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Liberal Arts Inspired Mathematics: A Report OR How to bring cultural and humanistic aspects of mathematics to the classroom as effective teaching and learning tools

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Liberal Arts Inspired Mathematics: A Report OR How to bring cultural and humanistic aspects of mathematics to the classroom as effective teaching and learning tools

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Work supported by the Research and Education Board at the University of Borås and by Stiftelsen Längmanska kulturfonden. Anders K. H. Bengtsson e-mail: anders.bengtsson@hb.se School of Engineering, University of Borås, Allégatan 1, SE-50190 Borås, Sweden.

Liberal Arts Inspired Mathematics: A Report
OR
How to bring cultural and humanistic aspects of mathematics to the classroom as effective teaching and learning tools

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Synopsis

This is the report of a project on ways of teaching university-level mathematics in a humanistic way. The main part of the project recounted here involved a journey to the United States during the fall term of 2012 to visit several liberal arts colleges in order to study and discuss mathematics teaching. Various themes that came up during my conversations at these colleges are discussed in the text: the invisibility of mathematics in everyday life, the role of calculus in American mathematics curricula, the "is algebra necessary?" discussion, teaching mathematics as a language, the transfer problem in learning, and the relationship between humanistic mathematics and mathematics as taught in liberal arts contexts.

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[1]This work was supported by the Research and Education Board at the University of Borås and by Stiftelsen Längmanska kulturfonden.

1. What is this?

This is a report of an ongoing quest for a way of teaching university-level mathematics in a humanistic way. The quest has not been very systematic and it has no particular method. Rather it is, and has been, guided by conversations and readings and practical experiments for many years. Furthermore, it is just during the last couple of years that I've understood that it is indeed a quest for a *humanistic mathematics teaching* or perhaps that such a phrase could be an appropriate characterization of it. Happily, I've come to realize that I'm not the first one to think along these lines.

A main part of the project was my Liberal Arts Colleges trip, the journey I made to the United States during seven weeks in the fall term of 2012 to visit liberal arts colleges to study and discuss mathematics teaching. Writing up a report of that trip is the concrete motivation behind the present text.

But there is more to discuss.

Though it remains to explain what humanistic mathematics could be—and I hope the present text will convey some of the meaning that could be given to the concept—I think there are at least two cornerstones to an attempt to teach university mathematics humanistically. One of them is a serious discussion with the students about the nature of mathematical objects. Another one is an acute awareness by both students and teachers that what is studied is a language. I will return to these topics towards the end of this text in Sections 6 and 7.

Sections 2 through 4 try to paint the context and the background to the project. A report of the Liberal Arts Colleges trip itself is in section 5. Continued analysis of the project can be found at my weblog *Mathematics as a Humanism*.[2]

I should get started, but I have to point out just a couple more things. First, this text is written in a personal voice that I think is appropriate for a project like this, as it mirrors the fundamental, humanistic nature of the project. Much of what follows is anecdotal and impressionistic. What conclusions I arrive at, I cannot support with solid empirical data, but I hope they will be worth considering nevertheless. This is still an inherently academic text, in the sense that it relates to education and scholarship, although the conventions of academic writing are not strictly adhered to. I readily admit to my readers that there is no way that I can refer to everything that has been written on this subject. Even though I have the feeling that I've read a lot, it is surely just a small fraction of everything that has been written. My reading has been mostly unsystematic and even leisurely, and I often find myself in the awkward position of not knowing from where I've picked up ideas. Furthermore, as will become clear, I don't have many answers to offer; I will rather pose questions that need further investigations. This is, in fact, a long-term quest, which, according to Merriam-Webster's Eleventh Collegiate Dictionary, is "*a chivalrous enterprise in medieval romance usually involving an adventurous journey*" [37].

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[2] http://libartinspmath.wordpress.com/, accessed on January 7, 2014.

2. A Paradoxical Situation

A little bit of reflection based on a—not too superficial—look at human history and on our present-day society, make it apparent that mathematics is at the center of it all. There are even theoretical physicists who play with the idea that the very bedrock of reality is mathematics [57], but we need not go as far out in speculation as that. It is enough to realize that mathematics is one of our languages. And it is not easy to learn, but once learned it is the same for everyone, except for the natural language that we have to wrap around it. This fact alone, the need for a *meta-language*, tells us that mathematics is part of human culture. If no meta-language had been needed, if just the symbolic syntax and semantics of formal mathematics had sufficed, then we would have been machines, not human beings.

But there is a visibility problem. Few people use more than the most trivial mathematics in their daily life or at their workplace. The applications of mathematics are invisible [54]. What is not invisible, however, is school mathematics. For most people, learning mathematics is a struggle. In school, mathematics and its esoteric language are highly visible. There may of course be the exceptional few who have been endowed with a natural talent for mathematics. For them, if they continue on to become mathematicians or mathematics teachers, mathematics becomes a natural language that they are, most of the time, unaware of. As teachers, they may be unaware of the fact that they speak and write a language foreign to their students. I will return to this later in the text.

I think this is at the center of the difficulties with mathematics teaching and learning and leads up to one of the main points of this report, *The Language Teaching Metaphor*, on which I'll write more in Section 6. But it is also strongly connected to a second major issue, *The Nature of Mathematical Objects*, and I discuss that in Section 7. These two issues are connected through the question: *What is the language of mathematics trying to say something about?*

3. Introduction and Intention

What's new—if anything?

Anyone interested in the kinds of questions I raise here will see that my thinking is not very original. Much, if not all, of what I write has been written before.[3]

In Sweden there has been an ongoing conversation—though not very widely known, as far as I understand—for a couple of decades about the cultural aspects of mathematics. A publication that opens up into this discourse is *Det matematiska kulturarvet* [3]. Another route is Mouwitz's doctoral dissertation on "Mathematics and Bildung" [42]. Some of the Swedish mathematicians who have written about language aspects of mathematics are Lennerstad [33] and Kiselman [28].[4]

What is perhaps new here is my focus on teaching and learning. The philosophy and humanistic aspects of mathematics are very interesting subjects in themselves, but my main focus is how they can be made the foundation of designing courses and teaching methods. I've seen very little written about that.

Constraints and opportunities

There are some assumptions that delimit my project.

I'm not teaching mathematics majors. We don't have that in Sweden, although a corresponding set of students could be those who go to university to study mathematics or to a mathematics-heavy educational program, like engineering physics. So my focus is not on students with particular talent and interest in mathematics. My focus is on students with no particular mathematics talent or interest. Some of them might even detest mathematics or just feel queasy about the subject. They have had mathematics for ten to twelve years in school, but their knowledge is weak.[5] Or perhaps I should phrase it like this: These students may have a lot of implicit and disconnected

[3]I will cite other authors when I have a specific reference, but mostly I am not in that situation. I know I have picked up from many sources, but I do believe that correct referencing in a subject like this is practically impossible. There are most likely many who have thought parallel thoughts more or less independently of each other, and probably they too have been influenced in the same way as I've been influenced.

[4]All these references are written in Swedish.

[5]If any such student is reading this, please don't be offended. I'm not saying you are dumb. I know myself what it means to have problems with mathematics. In seventh and eighth grades in school, I completely lost track of what was going on in the mathematics classes. I understood nothing. But I came back. I came back because I knew I wasn't dumb and because I had liked mathematics before. I just retraced a few years and started over again on my own. It was then that I really started to learn the language of mathematics.

knowledge in mathematics, but it doesn't make sense to them. It's like pieces of a jig-saw puzzle. These pieces need to be scattered on big table, all turned over with the right side up, then be put together into a coherent whole, providing context and adding in lost or never-found pieces. This is an opportunity.

I'm also not discussing pre-college or pre-university mathematics teaching. I'm primarily interested in teaching and learning mathematics at the college and university level. This is where I'm active. Important as earlier stages in mathematical education are, that's not something I can do anything about. My objective is to try to enhance learning for students I actually meet.

In Sweden this corresponds to ages 19 and above, that is, young adults and adults. In the US, students start college at around 18 years of age. So liberal arts experiences in mathematics teaching are highly relevant for the Swedish situation.

My basic assumption is that, *precisely because the students are adults, we can be explicit about teaching and learning methods*, we can use *reason* on a meta-level, so to speak. We can discuss teaching and learning explicitly with the students in a way that may be impossible, difficult, or otherwise inappropriate for younger students or children.

4. Background

Let me sketch the background to this project and in particular to my Liberal Arts Colleges trip. The first concrete idea came in the early fall of 2010 when I sat down to write a contribution to a Swedish anthology [7] about liberal arts education. The idea behind that was to discuss various aspects of the liberal arts tradition and how it could inspire Swedish higher education. All the authors had some personal experience with this American tradition, many having spent a term at an American college under the STINT *Excellence in Teaching Scholarship*.[6] I had volunteered to write about mathematics. But when I started to write, I realized I knew next to nothing about mathematics teaching at liberal arts colleges. During the writing process I learned about the corresponding Swedish discussion (referred

[6]STINT is the Swedish Foundation for International Cooperation in Research and Higher Education. I was at Skidmore College in Saratoga Springs, NY, during the fall term of 2004.

to above) on mathematics and *bildung* from the last decade. It became a learning experience.

So why was I at all interested in going in this direction? It goes back to a long-time interest in the history and philosophy of mathematics. Also there is my interest in the didactics of physics and mathematics. I worked for six years at a gymnasium (corresponding to school years 10–12). I then came into contact with learning theories based on meta-cognition, constructivism and Ference Marton's research on deep and surface learning strategies [34].

About ten years ago, a year or so after I started to work at the University of Borås, I planned and carried out a pedagogical experiment in mathematics together with two colleagues [1]. That project was motivated by discussions among mathematics teachers at the institute about the weak background knowledge in mathematics among the incoming students. I wanted to do something about it. The experiment did not go very well. Some of the ideas were good and I still believe in them, but our implementation was too weak and I now realize that we missed many important ideas that I just recently have become aware of. After that, I got the opportunity to study computer science for some years, and my teaching also turned to programming courses as well as basic courses in natural science (for students lacking that from high school).[7] In the last couple of years, I've moved back to teaching mathematics. And I see that not very much has changed, at least not for the better. Incoming students are still very weak in mathematics.

There is one special circumstance that is of importance. In Sweden we have educational programs in engineering that take three years. There are just two mathematics courses in general (linear algebra and calculus), occasionally supplemented with a third course, possibly in mathematical statistics. The standard length of a Swedish university course is seven weeks. That means that our three-year engineering students have one half-semester of mathematics (they take two courses in parallel), which is not that much. So available time is a scarce resource. It is of course almost impossible to go

[7]Studying computer science, in particular the theory of programming languages, turned out to be a valuable experience. Programming languages share many features with mathematics, and I came into contact with computer scientists' explicit thinking in terms of *syntax*, *semantics* and *pragmatics* of programming languages. It has gradually dawned on me that this is a fruitful way of thinking of mathematics too. More on that below in Section 6.

very deep into calculus in such a short time. Most engineering students have had some calculus in gymnasium, including at least a brief encounter with derivatives, but some haven't seen integrals. So the problem would seem to be unsolvable.[8]

Why then try a humanistic approach to engineering mathematics? The students that enter engineering programs have ten to twelve years of mathematics from school. That is indeed a lot, and they do have a lot of knowledge, but it's a kind of implicit knowledge. It's not really workable knowledge. Very many have problems with simple numerics and algebra. Functions are dim concepts. As I wrote above, it's like all the acquired knowledge from twelve years of school needs to be scattered on a large table, like jig-saw puzzle pieces, and then put together into a coherent whole, with new pieces added and context provided. That sounds like a humanistic endeavor.

Even though mathematics is a supporting topic subordinated to technology, at least in our educational context, that doesn't mean it has to be taught and learned that way. The question is: how can the scarce resource of time be used in an effective way? The students have to be engaged so that they are prepared to invest extra study time for home work.

A rationale for liberal arts[9]

In Europe, in connection to the Bologna process, there is a focus on employability as the overall outcome of education. University education in Sweden is almost exclusively vocational. Of course it cannot be denied that most people study in order to get a good and interesting job and a decent career. There's nothing strange or wrong about that.

At the same time, society becomes ever more complex, and humanity faces outstanding challenges. We know nothing about the future. Education should also prepare for that. Then there are democratic, humanistic and personal values connected to education that cannot be reduced to a career in the workplace. I'm sympathetic to the kind of generalist knowledge liberal

[8]Gymnasium courses in mathematics have just been changed again, hopefully for the better, and the first students having had the new courses will arrive at the university in 2014. It remains to see what difference it makes.

[9]For readers unfamiliar with the American liberal arts tradition, here are two references [47, 36]; the second one specifically discusses how ideas from the liberal arts can help improve Swedish higher education.

arts education fosters. Even from a purely pragmatic point of view, the cornerstones of a liberal education are good both for a personal career and for the bettering of society. Studying mathematics from that point of view should make sense.

Humanistic mathematics

When I first put pen to paper and began to write about this project, early in 2012, I was worried that the idea of mathematics as a humanism would sound strange. I had arrived at the idea in connection with a book-writing project with a colleague, but curiously enough, during all that time the very phrase "humanistic mathematics" never passed my eyes. So I wasn't aware of the fact that the term *humanistic mathematics* was fairly well-established in the US. It actually came as a pleasant surprise, although thinking about it, it would have been really strange if no one had thought along these lines before. Anyway, such was the extent of my ignorance just a year ago.

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Viewing mathematics as part of human culture is of course not new. I think most people with an interest in mathematics sooner or later come across the humanistic aspects of mathematics.

I asked myself when I had first come into contact with this view of mathematics. I thought about the time I had read Philip Davis and Reuben Hersh's *The Mathematical Experience*. I had found this book in Camden Market in London in 1986 when I was working at Queen Mary College as a post-doc research fellow. We used to go there on Sundays since it was within convenient walking distance from Kentishtown where we lived.

The question struck me while reading the doctoral dissertation *Matematik och bildning* by Lars Mouwitz, a Swedish mathematics teacher and scholar [42].[10] It then occurred to me that my first encounter with the cultural aspects of mathematics must have been in high school in the early seventies

[10] The word *bildning*, which is the same word as the German *Bildung*, has no direct translation in English, as far as I know. But it has the same connotations as *liberal arts*. In Swedish, *bildning* is something you could possess. Perhaps the words *erudition* and *wisdom* convey part of the meaning. Could one say that whereas *bildning* is an object, *liberal arts* is more of a process?

when I read all the popular science books in physics and mathematics that I could find at the local library. In particular, there was the anthology Σigma in six volumes which I must have bought sometime because it is still on my bookshelf. When I picked it out a while ago, I got hold of volume 6, and it randomly fell open at an excerpt from Oswald Spengler [53]. My eyes fell on the sentence "The mathematic [*sic*], then, is an art." This caught my attention and I read the full article which actually turned out to be very interesting, its strange context notwithstanding. Apparently, Spengler wrote a long section on the meaning of numbers in his *The Decline of the West*.[11]

Σigma was published in the US in 1956 under the title *The World of Mathematics*, and it proves that the awareness of the cultural aspects of mathematics goes at least that far back. The editor, James R. Newman had been working on the project since 1944. Strangely enough, some time ago, I found the English version in an antiquarian book shop nearby. It was in good condition, almost unread, and there was a bookmark in an article by Bertrand Russell with the title *Mathematics and the metaphysicians*.

A personal anecdote

Modern mathematics would be impossible without a symbolic language, without *signs*. Anyone with a talent and interest in mathematics picks up this language in a more or less painful process. Allow me a personal anecdote. When I went to school, it could have been in fourth grade, I found a mathematics book on a bookshelf at home. It was a six-hundred-page textbook for business mathematics containing elementary arithmetic, algebra, logarithms, practical geometry and trigonometry [17]. It was my father's book. I was intrigued by the chapter on algebra "figuring with letters". Right on the first page there was an expression

$$a + a + b + b$$

that mystified me. I don't know how many hours I spent contemplating it. The memory is clear and when I now, many years later, look at the page again, I remember (somewhat romantically) how I rushed home from school and sat at the kitchen table with a sandwich and glass of milk, trying to

[11] One of his quite intriguing ideas is that there are different mentalities in mathematics. He writes that the analytic geometry of Descartes (and Fermat) is conceptually different from the ancient Greek geometry. People saw different things.

understand what it could possibly mean. The explanations in the book are not bad, they are actually quite good (compared to many modern books), but I still couldn't understand how you could "add" letters. If you try to add a to b, what do you get? You get nothing! The plus sign asks you to add numbers, and I could do that. But to be asked to add a and b was, ... well what was it? At that time I couldn't put words to what was wrong with the request. Today I would call it a category mistake. Letters are for writing words, not for adding. When I tell this story to colleagues, they do not seem to recollect any similar problems. To me the process of understanding the symbolic language of mathematics was, though not really painful, an intellectual challenge. Or perhaps I was just too young.

How the College Odyssey came about

I enjoy traveling in the US. I have taken some short conference trips to the US. In 1997 my son Erik and I made a coast-to-coast trip by train and car. And then I spent four months in upstate New York in the fall of 2004. So I had had, in the back of my mind for some time, the idea of the next trip. When I started to write about mathematics in the anthology about Liberal Education, it occurred to me that I could actually go back to the US and study mathematics teaching in particular. This was in the fall of 2010. I wrote a letter to Sheldon Rothblatt who encouraged me, and directed me to Lynn Steen to whom I also wrote. From Professor Steen I got more encouragement as well as a couple of reading tips, one of them being his article about the invisibility of mathematics [54].

There the matter rested for a year. I didn't know how to proceed or how to finance the project. Eventually I got the idea of not worrying about funds but instead just writing to colleges and presenting the idea. And without concrete contacts in the US, nothing would come of it anyway. So for a month or two in the early fall of 2011, I struggled with an email letter to liberal arts colleges. Then one night in October, I systematically ran down the list of top-ranked liberal arts colleges and looked up email addresses of heads of mathematics departments. A few more weeks of procrastination went by before I summoned the courage to actually send off the emails. About half of those I had written answered and the answers were all very kind, some of them enthusiastic about the idea. Having so gained confidence that the project made sense, I got the very natural idea of simply asking my own institute to finance the project. It did. Then I applied and got additional funding from a Swedish foundation, Stiftelsen Längmanska kulturfonden. Suddenly it was all clear that the college trip was going to be.

5. Notes from the Colleges

I made the college trip to learn about mathematics teaching at liberal arts colleges. I must have had some preconceptions about it, but they were not very explicit or conscious. I knew from my visit to Skidmore College in 2004, and from conversations with other STINT-scholars, that the teaching at such [liberal arts] colleges was of high quality. At Skidmore, I had sat in on quite a few classes in all sorts of subjects.[12]

If there had been dramatic differences in the teaching from what I was used to, then certainly I would have noticed. But since my focus in 2004 was not on mathematics in particular, I had the feeling that there must be something more to learn. What are they doing and how? But I needed to formulate a more concrete question. You can't do research without a question. Gradually, the question, the key question as I called it, came out as folllows:

> *In what ways does the liberal arts environment influence the way mathematics is taught as compared to other colleges?*

But my thinking wasn't very clear. I had somehow conflated humanistic mathematics with mathematics as taught at liberal arts colleges. I embarked on the trip thinking that I would find examples of humanistic mathematics teaching at liberal arts colleges. That was not really to be.

In the following sections I will briefly review some of my experiences and conversations at the colleges I visited. As I had expected, each of the visits turned out to be different, and this is reflected in what follows below. Looking back at the conversations, I cannot really say that they revealed anything dramatic that I had never thought about before. Rather they deepened my understanding of the teaching of mathematics, they underlined and put into new perspectives things that I had observed, read, and thought about. I gained confidence that my thinking wasn't completely crazy. And a few things were definitely new.

[12] I in particular enjoyed Tad Kuroda's classes on Colonial America.

Several themes ran through all the visits. To just chronologically review them would not make any sense to the reader. Instead I have tried to organize the discussions logically and write about them under the heading of the particular college where, so to speak, my understanding began to jell. However, I have to begin with a general comment.

Historical and philosophical backdrop

The United States, just as Sweden and many other western countries, is not doing well in international tests in mathematics. There is a history of declining standards and watered-down content. In Sweden this is part of the folklore among university teachers. In part, possibly a large part, this is due to higher education being transformed from concerning only a small elite to including at least half of the population. This is of course a good development paralleling democratic and egalitarian ideals, but it has created problems for teaching and learning. The pedagogy of fifty or a hundred years ago is not likely to be suitable in modern societies. Then, most students were likely to sustain doubt and just carry on, and we don't know how much they actually learned or understood. Modern students need motivation, they voice discontent, and we as educators are indeed aware of how much (or little) they learn or understand.

There is a persistent tendency among educators to lament the present situation: things are bad and they are becoming worse; something must be done! Sometimes the quality of the students and their knowledge is lamented, sometimes the teachers, the administrators, the teaching schools, and the politicians, and often, all of them. An almost caricatural illustration—regarding the university and college teachers—of that is Morris Kline's book *Why the Professor Can't Teach* [31]. If his picture of American mathematics teaching in 1977 is correct, then surely the situation must be much better today.[13] Things are changing, sometimes for the better, and sometimes for the worse. My outlook is that "things" are "generally improving" over time. That does not mean that there aren't any problems, quite to the contrary, there are always problems; but as we try to fix them, things are improving. Unfortunately, solutions to old problems tend to create new problems. The opposite,

[13]The book can, and perhaps should, be read as written tongue-in-cheek, but Klein must have felt he had an urgent message to get across. I talk more about this little book in Section 6.

and quite common, view: "things" are "generally getting worse", is not tenable. If that was true, then looking back in history, we would see that things were always better before. Eventually we end up back in the caves [45].[14]

5.1. Beloit

Beloit College in Beloit, Wisconsin, was my first college visit and it turned out to be a very good start. I drove up from Clinton, Iowa, on a nice Sunday morning in early September. I had spent a few days in Iowa—a state I long had wanted to see—after flying in to Minneapolis. Now I crossed the Mississippi on an old rusty iron bridge, but I didn't see much of the great river. In Rockford I phoned Paul Campbell who was my contact at the college. We met outside the college guest-house, then we went to a "heritage day" nearby. Later I was invited to dinner at his house.

When Paul Campbell answered my letter in March, he directed my attention to the *Journal of Humanistic Mathematics*. He also attached to his reply an article he had written about calculus [9] and an answer to it [14].

At Beloit I immediately hit on several themes that were to resurface throughout my trip:

- Calculus
- The "Is Algebra Necessary" discussion
- Mathematics as a liberal art
- Mathematics as a humanism

Let me start with the last two items.

Mathematics as a liberal art, mathematics as a humanism

I had a short discussion with David Ellis on the Friday just before the seminar talk I gave at Beloit. Since time was short I simply tried out the key question in whatever early phrasing it had at that time. The answer was short and succinct:

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[14] A simplification of Popper's philosophy of science (and politics) is: The point is not to do things right; it is to see the problems and try to fix them. Then we get progress.

Mathematics **is** a liberal arts subject.

It seems that the term *liberal arts* is often equated with *the humanities* or with *humanism*, and mathematics is not naturally placed in these categories. Mathematics is more often thought of as standing closer to the natural sciences and technology, probably because of the spectacular success of applied mathematics.

Paul Campbell told me that he had looked up the words *humanity* and *humanism* in a dictionary and found:

> **humanities**:
>
> (a) The languages and literatures of ancient Greece and Rome; the classics.
>
> (b) Those branches of knowledge, such as philosophy, literature, and art, that are concerned with human thought and culture; the liberal arts.
>
> **humanism**:
>
> (a) A system of thought that rejects religious beliefs and centers on humans and their values, capacities, and worth.
>
> (b) Concern with the interests, needs, and welfare of humans.

In Swedish we don't have this distinction. The Swedish word corresponding to *humanity* has the meaning of the human species. This means that the term *humanism* is used both for a philosophy and a world view and for a wide-ranging set of academic disciplines. I now realize that this ambiguity must have been behind my feeling (when planning the project) that the phrase *mathematics as a humanism* might sound strange.[15] I now believe that thinking of mathematics as a humanism can thrive on this ambiguity.

However, there should be no doubt that mathematics is indeed a liberal art. In the old classification of the seven liberal arts, the first three were called the Trivium and the consisted of Grammar, Logic and Rhetoric. These were the "language" liberal arts. Logic has been a mathematical subject ever

[15]Clearly, *humanism* as defined above was what Jean-Paul Sartre had in mind when he gave the talk "L'existentialisme est un humanisme".

since George Boole, perhaps even since Leibniz, and certainly since Frege and Russell.[16] Grammar, widely interpreted not just as grammar of natural languages, also includes the formal grammars of programming languages and is by now a mathematical science. The next four liberal arts, the Quadrivium, consisted of Arithmetic, Geometry, Astronomy and Music. The first two have always been mathematics. Astronomy was mathematics at the time but has since moved into the natural sciences. Music, at the time of Pythagoras, was mathematical music, the harmony of the spheres.

With some anachronism, all of the classical liberal arts except Rhetoric were mathematical. Rhetoric, if it can be thought of as teaching, is also relevant for mathematics. Mathematics could easily be thought of not just as *a* liberal art, but *the* liberal art.

So the answer that David Ellis gave to my question, that mathematics is a natural subject at a liberal arts college, is a good answer. But it still begs the question. Really, in what way is this fact visible in the teaching done at liberal arts colleges today? My quest had just started.

5.2. Carleton and Macalester

I visited Carleton College and Macalester College during the same week in September. I stayed in Northfield, the home of Carleton and St. Olaf College. I did not visit St. Olaf but met with people from there. In contrast to the quite leisurely week in Beloit, the week in Northfield was intense. I had spent the weekend in Chicago, visiting my old friend Jean Capellos, who lived in the flat above ours in Kentishtown, London, in the 80s. On the way back to Northfield I visited a railway museum in Illinois. But I lingered too long, and it grew dark as I drove north on highway 61. I crossed the Mississippi at La Crosse, but did not see the river this time either.

My contacts in Northfield and St. Paul were Deanna Haunsperger from Carleton and Karen Saxe from Macalester. They had me scheduled from early morning till late night. The classes I went to and the people I spoke to are listed in the acknowledgments. At Macalester I was also invited to an outdoors lunch with the faculty at the college president's house. At Carleton I sat in on a morning of a mathematics faculty retreat.

[16] For a popular account, see [12].

Calculus, Sputnik Calculus, Reform Calculus and Calculus

Already after three colleges, it became clear to me that calculus was at center stage. I had discussed the Calculus Reform with Paul Campbell and Bruce Atwood at Beloit and now the subject cropped up again. I remembered reading about it in the *Mathematical Intelligencer* in the 1990s, but I had not thought about calculus and calculus courses in particular during the planning of the project. My thoughts were focused on algebra where the weakness of student skills and understanding are already apparent.

Now I saw calculus as an American preoccupation. Of course, the importance of calculus is obvious. It was the second major breakthrough (after analytic geometry) of Western mathematics in the 17^{th} century, after centuries of poring over the surviving manuscripts of the classics. The calculus of Newton and Leibniz then rapidly developed over the next two hundred years, together with mechanics, hydrodynamics, thermodynamics, and electrodynamics. It became the language of natural science and technology. And it spurred an enormous evolution in mathematics itself.

However, this was primarily a European development. American mathematics lagged far behind, both in teaching and research, up to the early twentieth century [54]. America was mainly a country of poor immigrants trying to build a civilization in the wilderness. Indeed, schools were set up from the very beginning and the first colleges were soon to follow. But they catered to the needs of a pragmatic pioneer society.

By 1900, calculus was a standard college subject [31], but few high-school students took algebra or any higher mathematics [60]. Then came the world wars, in particular WWII, and the rise of the US to a world power, and the influx of European mathematicians and scientists fleeing Nazi persecution. The US became a leading power in science.

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After the mathematics faculty retreat at Carleton, I talked with two young faculty members, Andrew Gainer-Dewar and Brian Shea, during lunch. The topic of Sputnik and calculus came up again. The shock of the Soviets putting up a satellite before the US (on October 4, 1957) lead to a huge increase in government money spent on education, in particular science and mathematics, through the National Defense Education Act. More and more high-school and college students began to take algebra and calculus in some form. At the turn of the millennium, roughly 700,000 students were enrolled in college-level calculus courses [52].[17]

Not surprisingly, the teaching methods that had worked well enough when higher mathematics was an elite subject did not work very well when it became a mass education subject. This became increasingly clear in the 1980s. By this time, the electronic calculator had been around for a while, and its impact had become noticeable in the mathematics classrooms. Perhaps more importantly, knowledge of research into how human learning actually comes about—as opposed to pedagogical ideologies based mainly on wishful thinking—was spreading among educators.[18]

The *Calculus Reform Movement* was an initiative led by the National Science Foundation, running from 1988 to 1994. It was based on recommendations from a small conference of mathematicians and teachers known as the Tulane Conference [13, 40]. The focus was both on content and teaching methods, in particular student-centered learning, project work, and writing, as well as the use of calculators and computers in the teaching and learning of mathematics.

After reading about the calculus reform, what it tried to accomplish, how it succeeded and failed, and the criticisms of it, I must say I'm confused. As an outsider there is no way I can do justice to this complicated historical process; a historical background is provided in the report [58]. Teaching calculus is difficult in whatever way we try to do it, and many students will fail for many different reasons. If there is one more or less traditional way of teaching, it is easy to see its shortcomings: too much blind drill focused on calculations and procedures, superficial and insufficient conceptual understanding, and so on. So it makes sense to propose a replacement. Let me quote from the preliminary evaluation in [19]:

> Institutions nationwide have implemented programs as part of the calculus reform movement, many of which represent fundamental changes in the content and presentation of the course. For example, more than half of the projects funded by NSF use

[17] More estimates of enrollment in US calculus courses can be found in David Bressoud's Launchings column entry for April 2007, "The Crisis of Calculus" [4].

[18] Perhaps there is an analogy here with the study of the scientific method, where thinkers such as Popper, Lakatos, Feyerabend, and Kuhn were more interested in how research is actually done, rather than in prescribing how it ought to be done.

> computer laboratory experiences, discovery learning, or technical writing as a major component of the calculus course, ideas rarely used prior to 1986 [reference removed]. The content of many reform courses focuses on applications of calculus and conceptual understanding as important complements to the computational skills that were the primary element of calculus in the past. It is believed by many that such change is necessary for students who will live and work in an increasingly technical and competitive society.

The reform proposals were controversial and new problems arose where they were adopted, (some examples of the implementation of the reform efforts are described in [51] and [52]). The two sides of the debate can be illustrated by references [43] and [29], which I don't think stand so far apart after all.[19] One might even get the impression that the long-term effects of the reform movement have been rather small. Or perhaps it would be more accurate to say that whatever worked well in the reform projects has been absorbed into today's mainstream calculus teaching. The main objectives of the movement—focus on conceptual understanding, more varied teaching methods that involve active student engagement, the use of modern technology—do not sound at all provocative today [46].

But the story continues to unfold of course [4]. There are still concerns in the mathematics community about calculus and how it is taught. David Bressoud at Macalester told me about a national survey of Calculus I instruction conducted by the MAA in 2010. One of the goals of the study is to improve calculus instruction across the US. Readers interested in this study can refer to [38] or check out the main website for the project.[20]

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In the light of all this, I reread Paul Campbell's article [9] about the problems of calculus teaching once again.[21] It does not lend itself to a short

[19] Mumford [43] refers to Lancelot Hogben [26], which I also read in my youth, a piece of very good liberal arts writing in mathematics.

[20] The website for the project is currently hosted at http://www.maa.org/programs/faculty-and-departments/curriculum-development-resources/characteristics-of-successful-programs-in-college-calculus, accessed on January 7, 2014.

[21] The follow-up discussions are also interesting, see [14, 56, 10].

summary, and I guess it must be read against the backdrop of the calculus reform movement, because as it says "Didn't we go through all this already in the 1990s in the 'calculus reform' movement?" Obviously, the calculus reform did not solve all problems for all time. Which of course is not to be expected. If I understand the article correctly, Campbell's main complaint concerns how calculus is taught. Let me quote two key sentences. The first one (from page 417) is in relation to the calculus reform:

> What I offer is a philosophical critique about how we should teach calculus so as to situate it in the mainstream of intellectual pursuits.

The second one is just at the beginning of the article (on page 416):

> - Either "intellectualize" and "pragmatize" calculus - return calculus to the world of ideas and applications [...];
>
>   or else
>
> - acknowledge that calculus is basically a utilitarian skills course, stop giving liberal arts credit for it, [...]

This is then elaborated in the text. I find it intriguing to think about goals of teaching calculus in this way. We should want to both intellectualize and pragmatize it. To intellectualize it is to teach it as a humanities subject, as Campbell does indeed write. But this is by no means in contradiction with the practical and applied aspects of the subject. I believe that this is one of the strong points of thinking about mathematics as a humanistic subject. First, it moves the focus to aspects of mathematics (the cultural, philosophical, historical ones) that are often forgotten in teaching. Second, it includes all the applications, because applying mathematics is a human endeavor.

It seems that the calculus reform movement did not do this. In what I have read about it, there is no mention of the humanistic aspects of mathematics. There is a curious sentence in the historical background text in the report of the calculus reform movement:

> While mathematics was always essential for most scientific disciplines, in 1960 calculus was still viewed by many outside the sciences and engineering as a liberal arts subject [58, page 10].

Thinking about this, I now see another meaning to humanistic mathematics. Humanism is opposed to any form of extremism or simplification of reality, and this is to be held also in pedagogy. The humanist sees the complexity of reality as a blessing.

5.3. Oberlin

My next college stop was Oberlin College in Oberlin, Ohio. I arrived there on a Sunday afternoon after having visited the Henry Ford Museum in Dearborn outside Detroit. I had traveled through Wisconsin and the Upper Peninsula and over the Mackinac Bridge. My contact was Susan Colley who picked me up at the Oberlin Inn for a dinner at her home. Her husband cooked a wonderful dinner of smoked fish, spinach and cornbread while we chatted over a beer. The discussion turned to the notorious "Is Algebra Necessary" article [22].

Is Algebra Necessary?

This discussion came up several times during my college trip, in particular in Beloit and Oberlin. It was initiated by a *New York Times* article by Andrew Hacker [22]. The article itself starts out with a picture of the ordeal that mathematics, and algebra in particular, represents for most American high school and freshmen college students each year. Then it says that:

> Nor is it clear that the math we learn in the classroom has any relation to the quantitative reasoning we need on the job.

This is by now a familiar sentiment. School mathematics is felt to be mostly irrelevant. I find the argument quite difficult to answer. One thing that could be said is that engineering students do need quite a lot of classic algebra and calculus. Studying natural science and technology without mathematics doesn't make any sense. Then it is another thing whether most of the engineering graduates actually use much explicit mathematics in their jobs. Many don't but some do.

Hacker poses the question

> What of the claim that mathematics sharpens our minds and makes us more intellectually adept as individuals and a citizen body?

and answers that traditional algebra instruction does not do that. We would hope that it does, but I'm dubious that it does.

This connects to something else that I've had in my mind without being able to formulate explicitly, until I just recently read an article by Leone Burton [8]. One thing I picked up from that article is the importance of keeping *content* apart from *process*, in this case, the content of courses and the process of mathematical thinking. Teaching content does not guarantee that the students pick up mathematical thinking. One of my colleagues, Magnus Lundin, has been saying this for many years: "It does not matter what we teach them as long as we teach them to think." The sense of this has gradually dawned on me. Of course we must choose relevant content, but often the focus is too much on the content itself, instead of on the processes of mathematical thought.

But this is a tricky question. We do teach problem solving and we do teach proofs, which are indeed examples of mathematical thinking. But I think that what we do is still too much cut-and-dried. We show *the results of problem solving*, solving problems in a linear way without errors, false starts, or re-tracings. We prove theorems as if they were straightforward mechanical deductions. The creative, tentative, and exploratory aspect of mathematical thought is downplayed.

Much of Hacker's article is anathema to many mathematics teachers. Myself, I'm not so offended by it. And the article actually ends on a very positive note as it asks for alternatives:

> The aim would be to treat mathematics as a *liberal art* [my emphasis], making it as accessible and welcoming as sculpture or ballet. If we rethink how the discipline is conceived, word will get around and math enrollments are bound to rise.

It seems that many readers of the article miss this point. The way mathematics is taught today is not optimal.

At Oberlin I had a lunchtime meeting with faculty where I described my project. The next day I had a similar meeting with a set of students, they were all mathematics majors.

5.4. Bryn Mawr

After Oberlin I drove through Ohio and Pennsylvania. I got the impression that the countryside in Ohio was scaled down, more European, compared to the vast fields of Iowa and Minnesota. It was beautiful. In Pennsylvania I crossed the Allegheny Mountains and stayed the night in the small town of Shawnee after descending Mount Ararat. It was the coziest little motel, with small rooms and no wifi, but a nice restaurant across the road with hearty Italian food and German October beer. In the misty morning I left Shawnee as the yellow school buses picked up kids waiting along the road. I continued on the Lincoln highway to Bryn Mawr outside Philadelphia.

My contact at Bryn Mawr was Paul Melvin. I had written to the three colleges Bryn Mawr, Swarthmore, and Haverford, and gotten positive answers. But as my list of colleges had grown a bit long, I got the idea of suggesting that I give a joint seminar at these three colleges. This was arranged by Paul Melvin, and on a Friday afternoon I held the seminar. Joshua Sabloff from Haverford College and Thomas Hunter from Swarthmore College, all of whom I had written to, were there.

What's retained and the transfer problem

The question about what students retain from a mathematical education came up in the discussion after my seminar. Thomas Hunter said that it was an experience, of having learned mathematics. Josh Sabloff quite strongly argued that it must be something more than an experience. A heated but good-humored debate followed, with Paul Melvin as a mediator. Partly it was a question of the meaning of words like *experience*. I listened. Josh Sabloff made up an example of a medical doctor who, ten years after his last mathematics course, had to read a scientific article about tests of some new drug. How would that doctor go about judging the evidence put forward in such an article? His answer was: using knowledge retained from a mathematical education.

Such retained knowledge[22] could consist of specific things such as reading tables and diagrams, parsing formulas, but perhaps more likely, of abstracted knowledge such as analytical and logical thinking, discriminating between what's important and not, and so on.

The discussion reminded me of an article by Underwood Dudley, arguing that algebra (mathematics) teaches us to think [15]. This is something we

[22] I'm filling out the discussion in retrospect with things that were not explicitly said, but was implicit in the trains of thought as I understood them.

all want to believe. I said, at Bryn Mawr, that it would be very interesting to have some kind of evidence that such deep, and tacit, knowledge is indeed what results from a good mathematics education. Then digging deeper into the texts that I acquired during the trip, I found an answer [27] to Dudley's article that referred to actual research on transfer. This article, which also contains useful references to work on transfer, concludes that:

> There appears to be no research whatsoever that would indicate that the kind of reasoning skills a student is expected to gain from learning algebra would transfer to other domains of thinking or to problem solving or critical thinking in general. The lack of such research evidence does not mean that such transfer does not occur or that algebraic reasoning might not have positive effects on problem solving and critical thinking.

The point is that there seems to be no research showing transfer of algebraic skills to other domains.[23]

When writing these sections I happened upon the article, mentioned above, by Leone Burton [8] that may be relevant to these questions. If I understand it correctly, Burton argues that mathematical thinking is not learned by learning mathematical content. Indeed, conventional mathematics teaching is mostly concerned with content, not the process of mathematical thinking. This should be relevant for the transfer discussion, since content is eventually forgotten, but the attitude of a mathematical approach to problem solving may be retained.

Granted that mathematics teaches us quantitative, analytical, and logical skills that may be retained, perhaps tacitly, long after the details are forgotten, it can perhaps be contended that it does not really matter what parts of mathematics are studied as long as the teaching is good and the studies are serious, and as long as the teaching and learning is focused on the processes of mathematical thinking. These are clearly very interesting and important issues to understand.

[23] Paul Campbell, who was also interested in the transfer problem, gave me an article [23] about a pedagogical experiment that indicated that pattern recognition within a narrow mathematical context seemed to transfer more easily from an abstract formulation to concrete representations than between different concrete representations.

The transfer problem and algebra

The connection between the "Is algebra necessary" discussion and transfer problem was also made in an article by Lynn Steen [55]. Steen writes that

> [...] what he [Hacker] really says is that it [algebra] is not working in the [American] curriculum.

Steen argues that algebra does not work as a vehicle to convey usable and transferable mathematical skills to the majority of students, most of whom go on to careers that do not use much (or any) explicit mathematics. They would be better served by a more varied mathematics curriculum along the lines of *quantitative literacy*.[24] The discussion is further commented upon by David Bresssoud in [5] who stresses that if we want transfer to occur, we must teach for transfer.

There is of course no question that students aiming for STEM (Science, Technology, Engineering, and Mathematics) careers need to study content relevant to such careers, but a move towards teaching the process of mathematical thinking, I do think, is needed.

These two articles also refer back to earlier discussions of this topic. I will return to one aspect of it later on, but for now, we have to move on.

5.5. Skidmore

After Bryn Mawr, I headed north for the last leg of my trip. I was going to Saratoga Springs. On route, I visited the Daniel Boone Homestead and the incredible Roadside Americana which is a huge model railway—an America in miniature. I also passed the Delaware Water Gap. This was a disappointment as there didn't seem to be any way of getting near enough to see anything of interest. Instead, I continued through the enormous mountain ranges of northeastern Pennsylvania. When I passed into New York it was late afternoon, and as I neared Saratoga it was already dark. The rain was

[24]Two ideal places to start investigating the notion of quantitative literacy are the MAA website on quantitative literacy at http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/quantitative-literacy, and *Numeracy*, the official journal of the National Numeracy Network, at http://scholarcommons.usf.edu/numeracy/, both sites last accessed on January 7, 2014.

pouring down but I recognized the Northway as if I had just driven on it yesterday. Although it was past nine, I met with Steve Goodwin, and we went for a downtown beer in a not too noisy place; it was Saturday evening.

Many liberal arts colleges have what is generally called *First Year Seminars* or variants thereof. These are special seminar courses taken by the freshman students[25] often as part of a wider *First Year Experience*[26]. The F.Y.S can be anything really, and many colleges offer seminars that are mathematical. A F.Y.S in mathematics should be an excellent opportunity to do something different in mathematics, and it seems that this is what is often done. Most of the students are not planning to study any more mathematics, apart from perhaps the distribution requirement. From an article by Susan Colley at Oberlin College [11], I learned about one way of conducting a first year seminar in mathematics. Many more examples can be found on the Internet.

This year at Skidmore, the mathematics F.Y.S was conducted by Mark Huibregste whom I knew from my 2004 stay at the College. The seminar was focused on Geometry. The students read Euclid; they were reading the first book at the time of my visit. There is an on-line edition with interactive pictures that was used in the seminar. To Swedish ears this must sound incredible. To many, it may also sound like a complete waste of time. I mean, Euclid, isn't that stuff 2500 years old? And I admit, I sat down a little prejudiced because I've never really liked geometry myself, in particular not that type of synthetic geometry.

But amazingly it worked. Even I understood the proofs and the general drift of the argument. From working with parallelograms and proving theorems on areas ... over the fifth (parallel) postulate ... to an elegant proof of the Pythagorean theorem. And the humanistic aspects of mathematics are

[25] In the American higher education system, the terms *freshman*, *sophomore*, *junior*, and *senior* are used, respectively, for students in the first, second, third, and fourth year of a four-year institution.

[26] What a *First Year Experience* is, is well-known in America, but for Swedish readers here's a short description: Most liberal arts colleges have what they call a planned *First Year Experience* (F.Y.E) for the new students. Besides offering a general introduction to college life, college-level studies, and a socialization experience, these are also aimed at introducing the students to the particular characteristics of a liberal education. Various seminars are offered centered on various subjects, among them mathematics. The seminars are often inter-disciplinary.

there too, as Mark pointed out. It's there in the comment that the Greeks did not have the real numbers and so they had no quantitative measure of area. It's there in the comment that some areas (lengths really) could not be measured by the only numbers the Greek knew about (the natural numbers). It's there in one aspect of mathematics that is often so hard to communicate: the logical character of the subject.

From this class I went away with the happy realization that studying Euclid is not at all useless. Here the strength of the liberal arts context shone through.

5.6. Bennington

During the week in Saratoga I drove the short distance into Vermont to visit Bennington College, very beautifully situated in the mountains outside the small town Bennington. Autumn colors were in full swing and the air was a bit chilly. I found some use for the pullover I had bought in Saratoga Springs the day before.

My contact at Bennington College was Andrew McIntyre. The college is focused on the humanities and it was the first in the country to include visual and performing arts in a liberal arts education. Mathematics is not a big subject at the college, and mathematics and science form a faculty together. As Andrew had told me in e-mail conversations, he had spent time re-thinking the mathematics curriculum and the specific course syllabi. The catalog course descriptions are distinctly different from most other descriptions I've seen. This made it interesting to me. It was Mark Huibregtse who had, in e-mail conversations in the spring, directed my attention to the mathematics curriculum of Bennington College.

The teaching method was also different. It was much more student-focused, more bottom-up, working from examples and exercises (which the students worked on themselves in groups and individually) towards general theory, but in a guided way, guided by lists of things to do which I thought of as roadmaps into the subject. I also talked to a student who was an Astronomy major. She had been studying at the University of Massachusetts Amherst, so she could compare the Bennington method to the teaching she was exposed to at a large research university. Her comments gave me additional insights into the teaching style at Bennington. I had actually tried something like this last spring in a multi-variable calculus course where I worked from examples towards the general theory using matlab for visualization. It worked better than a traditional "theory → examples" approach. What was lacking (in my class) was supportive course material. I did it on the run out of necessity, because standing in front of the students the third time I gave the course, I realized I had to communicate. Remember, the students I have are not mathematics majors. Their knowledge of single-variable calculus, even algebra, is weak.

5.7. Colby

I left Saratoga Springs on a Saturday morning. It was the Columbus weekend, Monday being a holiday (although it was not a college holiday), and I had planned to use three days for the trip up to Maine to visit Colby College and Bates College. It was fall foliage season. I had read about crowded roads, but saw nothing of that except the road leading up into the White Mountains. I turned around and skipped that part. I did however have some problems finding a room for one of the nights. A nice landlord at Nootka Lodge in Woodsville let me sleep in the "game room". He had phoned all the big chain motels and there were no rooms to be had in all of New York, New England, Maine and Canada. "Nothing in the whole Kingdom," he said.

At Colby I had the most intense afternoon on the whole trip. During four hours I talked to eight faculty members. But before that I had lunch with Scott Taylor. I learned that the department offers a course "Mathematics as a Liberal Art" for students who need to fulfill the distribution requirement. It is in general not taken by the mathematics majors, because there are so many other requirements to meet if you want to go to graduate school or an engineering school. I asked my key question, and again I got the answer that the differences are not that big. Most of the instructors themselves come from research universities, and the tenure track system[27] acts as a brake on experimentation with content and teaching methods. For instance, in student-centered teaching, there is a tension between letting the students be wrong most of the time, and showing them the correct procedures.

In the afternoon, I again asked my key question about what characterizes mathematics education at liberal arts colleges. I got somewhat more specific answers. I talked to Fernando Gouvêa. One difference, as compared to other

[27] Professors are provisionally hired for five years during which time their teaching and research are evaluated. If successful, they get tenure.

schools, he said, is the *Telos*, the goal. Education at liberal arts colleges is not vocational, and mathematics teaching is not primarily to prepare for graduate school (PhD studies). In particular, in upper-level courses there is more freedom to choose topics. And even in basic calculus classes, even if several parallel sections are taught by different professors, they can all have different books and they don't have the same exams.

Then I talked to Leo Lifshits. He was also very specific. He said that he always uses classroom time for teaching ideas, not techniques. For instance, in Calculus, he uses a standard textbook (Stewart) which has exercises online that are automatically corrected. So the students do these as homework. This frees up time for talking about the ideas underlying calculus. I asked if this really works with weak students, and he admitted that there is a self-selection. Students that cannot work like this, or do not want to work like this, take other calculus classes. Still it is an interesting way to work. This way of teaching forces students to think as opposed to doing routine manipulations. That forces a "crisis". Those who cannot deal with it leave.

But perhaps it can be done in a milder way, in a humanistic way. I'm sympathetic to the idea of challenging students' prejudices about what constitutes a mathematics class and what mathematics is about. That's really the fundamental reason why I'm off on this quest for the perfect mathematics class.

In the evening I had dinner with Jan Holly, her son, and Scott Lambert at a downtown café. It was a nice place, although downtown Waterville had looked a bit deserted when I arrived the day before. I now learned that the college had moved to its present location up on a hill, quite far from downtown, in the early twentieth century, and that the town really didn't identify with the college.

A conclusion of the many conversations I had at Colby College, of which I have reviewed but a few here, is that the personality, knowledge, and interests of the individual teacher are essential. What the liberal arts milieu provides is the freedom to express this in the courses, even though the tenure track system may occasionally hold back younger teachers.

5.8. Bates

From Waterville it is just a short drive down to Lewiston. I arrived early; I had planned for a stroll downtown and some reading at a nice coffeehouse.

But it was not to be. The downtown area looked deserted, and it was chilly and cold, so I gave it up. But I did find a nice restaurant just by the motel.

At Bates I had an all-too-short conversation with Bonnie Shulman that clarified an issue that I had mixed up in my thinking. It has to do with the relationship between *humanistic mathematics* and *liberal arts mathematics*—the very core of my project.

It was only when Paul Campbell at Beloit College sent me the link to the *Journal of Humanistic Mathematics* that I became aware of that "subculture," as Bonnie called it.

At the start of our conversation, Bonnie made it clear that if my key question is how the liberal arts context influences how mathematics is taught, then it is not humanistic mathematics that I'm interested in. This drastic way of putting it made it clear to me that liberal arts mathematics and humanistic mathematics are actually different things. I wasn't surprised by the fact itself, but I was surprised by myself having had it mixed up for so long. I said that, well, then what I'm really interested in is indeed humanistic mathematics. But I had thought that I could find it practiced at liberal arts colleges.

Nothing prevents a professor from teaching humanistic mathematics or teaching mathematics humanistically, and it is sometimes done, but perhaps not that often. The opportunity is there, but perhaps too often it is a missed opportunity. In this way, my conversation with Bonnie verified the impression I had been getting from other conversations all along, that the liberal arts environment may influence how mathematics is taught, but it does not have to.

Another thing that became apparent to me was that what you get at a liberal arts college is not so much the classes themselves being taught differently, but the context itself with its focus on breadth, depth, interdisciplinarity, critical thinking, writing, communication, and so on, rather than a focus on a vocation or profession.

The American movement for humanistic mathematics

It would be interesting to dig deeper into humanistic mathematics itself. What is it, really? I had planned to write about it under this heading, but I now realize that I don't know enough, and the text is already long as it is. The reader of this *Journal* is surely familiar with it (or alternatively, can quickly check its *About Page*.[28] Another good starting point is an article about humanistic mathematics education sent to me by Paul Campbell [6].

5.9. Amherst and Wellesley

As I drove down to Amherst in Massachusetts I felt that the trip was nearing its end. I decided to have a look at the Atlantic Ocean on the way and did so at Fortunes Rocks near Kennebunkport. Then I headed inland and arrived in Amherst in the early afternoon. After checking in to the College guest house, I strolled downtown and had coffee at a local Starbucks. It was Sunday. And this was a real college town, with a main street and many coffeehouses, restaurants, and bookshops.

The next day I met with Robert Benedetto who was my contact at the mathematics department. Rob had been very helpful during our mail conversation, providing useful information about New England and Amherst, when I was planning the trip. Our short conversation before I sat in on one of his calculus classes corroborated what I had learned throughout the trip: There is a certain self-selection of students to liberal arts colleges, the environment encourages more student participation than may be common at other schools, and there may be more interaction with faculty from other departments. But there is not a great difference in teaching styles and methodologies. Liberal arts as such are not explicitly discussed with the students.

In Rob's class, it dawned on me that what I've been seeing during all my classroom visits is mathematicians teaching mathematics.[29] This is of course obvious, but perhaps the sentence "it flows so easily from the pen" conveys the feeling I got.

After lunch I talked to David Cox who told me how calculus classes are organized at Amherst. Besides the standard sections running for one term, there is one section that is stretched out over two terms and another, more intense, section running for less than a term. I thought this was a simple and practical example of a humanistic approach to mathematics where the classes are adapted to the needs of the students.

[28] http://scholarship.claremont.edu/jhm/about.html, accessed January 12, 2014.

[29] The class was about the definitions of extreme values, which, as Rob stressed, have nothing to do with limits or derivatives. Calculus proper enters with the extreme value theorem for continuous functions on a closed interval. Then Fermat's theorem was treated, leading to a definition of critical points.

I had planned to continue to Wellesley after a few days, but it turned out to be almost impossible to find any reasonably-priced place to stay there. So I decided to stay in Amherst and just drive to Wellesley for a one-day visit to the college. This I did on a Wednesday, starting out early before dawn and arriving in Wellesley at around 8 am where I met with Stanley Chang for breakfast. It was Sheldon Rothblatt who had directed me to Stanley.

Wellesley is a women's college, and among its many prominent former students we find two US Secretaries of State, Madeleine Albright and Hillary Clinton. Over breakfast I asked Stanley what difference it makes to work at a women's college, and during my one-day stay at the college I thought I could detect the amiable atmosphere he had talked about.

As I drove back to Amherst in the evening, I felt that there was now no point in talking to more people. I had already learned more than I could possibly have hoped to learn. My conversations with Stanley, the classes I sat in on, the lunchtime conversation with several of the faculty and in particular with Alexander Diesl over coffee at the local Peet's coffeehouse all rounded off the experience.

In the end I stayed for eight days in Amherst. I fell into a daily routine of early morning writing, then breakfast at the Black Sheep Cafe, a stroll on the main street, then more work. In the afternoons I did some shopping out at the mall or some leisurely walking. One day I drove a few miles west and had a look at the Connecticut River. I was invited to Sheldon's Amherst home, and we talked, among other things, about my project and how it had turned out.

Since I would be teaching as soon as I got home (after a week of holiday in Paris), I had preparation work to do. And I began to get a little homesick, or perhaps more family-sick. For the last two days of the trip I found a place in Natick. I took the train to Boston one of the days. Had I done my homework I would have known about the JFK presidential library. That has to wait for next time.

6. Mathematical Language

Here I will try to discuss mathematics as a language. In several places I will draw analogies with computer science, in particular the theory of programming languages; the latter has some similarities with mathematics in that it relies on formalized languages and their relationships with reality.

Magic

One of my colleagues, Magnus Lundin, said something interesting at a meeting at our institute a while ago. It was an observation he had made. It was as if some students, or even many students, when they enter the mathematics classroom, shed[30] the natural logical and rational thinking they use in everyday life and in other academic subjects of study. Instead it is as if magic could solve the problems. We had been talking along these lines a few times before, but now he expressed it explicitly. The point got across. I realized that I've seen the same phenomenon. It connected to my own thinking about mathematics as a language and what I have started to name the *The Language Teaching Metaphor*. Let me back up a bit and try to explain my thoughts. It is certainly not a new observation that mathematics can be looked upon as a language. I don't think anyone would deny that. But I think there is much more to it than is normally surmised. One aspect is the invisibility paradox I briefly mentioned earlier in Section 2.

The invisibility paradox

It is obvious that our high-tech society couldn't exist without advanced mathematics. It's not just electronics, which relies on physics understood in terms of mathematical physics; it is also the logistics of administering energy, materials, and information that require mathematics and computation. This is all very well-known and acknowledged. A thorough and still up-to-date discussion can be found in Lynn Steen's 1985 article *Mathematics: Our Invisible Culture* [54]. Mathematics is central to our technology, society and culture, yet goes unnoticed most of the time. The applications are invisible; mathematics is built into our society and technology, and it is only visible to the engineers, economists and scientists (mathematicians included) who work on a day-to-day basis with maintaining and developing it. Even many students in engineering never use much mathematics in their jobs after graduating. Mathematics may be visible while studying at school and university but not after that. Indeed, for almost everyone, mathematics is highly visible

[30]Being a non-native speaker of English, I now and then look up words in a dictionary: "shed" = "to rid oneself of temporarily or permanently as superfluous or unwanted," according to Merriam-Webster's Eleventh Collegiate Dictionary [37].

in school but invisible outside school.[31]

One way of understanding the visibility of school mathematics is precisely by analogy with natural language, that is, any spoken and written human language. When you speak your own native language, then you are not, most of the time, conscious of that. It's just something you do. You're probably more conscious when writing since that is more difficult—it is not so instantaneous, and it is more reflective.[32] However, when you speak a foreign language, you are much more conscious of speaking.

Now, could it be that when mathematics teachers use the mathematical language (and here I'm thinking of both the formalism itself and the natural meta-language needed to communicate mathematics), they are not aware of the fact that they are using it? Could it be that they speak and write as if the students were as fluent as themselves? It's like in Foreignland long time ago when people couldn't understand that not everyone spoke Foreignese. But the typical student is not fluent. Mathematics is not a native language for most people.

Formal mathematics as taught in school is therefore highly visible to most people. But the mathematics that is built into our society and technology is almost entirely invisible. Of course it is not formulas that are built into technology. There are no formulas in a cell phone. But mathematics was needed when designing it, and it is certainly needed for the network infrastructure that makes it work. And all that is based on our knowledge of physics, described in mathematical language. But can I really understand how mathematics is built into a cell phone? Could I, in some detail, give a plausible explanation to a student asking me?

We use embedded mathematics all the time without being aware of it, but not formal mathematics. The latter seems to be used by almost no-one except mathematicians and a subset of scientists, engineers, economists, logisticians and the like.

[31] The invisibility paradox is one background to the "Is Algebra Necessary" discussion described in Section 5.3.

[32] The reflective nature of writing is indeed one reason it is worth writing at all.

Plato and Object Oriented Programming

This is very strange. It is as if mathematics is built into the very fabric of reality. Most languages (as far as I know) have nouns, adjectives and verbs. This corresponds to the fact that the world consists of *things* that have *properties* and can *perform actions*. This also, by the way, corresponds closely to the classes of Object-Oriented-Programming (OOP): classes are blueprints of things (abstract or concrete) that have properties and actions. It is a close step to think of classes as Plato's *ideas* and the objects (the instantiations of classes, still speaking OOP) as concrete physical things.

However, this view of reality and language can be questioned. Bonnie Schulman at Bates College gave me an article she had written, which, among other topics, discusses the noun-verb-adjective view of the world [50]. I haven't had time to think more deeply about this. Understanding the mappings between reality and our descriptions of it and the categories and language constructs we use must be important for mathematics teaching. This discussion leads naturally over to philosophy of mathematics, and questions about the nature of mathematical objects, see Section 7.

The symbols of mathematics

Suppose provisionally that mathematics is about ideas, ideas somehow connected to phenomena in reality. These ideas must be captured by some kind of formalism using some kind of symbols. The ideas of mathematics may start out as vague, but eventually they have to be made precise, or exact, because we want to prove beyond any reasonable doubt that our theorems, phrased in terms of these ideas, are correct, given the correctness of the underlying axioms or foundations.[33]

I have the impression that most mathematics instructors, if they think about it at all, consider the formalism itself and the symbols themselves, to be very exact. I don't think this is true. More and more I have come to think that the symbols and formalism of mathematics are inherently vague. How can a symbol, however elaborate and decorated by prefixes, suffixes, indices and what-not, capture the full body of a complex mathematical concept? Here's an example from topology.

[33] It must be realized that basing mathematics on secure foundations is an "after-the-action" reconstruction of mathematics; it is not in general how new mathematics is discovered. See for instance [25] and [50].

> Let K be a finite simplicial complex. The n-th homology group is denoted by $H_n(K)$ and defined by:
>
> $$H_n(K) = Z_n(K)/B_n(K).$$

Even if, or perhaps in particular if, you don't know what a *finite simplicial complex* is or what a *homology group* is, it should be clear that they are complicated objects all of whose properties cannot possibly be captured by the symbols K and $H_n(K)$. First, K is just a *name* for the finite simplicial complex under consideration. Secondly, the symbol $H_n(K)$ is a little bit more elaborate, but in another context it could stand for something entirely different, a function for instance. Explaining what $Z_n(K)$ and $B_n(K)$ stand for helps a little, but then the reader must rethink the meaning of /. There is a lot of conceptual understanding, based on many concrete examples, that is denoted by this piece of mathematical formalism.

Context is the key word here. Mathematical symbols and even whole formalisms say nothing without a context. The symbols have connotations given by that context. In order to even read a formula, parse it so to speak, and understand it, you must have some picture of the context. When you are learning new mathematics, part of the problem is that you don't have that context yet. In bad mathematics teaching the problem is aggravated by the lack of *narrative*, something that is often the case with mathematics textbooks. Providing the narrative is humanism. (That narrative is central to mathematics teaching soon becomes clear when you sit in on good college mathematics classes, as I did.)

My contention is that the symbols and formalism of mathematics cannot in principle capture all aspects of a mathematical concept. Therefore the formalism is inherently vague to a considerable extent. To make things worse, symbols are often used in slightly different ways in different contexts.

An analogue with computer science is useful here, too. In order for a computer (a program really) to work with a concept or an idea, it must be completely captured by the "symbol," in this case the appropriate data structure or class. Everything must be encoded. The program may have "background knowledge," but it has no imagination and it cannot use any information not programmed into it or learned in some way. This is drastically different from how we as human beings work. After all, we are not machines.

The language teaching metaphor

These observations bring us over to what I have started to name the *The Language Teaching Metaphor*. There is an analogy with computer science and there is an analogy with teaching natural language. Computer scientists, particularly when discussing the theory of programming languages, speak of *syntax*, *semantics* and *pragmatics* (cf.[41]). *Syntax* is precisely the syntax of the language, its grammar, its rules for how the words and phrases of the language can be put together without committing any errors. A program must be syntactically correct; otherwise the computer cannot run it. Likewise, a piece of written mathematics must be syntactically correct in order to make sense. The same holds for sentences and text written in a natural language. But as we proceed from computer programs to written mathematics to written natural language, we can tolerate a few errors in mathematics, and perhaps quite a few errors in natural language. We understand anyway (even though reading texts with lots of grammatical errors throws uncertainty on the meaning and it is often quite exhausting). The computer tolerates no syntactical errors at all.

With *semantics* we focus on the meaning of what is written. In a computer program, this is what the program is meant to do when it is running. A program can run, but it may not do the right thing, it may not do what it was intended to do. The same goes for a piece of written mathematics. You may require a function to be zero in order to find points where there could be a maximum or a minimum. You may solve the equation correctly. Everything is syntactically correct. But of course, the semantics is all wrong. To find candidate points for extrema, you should start by differentiating the function first.[34] Likewise, the semantics of written text is the very motivation of writing in the first place, to convey a message, to communicate.

Pragmatics has to do with how the language is used in practice. Different programming languages are used for different programming tasks, partly out of tradition, but more so because they were originally constructed with different applications in mind. The same goes for mathematics. Methods are chosen that are thought to be appropriate to a given problem and symbols are chosen in accordance with that.

[34] On the last calculus exam I gave, a student complained that he couldn't solve the equation $f(x) = 0$. It was an extremum problem.

Let me take an example of a pragmatic question in calculus. Let's say we are teaching calculus. What is the basic pragmatic question when teaching derivatives? I would say it is the following:

> What kinds of questions are derivatives the answer to? And how do we use the derivative in each such case?

If the topic is integrals, the corresponding questions are:

> What kinds of questions are integrals the answer to? And how do we use the integral in each such case?

The first parts of the questions "What kinds of problems ..." are humanistic questions. They are about concepts and classical problems, about history, philosophy, and culture. The second parts of the questions "And how do we use ..." are more about skills; they are practical and applied.

I think one problem with much of traditional mathematics teaching is that it is mostly concerned with the syntax. Symbols are manipulated according to the rules, sometimes with a grounding in semantics, but seldom in any pragmatic humanistic context.

But how is all this related to natural language and the teaching of natural language? I will take French as an example. A good course in French must consist of three things

1. The grammar of French;
2. The literature, culture, and history of France;
3. How French is used in practice in various circumstances such as reading, speaking, and writing.

It may not be entirely one-to-one, but I see a clear correspondence to:

1. Syntax;
2. Semantics;
3. Pragmatics.

I don't think French teachers think of their subject or their teaching as particularly *logical* or *rational*. These are not concepts naturally associated with natural language teaching, apart from grammar.[35] But I do think their teaching is highly logical and rational. And I don't think the students shed their logical and rational thinking and their common sense when they enter the French classroom. Studying language is a rational and humanistic endeavor.

How does it come about then that mathematics, a subject that ought to be the most logical and rational of all human studies, is typically studied by first leaving common sense outside the door and relying on pure magic? I think the reason is that mathematics has ever since first grade been disconnected from language. Mathematics is conceived of as something entirely different from language. Learning to read and write and learning to count are conceived of as totally disconnected activities.

It is said that mathematics is the language of nature. Well, aren't all human languages languages of nature and culture? Why else would we need them? The reader may now be curious. How would you teach mathematics as a language in this way? I will return to that question elsewhere [2].

Mathematics writing

I came across a hilariously funny book by Morris Kline [31]. Titled *Why the Professor Can't Teach*, it laments the poor quality of mathematics teaching and tries to explain why it was so—I used "was" because surely the situation must have improved by now. It was published in 1977. In a chapter about mathematics texts, Kline writes

> Many authors seem to believe that symbols express ideas that words cannot. But the symbolism is invented by human beings to express their thoughts. The symbols cannot transcend the thoughts. Hence, the thoughts should first be stated and then the symbolic version might be introduced where symbols are really expeditious. Instead, one finds masses of symbols and little verbal expression of the underlying thought.

This paragraph confirms what I have written above about the formalism and symbols of mathematics. Kline also writes

[35] I remember being extremely annoyed when the German teacher in school said that German grammar was as logical as mathematics. I can accept that today.

> Student interviews, discussions, and dialog quickly revealed that what the student sees when looking at a graph is not what the teacher sees. What students hear is not what the instructor thinks they hear. Almost nothing can be taken for granted. Students must be taught to read and interpret the text, a graph, an expression, a function definition, a function application. They must be taught to be sensitive to context, to the order of operations, to implicit parentheses, to ambiguities in mathematical notation, and to differences between mathematical vocabulary and English vocabulary when the same words are used in both. Interviews revealed that the frequent use of pronouns often masks an ignorance of, or even an indifference to, the nouns to which they refer. The weaker student has learned from his past experience, that an instructor will figure out what 'it' refers to and assume he means the same thing.

This is also my experience from many years of teaching and contemplating what's happening when teaching. In the preface to the book I'm writing with a colleague[36], we write "The only thing a teacher speaking in front of an audience can be sure of is that everyone thinks about something different than the speaker." And we continue to say that this is why good writing is so important in mathematics. The spoken word is transitory—it briefly passes through the lecture hall or the class room—while the text remains and can be read again and again. That is, in case there is something to read on the pages.

Kline laments, as many other authors do, the poor writing in mathematics textbooks. And often, even when the writing is not poor, it is brief. A mathematics textbook often consists mainly of formulas, figures, examples, and exercises, with just short segments of explanatory text in between. These explanations are submerged in all the rest. They do not stand out, and since they are often tersely written, they are not easy to understand. The connection between the text and the formulas is weak. It is as if the formulas, in the imagination of the author, in some magical way, speak for themselves—which they don't.[37]

[36] *Konsten att räkna*, Anders Bengtsson and Mats Desaix.

[37] One of the references I've lost is to a recent Swedish PhD dissertation in pedagogy which studies (among other things) how students jump between the formulas in mathe-

This is not to say that there aren't any well-written mathematics books. There are many popular books about mathematics that do a good job. There are also quite a few books written for liberal arts mathematics courses, for instance [62, 30, 16]. And an Internet search brings up many more, as well as several links to liberal arts mathematics courses. What is lacking, I believe, are well-written and readable textbooks for the standard university courses in algebra, linear algebra and calculus.[38]

There is also a problem with the expectations of the students. They are used to mathematics books where you skip the text entirely, often also the formulas, and you go directly to the examples and the exercises.[39] Why don't we assign a pack of books, as is common in social science and the humanities, at least a textbook, a popular book, and a liberal arts book? Doing that, we would have to teach that way and and we would be forced to examine the course that way, too.

Let me end this section by mentioning something I've read and heard a few times lately, the fear that school mathematics may go the way of Latin, that is, disappear. I just read one take on this in a *New York Times* article [20] that preceded the Hacker article, but discussed the same issues, arguing for applied mathematics in schools:

> Traditionalists will object that the standard curriculum [algebra - my insertion] teaches valuable abstract reasoning, even if the specific skills acquired are not immediately useful in later life. A generation ago, traditionalists were also arguing that studying Latin, though it had no practical application, helped students develop unique linguistic skills. We believe that studying applied math, like learning living languages, provides both usable knowledge and abstract skills.

Indeed! Study mathematics as a *living language* with a culture—like French!

matics books, not reading the text.

[38]By writing this I may be proven wrong.

[39]There is also an art to reading mathematics texts. At Colby College I got a reference to Simonson and Gouvêa's *How to Read Mathematics*, currently available at http://www.people.vcu.edu/~dcranston/490/handouts/math-read.html, last accessed on January 7, 2014.

7. Mathematical Reality

What is mathematics about? Is there a mathematical reality or a reality to mathematical concepts, and if so, then in that case, where is that reality located? Such questions, I presume, are seldom discussed in the classroom. I think they should. Integrating them into courses is one of my ideas about how to humanize mathematics education. But wouldn't that be a complete waste of time? Not if it opens new roads into an understanding of the esoteric formalism and language that mathematics uses to capture this very reality, whether it exists or not.

Isn't it very strange that a subject that deals with abstract objects seldom discusses with its students what these objects are, or where they happen to be? Is it a wonder that most people have problems with mathematics? Who wouldn't have problems? Wouldn't you have problems understanding a subject if you didn't know what the objects of the subject are or where they reside?

There is an old quip from Bertrand Russell:

> Mathematics may be defined as the subject where we never know what we are talking about, nor whether what we are saying is true.

This was written in 1901 [49] in the context of Russell's grand attempt at reducing all of mathematics to logic [48]. Perhaps it also captures, unintentionally, how many students view mathematics.

One common objection to this line of argument is: Students don't want it, they only want to know how to solve the problems. But how would they know what to want if they've been taught mathematics for years as trying to solve problems that make little sense except as routine exercises? It is we as teachers who give them the problems to solve, and we can give them other problems. No, I don't buy that objection.

———o———0———o———

Sometimes answers to questions about mathematical reality collapse down to a duality. Either mathematics exists "out there" and is discovered by the mind, or it's all a mental construction, and is consequently invented by the mind. However, there are many nuances and the question has a similarity to the old philosophical question from the Middle Ages about the existence of universal concepts. I got this notion from Lars Mouwitz's PhD dissertation [42]. Four distinct directions of thought crystallized (in my interpretation).

Universalia ante res: Universal concepts come first. This is the concept realism of Plato. Concepts are not invented, they exist before reality, and real things are copies of the concepts. Mathematical concepts are real (in this sense) and are discovered by the mind. This is Mathematical Platonism. It is sometimes jokingly said that most mathematicians are Platonists on weekdays due to the very strong feeling they have that they are working with real, existing objects. A critique against this view is based on the obvious problem of locating where the concepts actually reside.

Universalia in res: Universal concepts reside in real "things". This is Aristotle. Concepts exist in reality and are extracted by the mind—not invented, but discovered. They are built into things. Knowledge is empirical and abstracted. Mathematics becomes the language of nature.

Universalia post res: Universal concepts come after real "things". This sounds like a more modern view. There are no concepts where there are no minds. Concepts are invented by the mind based on empirical studies, but they don't exist in physical reality itself. Concepts are cultural phenomena, propagating through society (space) and history (time). An analogy would be the *memes* of Dawkins. In pedagogy and philosophy, this is constructivism. Concepts are created by the mind in the mind. A question is: How can concepts be private, yet commonly shared, correct, and useful? An answer could be: by social and cultural processes and communication. And our concept-forming minds are parts of reality, too. Mathematical concepts are social constructions describing phenomena in reality.

Nominalism: There are no concepts, only the things themselves. Concepts are just names. In mathematics this corresponds to formalism. Still jokingly, when the weekday Platonism of mathematicians is challenged, they resort to Sunday formalism. Mathematics is about nothing; it is just a play with symbols. Wittgenstein held the view that there is nothing beyond the signs and symbols; it's all language-games. Perhaps many students end up here. The mathematical formalism doesn't mean

> anything. The symbols are disconnected from reality. It becomes a meaningless and largely incomprehensible game.

To me, the first and last positions are too extreme. Platonism may be a beautiful idea, but is it scientifically plausible? Nominalism is too poor. A problem with full constructivism (mathematical concepts as social constructions) is the Wigner problem: How can it be that a human construction such as mathematics so closely models the behavior of physical reality [63]? There is no denying, I think, that mathematical principles seem to be built into physical reality. So some kind of compromise between the second and third viewpoint may be the most viable.

In his book *What is Mathematics, Really?* [24], Reuben Hersh argues for mathematics being a social construction, a part of human culture. If mathematics is a shared social construction, then that could account for the feeling that mathematics exists "out there somewhere," somewhere external to the individual mind, and it is somehow discovered. Yet it is invented by the minds.

This is an idea with precursors. There is an interesting article by Leslie White [61], in *The World of Mathematics* [39], that forcibly argues that mathematics is a cultural phenomenon. White, who was an American anthropologist, addresses the question of where the mathematical concepts and truths reside. Do they belong to the external physical world or are they human mental constructions? His text is interesting in many ways. He refers back to earlier discussions about this issue, and he is eloquent about his own point of view, that mathematics is purely a cultural phenomenon.

Much of what White writes makes perfect sense, but he carries the argument too far. In reducing all of mathematics to a purely cultural phenomenon, completely ruling out the role of the individual mathematician, and of physical reality, his concept of culture takes on a metaphysical character in itself.

Another way to phrase the question is to ask whether mathematics is discovered or invented. White reviews how mathematicians have the feeling that they are discovering something which is external to themselves, citing G.H. Hardy as an example (among others). On the other hand, it seems just as clear that mathematical concepts are human inventions. So the answer would be that mathematics is both discovered and invented. White clearly renounces any notion of a Platonic abstract realm where mathematics resides.

White's answer to the dichotomy of discovery versus invention is to claim that mathematics is a cultural phenomenon. This sounds reasonable, but he does it in an anthropological framework which I think goes too far. When thinking of mathematics as a mental phenomenon, there are of course two senses to the concept of "mental." Mental concepts in the individual human being and shared mental concepts of the species.

Here is how White starts (on page 2350):

> What we propose to do is to present the phenomenon of mathematical behavior in such a way as to make clear, on the one hand, why the belief in the independent existence of mathematical truths has seemed so plausible and convincing for so many centuries, and, on the other, to show that all of mathematics is nothing more than a particular kind of primate behavior.

Clearly it is too simple-minded to consider "external physical reality" and "internal mental reality" as the only possibilities for where mathematics could reside. The human culture is another possibility. White writes (still on page 2350):

> Mathematical truths exist in the cultural tradition into which the individual is born, and so enter his mind from the outside. But apart from cultural tradition, mathematical concepts have neither existence nor meaning, and of course, cultural tradition has no existence apart from the human species. Mathematical realities thus have an existence independent of the individual mind, but are wholly dependent upon the mind of the species.

This makes sense to me, but then he continues (on page 2351) with:

> Or, to put the matter in anthropological terminology: mathematics in its entirety, its 'truths' and its 'realities', is a part of human culture, nothing more.

It's the "nothing more" part I don't agree with. These quotes are from the beginning of the text. White then goes on to argue his case, and much of it makes sense, but as said, he takes the argument too far. Even physical theories like Maxwell's Electrodynamics and Einstein's General Relativity become purely cultural in White's view. Certainly, the detailed formulation of the theories of fundamental physics is dependent upon culture, but there is a basis in physical reality, independent of human culture, or any culture anywhere. But to White, culture is a *super-human* entity that could only be explained in terms of itself. The role of the individual is reduced to null. Discoveries are not made by individuals; they are something that happen to them. In this way, culture becomes metaphysical.

Reuben Hersh argues for mathematics being a cultural phenomenon, too. The argument is popularized in his book *What is Mathematics, Really?* [24]. Another reference is his article "Some Proposals for Reviving the Philosophy of Mathematics" in [59]. Hersh does not take the argument as far as White.

The arguments for mathematical concepts residing in our common human culture are convincing, but do they explain everything? We still have Wigner's problem, the problem of the *the unreasonable effectiveness of mathematics in the natural sciences* [63]. I cannot escape the feeling that there must be something in physical reality that serves as a basis for mathematics.

A perspective very different from the cultural basis view is put forward by Roger Penrose in his *The Road to Reality* [44]. Penrose is a Mathematical Platonist and his discussion on the interactions between three worlds—the Platonic mathematical world, the Physical world, and the Mental world—in the first chapter is very intriguing. Where is culture in that picture?

These questions remind me of Gottlob Frege's struggle in *The Foundations of Arithmetic* [18]. Frege wanted to define numbers independently of individual mental states so that mathematics would not become a part of psychology [21].

How about a compromise? Perhaps mathematics is both discovered *and* invented. The concepts are invented whereas the truths are discovered.

Discussions like these are closely related to issues about the foundations of mathematics and the related debates of more than a hundred years ago. The history of the major philosophical strands of logicism, formalism, and intuitionism is nicely summarized in Reuben Hersh's article in [25]. Another reference is Morris Kline's *Mathematics: The Loss of Certainty* [32], a book that would work well in a liberal arts-inspired mathematics course.

Discoveries in analysis such as, for instance, the existence of continuous but nowhere differentiable functions, and various problems with Fourier series showed that the geometric intuition underlying infinitesimal calculus was insufficient. This lead to the arithmetization of analysis and eventually to

Cantor's set theory. Then came Russell's paradox and the breakdown of Frege's system of logic and mathematics. The classical foundational programs of logicism, formalism, and intuitionism were all attempts to resurrect the certainty of mathematical knowledge. They all failed. Hersh writes that mathematics has no foundations and needs no foundations. This is a point of view that I think is quite controversial.

Certainty of knowledge has been a preoccupation of philosophers of all time, in particular since Descartes. Today, this preoccupation seems antiquated. Of course, our scientific knowledge of fundamental physical reality, for instance, is certain to a very high degree, but no one claims it to be 100% certain or even hopes for it ever to be. The modern scientist can live with uncertainty. Indeed, if you can't stand living with uncertainty, then you're no scientist at heart. Mathematics is likely to be even more certain than fundamental physics. But isn't it more interesting if there is an epsilon risk of error rather than zero risk? And historically, no paradox or inconsistency ever discovered has been able to destroy mathematics. The only consequence of Russell's paradox is: Don't deal with such silly ideas. Isn't it obvious from the very beginning that the idea of the set of all sets is ill conceived?

Anyway, my real question is not what is the best philosophy of mathematics, but instead whether these kinds of discussions can help in the teaching and learning of mathematics. In my view, they could infuse life into a subject that for many students seems very dry.

8. Concluding Remarks

To conclude, let me return to where I started, with the invisibility of mathematics as described in Lynn Steen's article [54]. Though written in 1985 from a distinctly American perspective, its contents are still relevant today. The main argument is that most of modern mathematics and its applications in technology and society are largely unknown to the general public. Living mathematics is invisible, forming an invisible culture. Known mathematics is old mathematics. In one drastic phrase (on page 6), Steen writes:

> In contrast [to the situation regarding science and technology], public vocabulary concerning mathematics is quite primitive: it

> is not a decade, not a century, but a millennium out of date.[40]

The third time I read the article, I had had the discussion with one of my colleagues referred to above about mathematics perceived as magic. It then struck me that Steen uses analogous imagery when describing the status of mathematics education. It is *wizardry*. One comment I sometimes get when discussing alternatives to the way we teach mathematics today is that students don't want that, they are not interested in why it works, they just want to know how to use it. I'm not saying that's not generally true. I'm saying that we as teachers need not succumb to such a naive view. I find support in Steen's writing:

> Yet, to be honest, this is the only level [cultural literacy as opposed to practical and civic literacy[41]] on which the arcane research of twentieth century mathematics can truly be appreciated—as an invaluable and profound contribution to the heritage of human culture.

A humanistic, liberal arts-inspired approach to mathematics may indeed be what is needed at the university level.

The paradoxical invisibility of mathematics in society, which I discussed earlier in Sections 2 and 6, leads to the awkward question: Why teach advanced mathematics at all? Besides the relatively few who study to become engineers or scientists or mathematicians, for a vast majority of people, advanced mathematics is superfluous.

I've lived with this question since I started to teach at the University of Borås. I don't think I thought about it before that. I remember one colleague saying that he never used the mathematics he learned at engineering school. I have other colleagues claiming that what little mathematics is needed in their applied courses can be picked up there, implying that mathematics courses are really unnecessary. I don't know how common sentiments like these are. This is certainly anecdotal evidence.

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[40] The text continues with "Explaining what is actually happening in contemporary mathematical science to the average layman is like explaining artificial satellites to a citizen of the Roman Empire who believed that the earth was flat."

[41] Here Steen is referring to Benjamin Shen's three aspects of literacy in science. I haven't been able to find a reference.

But there must be something wrong here. My institute has just recently gone through an evaluation process, which partly entailed writing descriptions of our educational programs. To me it seems that there is quite a lot of mathematics in at least some of the applied courses. I also had the experience about a year ago of working through a course in polymer chemistry to make the mathematics more comprehensible. It was clear that there was mathematics throughout the course, one example being the differential equations for reactions. Another example was the need for partial integration to calculate the average length of polymer chains.

Can it be that teachers in applied courses don't recognize the mathematics of their own education in the applied courses they teach? Is the mathematics of the applied courses so strongly tied to the application that we have a kind of backwards transfer problem where the teachers themselves don't understand where their knowledge comes from?

And this leads to the need for more communication between teachers in applied courses and the mathematics teachers. It's all about language, reality, and communication. Humanism, that is.

Acknowledgments

Classes & Conversations

During my college visits I had a lot of interesting conversations and had the opportunity to sit in on many classes. Here is a list. I learned a lot and my findings are scattered throughout the text above. Thanks to everyone!

Beloit College. Classes: *Discrete Structures* with Paul Campbell, *Calculus* with Ranjit Roy, *Calculus* with David Ellis, *Calculus* with Tatiana Dimitreava. The calculus classes were parallel sections, and they were very different.

Conversations: Paul Campbell, Bruce Atwood, Ranjit Roy, and David Ellis.

Carleton College. Classes: *Calculus with Problem Solving* with Deanna Haunsperger, *Real Analysis* with Gail Nelson, *Combinatorics* with Eric Egge, *Calculus* with Sam Patterson.

Conversations: Deanna Haunsperger, Stephen Kennedy, Andrew Gainer-Dewar, and Brian Shea.

Macalester College. Classes: *Number Theory* with David Bressoud, *Applied Calculus* with Chad Topaz, *Statistical Modelling* with Daniel Kaplan, *Math and Politics* with Karen Saxe, *Combinatorial Games* with Andrew Beverage.

Conversations: Karen Saxe and David Bressoud.

St. Olaf College. Conversations: Paul Zorn and Ted Vessey.

Oberlin College. Classes: *Vector Calculus* with Susan Colley, *Topics in Contemporary Mathematics* with Michael Raney, *Foundations of Analysis* with Michael Henle, *Calculus* with Michael Raney.

Conversations: Susan Colley, Michael Raney, Jim Walsh, Kevin Woods, and Robert Young.

Bryn Mawr College. Conversations: Paul Melvin, Thomas Hunter (Swarthmore College), and Joshua Sabloff (Haverford College).

Skidmore College. Classes: First Year Seminar with Mark Huibregste.

Bennington College. Classes: *Linear Algebra* with Andrew McIntyre, *Introduction to Fundamental Mathematics* with Andrew McIntyre, *Dynamical Systems* with Michael Reardon.

Conversations: Andrew McIntyre and Michael Reardon.

Colby College. Classes: *Topics in Real Analysis* with Fernando Gouvêa, *Introduction to Topics in Abstract Mathematics* with Scott Lambert.

Conversations: Richard Fuller, Scott Taylor, Fernando Gouvêa, Ben Mathes, Otto Bretscher, Justin Sukiennik, Andreas Malmendier, Leo Livshits, Jan Holly, and Scott Lambert.

Bates College. Classes: *Multivariable Calculus* with Benjamin Weiss.

Conversations: Bonnie Shulman, Meredith Greer, Catherine Buell, Pallavi Jayawant, and Peter Wong.

Amherst College. Classes: *Calculus* with Rob Benedetto.

Conversations: Robert Benedetto and David Cox.

Wellesley College. Classes: *Analysis* with Stanley Chang, *Abstract Algebra* with Andrew Schultz.

Conversations: Stanley Chang, Alexander Diesl, Andrew Schultz, Ismar Volić, Karen Lange, and Jonathan Tannenhauser

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References

[1] Anders Bengtsson, Jim Arlebrink, and Mats Desaix, Försöksprojekt i matematik vid ingenjörshögskolan i Borås, *Rapport från Centrum för lärande och undervisning*, Nr 3:2002, 2002.

[2] Anders Bengtsson, *How to teach engineering school mathematics humanistically,* in preparation.

[3] Gunnar Berg (editor), *Det matematiska kulturarvet*, Number 71-72, Dialoger 2005.

[4] David M. Bressoud, "The crisis of calculus," *Launchings*, April 2007, available at http://www,maa,org/external_archive/columns/launchings/launchings_04_07,html, accessed on January 7, 2014.

[5] David M. Bressoud, "Teaching and learning for transference," *Launchings*, September 2012, available at http://launchings.blogspot.com/2012/08/teaching-and-learning-for-transference.html, accessed on January 8, 2014.

[6] Stephen I. Brown, "Towards humanistic mathematics education," in *International Handbook of Mathematics Education*, Alan J. Bishop *et al*, eds., Kluwer Academic Publishers, Dordrecht / Boston, 1996, pages 1289–1331. Article is also available at http://mumnet.easyquestion.net/sibrown/sib003.htm, accessed on April 22, 2013.

[7] Anders Burman and Patrik Mehrens (editors), *Det goda lärandet*, Studentlitteratur, Lund, 2011.

[8] Leone Burton, "Mathematical thinking: The struggle for meaning," *Journal for Research in Mathematics Education*, **15**(1):35–49, 1984.

[9] Paul J. Campbell, "Calculus is crap," *Journal of Undergraduate Mathematics and its Applications*, **27**(4):415–430, 2006.

[10] Paul J. Campbell, "Where's the beef?" *Journal of Undergraduate Mathematics and its Applications*, **32**(4):277–278, 2011.

[11] Susan J. Colley, "What is mathematics and why won't it go away?" *PRIMUS*, **21**(3):211–224, 2011.

[12] Martin Davis, *The Engines of Logic*, W, W, Norton & Company, New York, London, 2000.

[13] Ronald G. Douglas (editor), *Toward a Lean and Lively Calculus: Report of the Conference/Workshop to Develop Curriculum and Teaching Methods for Calculus at the College Level,* Mathematical Association of America, 1986, MAA Notes Number 6.

[14] Underwood Dudley, "Calculus isn't crap," *Journal of Undergraduate Mathematics and its Applications*, **29**(1):1–4, 2008.

[15] Underwood Dudley, "What is mathematics for?" *Notices of the AMS*, **57**(5):608–613, 2010.

[16] Martin J. Erickson, Donald Bindner, and Joe Hemmeter, *Mathematics for the liberal arts*, Wiley, 2013.

[17] Erik Eriksson (översättning och omarbetning) Vilhelm Jörgensen, *Tekno's Stora räkneboken*, Teknografiska institutet, Stockholm, 1956.

[18] Gottlob Frege, *Die Grudlagen der Arithmetik (The Founadtions of Arithmetic)*, Verlag vo Wilhelm Koebner (Basil Blackwell, Oxford), 1884 (1968).

[19] Susan L. Ganter, *An Evaluation of Calculus Reform: A Preliminary Report of a National Study*, Volume **49** of [35], 1999.

[20] Sol Garfunkel and David Mumford, "How to fix our math education," *New York Times*, August 24, 2011, available at http://www.nytimes.com/2011/08/25/opinion/how-to-fix-our-math-education.html?_r=0, accessed January 8, 2014.

[21] D, A. Gillies, *Frege, Dedekind, and Peano on the Foundations of Arithmetic?* Van Gorcum, Assen, The Netherlands, 1982.

[22] Andrew Hacker, "Is algebra necessary," *New York Times*, July 28, 2012.

[23] Andrew F. Heckler, Jennifer A. Kaminski, and Vladimir M. Sloutsky, "The advantage of abstract examples in learning math," *Science*, 25 April 2008: 454–455, 2008.

[24] Reuben Hersh, *What is Mathematics, Really?* Oxford University Press, Oxford UK, 1997.

[25] Reuben Hersh, "Some proposals for reviving the philosophy of mathematics," In Tymoczko [59], pages 9–28.

[26] Lancelot Hogben, *Matematik för millioner (Mathematics For Millions)*, Natur och Kultur, Helsingfors, 1938.

[27] Peter Johnson, "Does algebraic reasoning enhance reasoning in general? A response to Dudley," *Notices of the AMS*, **59**(9):1270–1271, 2012.

[28] Christer Kiselman, Matematikens två språk, *U,U,D,M, Report 2005:25*, 2005, Department of Mathematics, Uppsala University.

[29] David Klein and Jerry Rosen, "Calculus reform—for the $millions," *Notices of the AMS*, **44**(10):1324–1325, 1997.

[30] Morris Kline, *Mathematics for liberal arts*, Addison-Wesley, Reading MA, 1967, Republished as *Mathematics for the Nonmathematician*, Dover Publications, 1985.

[31] Morris Kline, *Why the Professor Can't Teach*, St, Martins Press, New York, 1977.

[32] Morris Kline, *Mathematics, The Loss of Certainty*, Oxford University Press, New York, 1980.

[33] Håkan Lennerstad, Matematikens dubbelnatur - undflyende innehåll, självtillräckligt språk, In *Bildning och demokrati i matenatikutbildning*, volume 14 of *Utbildning och demokrati*, pages 27–55, 2005.

[34] Ference Marton (editor), *Fackdidaktik, Vol, 3, Matematik, naturorienterande ämnen*, Studentlitteratur, Lund, 1986.

[35] Mathematical Association of America, *Assessment Practices in Undergraduate Mathematics*, Volume 49, MAA, 1999.

[36] Patrik Mehrens, *Learning from Liberal Arts Education: Ideas for the Improvement of Undergraduate Education in Sweden*, UPI Pedagogisk utveckling, Uppsala University, Uppsala, 2006.

[37] Merriam-Webster's Collegiate Dictionary, 11th edition, Merriam-Webster Incorporated, Springfield MA, 2003.

[38] Vilma Mesa, David M. Bressoud, Marilyn P. Carlson and Chris Rasmussen, "Description of and selected results from the MAA national study of calculus," submitted, version of September 18, 2012, available at http://www.maa.org/sites/default/files/pdf/cspcc/CSPCC4IJMEST-12-09-18.pdf, accessed on January 8, 2014.

[39] James R. Newman, editor, *The World of Mathematics*, Simon and Schuster, New York, 1956.

[40] L.C. Moore and David A. Smith, "Review of: Toward a lean and lively calculus," *The College Mathematics Journal*, **18**(5):439–442, 1987.

[41] Charles Morris, *Foundations of the Theory of Signs*, University of Chicago Press, Chicago, 1938.

[42] Lars Mouwitz, *Matematik och bildning*, Dialoger 2006, PhD dissertation.

[43] David Mumford, "Calculus reform—for the millions," *Notices of the AMS*, **44**(5):559–563, 1997.

[44] Roger Penrose, *The Road to Reality*, Jonathan Cape, London, 2004.

[45] Karl Popper, *The Open Society and its Enemies*, volume 2, Routledge & Kegan Paul Ltd, London, 1945.

[46] A. Wayne Roberts (editor), *Calculus, The Dynamics of Change*, Mathematical Association of America, 1994, MAA Notes Number 39.

[47] Sheldon Rothblatt, *The Living Arts, Comparative and Historical Reflections on Liberal Education*, The Academy in Transition, Association of American Colleges and Universities, Washington DC, 2003.

[48] Bertrand Russell, *The Principles of Mathematics*, Allen & Unwin, London, 1903, 1937 (2:nd edition).

[49] Bertrand Russell, "Mathematicians and the metaphysicians," In Newman [39], pages 1576–1590.

[50] Bonnie Shulman, "What if we change our axioms? A feminist inquiry into the foundations of mathematics," *Configurations*, **3**:427–451, 1996, The John Hopkins University Press and the Society for Literature and Science.

[51] Joel Silverberg, *Does Calculus Reform Work?* in [35], 1999.

[52] David A. Smith, "Renewal in collegiate mathematics education," *Documenta Mathematica*, Extra Volume **ICM:777–786**, 1998, available at `http://www.math.duke.edu/~das/essays/renewal/index.html`, accessed on January 8, 2014.

[53] Oswald Spengler, "Meaning of numbers," in Newman [39], pages 2315–2347.

[54] Lynn A. Steen, "Mathematics: Our invisible culture," 1985, available at `http://www.stolaf.edu/people/steen/Papers/85invisible.pdf`, accessed on January 8, 2014.

[55] Lynn A. Steen, "Reflections on mathematics and democracy," August 2012, available at http://www.maa.org/publications/periodicals/maa-focus/reflections-mathematics-and-democracy-0, accessed on January 8, 2014.

[56] Keith Stroyan, "Calculus and DNA," *Journal of Undergraduate Mathematics and its Applications*, **32**(2):93–98, 2011.

[57] Max Tegmark, "The mathematical universe," *Foundations of Physics*, Volume **38** (2008), pages 101–150, also available as `arXiv:0704.0646v2`.

[58] Alan C. Tucker and James R. C. Leitzel, *Assessing Calculus Reform Efforts*, Mathematical Association of America, 1994.

[59] Tomas Tymoczko, editor, *New Directions in the Philosophy of Mathematics*, Princeton University Press, Princeton NJ, 1998.

[60] Jacob Vigdor, "Solving America's math problem," *Education Next*, Volume **13** Issue 1 (Winter 2013), available at http://educationnext.org/solving-america%E2%80%99s-math-problem/#, accessed on January 8, 2014.

[61] Leslie A. White, "The locus of mathematical reality: An anthropological footnote," In Newman [39], pages 2348–2364.

[62] Alfred North Whitehead, *An Introduction to Mathematics*, Oxford University Press, London, 1911, republished 1990.

[63] Eugene Wigner, "The unreasonable effectiveness of mathematics in the natural sciences," *Comm, Pure Appl, Math*, **13**:1, 1960.