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Melvin Henriksen
Harvey Mudd College

Ralph Kopperman
City College of New York

Frank A. Smith
Kent State University

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Ordered products of topological groups

BY M. HENRIKSEN
Harvey Mudd College, Claremont, CA 91711, *U.S.A.*
R. KOPPERMAN
City College of New York, New York, NY 10031, *U.S.A.*
AND F. A. SMITH
Kent State University, Kent, OH 44242, *U.S.A.*

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1. *Introduction*

The topology most often used on a totally ordered group $(G, <)$ is the interval topology. There are usually many ways to totally order $G \times G$ (e.g., the lexicographic order) but the interval topology induced by such a total order is rarely used since the product topology has obvious advantages. Let $\mathbb{R}(+)$ denote the real line with its usual order and $Q(+)$ the subgroup of rational numbers. There is an order on $Q \times Q$ whose associated interval topology is the product topology, but no such order on $\mathbb{R} \times \mathbb{R}$ can be found. In this paper we characterize those pairs G, H of totally ordered groups such that there is a total order on $G \times H$ for which the interval topology is the product topology.

Throughout $(G, <_G)$ will denote a group G with identity element e that is totally ordered by a relation $<_G$ (abbreviated by $<$ whenever the group G is clear from the context) compatible with the multiplication of G. More precisely, if we let $P(G) = \{g \in G\colon e <_G g\}$, we require that

(α) $P(G)\,P(G) \subseteq P(G)$
(β) $P(G) \cap P(G)^{-1} = \varnothing$
(γ) $P(G)\,g = gP(G)$ for each $g \in G$, and
(δ) $P(G) \cup P(G)^{-1} \cup \{e\} = G$.

Also $a <_G b$ if and only if $a^{-1}b$ or ab^{-1} is in $P(G)$. Any such order $<_G$ is called a *group order* on G. If a subset P of G satisfies (α), (β), (γ) and (δ), and we let $a <_G b$ mean $a^{-1}b \in P$, then $<_G$ is a group order on G for which $P = P(G)$. See [**3**] or [**1**], where the above is formulated in terms of $G^+ = P(G) \cup \{e\}$.

Suppose $<_G$ is a group order on G and $<_H$ is a group order on H. A group order $<$ on $G \times H$ such that $(e, e) < (a, e)$ if and only if $e <_G a$ and $(e, e) < (e, b)$ if and only if $e <_H b$ is said to *extend the orders of G and H*. Note that if $<$ extends the orders on G and H, then $P(G) \times P(H) \subseteq P(G \times H)$. If $<_G$ is a group order on G, then the collection of all open intervals $\langle g_1, g_2 \rangle$ of G where $g_1 < g_2$ are in G, forms a base for a topology $\tau(<_G) = \tau(G)$, called the *interval topology* on G. Note that the symmetric open intervals $\{\langle g^{-1}, g \rangle : e < g\}$, form a base of neighbourhoods of e, and that the map $(a, b) \to ab^{-1}$ on $G \times G$ to G is continuous, whence $(G, \tau(<_G))$ is a topological group. See [**7**], chapter VII.

If the group order $<$ on $G \times H$ extends the orders of G and H, and if the restriction of the interval topology $\tau(<)$ on $G \times H$ to $G \times \{e\}$ is homeomorphic to the topology $\tau(<_G)$ under the map $(g, e) \to g$, we say that $<$ is *topologically compatible with the order of G*. Topological compatibility with the order of H is defined similarly.

In this paper, we determine precisely when the product of two totally ordered groups admits an order topologically compatible with each of the factors.

A totally ordered group $(G, <)$ is said to be *densely ordered* if $g_1 < g_2$ in G implies there is a $g_3 \in G$ such that $g_1 < g_3 < g_2$. Since $g_1 < g_2$ if and only if $e < g_1^{-1} g_2$, it is clear that $(G, <)$ is densely ordered if and only if $P(G)$ has no least element. If $(G, <)$ is not densely ordered, it is said to be *discretely ordered*. It is easy to show that $\tau(G)$ is the discrete topology if and only if the order on G is discrete.

It turns out that if either G or H is discretely ordered, then $G \times H$ admits an order topologically compatible with the orders of G and H. If the orderings on G and H are dense and archimedean, then we may identify G and H with subgroups of the additive group $\mathbb{R}(+)$ of real numbers. We show below that under these hypotheses, $G \times H$ admits an order topologically compatible with the orders of G and H if and only if not every real number is of the form g/h, where $g \in G$ and $0 \neq h \in H$. We use this latter result to characterize, more generally, those densely ordered groups G, H for which $G \times H$ admits an order topologically compatible with the orders of G and H, but this result is too complicated to state at this point; see Section 4.

2. *Preliminary results and the hiding maps*

If every element of the set A is also in the set B, we write $A \subseteq B$, and if the inclusion is proper we write $A \subset B$.

The *lexicographic order* on $G \times H$ *with G dominating* is the order $<$ such that $(g_1, h_1) < (g_2, h_2)$ if $g_1 <_G g_2$ or $g_1 = g_2$ and $h_1 <_H h_2$. The *lexicographic order* on $G \times H$ *with H dominating* is defined similarly. Note that each of these orders extends the orders on G and H.

2·1. PROPOSITION. *If $(G, <_G)$ and $(H, <_H)$ are totally ordered groups, one of which is discretely ordered, then $G \times H$ admits an order $<$ topologically compatible with the orders of G and H.*

Proof. Suppose $<$ is the lexicographic order on $G \times H$ with G dominating, where $<_G$ is discrete. If l is the least element of $P(G)$, then (e, e) is the only element of the open interval $\langle (l^{-1}, e), (l, e) \rangle$ of $(G \times \{e\}, <)$. So $<$ induces the discrete topology on $G \times \{e\}$. Since $\langle l^{-1}, l \rangle = \{e\}$, $<_G$ also induces the discrete topology on G. Thus $\tau(<_G)$ is homeomorphic to $\tau(<)$ restricted to $G \times \{e\}$. Since $\{e\} \times H$ is a convex subgroup of $G \times H$, the order $<'$ obtained by restricting $<$ to $\{e\} \times H$ is such that $(\{e\} \times H, <')$ and $(H, <_H)$ are order isomorphic. Hence $<$ is topologically compatible with the orders of G and H. In the case when, instead, $<_H$ is discrete, the proof is similar.

Dense orders on groups are characterized as follows.

$$(G, <) \text{ is densely ordered if and only if } [P(G)]^2 = P(G). \tag{1}$$

To see this, assume first that $P^2 = P$ and $g \in P$. Then $g = pq$ for some $p, q \in P$. Since $e < p$, we have $e < q < pq = g$, so P has no least element and $<$ is a dense order on G. Conversely, if P has no least element and $g \in P$, there is an $f \in P$ such that $f < g$. Then $e < f^{-1}g$ and $g = f(f^{-1}g) \in P^2$. So $P = P^2$ and (1) holds.

An *upper filter* in a densely ordered group $(G, <)$ is a subset U of G such that $UP = U$. Thus $\varnothing$ and G are always upper filters in G, as is gP for each $g \in G$ by (1). Let $\mathscr{U}(G)$ denote the set of upper filters on G.

It is an exercise to verify

$$g_1 < g_2 \quad \text{if and only if} \quad g_2 \in g_1 P. \tag{2}$$

Hence
$$\text{If} \quad U \leqslant \mathscr{U}(G), g_1 \in U \quad \text{and} \quad g_1 < g_2 \quad \text{then} \quad g_2 \in U. \tag{3}$$

An upper filter of the form gP for some $g \in P$ is called a *principal upper filter*. Since each non-empty $U \in \mathscr{U}(G)$ is the union of principal upper filters,

$$\text{Each} \quad U \in \mathscr{U}(G) \quad \text{is open in the interval topology of} \quad G. \tag{4}$$

If $(G, <)$ is densely ordered, then the set of principal upper filters of G is dense in $\mathscr{U}(G)$ in the following sense.

2·2. Lemma. *Suppose $(G, <)$ is densely ordered, and U and V are distinct elements of $\mathscr{U}(G)$. Then*

(*a*) *$U \subset V$ or $V \subset U$, and*

(*b*) *there is a principal upper filter of G strictly between U and V.*

Proof. (*a*) Either $U \subset V$ or there is a $u \in U \backslash V$. By (3), if the latter holds, we cannot have $u \geqslant v$ for any $v \in V$. So $u < v$ for each $v \in V$, and hence $V \subseteq uP \subseteq UP = U$. Hence $V \subset U$ since $V \neq U$.

(*b*) Suppose $V \subset U$ and $g \in U \backslash V$. Then $gP \subseteq UP = U$ and $g \in U \backslash gP$. Thus $V \subseteq gP \subseteq U$. If $V \neq gP$, we are done; otherwise, since G is densely ordered, $gP \cup \{g\}$ is not open. So there is an $m \in U \backslash (gP \cup \{g\})$. Thus $mP \subseteq U$, $m \in U \backslash mP$, and $m < g$. So $g \in mP$ and hence mP lies properly between V and U since $g \in mP \backslash V$.

Our last lemma showed that $\mathscr{U}(G)$ is totally ordered under set inclusion. Although it is an abuse of notation, we let $(\mathscr{U}(G), <_G)$ denote $\mathscr{U}(G)$ under the ordering defined by letting $U <_G V$ mean $V \subset U$.

2·3. Proposition. *If $(G, <)$ is a totally ordered group, then under the operation of set multiplication $(\mathscr{U}(G), <_G)$ is a totally ordered monoid with identity element eP. Moreover, the map $\alpha: G \to \mathscr{U}(G)$ given by $\alpha(g) = gP$ is an order-preserving monomorphism of G onto a dense subset of $\mathscr{U}(G)$.*

Proof. If U and V are in $\mathscr{U}(G)$, then $(UV)P = U(VP) = UV$, and $(eP)U = PU = UP = U$ by (γ) of Section 1. So $\mathscr{U}(G)$ is a monoid. Clearly $U \subseteq U'$ and $V \subseteq V'$ imply $UV \subseteq U'V'$, whence $\mathscr{U}(G)$ is a totally ordered monoid.

If $g_1 < g_2$ in G, then $g_2 \in g_1 P$ by (2), whence by (1), $g_2 P \subseteq g_1 P^2 = g_1 P$, so α is order preserving. But $g_2 \notin g_2 P$, so $g_2 P \neq g_1 P$ and α is a monomorphism. It is immediate from Lemma 2·2(*b*) that $\alpha[G]$ is dense in $\mathscr{U}(G)$.

The following characterization of topological compatibility is the major tool in solving the problem posed in the introduction.

2·4. Theorem. *Suppose $<$ is a group order on the product $G \times H$ of two densely ordered groups that extends the orders of G and H. Then:*

(*a*) *$\tau(G)$ and $\tau(H)$ are weaker than the order topologies induced on $G \times \{e\}$ and $\{e\} \times H$ by $<$;*

(*b*) *$<$ is topologically compatible with the orders on G and H if and only if, whenever*

$(g,h) \in P(G \times H)$, there are $g^* \in G$ and $h^* \in H$ such that $(e,e) < (g^*,e) < (g,h)$ and $(e,e) < (e,h^*) < (g,h)$;

(c) If $<$ is topologically compatible with the orders of G and H, then G and H are totally disconnected.

Proof. (*a*) If $g_1 < g_2$ in G, then $\langle g_1, g_2\rangle \times \{e\} = \langle (g_1,e),(g_2,e)\rangle \cap (G\times\{e\})$, so $\tau(G)$ is weaker than the order topology induced on $G \times \{e\}$ by $<$. The proof for $\tau(H)$ is similar.

(*b*) Suppose $<$ is topologically compatible with the orders of G and H and $(e,e) < (g,h)$. Since (e,e) is in the open interval $\langle (g^{-1},h^{-1}),(g,h)\rangle$, the topological compatibility implies there are g_1^*, g_2^* in G such that $(e,e) \in \langle g_1^*, g_2^*\rangle \times \{e\} \subset \langle (g^{-1},h^{-1}),(g,h)\rangle$. Thus in particular, $(e,e) < (g_2^*,e) \leqslant (g,h)$. Since $<_G$ is a dense order, there is a $g^* \in G$ such that $e < g^* < g_2^*$. Then $(g^*,e) \in \langle (g^{-1}, h^{-1}), (g,h)\rangle$. Thus $(e,e) < (g^*,e) < (g,h)$. An element $h^* \in H$ such that $(e,e) < (e,h^*) < (g,h)$ can be produced similarly.

Suppose next that whenever $(e,e) < (g,h)$, there are $g^* \in G$, $h^* \in H$ satisfying the inequalities in (*b*), and suppose $(e,e) \in \langle (g_1,h_1),(g_2,h_2)\rangle$. By assumption, there is a $g_2^* \in G$ such that $(e,e) < (g_2^*,e) < (g_2,h_2)$ and a $g_1^* \in G$ such that $(e,e) < (g_1^{*-1},e) < (g_1^{-1},h_1^{-1})$. Thus $(e,e) \in \langle (g_1^*,e),\ (g_2^*,e)\rangle \subset \langle (g_1,h_1),\ (g_2,h_2)\rangle$. So the restriction of $\tau(<)$ to $G \times \{e\}$ is weaker than $\tau(G)$. Similarly, it is weaker than $\tau(H)$. Thus, by (*a*), $<$ is topologically compatible with the orders on G and H.

(*c*) It suffices to show that the component of e in each of G and H is $\{e\}$. If $e <_G g$, then there is an $h \in H$ such that $(e,e) < (e,h) < (g,e)$. Thus $e \in \{k \in G\colon (k,e) < (e,h)\}$ and $g \in \{k \in G\colon (k,e) > (e,h)\}$, so there is a partition of G into disjoint open sets, one containing e and the other g. Thus the complement K of e contains no positive element and it follows that $K = \{e\}$. Similarly, the component of e in H is $\{e\}$.

For any set A, let $\exp A$ denote the family of all subsets of A.

2·5. DEFINITION. *Suppose $(G, <_G)$ and $(H, <_H)$ are densely ordered groups and $<$ is an order on $G \times H$ that extends the orders on G and H. For each $a \in G$, let $\phi(a) = \{h \in H\colon (e,e) < (a,h)\}$. Then $\phi\colon G \to \exp H$ is called the map that hides G from H in* $\exp H$*, or the hiding map.*

If $g, a \in G$, we abbreviate $a^{-1}ga$ by g^a. The terminology 'hiding map' will be justified in part (*c*) of the following lemma.

2·6. LEMMA. *Suppose $<$ is a group order on the product $G \times H$ of two densely ordered groups. Then:*

(a) If $<$ extends the orders of G and H, and $\phi\colon G \to \exp H$ is the hiding map, then for $a, b \in G$ and $h \in H$

(i) *$\phi(a^{-1}) \cup \phi(a)^{-1} = H$ if $a \neq e$, and $\phi(e) \cup \phi(e)^{-1} = H \backslash \{e\}$,*

(ii) *$\phi(a^{-1}) \cap \phi(a)^{-1}$ is empty,*

(iii) *$\phi(a)\phi(b) \subseteq \phi(ab)$,*

(iv) *$\phi(a)^h = \phi(a) = \phi(a^b)$,*

(v) *$\phi(e) = P(H)$,*

(vi) *$a \in P(G)$ implies $e \in \phi(a)$, and*

(vii) *$a <_G b$ implies $\phi(a) \subseteq \phi(b)$;*

(b) If $\phi\colon G \to \exp H$ satisfies (i) *through* (vi), *and we let $P(G \times H) = \{(a,h) \in G \times H\colon h \in \phi(a)\}$, then $P(G \times H)$ defines a group order $<$ on $G \times H$ that extends the orders of G and H;*

(c) *If $<$ is topologically compatible with the orders of G and H, then $\phi(g)$ is never a principal upper filter unless $g = e$. That is, $\phi[G] \cap \alpha[H] = \{P(H)\}$.*

Proof. (a) It is clear from the definition of ϕ that $\phi(e) = P(H)$, so (v) holds. If $a \neq e$, then $(e,e) < (a^{-1},h)$ or $(e,e) < (a^{-1},h)^{-1} = (a,h^{-1})$, so $h \in \phi(a^{-1}) \cup \phi(a)^{-1}$. Since $\phi(e) = P(H)$, $\phi(e) \cup \phi(e)^{-1} = H\backslash\{e\}$, and (i) holds.

If $h \in \phi(a^{-1}) \cap \phi(a)^{-1}$, then $(e,e) < (a^{-1},h)$ and $(e,e) < (a,h^{-1}) = (a^{-1},h)^{-1}$, contrary to (β) of Section 1. So (ii) holds.

If $h \in \phi(a)$ and $j \in \phi(b)$, then $(e,e) < (a,h)$ and $(e,e) < (b,j)$, whence $(e,e) < (a,h)(b,j) = (ab,hj)$. Thus (iii) holds.

To see (iv), note that for $k \in G$, $k \in \phi(a)$ if and only if $(e,e) < (a,k)$ if and only if $(e,e) < (a,k)^{(b,h^{-1})}$. This is (γ) of Section 1. Thus

$$\phi(a) = \phi(a^b)^{h^{-1}}. \tag{5}$$

Letting successively $b = e$ and $h = e$ in (5) yields (iv).

Since the order of $G \times H$ extends the order of H, $\phi(e) = P(H)$, and (vi) restates the assumption that $<$ extends the order of G. So (vi) holds.

If $a <_G b$, then $(a,e) < (b,e)$ since $<$ extends the order of G. So if $h \in \phi(a)$, then $(e,e) < (a,h) = (a,e)(e,h) < (b,e)(e,h) = (b,h)$ whence $h \in \phi(b)$. Thus (vii) holds and the proof of (a) is complete.

(b) To show that $<$ is a group order, we will verify that (α), (β), (γ) and (δ) of Section 1 hold. Suppose (a,h) and (b,k) are in $P(G \times H)$; then $h \in \phi(a)$ and $k \in \phi(b)$, so by (iii), $hk \in \phi(a)\phi(b) \subseteq \phi(ab)$. Thus $(ab,hk) \in P(G \times H)$ and (α) holds.

If $(a,h) \in P(G \times H)$, then $h \in \phi(a)$, so $h^{-1} \in \phi(a)^{-1}$. If also $(a,h) \in P(G \times H)^{-1}$, then $(a^{-1},h^{-1}) \in P(G \times H)$, whence $h^{-1} \in \phi(a^{-1})$ as well as $\phi(a)^{-1}$, contrary to (ii). This contradiction establishes (β).

That (γ) holds follows from (5), and that (i) implies (δ) is an exercise.

By (v), $e <_H h$ if and only if $h \in \phi(e)$ if and only if $(e,e) < (e,h)$. So $<$ extends the order of H. Also, if $e <_G a$, then $e \in \phi(a)$ by (vi), so $(e,e) < (a,e)$. Thus $<$ extends the order of G as well as that of H. This completes the proof of (b).

(c) By (vi) and the definition of α, $\phi(e) = \alpha(e) = eP(H) = P(H)$, so $P(H) \in \phi[G] \cap \alpha[H]$. Clearly h is the greatest lower bound of $\alpha(h) = hP(H)$, while, as will be shown next, $\phi(g)$ fails to have a greatest lower bound if $g \neq e$.

For, if $h \in \phi(g)$, then $(e,e) < (g,h)$, and by Theorem 2·4 (b), there is an $h^* \in H$ such that $(e,e) < (e,h^*) < (g,h)$. Thus $(e,e) < (g,h(h^*)^{-1})$. So $h(h^*)^{-1} \in \phi(g)$, and $h(h^*)^{-1} < h$. Thus h is not a lower bound of $\phi(g)$. If $h \notin \phi(g)$, then, since $g \neq e$, $(e,e) < (g,h)^{-1} = (g^{-1},h^{-1})$. Using Theorem 2·4 (b) again, there is an $h^* \in H$ such that $(e,e) < (e,h^*) < (g^{-1},h^{-1})$. So $(e,e) < (g^{-1},h^{-1}(h^*)^{-1}) = (g,h^*h)^{-1}$. Thus $h^*h \notin \phi(g)$. Also $h < h^*h$, showing h is not a greatest lower bound for $\phi(g)$. Thus no $h \in H$ can be a greatest lower bound for $\phi(g)$.

Hence (c) holds and the proof of the lemma is complete.

This next example illustrates that the hiding map may assume H or $\varnothing$ as values, that not every value need be open, and that $\phi(a)\phi(b)$ need not equal $\phi(ab)$ even if both $\phi(a)$ and $\phi(b)$ are non-empty.

2·7. *Example.* Let $\mathbb{R}_1(+)$ and $\mathbb{R}_2(+)$ denote two copies of the additive group of real numbers with its usual order, and let $<$ denote the lexicographic order of $\mathbb{R}_1 \times \mathbb{R}_2$ with $\mathbb{R}_1$ dominating (clearly $<$ extends the orders of $\mathbb{R}_1$ and $\mathbb{R}_2$). For each $g \in \mathbb{R}_1$, let

$\phi_1(g) = \{h \in \mathbb{R}_2 : (0,0) < (g,h)\}$, so ϕ_1 is the hiding map of $\mathbb{R}_1$ in $\exp \mathbb{R}_2$. Routine calculations show that $\phi_1(g) = \mathbb{R}_2$ if $g > 0$, $\phi_1(0) = P(\mathbb{R}_2)$, and $\phi_1(g) = \varnothing$ if $g < 0$.

Let $\phi_2 : \mathbb{R}_2 \to \exp \mathbb{R}_1$ denote the hiding map of $\mathbb{R}_2$ into $\exp \mathbb{R}_1$, so for each $h \in \mathbb{R}_2$, $\phi_2(h) = \{g \in \mathbb{R}_1 : (0,0) < (g,h)\}$. It is easy to see that $\phi_2(h) = G^+ = P(G) \cup \{0\}$ if $h > 0$, and $\phi_2(h) = P(G)$ if $h \leqslant 0$. In particular, $\phi_2(h)$ fails to be open if $h > 0$. Moreover, $\phi_2(-1) + \phi_2(2) = P(G) + G^+ = P(G) \subset G^+ = \phi_2(1)$.

Much more can be said about the hiding map when the order on $G \times H$ is topologically compatible with the order of each of its factors.

2·8. THEOREM. *If $(G, <_G)$ and $(H, <_H)$ are densely ordered groups, $<$ is an order on $G \times H$ that extends the orders on G and H, and $\phi : G \to \exp H$ is the hiding map, then the following are equivalent:*

(i) *$<$ is topologically compatible with the orders on G and H;*

(ii) *$\phi[G] \subset \mathscr{U}(H)$ and ϕ is continuous with respect to the interval topologies on G and $\mathscr{U}(H)$.*

Moreover, if (ii) *holds, then there is a $g \neq e$ in G such that $\phi(g)$ is a non-empty proper subset of H.*

Proof. Suppose (ii) holds and $(e,e) < (g,h)$. Elements g^* and h^* satisfying the conditions of Theorem 2·4(*b*) will be produced. Since $\phi(g) \in \mathscr{U}(H)$, $h \in \phi(g) = \phi(g)P$, so $h = kh^*$ for some $k \in \phi(g)$ and $h^* \in P$. Hence $(e,e) < (e,h^*) < (g,k)(e,h^*) = (g,h)$. By the continuity of ϕ, since $hP \subset \phi(g)P = \phi(g)$, there is a neighbourhood $\langle g_1, g_2 \rangle$ of g in G such that if $g_1 < g' < g_2$, then $\phi(g') \supset hP$. Thus $q \in \phi(g')$ for some $q \leqslant_H h$. If $q = h$, then $h \in \phi(g')$. If $q <_H h$, then $h = qp$ for some $p \in P$. Hence $h \in qP \subseteq \phi(g')P = \phi(g')$, and we have $h \in \phi(g')$. Since $<_G$ is a dense order, there is a $k \in G$ such that $g_1 <_G k <_G g$. Thus $e <_G gk^{-1} = g^*$; and $(e,e) < (k,h)$ whence $(e,e) < (g^*,e) < (g^*,e)(k,h) = (g,h)$. So, by Theorem 2·4(*b*), $<$ is topologically compatible with the orders of G and H.

In the proof of 2·6(*c*), it was shown that $\phi(g) \in \mathscr{U}(H)$.

To establish the continuity of ϕ, we begin by showing:

$$\text{if} \quad (e,e) \in \langle (g,h_1),(g,h_2) \rangle, \quad \text{then there are} \quad g_1, g_2 \quad \text{in} \quad G \quad \text{such that}$$
$$g \in \langle g_1, g_2 \rangle \quad \text{and if} \quad k \in \langle g_1, g_2 \rangle, \quad \text{then} \quad (e,e) \in \langle (k,h_1),(k,h_2) \rangle. \tag{6}$$

To establish (6), we begin by using Theorem 2·4(*b*) to find $g_1^* < g_2^*$ in G such that $(g,h_1) < (g_1^*,e) < (e,e) < (g_2^*,e) < (g,h_2)$. Let $g_1 = gg_2^{*-1}$ and $g_2 = gg_1^{*-1}$. Since $g_1^* <_G e <_G g_2^*$, $g_1 = gg_2^{*-1} <_G g <_G gg_1^{*-1} = g_2$. If $g_1 <_G k <_G g_2$, then $(k,h_1) < (g_2,h_1) = (g,h_1)(g_2^{*-1},e) < (e,e) < (g,h_2)(g_1^{*-1},e) = (g_1,h_2) < (k,h_2)$, and (6) holds.

Now suppose $\langle U_1, U_2 \rangle$ is a neighbourhood of $\phi(g)$ in $\mathscr{U}(H)$; that is suppose $U_2 \subset \phi(g) \subset U_1$. We wish to find a neighbourhood $\langle g_1, g_2 \rangle$ of g in G such that if $g_1 < k < g_2$, then $\phi(k) \in \langle U_1, U_2 \rangle$. Choose $h_2 \in \phi(g) \backslash U_2$, whence $(e,e) < (g,h_2)$. If $h_1' \in U_1 \backslash \phi(g)$, then $(g,h_1') \leqslant (e,e)$. If $g \neq e$, then $(g,h_1') < (e,e)$. If $g = e$, there is an $r \in U_1 = U_1 P$ such that $r \leqslant_H e$. Then $r = h_1 p$ for some $h_1 \in U$ and $p \in P$, and $(g,h_1) < (e,e) < (g,h_2)$. By (6), there is a neighbourhood $\langle g_1, g_2 \rangle$ of g such that if $g_1 < k < g_2$, then $(k,h_1) < (e,e) < (k,h_2)$; that is, $h_1 \in U_1 \backslash \phi(k)$ and $h_2 \in \phi(k) \backslash U_2$, whence $\phi(k) \in \langle U_1, U_2 \rangle$. Thus ϕ is continuous at g, and the equivalence of (i) and (ii) is established.

If (ii) holds and $h \in P(H)$, then $h^{-1} <_H e <_H h$, whence $hP \subseteq eP = \phi(e) \subseteq h^{-1}P$. Now $e = h^{-1}h \in h^{-1}P$ and $e \notin eP$, so $\phi(e) \neq h^{-1}P$. Also, since $h \in \phi(e)$ and $h \notin hP$, the latter is included properly in $\phi(e)$. Hence $\phi(e) \in \langle h^{-1}P, hP\rangle$, which we call U. Since ϕ is continuous, there is a $k \in G$ such that $V = \langle k^{-1}, k\rangle$ is a neighbourhood of e and $g \in V$ implies $\phi(g) \in U$. Clearly $\phi(g)$ is a non-empty proper subset of H and since $<_G$ is a dense order, we may assume that $g \neq e$. This completes the proof of Theorem 2·8.

3. *Topologically compatible pairs; the Archimedean case*

Recall that a totally ordered group G is said to be *archimedean* if $a \in P(G)$ implies $\{a^n : n = 1, 2, 3, \ldots\}$ has no upper bound.

3·1. Proposition. *If $<$ is an order on the product $G \times H$ of two densely ordered archimedean groups that is topologically compatible with the orders of G and H, and if $\phi: G \to \mathscr{U}(H)$ is the hiding map, then ϕ is a monomorphism of G onto a subgroup of $\mathscr{U}(H)$.*

Proof. By Theorem 2·8, $\phi[G] \subset \mathscr{U}(H)$, and by Lemma 2·6($a$) and Theorem 2·8 again, there is an $a \in G$ such that both $\phi(a)$ and $\phi(a^{-1})$ are non-empty proper subsets of H. Choose $h \in \phi(a^{-1})\,\phi(a)$. It will be shown by induction that

$$\text{if}\quad \phi(a)\quad \text{and}\quad \phi(a^{-1})\quad \text{are non-empty, then for each positive integer } m, \text{ there is a}\quad p \in \phi(a^{-1})\,\phi(a)\quad \text{such that}\quad p^m \leqslant h. \tag{7}$$

Note first that $\phi(a^{-1})\,\phi(a) \subseteq \phi(e) = P(H)$ by Lemma 2·6(a).

If $m = 1$, take $p = h$.

Next assume that (7) holds for the positive integer m; more precisely pick $j \in \phi(a^{-1})$, $k \in \phi(a)$ such that $(jk)^m \leqslant h$. Then $jk \in P(H)$, and by (1), there are $p, q \in P(H)$ such that $jk = pq$. If also $p \leqslant q$ then $p^2 \leqslant jk$. Since H is archimedean, there is a positive integer s such that $jk \leqslant_H p^s$, whence $p^s k^{-1} \geqslant j = \phi(a^{-1})$. Since $\phi(a^{-1}) \in \mathscr{U}(H)$, $p^s k^{-1} \in \phi(a^{-1})$. Now $p^0 k^{-1} = k^{-1} \in \phi(a)^{-1}$, so by Lemma 2·6($a$), $p^0 k^{-1} \notin \phi(a^{-1})$. Hence there is a least positive integer r such that $p^r k^{-1} \in \phi(a^{-1})$. Then $p^{r-1}k^{-1} \in \phi(a)^{-1}$ and $(p^{r-1}k^{-1})^{-1} \in \phi(a)$, so $p = (p^r k^{-1})\,(p^{r-1}k^{-1})^{-1} \in \phi(a^{-1})\,\phi(a)$, and $p^{m+1} \leqslant p^{2m} \leqslant (jk)^m \leqslant h$. If, instead, $q < p$, then $q^2 \leqslant jk$ and a similar argument yields $q \in \phi(a^{-1})\,\phi(a)$ and $q^{m+1} \leqslant h$. Thus (7) holds.

Next, we show that

$$\phi(a^{-1})\,\phi(a) = \phi(e)\quad \text{if}\quad \phi(a)\quad \text{is a non-empty proper subset of}\quad H. \tag{8}$$

By Lemma 2·6(a), $\phi(a^{-1})\,\phi(a) \subseteq \phi(e)$. Suppose $q \in \phi(e) = P(H)$. Since H is archimedean, there is a positive integer t such that $h \leqslant q^t$, and by (7), there is a $p \in \phi(a^{-1})\,\phi(a)$ such that $p^t \leqslant h \leqslant q^t$. Hence $p \leqslant q$. By Theorem 2·8 and Proposition 2·3, $\phi(a^{-1})\,\phi(a) \in \mathscr{U}(H)$, so $q \in \phi(a^{-1})\,\phi(a)$ and (8) holds.

Our next task is to verify

$$\phi(b^{-1})\,\phi(b) = \phi(e) = \phi(b)\,\phi(b^{-1})\quad \text{for any}\quad b \in G. \tag{9}$$

By Theorem 2·8, there is an $a \in P(G)$ that satisfies the hypothesis of (8). If $b \in P(G)$, then since G is archimedean, there is a positive integer n such that $b <_G a^n$. By Lemma 2·6(a), $\phi(b) \subseteq \phi(a^n)$. If $\phi(a^n) = H$, then for any $h \in H$, $(a, h)^n = (a^n, h^n) > (e, e)$, and by ([1], 12·12), $(a, h) > (e, e)$, so $\phi(a) = H$, contrary to the choice of a. Hence $\phi(b)$ is a proper subset of H and is non-empty since it contains $\phi(e) = P(H)$. So $\phi(b^{-1})\,\phi(b) = \phi(e)$ by (8). By Lemma 2·6(a), $\phi(b^{-1})$ is also a non-empty proper subset of H, so (8) may also be used to show that $\phi(b)\,\phi(b^{-1}) = \phi(e)$, and may be used again to show that $\phi(b^{-1})\,\phi(b) = \phi(e) = \phi(b)\,\phi(b^{-1})$ if $b <_G e$; that these latter equalities hold if $b = e$ is the content of (1). Thus (9) holds; and we know that for each $b \in G$ $\phi(b^{-1})$ is the inverse of $\phi(b)$.

Next, suppose $a, b \in G$ are arbitrary. By Lemma 2·6 (a), $\phi(a)\phi(b) \subseteq \phi(ab)$. If this inclusion is proper, and $<$ denotes the order of $\mathscr{U}(H)$, then $\phi(ab) < \phi(a)\phi(b)$, so by (9),

$$\phi(e) < \phi(a)\phi(b)\phi((ab)^{-1}) = \phi(a)\phi(b)\phi(b^{-1}a^{-1}) \leqslant \phi(a)\phi(b)\phi(b^{-1})\phi(a^{-1}) = \phi(e).$$

This contradiction shows that ϕ is a homomorphism. So if we can show

$$\phi(g) = \phi(e) \quad \text{implies} \quad g = e, \tag{10}$$

we may conclude that ϕ is a monomorphism.

If $\phi(g) = \phi(e)$, then since ϕ is a homomorphism, $\phi(e) = \phi(g)\phi(g^{-1}) = \phi(g^{-1})$, so $e \notin \phi(e) = \phi(g)^{-1} \cup \phi(g^{-1})$, contradicting Lemma 2·6 (i) unless $g = e$.

By a well-known theorem of Hölder, every archimedean ordered group is isomorphic to a subgroup of $\mathbb{R}(+)$. If G and H are subgroups of $\mathbb{R}(+)$, let

$$G * H = \{g/h\colon g \in G,\, h \in H\backslash\{0\}\}. \tag{11}$$

3·2. THEOREM. *Suppose G and H are densely ordered subgroups of $\mathbb{R}(+)$. Then $G \times H$ admits an order $<$ topologically compatible with the orders on G and H if and only if $G * H \neq \mathbb{R}$. When $<$ is such an order, $(G \times H, <)$ is archimedean.*

Proof. Suppose first that there is an $a \in \mathbb{R}\backslash G * H$, and let $P = P(G \times H) = \{(g, h) \in G \times H\colon ah <_{\mathbb{R}} g\}$. We will show that P defines a group order on $G \times H$ by verifying that (α), (β), (γ) and (δ) (rewritten in additive notation) of Section 1 hold.

Suppose (g, h) and (g', h') are in P; then $0 <_{\mathbb{R}} (g - ah) + (g' - ah') = a(g + g') - a(h + h')$. So ($\alpha$) holds. If $(g, h) \in P \cap (-P)$, then $ah <_{\mathbb{R}} g$ and $a(-h) <_{\mathbb{R}} (-g)$. Since this cannot hold, (β) follows. The commutativity of $\mathbb{R}(+)$ implies (γ). If $g = ah$, then $g = h = 0$ or $a \in G * H$ by (11). Hence $g <_{\mathbb{R}} ah$ or $ah <_{\mathbb{R}} g$ and (δ) holds. So $P(G \times H)$ defines a group order.

Suppose $(0, 0) < n(g, h) < (x, y)$ for some $g, x \in G$, $h, y \in H$, and $n = 1, 2, \ldots$. Then $0 >_{\mathbb{R}} (ah - g)$ and $(x - ng) >_{\mathbb{R}} a(y - nh)$ or $0 > n(ah - g) >_{\mathbb{R}} ay - x$ whenever n is positive. Since $\mathbb{R}(+)$ is archimedean, this cannot hold, so $<$ is an archimedean order on $G \times H$.

We will show that $<$ is topologically compatible with the orders of G and H by verifying the conditions of Theorem 2·4 (b). If $(0, 0) < (g, h)$, then $r = g - ah \in P(\mathbb{R})$. It is routine to verify that $(0, 0) < (r/2, 0) < (g, h)$ and $(0, 0) < (0, -r/2a) < (g, h)$.

Suppose, conversely, that the order $<$ on $G \times H$ is topologically compatible with the orders on each of its factors. By Theorem 2·8 and Proposition 3·1, ϕ is a continuous monomorphism onto a subgroup of $\mathscr{U}(H)$. Thus $\phi\colon G \to \mathbb{R}(+) = \mathscr{U}(H)\backslash\{\varnothing, H\}$ (by the density of H). ϕ is order-preserving by 2·6 (vii), so by ([1], 12·2·1), there is an $a \in \mathbb{R}$ such that $\phi(g) = ag$ for each $g \in G$. If $a = g'/h'$ for some $g' \in G$ and $0 \neq h' \in H$, then $\phi(g') = \{h \in H\colon (g', h) > (0, 0)\} = \{h \in H\colon ah <_{\mathbb{R}} g'\}$, and clearly $h' \notin \phi(g') \cup \phi(-g')$, contrary to Lemma 2·6 (a). Hence $G * H \neq \mathbb{R}$, and the proof of the Theorem is complete.

The next theorem, which is due to Fred Galvin, provides an ample supply of pairs G, H of subgroups of $\mathbb{R}(+)$ such that $G * H = \mathbb{R}$.

Let Z, respectively Q, denote the additive groups of integers, respectively rational numbers. If $a \in \mathbb{R}$, let $G_a = \{ag/n : 0 \neq n \in Z,\, g \in G\}$. Clearly G_a is a subgroup of $\mathbb{R}(+)$ which will contain Q if $ag = 1$ for some $g \in G$.

3·3. LEMMA. *If G, H are subgroups of $\mathbb{R}(+)$, and a, b are non-zero real numbers such that $G_a * H_b = \mathbb{R}$, then $G * H = \mathbb{R}$.*

Proof. If $x \in \mathbb{R}$, then by assumption there are non-zero $n, m \in Z$, $g \in G$, and $0 \neq h \in H$ such that

$$\frac{xa}{b} = \frac{(ag/n)}{(bh/m)} = \frac{mg}{nh}\frac{a}{b}.$$

Hence $x = mg/nh \in G * H$, so $\mathbb{R} \subseteq G * H \subseteq \mathbb{R}$, and the lemma holds.

3·4. THEOREM (Galvin). *There is a proper subgroup G of $\mathbb{R}(+)$ such that whenever H is a non-zero subgroup of $\mathbb{R}(+)$, $G * H = G * G = \mathbb{R}$.*

Proof. If t is irrational, there is by Zorn's lemma a subgroup G of $\mathbb{R}(+)$ containing Q and maximal with respect to avoiding t. We now show that $G * Q = \mathbb{R}$. Note first that $G \subseteq G * Q$ since $Q \subseteq G$. For any $x \in R/G$, there is by definition of G a non-zero $n \in Z$ and a $g \in G$ such that

$$\text{(i)}\quad nx + g = t.$$

If $2nx \in G$, then $x \in G * Q$. Otherwise, the definition of G yields an $m \neq 0$ in Z and an $h \in G$ such that

$$\text{(ii)}\quad m(2nx) + h = t.$$

Subtracting (ii) from (i) yields

$$n(1-2m) + (g-h) = t, \quad \text{so} \quad x = \frac{(h-g)}{n(1-2m)} \in G * Q.$$

Thus $G * Q = R$.

Let H denote any non-zero subgroup of $\mathbb{R}(+)$, and choose $k \neq 0$ in G. For $a = 1/k$ and $b = 1$, we have $Q \subseteq H_a$ and $G \subseteq G_b$, so $\mathbb{R} = G * Q \subseteq G_b * H_a$. Then by Lemma 3·3, $G * H = \mathbb{R} = G * G$.

4. *Topologically compatible pairs: the general case*

For the balance of this paper, G and H will denote infinite densely ordered groups unless the contrary is stated explicitly.

Recall that a subset K of G is called *convex* if $x_1 \leqslant g \leqslant x_2$, where $g \in G$ and $x_1, x_2 \in K$, implies $g \in K$. If $T \subseteq G$, let $cn(T)$ denote the intersection of all of the convex normal subgroups of G that contain T. It is not difficult to verify that $cn(T) = \{g \in G$: for some $t \in T$, $a \in G$, and positive integer n, $|g| < |t^n|^a\}$. By the set $ni(T)$ of *normal infinitesimals relative to* T, we mean the union of all the convex normal subgroups of G disjoint from T. It is an exercise to verify that $ni(T) = \{g \in G$: if $a \in G$, n is a positive integer, and $t \in T$, then $|g^n|^a < |t|\}$. By the *cardinal index of archimedeanness* $cia(G)$, we mean the least cardinal number of a subset S of G such that $ni(S) = \{e\}$. We call $\bigcap\{cn(g) : e \neq g \in G\}$ the *order kernel* $S(G)$ of G. Clearly $S(G)$ is a convex normal subgroup of G. If $F \subseteq P(G)$ is finite, then $ni(F) = ni(f)$, where f is the smallest element of F, and it follows easily that $S(G) = \{e\}$ if and only if $cia(G) > 1$. It is clear, also, that if $cia(G) = 1$, then $S(G) = cn(g)$ for any $e \neq g$ in $S(G)$. We summarize the above in the following proposition.

4·1. PROPOSITION. *If* $(G, <)$ *is a densely ordered group and* $S(G)$ *is the order kernel, then*:

(*a*) $cia(G) > 1$ *if and only if* $cia(G)$ *is infinite*;
(*b*) $S(G) = \{e\}$ *unless* $cia(G) = 1$.

For any $g \in G$, let $c(G)$ denote the smallest convex subgroup of G containing g. Note that G is archimedean if and only if $G = c(g)$ whenever $e \neq g \in G$.

The proof of the following lemma was simplified as a result of discussion with A. Rhemtulla.

4·2. LEMMA. *If a and b are distinct positive elements of the order kernel* $S(G)$ *of a densely ordered group, and* $S(G)$ *is not archimedean, then there are* y, z *in* G *such that* $a^y < b < a^z$.

Proof. Suppose

(*) there is an $x \in S(G) \cap P(G)$ such that for each $g \in G$, there is a positive integer n such that $x^g < x^n$.

Then $c(x) = cn(x) = S(G)$. If, for some $y \in S(G) \cap P(G)$, $x \notin c(y)$, then $y^m < x$ for every positive integer m; for each $g \in G$, choose n such that $x^g < x^n$. Thus $(y^{nm})^g < x^g < x^n$, so $(y^m)^g < x$, contrary to the fact that $x \in cn(y)$. This contradiction shows that $c(y) = c(x) = S(G)$ for each x, $y \in P(G) \cap S(G)$, and hence that $S(G)$ is archimedean. Thus (*) fails.

Assume without loss of generality that $a < b$. Since $b \in cn(a)$, for some positive integer m and $h \in G$, $b < (a^m)^h \leqslant (a^g)^h = a^{gh}$, where g is the element of G whose existence is guaranteed by the failure of (*). Taking $y = e$ and $z = gh$, the conclusion of the lemma follows.

Most of the remainder of this paper is devoted to establishing:

4·3. THEOREM. *If* G *and* H *are densely ordered groups with order kernels* $S(G)$ *and* $S(H)$, *then there is an order* $<$ *on* $G \times H$ *topologically compatible with the orders of* G *and* H *if and only if both of the following hold*:

(*a*) $cia(G) = cia(H)$, *and*

(*b*) $S(G)$ *and* $S(H)$ *are central*, (*thus archimedean*) *and we may identify them with subgroups of* $\mathbb{R}(+)$ *in such a way that* $S(G) * S(H) \neq \mathbb{R}$.

As in [**6**], pp. 266–271 and 274–275, we identify each ordinal α with its well-ordered set of predecessors, and we identify each cardinal m with the ordinal minimal with respect to being in one-one correspondence with a set of cardinality m.

To prove that if $G \times H$ admits an order topologically compatible with the orders of G and H, then $cia(G) = cia(H)$, we begin by showing:

4·4. *If both* $cia(G)$ *and* $cia(H)$ *exceed* 1, *then* $cia(G) = cia(H)$.

To verify this, begin by letting $T = \{g_\alpha : \alpha < cia(G)\} \subseteq P(G)$ be a set such that $ni(T) = \{e\}$. By Theorem 2·4(*b*), for each $\alpha < cia(G)$, there is an $h_\alpha \in H$ such that $(e, e) < (e, h_\alpha) < (g_\alpha, e)$. Suppose $cia(G) < cia(H)$. Then there is an $h \in P(H)$ such that $h < h_\alpha$ for each $\alpha < cia(G)$ since $ni(\{h_\alpha : \alpha < cia(G)\}) \neq \{e\}$. Thus $(e, e) < (e, h) < (e, h_\alpha) < (g_\alpha, e)$ for each $\alpha < cia(G)$. By Theorem 2·4(*b*), there is a $g \in G$ such that $(e, e) < (g, e) < (e, h)$, so $e <_G g <_G g_\alpha$ for each $\alpha < cia(G)$. Since $cia(G) > 1$, $ni(g) \neq \{e\}$, so for some $f \in P(G)$, $(f^n)^a < g$ for each positive integer n and $a \in G$. Thus $(f^n)^a < g_\alpha$ for each α, whence $f \in ni(T)$ contrary to the definition of T. We conclude that $cia(G) \geqslant cia(H)$ if $cia(G) > 1$. Similarly $cia(H) \geqslant cia(G)$ if $cia(H) > 1$, so 4·4 holds.

4·5. *$S(G)$ is archimedean.*

We may assume that $cia(G) = 1$. Let $\phi: G \to \mathscr{U}(H)$ denote the hiding map determined by $<$ as in Lemma 2·6 (and Theorem 2·8). By this latter theorem ϕ is continuous. Suppose $h <_H e$, in which case $h \notin \phi(e)$. By the continuity of ϕ, there is a $g \in P(G)$ such that $h \notin \phi(g)$. Since $cia(G) = 1$, there is an $a \in S(G) \cap P(G)$ such that $a \leqslant_G g$ and $h \notin \phi(a)$. If $S(G)$ fails to be archimedean, and $b \in S(G) \cap P(G)$, there are, by Lemma 4·2, y, z in G such that $a^y < b < a^z$. By Lemma 2·6, $\phi(a^y) \subseteq \phi(b) \subseteq \phi(a^z) = \phi(a)$, so $\phi(a) = \phi(b)$. It follows that if $h <_H e$, then $h \notin \phi(b)$ for any $b \in S(G) \cap P(G)$. Using Lemma 2·6 again, $e <_G b$ implies $\phi(e) \subseteq \phi(b)$, so $P(H) \subseteq \phi(b)$. Also since $<$ extends the order of G, $e \in \phi(b)$. Thus $\phi(b) = P(H) \cup \{e\}$ fails to be open, contrary to the density of the order of H. This contradiction establishes 4·5.

Next, we show

4·6. *If $(e,e) < (e,h) < (g,e)$ and $g \in S(G)$, then $h \in S(H)$; thus if $cia(G) = 1$, then $cia(H) = 1$.*

To see this, assume on the contrary that there is a $c \in ni(h) \cap P(H)$. Then $(e,e) < (e,c)$, so by Theorem 2·4(*b*), there is a $k \in G$ such that $(e,e) < (k,e) < (e,c)$. It follows from the definition of $ni(h)$ that for any $(x,y) \in G \times H$,

$$(e,e) < ((k,e)^n)^{(x,y)} < ((e,c)^n)^{(x,y)} < (e,h) < (g,e).$$

So if $x \in G$, then $(k^n)^x < g$, and hence $k \in ni(g) \cap P(G)$, contrary to the assumption that $g \in S(G)$. Thus $h \in S(H)$.

From this note that if $cia(G) = 1$ and $g \in P(G)$ is such that $ni(g) = \{e\}$, then as in the above $(e,e) < (e,h) < (g,e)$ for some $h \in H$. Thus $ni(h) = \{e\}$ and hence $cia(H) = 1$.

Clearly if $<$ is topologically compatible with the orders of G and H, then $<$ restricted to $S(G) \times S(H)$ is topologically compatible with the orders of the (archimedean) subgroups $S(G)$ and $S(H)$. So (*b*) will follow from Theorem 3·2 if we can show that each of $S(G)$ and $S(H)$ is central.

Denote by ψ the hiding map of $S(G)$ into $S(H)$. By Lemma 2·6(iv) and the definitions of ϕ and ψ, $\psi(g) = \phi(g) \cap S(H) = \phi(g^x) \cap S(H) = \psi(g^x)$ for each $g \in S(G)$ and $x \in G$. Since $S(G)$ is a normal subgroup of G, it follows from the last paragraph that the hypothesis of Proposition 3·1 is satisfied by $\psi: S(G) \to S(H)$. Thus ψ is a monomorphism and hence $g = g^x$, and we may conclude that $S(G)$ is central. A similar argument applied to the hiding map of $S(H)$ into $S(G)$ shows that $S(H)$ is also central. This completes the proof that (*b*) holds.

We turn now to establishing the sufficiency of conditions (*a*) and (*b*). Choose any subset S of $P(G)$ or cardinality $cia(G)$ such that $ni(S) = \{e\}$. First we assume $cia(G) > 1$ and establish the existence of a 'valuation'. Write $S = \{s_\alpha : \alpha < cia(G)\}$, let $N(g) = \{\alpha < cia(G) : g \in ni(s_\beta : \beta < \alpha)\}$, and define $v = v_G : G \to cia(G) \cup \{\infty\}$ by letting

$$v(g) = \begin{cases} \sup N(g) & \text{if } g \neq e \\ \infty & \text{if } g = e \end{cases}. \tag{12}$$

We establish first

4·7. *If $g \in G$, then $g \in ni(s_\alpha : \alpha < v(g))$.*

For, if $\beta < v(g)$, then $\beta < \sup\{\alpha : g \in ni(s_\beta : \beta < \alpha)\}$, so $g \in ni(s_\gamma : \gamma < \delta)$ for some $\beta < \delta$. Thus, for each $a \in G$, integer n, and $\gamma < \delta$, $(g^n)^a < s_\gamma$; in particular, for each such a and n, $(g^n)^a < s_\beta$. Since $\beta < v(g)$ is arbitrary, 4·7 holds.

4·8. Lemma. *For any g, h in G:*

(i) $v(gh) \geqslant \min(v(g), v(h))$;
(ii) $v(g^h) = v(g)$;
(iii) $v(g) = v(g^{-1})$;
(iv) $v(g) = \infty$ *if and only if* $g = e$;
(v) *If* $v(g) < v(h)$, *then* $v(gh) = v(g)$;
(vi) *If* $g \geqslant h > e$, *then* $v(g) \leqslant v(h)$;
(vii) *If* $g > e$ *and* $gh < e$ (*or* $hg < e$), *then* $v(h) \leqslant v(g)$;
(viii) *If* $g > e$ *and* $h > e$, *then* $v(gh) = \min v(g), v(h))$.

Proof. If $\delta = \min(v(g), v(h))$, then both g and h are in the group $ni(s_\alpha\colon \alpha < \delta)$ as is their product. So $\min(v(g), v(h)) \leqslant \sup\{\alpha\colon gh \in ni(s_\beta\colon \beta < \alpha)\} = v(gh)$. So (i) holds.

To see (ii) and (iii), note first that both g^{-1} and g^h are in the normal subgroup $ni(s_\alpha\colon \alpha \leqslant v(g))$, so each of $v(g^{-1})$ and $v(g^h)$ is $\geqslant v(g)$. So $v(g) = v((g^{-1})^{-1}) \geqslant v(g^{-1})$ and $v(g) = v((g^h)^{h^{-1}}) \geqslant v(g^h)$. Thus $v(g^{-1}) = v(g) = v(g^h)$, and (ii) and (iii) hold.

By definition, $v(e) = \infty$. If $v(g) = \infty$, then by 4·7, $g \in ni(s_\alpha\colon \alpha < v(g)) = ni(S)$, so $g = e$ and (iv) holds.

Suppose $v(g) < v(h)$ and $v(g) < v(gh)$. Then, by (i) and (iii), $v(g) = v((gh)h^{-1}) \geqslant \min(v(gh), v(h^{-1})) = \min(v(gh), v(h)) > v(g)$. Hence $v(g) < v(h)$ implies $v(gh) \leqslant v(g)$, whence $v(gh) = v(g)$ by (i). Thus (v) holds.

That (vi) holds is immediate from the definition of v. If $g > e > gh$, then $h^{-1} = h^{-1}g^{-1}g > g > e$. By (iii) and (vi), $v(h) = v(h^{-1}) \leqslant v(g)$ and (vii) holds.

If g and h are in $P(G)$, then $g < gh$ and $h < gh$. So by (vi), $v(gh) \leqslant v(g)$ and $v(gh) \leqslant v(h)$. Thus by (i) $v(gh) = \min(v(g), v(h))$ and (viii) holds. This completes the proof of the lemma.

Suppose $cia(G) = cia(H) > 1$ and consider the maps $v_G\colon G \to cia(G) \cup \{\infty\}$ and $v_H\colon H \to cia(H) \cup \{\infty\}$ as defined above. We define an order $<$ on $G \times H$ as follows:

$$\begin{aligned}(e,e) < (g,h) \quad &\text{if} \quad v_G(g) < v_H(h) \quad \text{and} \quad g \in P(G)\\ &\text{or} \quad v_H(h) \leqslant v_G(g) \quad \text{and} \quad h \in P(H).\end{aligned} \tag{13}$$

To show that $(G \times H, <)$ is a totally ordered group, we will verify that (α), (β), (γ) and (δ) of Section 1 hold.

Suppose $(g,h) \neq (e,e)$. By Lemma 4·8 (iv), $\min(v_G(g), v_H(h)) < \infty$. If $v_G(g) < v_H(h)$, then $v_G(g) = \min(v_G(g), v_H(h))$, so $g \neq e$. If $g \in P(G)$, then $(g,h) > (e,e)$, while if $g^{-1} \in P(G)$, then $(g,h)^{-1} = (g^{-1}, h^{-1}) > (e,e)$. We proceed similarly if $v_H(h) \leqslant v_G(g)$ and conclude that $P(G\times H) \cup P(G \times H)^{-1} \cup \{(e,e)\} = G \times H$, so ($\delta$) holds.

If both (g,h) and $(g,h)^{-1} = (g^{-1}, h^{-1})$ are in $P(G \times H)$, and $v_G(g) < v_H(h)$, then both g and g^{-1} are in $P(G)$. Similarly, if $v_H(h) \leqslant v_G(g)$, then both h and h^{-1} would be in $P(H)$. Hence (β) holds.

That (γ) holds is an exercise.

To verify (α), we must consider several cases under the assumption that (g,h) and (g',h') are elements of $P(G \times H)$. Suppose first that $v_G(g) < v_H(h)$ and $v_G(g') < v_H(h')$; then both g and g' are in $P(G)$, and by Lemma 4·8 (viii) and (i)

$$v_G(gg') = \min(v_G(g), v_G(g')) < \min(v_H(h), v_H(h')) \leqslant v_H(hh').$$

Hence $(gg', hh') \in P(G \times H)$.

A similar argument yields the same conclusion if both $v_H(h) \leqslant v_G(g)$ and $v_H(h') \leqslant v_G(g')$.

Suppose next that $v_G(g) < v_H(h)$ and $v_H(h') \leqslant v_G(g')$, in which case $g \in P(G)$ and

$h' \in P(H)$. (The remaining case, in which these inequalities are reversed, follows from this one and (γ).)

(i) Suppose also that $v_G(gg') < v_H(hh')$. If $gg' \leqslant_G e$, then by Lemma 4·8(iii), (vii), $v_G(g') \leqslant v_G(g)$, so $v_H(h') \leqslant v_G(g') \leqslant v_G(g) < v_H(h)$. By Lemma 4·8(v), $v_H(hh') = v_H(h') \leqslant \min(v_G(g), v_G(g')) \leqslant v_G(gg')$. This contradiction shows that $gg' > e$; thus $(gg', hh') \in P(G \times H)$.

A similar argument applies if, instead of (i), we have

$$\text{(ii)} \quad v_H(hh') \leqslant v_G(gg').$$

So (α) holds and we conclude that $(G \times H, <)$ is a (densely) ordered group.

To show that $<$ is topologically compatible with the orders of G and H, we must by Theorem 2·4(*b*), when given $(g,h) \in P(G \times H)$, find $g^* \in G$ and $h^* \in H$ such that $(e,e) < (g^*, e) < (g,h)$ and $(e,e) < (e,h^*) < (g,h)$. Either

$$\text{(i)} \quad v_G(g) < v_H(h) \quad \text{and} \quad g \in P(G), \quad \text{or}$$

$$\text{(ii)} \quad v_H(h) \leqslant v_G(g) \quad \text{and} \quad h \in P(H).$$

If (i) holds, then since $cia(G)$ is infinite, there is a $g^* \in P(G)$ such that $v_G(g) < v_G(g^*)$, whence $g^* < g$ by Lemma 4·8(vi). By parts (iii) and (v) of this lemma, $v_G(gg^{*-1}) = v_G(g) < v_H(h)$. Thus $gg^{*-1} \in P(G)$, so $(g,h)(g^*, e)^{-1} = (gg^{*-1}, h) > (e,e)$, and $(e,e) < (g^*, e) < (g,h)$. Also, let $h^* \in P(H)$, $v_H(h) \leqslant v_H(h^*)$. Then $v_H(h) = v_H(h^{*-1})$, so $v_H(hh^{*-1}) \geqslant \min(v_H(h), v_H(h^*)) = v_H(h) > v_G(g)$. From the last sentence, we conclude $(e,e) < (e,h^*) < (g,h)$.

In case (ii) holds, the argument is similar, reversing the roles of g and h, and of G and H. Thus the order $<$ defined in (13) is topologically compatible with the orders of G and H in case $cia(G) = cia(H) > 1$.

To complete the proof of Theorem 4·3, assume that $cia(G) = cia(H) = 1$ and define an order on $G \times H$ as follows:

By Theorem 3·2, there is a group order $<$ of $S(G) \times S(H)$ topologically compatible with the orders induced on $S(G)$ and $S(H)$ by the ordering of $\mathbb{R}(+)$. We let

$$\begin{aligned} &(g,h) \in P(G \times H) \quad \text{if:} \\ &g \in S(G),\, h \in S(H) \quad \text{and} \quad (g,h) > (e,e), \quad \text{or} \\ &g \notin S(G) \quad \text{and} \quad g \in P(G), \quad \text{or} \\ &g \in S(G),\, h \notin S(H), \quad \text{and} \quad h \in P(H). \end{aligned} \tag{14}$$

Since this order on $G \times H$ extends the order $<$ on $S(G) \times S(H)$ given above, we will denote it by $<$ as well. To show that it is a group order, we will verify (α), (β), (γ) and (δ) of Section 1.

Suppose $(g,h) \neq (e,e)$. If $g \notin S(G)$, then $g > e$ and $(g,h) > (e,e)$, or $g^{-1} > e$ and $(g,h)^{-1} = (g^{-1}, h^{-1}) > (e,e)$. If $g \in S(G)$ and $h \notin S(H)$, a similar proof shows that $(g,h) > (e,e)$ or $(g,h)^{-1} > (e,e)$. The same conclusion holds if $g \in S(G)$ and $h \in S(H)$ since $<$ is a group order on $S(G) \times S(H)$. Hence (δ) holds.

Suppose both (g,h) and $(g,h)^{-1}$ are in $P(G \times H)$. Then $g \notin S(G)$ or $h \notin S(H)$. If $g \notin S(G)$, then both g and g^{-1} are in $P(G)$ by the definition of $P(G \times H)$. Hence $g \in S(G)$, whence $h \notin S(H)$ and the definition of $P(G \times H)$ would yield both h and h^{-1} in $S(H)$. This contradiction shows that (β) holds.

Since $S(G)$ and $S(H)$ are central subgroups of G and H respectively, it follows easily that (γ) holds.

The proof that (α) holds may be carried through by cases in a straightforward way. We omit the details since they are similar to those given for the order of (13).

Once more, we apply Theorem 2·4(b) to show that $<$ is topologically compatible with the orders of G and H. Suppose $(g,h) \in P(G \times H)$. If $g \in S(G)$ and $h \in S(H)$, there is a $g^* \in G$ such that $(e,e) < (g^*,e) < (g,h)$ by Theorem 3·2. If $g \notin S(G)$, then $g \in P(G)$. Since $cia(G) = 1$, there is a $g^* \in S(G) \cap P(G)$ by Proposition 4·1, and $(e,e) < (g^*,e) < (g,h)$ since $g \notin cn(g^*) \subset S(G)$. In case $g \in S(G)$ and $h \notin S(H)$, clearly $(e,e) < (g,e) < (g,h)$. A similar argument by cases will produce an element $h^* \in H$ such that $(e,e) < (e,h^*) < (g,h)$. This completes the proof of Theorem 4·3.

We conclude with some remarks, examples and open problems (which we confine to the case when G and H are densely ordered).

By Theorems 4·3 and 3·2, given two archimedean densely ordered groups G, H, there is an order on $G * H$ topologically compatible with the orders of G and H if and only if there are embeddings ϕ of G into $\mathbb{R}$ and ψ of H into $\mathbb{R}$ such that $\phi(G) * \psi(H) \neq \mathbb{R}$. Moreover, by ([1], 12·2·1), if this latter holds and ϕ', ψ' are embeddings of G, respectively H into $\mathbb{R}(+)$, then there are nonzero real numbers a, b such that $\phi(G) = a\phi'(G)$ and $\psi(G) = b\psi'(G)$. So, as in the argument given in the proof of Lemma 3·3, $\phi'(G) * \psi'(G) = \mathbb{R}$. This comment inspires the following:

PROBLEM. *Find internal characterizations of densely ordered archimedean groups G, H, for which there is an embedding ϕ of G into $\mathbb{R}(+)$ and ψ of H into $\mathbb{R}(+)$ such that $\phi(G) * \psi(H) \neq \mathbb{R}$. Do the same in case $\phi(G) * \phi(G) \neq \mathbb{R}$.*

In [7] an ordered group is called *0-simple* if it has no proper normal convex subgroups other than $\{e\}$. Clearly, any infinite archimedean ordered group is 0-simple. In ([7], chapter 1, section 2, example 8), an example is given of an 0-simple non-abelian ordered group, and in ([8], corollary 2·6·9), it is shown that every solvable 0-simple group is archimedean.

Clearly if G is 0-simple, then $cia(G) = 1$ and $G = S(G)$. So by Theorem 4·3, if G is 0-simple, but not archimedean, there cannot be a densely ordered group H and an order $<$ on $G \times H$ that is topologically compatible with the orders of G and H. It seems natural to ask: If $cia(G) = 1$ then must $S(G)$ be 0-simple?

A negative answer to this question follows.

4·9. *Example.* Let B denote the direct sum of countably many copies of $Q(+)$ indexed by Z, that is $B = \{f: Z \to Q : f(k) = 0$ for all but finitely many $k \in Z\}$. Order B lexicographically with left-most non-zero coordinate dominating. Let $\mathbf{0}$ denote the zero-function, and for any $i \in Z$, let $f_i \in B$ be defined by letting $f_i(k) = f(k-i)$ for each $k \in Z$. Let $G = \{(k,f): k \in Z, f \in B\}$, and let $(k,f)(k',f') = (k+k', f_k + f')$. It is routine to verify that G is a group (with identity element $(0,\mathbf{0})$ and where $(k,f)^{-1} = (-k, -f_k)$; indeed G is the wreath product of $Q(+)$ and $Z(+)$; see [4]). Order G lexicographically with first coordinate dominating. It is routine to verify that $(0,\mathbf{0})$ and $\{(0,f): f \in B\}$ are the only proper convex normal subgroups of G, so G is not 0-simple but $cia(G) = 1$.

Finally, we give an example of a totally ordered group G such that $S(G)$ is archimedean but not central.

4·10. *Example.* Let T denote a subfield of $\mathbb{R}$ and let

$$G = \left\{ \begin{bmatrix} r & a \\ 0 & 1 \end{bmatrix} : r, a \in T \quad \text{and} \quad r > 0 \right\}.$$

If we let
$$\begin{bmatrix} r & a \\ 0 & 1 \end{bmatrix} \in P(G) \quad \text{if} \quad r > 1 \quad \text{or} \quad r = 1 \quad \text{and} \quad a > 0,$$

then under the operation of matrix multiplication, G is a totally ordered group as is noted in [7], p. 4. The following facts are easily verified.

(i) $S(G) = cn\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in T\right\}$.

(ii) The map $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \to a$ is an isomorphism of $S(G)$ into $\mathbb{R}(+)$, so $S(G)$ is archimedean.

(iii) For any $r, a, b \in T$,
$$\begin{bmatrix} r & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r & ra+b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r & a+b \\ 0 & 1 \end{bmatrix},$$

so $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is not in the centre of G unless $a = 0$. Thus $S(G)$ is not central.

By Theorem 4·3, for any totally ordered group H, there cannot be an order on $G \times H$ topologically compatible with the orders of G and H.

We close with reference to two papers related to our work, but without any obvious relationship with the above; namely [2] and [5]. In fact, hearing a lecture by E. Hewitt inspired this work.

REFERENCES

[1] A. Bigard, K. Keimel and S. Wolfenstein. *Groupes et Anneaux Réticulés*, Lecture Notes in Math. vol. 608 (Springer-Verlag, 1977).
[2] S. Eilenberg. Ordered topological spaces. *Amer. J. Math.* **63** (1941), 39–45.
[3] L. Fuchs. *Partially Ordered Algebraic Systems* (Pergamon Press, 1963).
[4] M. Hall. *The Theory of Groups* (Macmillan, 1959).
[5] E. Hewitt and S. Koshi. Orderings in locally compact groups and the theorems of F. and M. Riesz, *Math. Proc. Cambridge Philos. Soc.* **93** (1983), 441–487.
[6] J. Kelley. *General Topology* (Van Nostrand, 1955).
[7] A. Kokorin and V. Kopytov. *Fully Ordered Groups* (John Wiley and Sons, 1974).
[8] R. Mura and A. Rhemtulla. *Orderable Groups*. Lecture Notes in Pure and Appl. Math. **27** (Marcel Dekker, 1977).