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Voter Compatibility In Interval Societies

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Abstract

In an interval society, voters are represented by intervals on the real line, corresponding to their approval sets on a linear political spectrum. I imagine the society to be a representative democracy, and ask how to choose members of the society as representatives. Following work in mathematical psychology by Coombs and others, I develop a measure of the compatibility (political similarity) of two voters. I use this measure to determine the popularity of each voter as a candidate. I then establish local "agreeability" conditions and attempt to find a lower bound for the popularity of the best candidate. Other results about certain special societies are also obtained.

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Chapter 1

Introduction

In voting situations, a political spectrum can often be represented by a line. Political preferences of voters may be imagined to be segments of this line—generally someone will have a range of positions they approve of rather than only one specific ideal point. Then, one approach to political decision-making is to seek out points on this spectrum approved by many voters (see Berg et al. (2010)). This approach and related work has led to interesting geometric combinatorial results. For example, if every pair of voters has a point in common, Helly's theorem from convex geometry states that some point is shared by all voters. Berg et. al show a result for a relaxation of these conditions: a society is defined to be (k, m)-agreeable if among any m voters there is a subset of k voters who share a point in common. They provide a guaranteed minimum number of voters that must share a point in common in a (k, m)-agreeable society.

In his 2012 thesis, Burkhart (2012) uses tolerance graphs to explore agreeability where voters have an intermediate level of approval between complete approval and complete disapproval.

I have begun working on a different approach. Rather than attempt to find a *point* of high agreement, I develop a notion of good *representatives*. That is, I wish to find members of a society whose views are highly compatible with many other voters. I develop a notion of *compatibility* that assigns to any pair of voters a value between 0 and 1 measuring their similarity. I use this compatibility to determine the most popular voter, that is, the one with the highest average compatibility with the other voters in the society. My goal is to develop guarantees of good candidates similar to those shown by Berg et. al.

Whereas previous research has corresponded to *approval voting* soci-

eties, my popularity system does not appear to precisely correspond to any existing voting system. This is because voting systems typically rely on voters providing only a ranked list of their candidate preference, rather than the extent to which they approve each candidate (for one exception, see Brams (1975)). However, results relating to *positional voting* systems may still be relevant. In a positional voting system, each candidate receives points based on their position in each voter's ranked list of candidates, but the points assigned for each ranking are the same for each voter (Hanusa (2009)).

Chapter 2 is a more detailed review of previous work. In Chapter 3, I provide several important definitions, including developing and justifying my measure of voter compatibility. In Chapter 4, I show some results concerning minimal popularity and ideal candidates in a class of societies I call *accordion societies*. In Chapter 5, I discuss unanswered questions and new directions for future inquiry.

Chapter 2

Background

In this chapter I provide an overview of research involving voter approval sets and intersection, as well as some general background on voting problems.

2.1 Geometric Voting

In geometric voting models, a political spectrum is modeled as a space, e.g. $\mathbb{R}^n$ or a circle. Then, voters' approval sets are modeled as convex subsets of the political space. Past research has generally sought results regarding points in the intersections of many approval sets.

In an *interval society*, the political spectrum is the real line. Then we give the following definition:

Definition 1. A *society* is a political spectrum together with a set of voters. A voter is represented by a closed interval of $\mathbb{R}$, which may be thought of as a voter's *approval set* within the reals. We say that a *voter* $v = [\ell_v, r_v]$ has size $|v| = (r_v - \ell_v)$.
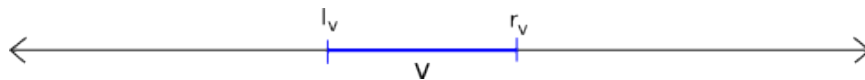

Figure 2.1   A linear spectrum with voter v's approval set

A voter is thought of as approving of any point within that voter's approval set. A concept of interest in most of this work is the *agreement number* of a society, which denotes the maximum number of voters that intersect at a single point.

One easy and useful result follows from Helly's theorem. It states:

Theorem 2 (Helly). *Given n convex sets in $\mathbb{R}^d$ where $n > d$, if every $d + 1$ of them intersect at a common point, then they all intersect at a common point.*

This was interpreted by Berg et al. (2010) to obtain the following (in the case where $d = 1$):

Theorem 3. *In any society where every pair of voters has at least one point in common, there exists a point shared by all voters.*

In *Voting In Agreeable Societies*, Berg et al. (2010) provide guarantees regarding the intersection of many voters. The local condition they use is as follows: suppose that out of every set of m voters, some k of those voters share a point in common. A society with this property is said to be (k, m)-*agreeable*. Then their main result is as follows:

Theorem 4. *Let $2 \leq k \leq m$. In a (k, m)-agreeable society of n voters, there is a candidate who has the approval of at least $n(k - 1)/(m - 1)$ voters.*

Note that in this work "candidate" is taken to mean a point on the spectrum.

Berg et. al. also extend their work to $\mathbb{R}^d$ where approval sets are now convex subsets of $\mathbb{R}^n$, providing analogous results about agreement in (k, m)-agreeable societies. Results are also found for the restricted case of d-box societies, where approval sets are now *boxes*, or Cartesian products of d intervals.

Eschenfeldt (2012) also studied agreement in box societies. He proves a Turán-type result for circular arc societies; a minimum agreement number is guaranteed given the number of pairs of voters with points in common. He also considers results related to the projections of approval boxes onto the coordinate axes.

Arcs of a circle were used as the model for voter preference sets in Carlson et al. (2010) and Hardin (2010). Here several convenient features of interval societies on a line no longer apply, including Helly's theorem. Inspired by results by Turán in interval intersection graphs, Carlson et al. (2010) suppose that $e_{\max}$ pairs of arcs intersect (that is, $e_{\max}$ edges are present in the intersection graph of the arcs). Then, let M denote the maximum and m the minimum number of arcs containing a given point. The authors proved a lower bound on M given $e_{\max}$ and m. They did this by constructing a society $\mathcal{A}_{max}$ which they proved minimized M given $e_{\max}$ and m.

In Burkhart (2012), voters are instead modeled by tolerance intervals, with an approval interval contained in a larger interval denoting a "maybe"
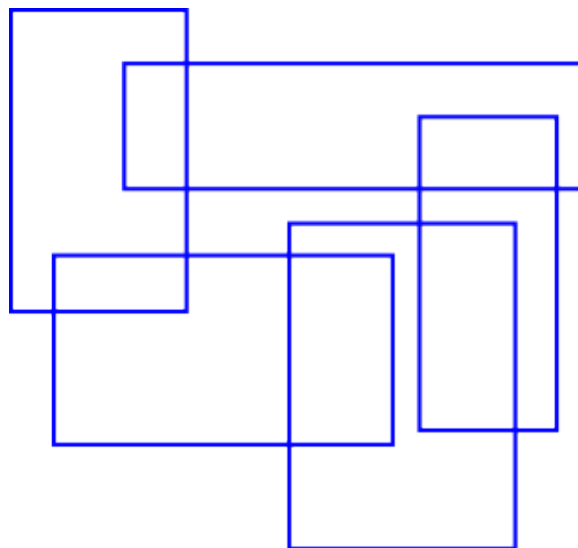
Figure 2.2   A 2-box society.

region. Then two voters are said to intersect if their approval regions intersect, or if the approval region of one intersects the "maybe" region of the other. The author uses results about tolerance graphs to bound agreement number below. He achieves improvements in these bounds by applying bounds on the relative sizes of voter intervals and the relative sizes of approval and "maybe" regions.

In *Basic Geometry of Voting*, Saari (1995) approaches voting problems through a geometric lens. Rather than modeling a political space geometrically, Saari uses discrete preference rankings to describe possible election results in geometric terms.

Some seminal work in geometric or spatial voting problems was done by Steven J. Brams. In *Spatial Models of Election Completion* Brams (1979), Brams models elections by supposing voters are points "distributed along a left-right continuum", justifying this model with examples of elections where candidate positions on one or a few key issues were the driving force of a campaign. Here Brams takes the usual position of making candidates into points, and focuses much of his attention on two-candidate races. However, in Chapter 8 Brams (1979) introduces the idea of candidates as intervals. Brams points out that candidates are often known to state their positions on issues vaguely, effectively creating a range of positions which the candidate could be believed to take. Brams offers several
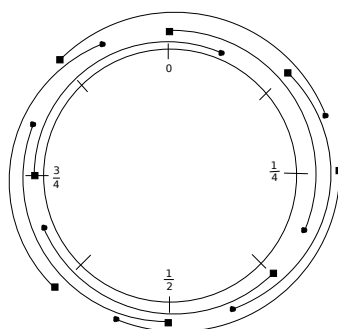
Figure 2.3 A circular arc society

different potential voter responses to such a "fuzzy" position, including assuming that the candidate's true position is the point in the range closest to their own, or instead assuming that it is the farthest point. He does not propose a probablistic model where voters make judgments based on estimating the probabliity that a candidate agrees with them. Brams points to Nixon's presidential primary victory in 1968 and George McGovern's general election loss as examples of varying effects of ambiguous positions.

Brams explores another interesting area of voting theory in *Game Theory and Politics* Brams (1975). Brams distinguishes between "qualitative" and "quantitative" voting games. In quantitative voting games, voters do not simply provide a preference ranking of the candidates, but may express approval with varying levels of intensity according to their own preference. For example, a voter may be given six votes to distribute as they choose, with the option of giving multiple votes to the same candidate. Brams considers elections to representative bodies such as multiple-person committees, and adds the additional condition that a minority faction of the voters should be able to guarantee roughly proportional representation on such a committee. He then shows an interesting result: these two conditions, proportional reperesentation and varied intensity voting, are actually interdependent. That is, voting systems in which voters cannot choose how strongly to support a candidate do not in general admit the guaranteed representation of minority factions. This is relevant to my work since I am also proposing a system where voters register different levels of support for their chosen candidates.

2.2   Mathematical Psychology

Since much of my paper is devoted to developing a concept and measurement of voter agreement, some discussion of theories of measurement is warranted. In Chapter 3, I will explain how these ideas relate to my model.

Mathematical Psychology (Coombs et al. (1970)) describes frameworks for applying mathematical models to psychology in several different ways. This book includes discussions of measurement, utility, and decision making that are all relevant to my work.

Coombs defines measurement as "The process by which the scientist represents properties by numbers." Several problems and desired properties present themselves with this process. Measurements should respect certain relationships between the objects they measure. We should know the extent to which a certain measurement is unique. And our measurement should yield meaningful inferences about the real-world properties we measure.

The representation problem refers to the evaluation of a measurement by checking whether it respects binary relations between the objects it measures. For example, a weight measurement reflects an "is heavier than" relation that may be checked empirically with a balance scale. Coombs et al provide a theorem that states that for any relation between a collection of objects, a measurement (function from the collection to the real numbers) respecting that relation exists if and only if the relation is transitive.

The uniqueness condition refers to the set of different functions which would be a valid measure for an empirical property. For example, if our representations only preserve an order relation, then any representation with the same order is valid. On the other hand, if a measurement is an *interval* scale (preserving "distance" between quantities), then admissible measurements are related by positive linear transformations. Weight is an interval scale by this definition, but a scale defined by asking people to describe the relative loudness of sounds may not be.

Coombs et. al. describe meaningfulness problems as those related to the kinds of interpretations which may be validly extracted from a measurement (for example, we must have an interval scale to meaningfully take an average). They define a statement as *formally meaningful* if its truth is not changed by admissible transformations of a measurement. For example, the average weight of a collection of objects is the same, up to unit conversion, in grams or pounds. They also discuss the problem in psychology of justifying a numerical measurement. The authors offer the primary justifications of using a numerical scale to predict some dependent variable, or

of providing descriptive statistics with respect to the population as a whole (such as IQ).

The authors conclude their discussion of psychological measurement by providing some examples of different types of scales. These include orders; *semiorders* where the transitivity condition is relaxed; *bisection systems* where a "halfway between" criterion provides a sort of interval scale; and systems where multiple independent variables are applied to one dependent variable.

In Chapter 5 of *Mathematical Psychology* Coombs et al. (1970) discuss individual decision making and mathematical theories of utility. The basic conflicts in individual decision making are incomplete knowledge of external outcomes on the one hand, and unsure internal preferences on the other.

An interesting outcome in decision making is that even in easily measureable decisions (with monetary awards, for example), a straightforward expected value is not adequate to predict decision making. We must instead use expected *utility*, where utility is a numerical measurement of outcomes. Utility theory was axiomatized by von Neumann and Morgenstern in 1971, providing the conditions under which a utility function that adequately predicts behavior in gambles exists.

I will next define some important terms and develop my concept of voter agreement. Since I am seeking good representatives rather than popular points, my concept of voter agreement will be significantly different from the definitions used by previous authors. I will also develop some new concepts analogous to agreement proportion and agreeability.

Chapter 3

The Compatibility Metric

Recall:

Definition 5. A *society* is a political spectrum (in our case, $\mathbb{R}$) together with a set of voters. A voter is represented by a closed interval of $\mathbb{R}$, which may be thought of as a voter's *approval* set within the reals. We say that a *voter* $v = [\ell_v, r_v]$ has size $|v| = (r_v - \ell_v)$.

The problem of choosing representatives in such a society has important differences from the related problem of choosing point positions. A voter may not simply be satisfied that a candidate has any agreement with them at all; voters have a range of opinions that they wish to see represented, and wish to vote for a candidate that they can trust to do things they approve of. I seek to develop a measure of voter compatibility that reflects this and allows interesting choices of candidates, not trivial ones. It is also desirable to have compatibility be symmetric and normalized (that is, take on values between 0 and 1). The basic idea of compatibility should be a measurement of *the similarity of two voters*.

A naive measure of this compatibility would simply measure the overlap of two voters. That is, for voters u and v, we could have their compatibility equal $|u \cap v|$.
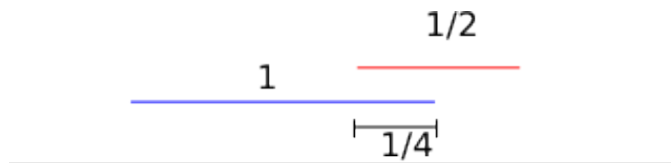

Figure 3.1   An example pair of voters

Under this approach, the pair of voters above has compatibility $\frac{1}{4}$, as does any pair of voters of any size with the same overlap.

However, this approach fails to take into account that voters disapprove of positions outside their approval intervals. A voter large enough to contain all the other voters would be the most popular candidate under this measure, but a voter might not want to elect a candidate that could take on any position in the entire spectrum.
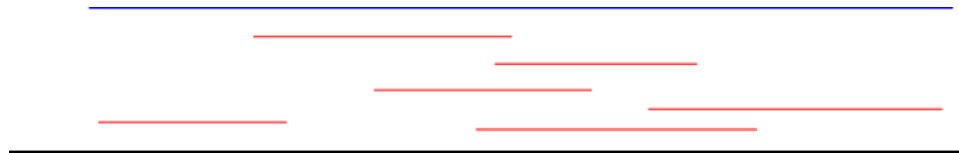

Figure 3.2   In a society where compatibility is measured by overlap size, the blue voter wins.

Another option is to compare the overlap to the candidate's entire interval—a voter may wish to vote for a candidate if the voter approves of most of the candidate's approval interval. We could imagine the compatibility of voter v with a candidate interval c as

$$C_c(v,c) = \frac{|v \cap c|}{|c|}.$$

Under this measure, the red voter in Fig. 3.1 has compatibility $\frac{1}{2}$ with the blue voter as a candidate, while the blue voter has compatibility $\frac{1}{4}$ with the red voter as a candidate.

This resolves the issue above. However, this compatibility measure invites the opposite problem: a very small candidate whose approval set intersects or is contained by many voters would be the most popular, despite the fact that any voter would have few of its positions represented by this candidate.

Figure 3.3   A society where a very small voter is the most popular under the C_c measure.

As comparing the overlap to the voter's interval size has similar problems to simply using the overlap size, I propose using the average of these

two approaches:

Definition 6. The *compatibility* of two voters u and v is defined to be

$$C(u,v) = \frac{|v \cap u|}{2} \left(\frac{1}{|v|} + \frac{1}{|u|} \right).$$

For example, the two voters in Fig. 3.1 have compatibility $\frac{3}{8}$.
Some facts are immediately apparent about this measure.

- It is symmetric, which matches the intuitive sense that voter v's willingness to vote for candidate u should match u's willingness to vote for v; it is a measure of voter compatibility. Thus $C(u,v) = C(v,u)$.

- It rewards candidates for having similar sizes. If $|u|/|v| = r \leq 1$, then their compatibility is bounded above by $\frac{1}{2}(1+r)$.

- Two voters have compatibility 0 if and only if they are disjoint; they have compatibility 1 if and only if they are identical.

If we suppose that a voter or candidate's decision on any issue is chosen randomly from their interval, we can interpret the compatibility measure as the probability that voter v chooses a position voter u finds acceptable averaged with the probability that u chooses a position v finds acceptable.

The compatibility measure can be examined in terms of the measurement properties described by Coombs et al. (1970). Because empirical data is outside the scope of this project, the representation problem cannot be directly addressed. However, taking the probablistic interpretation above we may consider compatibility to be an interval scale, implying that it is unique up to positive linear transformations. Considering compatibility as an interval scale also allows averages to be taken, a fact that will be used in definining popularity.

I also propose a concept of *popularity* which grows out of pairwise compatibility. If a candidate is compatible with many voters in a society, that candidate would make a good representative for the society. So that societies with different numbers of voters may be compared, we use an average compatibility to measure popularity.

Definition 7. A voter v in an n-voter society $\mathbb{V}$ has *popularity*

$$P(v) = \frac{1}{n-1} \sum_{u \neq v \in V} C(u,v).$$

This is the voter's average compatibility with all voters in the society besides itself.

Finally, we introduce two ideas of "agreeable" societies under this idea of compatibility. These are meant to be analogous to the agreeable societies studied in previous work on interval societies (Berg et al. (2010)).

Definition 8. A society $\mathbb{V}$ is *ε-overlapping*, or an *ε-ovelap society*, if for all $u, v \in \mathbb{V}$,

$$|u \cap v| \geq \varepsilon.$$

This agreeability measure is convenient in that its geometric implications are relatively obvious. However, it does not work well as a measure of a highly compatible society. This can be seen by the fact that any compatibility in $(0, 1]$ can be achieved by a pair of voters with any fixed $\varepsilon > 0$ overlap. In fact, any ε-overlap society can be written equivalently (in terms of compatibility) as a unit overlap society by scaling each voter by $\frac{1}{\varepsilon}$.

The following alternate agreeability measure attempts to resolve these issues by invoking pairwise compatibility directly.

Definition 9. A society $\mathbb{V}$ is *ε-compatible* if for all $u, v \in \mathbb{V}$,

$$C(u, v) \geq \varepsilon.$$

This measure, while closer to what is meant intuitively by an "agreeable" society, is harder to work with. As a result my present work concerns ε-overlap societies.

Next I will explore how to bound candidate popularity in these agreeable societies. This problem will be approached primarily by constructing societies which I hope will minimize average or maximum popularity while maintaining the overlap or compatibility conditions.

Chapter 4

Minimum Popularity and Accordion Societies

One of my overall goals is to develop guarantees regarding the best representative in society. Specifically I wish to find ways to give a lower bound for this best candidate's popularity. Clearly no nonzero bound exists in general; in a society where all voters are disjoint, every voter has popularity zero. But local conditions may be imposed that change this bound. I will investigate the ε-overlap societies defined in the previous chapter. Recall that in these societies, an interval of length at least ε is shared by all voters.

4.1 Unit Interval Societies

For simplicity's sake I first constrain my attention to societies where every voter is of unit length. Then in order to demonstrate the minimal popularity of the best candidate in such a society, I construct a society where the maximum candidate popularity is minimized for all ε-overlap societies.

Theorem 10. *For a unit interval society $\mathbb{V}$ with n voters, suppose that each pair of voters has an overlap of at least ϵ. Then there exists some voter $c \in \mathbb{V}$ with popularity at least*

$$P(c) \geq \frac{n}{2}(1 + \epsilon) - 1.$$

Furthermore, there exist societies where the most popular candidate has exactly this popularity—that is, the bound is tight.
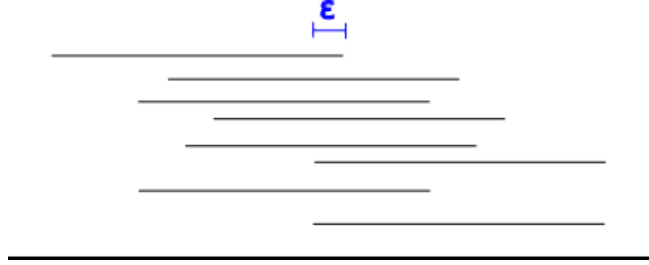
Figure 4.1 An ε-overlap society with unit intervals

Proof. To see this, first note that by Helly's theorem (Thm. 2) there exists some point a such that a is in the approval interval of every voter. Furthermore, since the approval sets are convex, a is involved in the ϵ overlap of each pair of intervals. Then, we may choose a so that for each voter interval $[l_i, r_i]$, we have

$$a - l_i \geq \epsilon/2,$$
$$r_i - a \geq \epsilon/2.$$

Now, order voters left to right, and consider $\mathbb{V}_l \subset \mathbb{V}$ and $\mathbb{V}_r \subset \mathbb{V}$ consisting of those voters with over half of their interval to the left and right of a, respectively. Note that $\mathbb{V}_l \cap \mathbb{V}_r = \emptyset$. If any voters have a fall exactly at the midpoint of its interval, one such voter is our popular candidate, c. Otherwise, if $|\mathbb{V}_l| > |\mathbb{V}_r|$ choose the rightmost member of $\mathbb{V}_l$, if $|\mathbb{V}_r| > |\mathbb{V}_l|$ choose the leftmost member of $\mathbb{V}_r$, and if the sets are the same size choose either of these.

Now consider the popularity of c. Suppose without loss of generality that $c \in \mathbb{V}_l$. Then for each $v \neq c$ with $v \in \mathbb{V}_l$,

$$v \cap c \geq \frac{1}{2}(1 + \epsilon).$$

Next, consider any $u \in \mathbb{V}_r$. It is possible that $(\frac{1}{2}(1 + \epsilon) - u \cap c) = \delta > 0$. However, because we know that the left endpoint of v is no further right than $a - \epsilon/2$, we must have that for l_c, the left endpoint of c,

$$l_c \leq a - \frac{\epsilon}{2} - \frac{1}{2} - \delta.$$

For any other member of $\mathbb{V}_l$, each member of $\mathbb{V}_l$ has its left endpoint lie between $(a + \epsilon/2 - 1)$ and l_c (since we have chosen c to be the rightmost interval in $\mathbb{V}_l$). Thus we see that for any interval $v \in \mathbb{V}_l$,

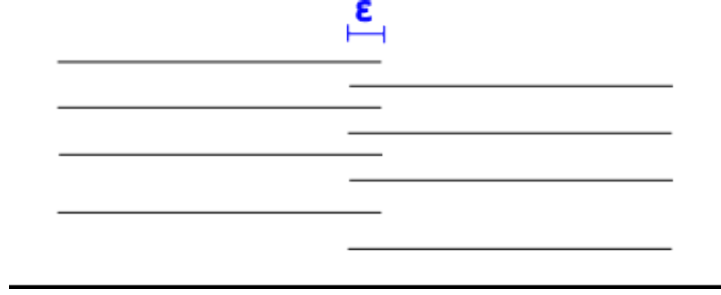$$c \cap v \geq \frac{1}{2}(1 + \epsilon) + \delta.$$

Figure 4.2 The 8-voter ϵ-accordion society

Because $|\mathbb{V}_l| \geq |\mathbb{V}_r|$, we may balance out the popularity loss from each member of $\mathbb{V}_r$ with a corresponding gain from a member of $\mathbb{V}_l$, so that the average popularity of c is at least $\frac{1}{2}(1 + \epsilon)$. Summing over all of $\mathbb{V}$ and subtracting the intersection of c with itself supplies the desired popularity of c.

The bound proved above is a tight bound. To see this, consider the ϵ-*accordion society*. I define this society to be one where the voters are evenly divided into left side and right side voters, with the left side voters containing only the required overlap interval and points to its left, while the right side voters contain only the overlap interval and points to its right.

In an ϵ-accordion society with an even number n of voters, each voter has an intersection of 1 with $\frac{n}{2} - 1$ voters and an intersection of ϵ with $\frac{n}{2}$ voters, for a total popularity of $\frac{1}{2}(1 + \epsilon) - 1$. □

4.2 Varied-Size Accordion Societies

It seems that it is unavoidable for each voter to have a high degree of compatibility with many other voters in a ε-overlap society, since even when such a society is polarized there are still many voters on a voter's "side". One way to decrease the average compatibility is to have these voters vary in size, since voter compatibility is bounded above as a function of the ratio of two voters' sizes. In such a society, we suppose that a pair of voters $v_{i,L}$ and $v_{i,R}$ of the same size exist on either side of the overlap region for each distinct voter size.

Now rather than a compatibility of 1 between all voters on the same side, their compatibility takes on values between $\frac{1}{2}$ and 1. And as larger voters are added to a society, a fixed voter v_i has its compatibility with large voters on the opposite side approach zero.
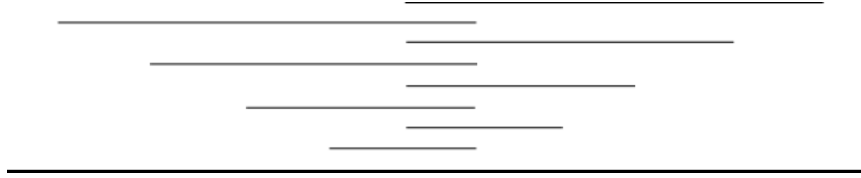
Figure 4.3 A ε-overlap accordion society with voters of varying sizes

To perform calculations on the popularity of voters in such a society, we fix an overlap size of 1 (since this is equivalent to a ε-overlap society scaled by a factor of $\frac{1}{\varepsilon}$). Then the following bound may be calculated:

Theorem 11. *Let $\mathbb{V}$ be a unit overlap society of size n in which the unit overlap region is shared by pairs of voters $v_{i,L}$ and v_{i_R} with $|v_{i,L}| = |v_{i,R}| = |v_i|$. Suppose each voter $v_{i,L}$ has its right endpoint at the right endpoint of the overlap interval, and voter $v_{i,R}$ has its left endpoint at the left endpoint of the overlap interval.*
Let

$$M = \sum_{k=1}^{n/2} \frac{1}{|v_k|}.$$

Then the popularity of a voter v is maximized when

$$|v| = \sqrt{\frac{n}{2M}}.$$

Proof. Note that our numebr $2M$ is simply the sum of the reciprocals of $|v|$ for all $v \in \mathbb{V}$. The voters are grouped into $|v_i|$ pairs for notational convenience.

Consider a voter $v_{i,L}$ (the calculation is the same for its opposite voter $v_{i,R}$).

For each of the $n/2$ voters on the left side of the overlap region,

$$C(v_{i,L}, v_{k,L}) = \frac{1}{2}\left(1 + \frac{|v_i|}{|v_k|}\right).$$

For each of the $n/2$ voters on the right side,

$$C(v_{i,L}, v_{k,R}) = \frac{1}{2}\left(\frac{1}{|v_i|} + \frac{1}{|v_k|}\right).$$

We take the sum of these (and subtract 1 to account for overcounting the compatibility of $v_{1,L}$ with itself), to find

$$P(v_{i,L}) = \frac{1}{n-1}\left(\sum_{k=1}^{n/2}\left[\frac{1}{2}\left(1 + \frac{|v_i|}{|v_k|}\right) + \frac{1}{2}\left(\frac{1}{|v_i|} + \frac{1}{|v_k|}\right)\right] - 1\right).$$

We simplify this to obtain

$$P(v_{i,L}) = \frac{|v_i| + 1}{2} \sum_{k=1}^{n/2} \frac{1}{|v_k|} + \frac{n}{4} \left(1 + \frac{1}{|v_i|}\right) - 1.$$

Then, this may be rewritten

$$P(v_{i,L}) = \frac{M}{2}(|v_i| + 1) + \frac{n}{4} \left(1 + \frac{1}{|v_i|}\right) - 1,$$

leaving only constants and terms that depend on $|v_i|$. Maximizing the voter's popularity is now a matter of finding a zero of the derivative; that is, our ideal candidate c satisfies

$$\frac{M}{2} - \frac{n}{4} \left(\frac{1}{|c|^2}\right) = 0.$$

Some algebraic manipulation leads to the desired result.   □

Note that a voter of the ideal size does not necessarily exist in the society. Then this result may be interpreted as giving the size of an extra "weirdo" candidate that could be added to the society on either the left or right side of the overlap region.

We can use this result to calculate the ideal voter size for different types of accordion societies. I will provide some examples. First I consider a society with linearly increasing voter sizes.

Corollary 12. *Let* $\mathbb{V}$ *be a society of size n as described in Theorem 11 where* $|v_k| = k$. *Then for large values of n, the ideal voter v has size*

$$|v| \approx \sqrt{\frac{n}{2\ln(n)}}.$$

Proof. The result follows from using

$$M = \sum_{k=1}^{n/2} \frac{1}{k} \approx \ln(n/2).$$

Since $\ln(n/2) = \log(n) + \ln(\frac{1}{2})$, the additive factor may be ignored when describing the result asymptotically.   □

So a linearly increasing society with 100 voters of lengths 1 to 50 would have an ideal candidate size of approximately 3.29, while a society with 1000 voters of lengths 1 to 500 would have an ideal candidate size of approximately 8.51. With 10,000 voters, the ideal candidate size reaches about 23.3.

Next, I consider a society where voters' size increases exponentially rather than linearly.
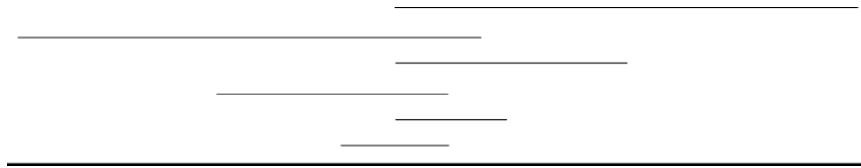

Figure 4.4   An exponentially increasing varied-size accordion society

Corollary 13. *Let $\mathbb{V}$ be a society of size n as described in Theorem 11 where $|v_k| = 2^k$. Then for large values of n, the ideal voter v has size*

$$|v| \approx \sqrt{\frac{n}{2}}.$$

Proof. Again we begin by calculating, for large values of n,

$$M = \sum_{k=1}^{n/2} \frac{1}{2^k} \approx 1,$$

giving the desired result by Theorem 11.   □

So if our society had 100 voters of lengths 1 to 2^{50}, the ideal candidate size would be approximately 7.07. If it had 1000 voters of lengths 1 to 2^{500}, the ideal candidate size would be approximately 22.36. In a 10,000 voter society with voters of length 1 to 2^{5000}, the ideal candidate has size approximately 70.7.

It remains to be shown that one of the voters of the two closest sizes to the ideal is the actual most popular voter, and to determine this voter's popularity. This could by done by demonstrating that popularity is monotonic on either side of the ideal candidate size. There are other related inquiries and further work that I will discuss in the next chapter.

Chapter 5

Further Questions

There are several unfinished approaches to my compatibility scheme, particularly involving accordion societies. In addition to determining the popularity of the ideal candidate, it should be shown that one of the voters with size close to the ideal will be the most popular candidate in practice and find this voter's popularity. Finally, it remains to be shown that under some restriction these accordion societies have the least popular best candidates for ε-overlap societies.

Then, this work could be applied to ε-compatibility societies. By bounding the maximum size of voters, an ε-overlap society becomes a compatibility society. Then it could be possible to find a scheme for varying voter size in an accordion arrangement that minimizes the maximum popularity. One related problem that may be useful for this is exploring minimizing average popularity (which is equivalent to minimizing average compatibility). This could lead to a result guaranteeing candidate popularity in ε-compatible societies.

An important direction for future research will be expanding my results to conditions more realistic than ε-compatibility. For example, it may be desirable to develop a condition analogous to (k, m)-agreeability.

Finally, some exploration of the relationship of my popularity rating to known or possible voting systems should be done. This work will involve more research in voting theory.

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