions discussed earlier help us illuminate and construct these paradoxes.

The breaking up of the profile space into smaller pockets actually makes our job of constructing profiles much easier; for example in the $n = 2$ case, rather than have to try and build a 24-dimensional profile, we can focus on building an 8-dimensional profile that will behave differently under different weighting vectors.

A simple example of a paradoxical profile is shown below. The profile $\mathbf{p}$ contains votes from only one pocket, to simplify the example. Under the Borda count which uses weighting vector $\mathbf{w} = [3, 2, 1, 0]$, the profile gives the following result:

$$T_{\textbf{Borda}} : \begin{array}{c} WXYZ \\ XZWY \\ ZYXW \\ YWZX \\ XWZY \\ WYXZ \\ YZWX \\ ZXYW \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \rightarrow \begin{array}{c} W \\ X \\ Y \\ Z \end{array} \begin{bmatrix} 27 \\ 25 \\ 23 \\ 21 \end{bmatrix}.$$

Thus, under the Borda count, committee W receives the most points and wins the election. However, under the plurality scoring method which uses weighting vector $\mathbf{w} = [1, 0, 0, 0]$, the profile gives a different result:

$$T_{\textbf{Plurality}} : \begin{array}{c} WXYZ \\ XZWY \\ ZYXW \\ YWZX \\ XWZY \\ WYXZ \\ YZWX \\ ZXYW \\ . \\ . \\ . \\ . \\ . \end{array} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \rightarrow \begin{array}{c} W \\ X \\ Y \\ Z \end{array} \begin{bmatrix} 3 \\ 6 \\ 6 \\ 1 \end{bmatrix}.$$

Under the plurality method, committees X and Y actually tie for the win, a completely different result than when the Borda count is used.

These paradoxes were constructed by finding the effective spaces $E(T_{\mathbf{w}})$ and $E(T_{\mathbf{v}})$ for different voting procedures which use weighting vectors $\mathbf{w}$ and $\mathbf{v}$, respectively. As an abstract example, suppose $U_1 \ncong U_2$ are irreducible submodules of P, and $E(T_{\mathbf{w}}) = U_1$ but $E(T_{\mathbf{v}}) = U_1 \oplus U_2$. We construct a profile $\mathbf{p} = \mathbf{u_1} + \mathbf{u_2}$ where $\mathbf{u_1} \in U_1$ and $\mathbf{u_2} \in U_2$. We choose $\mathbf{u_1}$ such that $T_{\mathbf{w}}(\mathbf{u_1})$ and $T_{\mathbf{v}}(\mathbf{u_1})$ produce results vectors indicating a victory for committee W, but with a small amount of points. We choose $\mathbf{u_2}$ such that $T_{\mathbf{v}}(\mathbf{u_2})$ produces a results vector indicating a victory for committee X with a large amount of points. Then when we compute $T_{\mathbf{w}}(\mathbf{p})$, we will produce a results vector that indicates a victory for W, since $T_{\mathbf{w}}(\mathbf{u_2}) = \mathbf{0}$. However, when we compute $T_{\mathbf{v}}(\mathbf{p})$, we produce a results vector giving a small amount of points to W plus a results vector granting a large amount of points for X, resulting in a victory for X overall.

Such constructions of paradoxes demonstrate how once the voters place their votes, the results can very much be out of their hands, assuming the procedure is announced afterwards or changed from the original. However, the discovery that different pockets have different effective spaces suggests a different kind of paradox. The discovery suggests that voters may have much greater control over the election results than they might realize. Whether they realize it or not, by simply voting for a different *kind* of full ranking, they vote for a different disjoint pair partition and thus change the effective space of the pocket to which their full ranking belongs. In a sense, they change the kind of data they input into the election. Of course, the data input by a single voter is easily understood: they simply add a permutation of the weighting vector to the results space. Where this kind of paradox may arise is when a large group of voters votes in a certain manner. If a large group of voters decides to vote for a certain kind of disjoint pair partition, it may produce vastly different results than an election in which they vote for a slightly different disjoint pair partition.

As an example, consider the profile we saw earlier, scored under the Borda count:

$$T_{\mathbf{Borda}} : \begin{array}{c} WXYZ \\ XZWY \\ ZYXW \\ YWZX \\ XWZY \\ WYXZ \\ YZWX \\ ZXYW \\ . \\ . \\ . \\ . \\ . \end{array} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \rightarrow \begin{array}{c} W \\ X \\ Y \\ Z \end{array} \begin{bmatrix} 27 \\ 25 \\ 23 \\ 21 \end{bmatrix}.$$

As a thought experiment, suppose each of the voters do not care too much about who is listed third and fourth in their full rankings. If we go through and swap the third and fourth committees in each of the full rankings in the first orbit, we obtain all of the full rankings in the second orbit, so the profile becomes

$$[\mathbf{0}|2,3,0,3,3,1,3,1|\mathbf{0}]^t.$$

Scoring this profile with the Borda count gives

$$T_{\mathbf{Borda}} : \begin{array}{c} . \\ WXZY \\ XZYW \\ ZYWX \\ YWXZ \\ XWYZ \\ WYZX \\ YZXW \\ ZXWY \\ . \end{array} \begin{bmatrix} \mathbf{0} \\ 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ \mathbf{0} \end{bmatrix} \rightarrow \begin{array}{c} W \\ X \\ Y \\ Z \end{array} \begin{bmatrix} 22 \\ 30 \\ 26 \\ 18 \end{bmatrix}.$$

By shifting all of the information in the profile to another orbit, we completely change the result of the election. This example may seem unsurprising and unimpressive, but the intent here is to suggest that when we extend this situation to larger cases, this behavior becomes more interesting. For example, in the $n = 3$ case, voters generate full rankings of eight committees. Suppose a group of voters nonchalantly swap their fifth and sixth ranked committees because their preferences become more arbitrary towards the middle of their full rankings. This swap shifts their full rankings to a completely different orbit with a different disjoint pair partition and possibly a different effective space. It may be the case that a small, nonchalant swap causes a world of difference in the results.

Chapter 5

Conclusion

In this thesis, we have constructed an algebraic framework with which we can try to understand the mathematics of a specific committee election procedure. The wreath product group enabled us to produce decompositions of the profile and results spaces which demonstrated that profiles and results vectors can be viewed as combinations of pieces from different irreducible submodules. These irreducible submodules each represent different kinds of preferences about the committees—preferences that are based on the candidates who make up the committees. This insight suggests that this framework may be useful in constructing or understanding voting procedures that strive to represent voters' opinions on the candidates within the committees. If we can find which irreducible submodules are in the effective space of a voting procedure, we may be able to see how the procedure relates with the way candidates make up the committees.

The study of committee elections is still an undeveloped and daunting world. Various methods have been proposed to tackle the problem of finding a fair and representative committee election procedure, but so far there does not seem to be a consensus on how to run committee elections in general. This research is making an effort to address positional voting procedures by starting with a simple, specific kind of committee election. Using our profile space decomposition, we discussed some voting paradoxes that can arise in committee elections. In the future, we may seek to expand our focus to elections which allow $k > 2$ candidates from each department, perhaps demanding the use and understanding of the representation theory of the wreath product group $S_k[S_n]$. We may attempt to apply our framework to various positional voting procedures that use only partial rankings, to try and understand them using our irreducible $\mathbb{Q}S_2[S_n]$-submodules. We may even try to use our findings to construct our own voting procedure to fairly represent the voters' opinions on the candidates and committees.

Appendix A

Conjugacy Classes of the Wreath Product Group

In this appendix we describe the conjugacy classes of the wreath product group and connect them with the double partitions which index the irreducible characters of $S_2[S_n]$. Here we also indicate which conjugacy classes of $S_2[S_3]$ were indexed by the roman numerals in Chapter 3. All definitions in this section relating to conjugacy classes are taken directly from Bayley (2006).

One common interpretation of the wreath product group $S_2[S_n]$ is to view the elements as signed permutations of the set $\{1, \ldots, n\}$. In other words, each element of the wreath product group permutes the numbers 1 through n and also assigns a $\pm$ sign to each number. We can connect this with our election structure: each element of the wreath product group permutes the n departments, and either fixes or swaps each pair of candidates.

We demonstrate the conjugacy classes of $S_2[S_2]$ and show how to generalize to larger n. Let our two departments be A and B. We can think of our four candidates as occupying four positions: 1, $\bar{1}$, 2, and $\bar{2}$. The candidates in positions 1 and $\bar{1}$ occupy one department, and the candidates in 2 and $\bar{2}$ occupy the other. Thus we can think of a permutation $(1\bar{1})$ as swapping the two candidates in one of the departments. In talking about actions on committees, we can think of 1 and 2 as the positions that are on the committee, while $\bar{1}$ and $\bar{2}$ are the positions that are off the committee. For example, recall our committees from Chapter 2; as an example, committee W is $\{a_1, b_1\}$. The action of the permutation $(1\bar{1})$ gives $(1\bar{1})$ $\{a_1, b_1\} = \{a_2, b_1\}$. As another example, the permutation $(1\bar{2}\bar{1}2)$ gives $(1\bar{2}\bar{1}2)\{a_1, b_1\} = \{b_1, a_2\}$.

The entire group $S_2[S_2]$ consists of the following permutations:

$$\{(), (2\bar{2}), (12)(\bar{1}\bar{2}), (12\bar{1}\bar{2}), (1\bar{1}), (1\bar{1})(2\bar{2}), (1\bar{2}\bar{1}2), (1\bar{2})(2\bar{1})\}.$$

To connect these elements with the double partitions, we define stable and antistable cycles. Much like the permutations in the symmetric group, the elements of the wreath product group are products of cycles. Let c be a cycle, say $c = (a_1 a_2 \dots a_k)$ with $a_i \in \{1, \bar{1}, \dots, n, \bar{n}\}$. Define $\bar{c} = (\bar{a}_1 \bar{a}_2 \dots \bar{a}_n)$, and $\bar{\bar{a}}_i = a_i$. If $\bar{c} = c$, then c is *stable*. If $\bar{c} \neq c$, then c is *antistable*.

The connection between a wreath product group element σ and its corresponding double partition (μ, ν) is as follows. The partition μ is obtained by taking each stable cycle in σ and dividing its length in two; we then arrange the resulting numbers in decreasing order to obtain the partition. For example, the element $(1\bar{1})(2\bar{2})$ consists of two stable cycles, so its double partition is $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing)$. The partition ν is obtained by taking each pair of antistable cycles (they always come in pairs) and taking the length of one half of the pair; we then arrange these numbers in decreasing order to obtain the partition. For example, the element $(1)(\bar{1})(2)(\bar{2})$ consists of two pairs of antistable cycles, so its double partition is $(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix})$.

As another example, the element $(2\bar{2})(1)(\bar{1})$ consists of one stable cycle and a pair of antistable cycles. Thus its double partition is $(\square, \square)$.

The elements are grouped into their conjugacy classes based on their associated double partitions. We can thus reindex the character table of $S_2[S_2]$ as follows:

	$(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix})$	$(\square\square, \varnothing)$	$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing)$	$(\square, \square)$	$(\varnothing, \square\square)$
$\chi_{(\square\square, \varnothing)}$	1	1	1	1	1
$\chi_{(\varnothing, \begin{smallmatrix}\square\\\square\end{smallmatrix})}$	1	1	1	−1	−1
$\chi_{(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing)}$	1	−1	1	1	−1
$\chi_{(\varnothing, \square\square)}$	1	−1	1	−1	1
$\chi_{(\square, \square)}$	2	0	−2	0	0

The same process can be performed to find the conjugacy classes of $S_2[S_n]$ for all values of n. The double partitions that correspond with the roman numerals in Chapter 3 are shown here, along with a conjugacy class representative for each.

It may be of interest to note that the only conjugacy classes which had nonzero character values in the results module (I, VIII, and IX) are the

Numeral	Double Partition	Class Rep
I	$(\varnothing, \yng(1,1,1))$	$(1)(\bar{1})(2)(\bar{2})(3)(\bar{3})$
II	$(\yng(1,1,1), \varnothing)$	$(1\bar{1})(2\bar{2})(3\bar{3})$
III	$(\yng(1), \yng(2))$	$(12)(\bar{1}\bar{2})(3\bar{3})$
IV	$(\yng(3), \varnothing)$	$(123\bar{1}\bar{2}\bar{3})$
V	$(\yng(1), \yng(1,1))$	$(1)(\bar{1})(2)(\bar{2})(3\bar{3})$
VI	$(\yng(2), \yng(1))$	$(1\bar{2}\bar{1}2)(3)(\bar{3})$
VII	$(\yng(1,1), \yng(1))$	$(1\bar{1})(2\bar{2})(3)(\bar{3})$
VIII	$(\varnothing, \yng(2,1))$	$(12)(\bar{1}\bar{2})(3)(\bar{3})$
IX	$(\varnothing, \yng(3))$	$(123)(\bar{1}\bar{2}\bar{3})$
X	$(\yng(2,1), \varnothing)$	$(12\bar{1}\bar{2})(3\bar{3})$

classes with no stable cycles. It turns out the same is true for the $n = 2$ case.

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