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Lines in Tropical Quadrics

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Abstract

Classical algebraic geometry is the study of curves, surfaces, and other varieties defined as the zero set of polynomial equations. Tropical geometry is a branch of algebraic geometry based on the tropical semiring with operations minimization and addition. We introduce the notions of projective space and tropical projective space, which are well-suited for answering enumerative questions, like ours. We attempt to describe the set of tropical lines contained in a tropical quadric surface in $\mathbb{TP}^3$. Analogies with the classical problem and computational techniques based on the idea of a tropical parameterization suggest that the answer is the union of two disjoint conics in $\mathbb{TP}^5$.

Contents

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Chapter 1

Introduction

Classical algebraic geometry is the study of geometric objects that can be defined through algebraic formulas, such as conics and hyperplanes. A prominent area of the subject is enumerative geometry, the study of counting the number of a particular geometric object. For instance, one may ask how many points are in the intersection of two lines, or how many conics are contained in a particular surface. This thesis is an attempt to answer an enumerative question, namely:

Question 1. How many tropical lines are contained in a general, smooth tropical quadric surface in $\mathbb{TP}^3$? In particular, what is their structure as a subset of the tropical Grassmannian?

Classical quadric surfaces in $\mathbb{P}^3$ have two distinct rulings of lines that form conics in the Grassmannian, so it is of particular interest to find to what extent the analogue holds in the tropical setting. One rephrasing of this question is: do all points in a smooth tropical quadric surface have two distinct tropical lines through them that are contained in the surface?

Vigeland Vigeland (2010) used a purely combinatorial approach to show that the answer to this question is yes for points contained in the 2-dimensional bounded region of a smooth quadric surface. We take a more algebraic approach.

In order to answer this question, we first introduce tropical geometry and what it means for a set to be a tropical variety (Section 2). Then, we define projective space, show how it can provide consistant answers to enumerative problems, and extend our definition to the tropical setting (Section 3). In Section 4, we explain how to define the structure of a set of lines, namely, we define the Grassmannian to be the set of lines (or more general linear spaces) and embed it in a space we do understand using the Plücker embedding. Section 5 solves this problem in the classical (non-tropical) setting using the background developed in previous sections and Section 6 describes how the classical result may be used to answer our main question. Section 7 discusses what it means for a tropical surface to be smooth and how we use properties of smoothness to restrict our attention to particular tropical quadrics. In Section 8, we develop of method of tropical parameterization, which is used to construct the key computer program of Section 9, which details our exploration of tropicalized conics in the tropical Grassmannian. This program failed to find a pair of intersecting conics in 157,000 examples of possibly smooth surfaces, providing evidence for the tropical analogue. Section 10 describes in detail two examples run by our computer program (one with intersecting conics and one with disjoint conics) and Section 11 discusses potential plans for future research.

Chapter 2

Tropical Geometry

Tropical geometry is a developing field of algebraic geometry that is often called "piecewise linear" geometry. To see why, we define tropical curves and surfaces through two different approaches in this section.

2.1 Tropical Hypersurfaces

First, we consider the tropical semiring: $\mathbb{R} \cup \{\infty\}$ with the operations $\odot$ and $\oplus$ defined by $x \odot y = x + y$ and $x \oplus y = \min\{x, y\}$. For example, $3 \odot (2 \oplus 8) = 3 \odot 2 = 5$. We refer to $\odot$ as tropical multiplication and $\oplus$ as tropical addition and may write $x \odot x \odot \ldots \odot x = x^n$ as an abbreviation of repeated tropical multiplication.

Under these operations, $\mathbb{R} \cup \{\infty\}$ satisfies all the axioms of a field except that not all elements have an additive inverse. To see this, observe that $x \odot 0 = x$ and $x \oplus \infty = x$ for all $x \in \mathbb{R} \cup \{\infty\}$. Thus, 0 is the multiplicative identity and ∞ is the additive identity. However, given a real number x, there is no real number y such that $\min\{x, y\} = \infty$.

Since $\mathbb{R} \cup \{\infty\}$ is "almost" a field, we can define polynomials over it. We now drop the $\cup\{\infty\}$ and use $\mathbb{R}$ to refer to the tropical semiring in order to simplify notation.

Definition 2. A *tropical monomial* in variables $x_1, \ldots, x_n$ is an element of the form $x_1^{i_1} \odot x_2 \odot \ldots \odot x_n^{i_n}$, written as $x_1 x_2 \ldots x_n$ for short. We say a tropical monomial has degree $i_1 + \ldots i_n$. A *tropical polynomial* is a tropical linear combination of tropical monomials and has degree equal to the maximum degree of these monomials.

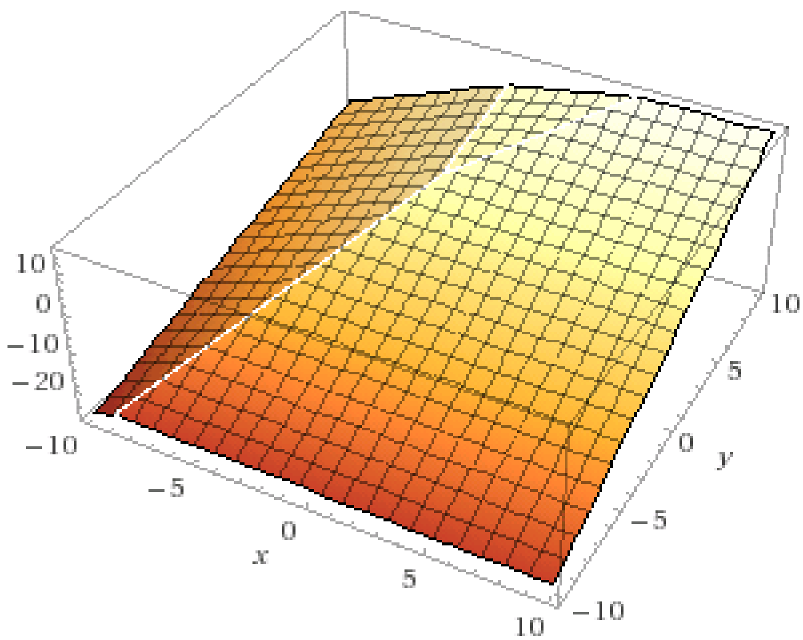

Figure 2.1 The graph of $f(x,y) = 3 \odot x^2y \oplus -5 \odot y^2 \oplus 1xy$. Source: Wolfram-Alpha.

Example 3. Consider the tropical polynomial in two variables $f(x,y) = 3 \odot x^2y \oplus -5 \odot y^2 \oplus 1xy$. We may write this in terms of usual arithmetic as $f(x,y) = \min\{3+2x+y, -5+2y, 1+x+y\}$. Graphing f in 3 space, we see that it is the minimum of three planes, hence continuous, concave, and piecewise linear.

We are now ready to define tropical curves through tropical polynomials.

Definition 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a tropical polynomial. Following Maclagan and Sturmfels (In progress), we define the *tropical hypersurface* $\mathcal{T}(f)$ to be the set of points p in $\mathbb{R}^n$ such that in computing $f(p)$, the minimum of the summands of f is attained at least twice. Equivalently, these are the points where f is not differentiable. We call these points the *corner locus* of f.

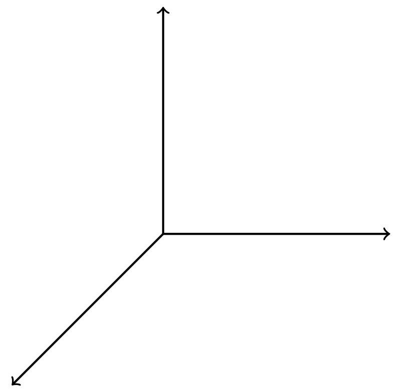

Figure 2.2 The tropical graph of $x \oplus y \oplus 0$

Example 5. As a simple example, consider the function $f(x,y) = x \oplus y \oplus 0 = \min\{x,y,0\}$. When x and y are negative, the minimum is attained twice if and only if $x = y$, so $V(f)$ contains the ray $x = y$ beginning at the origin of $\mathbb{R}^2$ and extending into the third quadrant. When $x > 0$, the minimum is attained twice if and only if $y = 0$, so $V(f)$ also contains the x-axis for positive x and similarly the y-axis for positive y. We also get that $(0,0) \in V(f)$ since the minimum is attained three times at the origin.

This approach is fine for defining hypersurfaces, that is, the corner locus of a tropical polynomial. However, we would often like to define other tropical objects such as conics in 3-space, or planes of codimension greater than 1. It is easier to define these objects as the "tropicalization" of their classical analogues.

2.2 Tropicalizations

Let us begin our description of this approach by discussing the classical version of such objects. In the classical case, the term we typically use to describe conics and hypersurfaces and the like is "affine variety." For the following, we let K be an algebraically closed field and $K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K.

Definition 6. Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal. Then we define the *zero locus* of I to be the set

$$V(I) = \{(x_1, \ldots, x_n) \in K^n | f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}.$$

We say that $V(I)$ is an *affine variety*.

The polynomial ring $K[x_1, \ldots, x_n]$ is Noetherian, which for our purposes means that all of its ideals are finitely generated. Since every element of an ideal in a commutative ring may be written as a linear combination of its generators, the zero locus of an ideal is equal to the zero locus of its generators.

Example 7. Let $K = \mathbb{R}$, $f = 1 - x^2 - y^2 \in K[x, y]$ and I be the ideal generated by f, that is, $I = (f)$. Then,

$$\begin{aligned} V(I) &= V(f) \\ &= \{(x, y) \in \mathbb{R}^2 | 1 - x^2 - y^2 = 0\} \\ &= \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}. \end{aligned}$$

Thus, $V(I)$ is the unit circle in $\mathbb{R}^2$. (Note that $\mathbb{R}$ is not algebraically closed, but we may still use it to provide intuition in some circumstances.)

Before we move on to tropical varieties, we introduce one last important concept involving classical varieties: the coordinate ring of a variety. Here we just give enough of an idea to motivate future definitions, but for those seeking a more rigorous development of the coordinate ring, we recommend Reid (1988). The coordinate ring of K^n is $K[x_1, \ldots, x_n]$. We may describe the coordinate ring as the set of all rational functions on V that are defined at all points of V. Since K is assumed to be algebraically closed, it is not hard to show that all denominators must be constant and that such functions (called *regular*) on K^n may only use positive powers of $x_1, \ldots, x_n$.

However, we will soon consider the set $(K \setminus \{0\})^n$, denoted $(K^*)^n$. Since no coordinates of $(K^*)^n$ are 0, we may adjoin inverse powers of the x_i and the coordinate ring is $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

The first step in tropicalizing an affine variety is to define a valuation. Intuitively, a valuation is a map that transfers the classical operations of multiplication and addition into their equivalent tropical operations.

Definition 8. Maclagan and Sturmfels (In progress) A valuation is a map $val : K \to \mathbb{R}$ such that:

(a) $val(a) = \infty$ if and only if $a = 0$

(b) $val(ab) = val(a) + val(b)$

(c) $val(a + b) \geq \min\{val(a), val(b)\}$

We remark that any valuation map $val : K \to \mathbb{R}$ naturally extends to a valuation map $val : K^n \to \mathbb{R}^n$ by $(a_1, \ldots, a_n) \mapsto (val(a_1), \ldots, val(a_n))$.

Example 9. Let $v : \mathbb{C} \to \mathbb{R}$ be defined by

$$v(a) = \begin{cases} \infty & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then v is a valuation map called the trivial valuation.

Example 10. Consider the field $\mathbb{Q}$ and let p be a prime. Any element of $\mathbb{Q}$ may be written as $\frac{p^k a}{b}$, where $k \in \mathbb{Z}$ and p divides neither a nor b. Then,

$$val(\frac{p^k a}{b}) = k$$

is a valuation, known as the p-adic valuation. Maclagan and Sturmfels (In progress) If we introduce a subscript to denote p, then $val_3(9/23) = 2$ and $val_5(7/5) = -1$.

Example 11. For reasons that will be clear later, we want a valuation map with image dense in $\mathbb{R}$. We would also like to work over an algebraically closed field to strengthen the transfer of results from the classical to the tropical setting. For both these reasons, we introduce the field of *Puiseux series*, $\overline{\mathbb{C}(t)}$, the algebraic closure of the field of rational functions on $\mathbb{C}$. We may equivalently define Puiseux series as the set of formal power series

$$c_1 z^{q_1} + c_2 z^{q_2} + \ldots$$

where $c_i \in \mathbb{C}$, $q_i \in \mathbb{Q}$ and have bounded denominators, and $q_1 < q_2 < \ldots$. From this definition, we may easily define a valuation map by $c_1 z^{q_1} + c_2 z^{q_2} + \ldots \mapsto q_1$, the lowest exponent in the power series expansion. In the study of tropical geometry, it is common to take the field K to be the field of Puiseux series and that is what we will do from here on. Maclagan and Sturmfels (In progress)

Returning to the idea of tropicalization, we define a tropical variety following Richter-Gebert et al. (2005).

Definition 12. Let $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an ideal, where $V(I) \subseteq (K^*)^n$. Then,

$$\mathcal{T}(I) = \overline{val(V(I))}$$

is a *tropical variety*. If $V(I)$ is a conic, we call $\mathcal{T}(I)$ a tropical conic and so forth.

Observe that the deletion of 0 from K removes the possibility of a coordinate of ∞ in $\mathcal{T}(I)$. We would like to show that in some way, this definition of tropical variety corresponds to the definition of tropical hypersurface we gave earlier. In order to do so, we will need the following translation between classical and tropical polynomials.

Definition 13. Let $f = \Sigma_{u \in \mathbb{N}^{n+1}} c_u x^u$ be a polynomial in $K[x_1, \ldots, x_n]$. Then, the *tropicalization* of f is the function $trop(f) : \mathbb{R}^n \to \mathbb{R}$ by $w \mapsto \min\{val(c_u) + w \cdot u : c_u \neq 0\}$.

Example 14. Take $f = (1+t)x - 2t^{-1}y + t^3x^2y \in K[x, y]$. Then, $trop(f)(x, y) = \min\{1 + x, -1 + y, 3 + 2x + y\} = 1x \oplus -1y \oplus 3x^2y$.

Theorem 15. *Let I be an ideal of $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then*

$$\overline{val(V(I))} = \cap_{f \in I} \mathcal{T}(trop(f)).$$

That is, a tropical variety defined through a classical ideal is the intersection of all corner locuses of the tropicalized ideal.

This theorem, found in Maclagan and Sturmfels (In progress), essentially says that both notions of a tropical variety are equivalent. For that reason, it is often called the Fundamental Theorem of Tropical Geometry. We introduce the first notion, based on corner locus, to provide intuition and background for applications of tropical geometry. But, the second characterization via classical varieties will prove the most useful for our purposes.

Chapter 3

Projective Space

In this section, we introduce the notion of projective space and extend our definition of a tropical variety to tropical projective space. Projective space is similar to affine space, except it is slightly "bigger" so that enumerative questions have consistent answers. For instance, we will show later in this section that any two distinct lines in 2-dimensional projective space intersect exactly once; they cannot be parallel like in affine space.

3.1 Defining $\mathbb{P}^n$

We begin with the definition of projective space.

Definition 16. Let K be a field with multiplicative group $K^* = K \setminus \{0\}$. Define an equivalence relation $\sim$ on $K^{n+1} \setminus \{(0, \dots, 0)\}$ by

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \text{ if } (x_0, \dots, x_n) = (\lambda y_0, \dots, \lambda y_n) \text{ for some } \lambda \in K^*.$$

Then, n-dimensional projective space over K is $\mathbb{P}^n = K^{n+1} \setminus \{(0, \dots, 0)\} / \sim$. Elements of $\mathbb{P}^n$ are written $(x_0 : \dots : x_n)$ with a colon instead of a comma.

We claim that projective space is much like affine space. First, there is a natural inclusion $\iota : K^n \to \mathbb{P}^n$. To see this, let $U_0 = \{(x_0 : \dots : x_n) \in \mathbb{P}^n | x_0 \neq 0\}$ and define the following two maps, one in each direction. Let $\pi : U_0 \to K^n$ by

$$(x_0 : \dots : x_n) \mapsto (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$$

and let $\iota : K^n \to U_0$ by

$$(x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n).$$

By considering the equivalence relation used to define $\mathbb{P}^n$, one may see, perhaps unintuitively, that π is well-defined and ι is surjective. After this, it is not hard to see that the two maps are inverses, so K^n is naturally included in $\mathbb{P}^n$. One may extend this construction to the sets $U_i = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n | x_i \neq 0\}$. We call such sets U_i the *standard open sets* of $\mathbb{P}^n$.

Second, zero sets of polynomials in affine space naturally extend to zero sets of polynomials in projective space.

Definition 17. A polynomial is *homogeneous* if it is the sum of monomials of equal degree.

Given a d-degree polynomial $f \in K[x_1, \ldots, x_n]$, we define the *homogenization* of f with respect to x_0 to be the polynomial $f_h = x_0^d f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in K[x_0, \ldots, x_n]$.

The first thing to notice is that while polynomials are generally not well-defined on $\mathbb{P}^n$ (i.e., we do not have $f(\vec{x}) \neq f(\lambda \vec{x})$ for all λ), the zero sets of homogeneous polynomials g on $\mathbb{P}^n$ are well-defined. That is, for $\lambda \in K^*$,

$$g(x_0 : \ldots : x_n) = 0 \text{ if and only if } g(\lambda x_0 : \ldots : \lambda x_n) = 0$$

since $g(\lambda x_0 : \ldots : \lambda x_n) = \lambda^d g(x_0 : \ldots : x_n)$. Because of this, we may define a *projective variety* to be the zero set of an ideal generated by homogeneous polynomials.

Also, the homogenization of a polynomial on U_0 is essentially the same as the original polynomial. If we take the representatives in U_0 with first entry equal to one, we find that $f_h(1 : x_1 : \ldots : x_n) = f(x_1, \ldots, x_n)$, so the inclusion defined by ι respects zero sets of polynomials, provided that we take the homogenization of such polynomials.

3.2 Enumerative Properties and $\mathbb{TP}^n$

Now, we demonstrate one of the critical properties of projective space, that enumerative geometric questions have consistent answers. Take the simplest nontrivial enumerative question possible: in how many points do two distinct lines in the plane intersect? In affine space, the answer is usually one, except it is zero when the two lines are parallel. We give three equivalent definitions of a projective line in $\mathbb{P}^3$ then prove a proposition regarding $\mathbb{P}^2$, which shows that the answer is always one in the projective case.

Definition 18. A *projective line* is the image of a map $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ defined by:

$$(x_0 : x_1) \mapsto (a_0x_0 + a_1x_1 : b_0x_0 + b_1x_1 : c_0x_0 + c_1x_1 : d_0x_0 + d_1x_1)$$

where the coefficients $a_i, \ldots, d_i \in \mathbb{C}$ and the vectors $(a_0, \ldots, d_0)$ and $(a_1, \ldots, d_1)$ are linearly independent to ensure that ϕ is well-defined and more than a single point.

Proposition 19. *A subset $L \subseteq \mathbb{P}^3$ is a line if and only if $L = H/\sim$ for some 2-dimensional plane $H \subset K^4$ through the origin, where $\sim$ is the equivalence relation used to define projective space.*

Proof. Let $H = \operatorname{span}\{(a_0, \ldots, d_0), (a_1, \ldots, d_1)\}$. Then, we may write $H/\sim$ as

$$\{(a_0x_0 + a_1x_1, b_0x_0 + b_1x_1, c_0x_0 + c_1x_1, d_0x_0 + d_1x_1) | (x_0, x_1) \in K^2 \setminus \{(0,0\}\}/\sim$$

Scaling a point (x_0, x_1) corresponds to scaling a point of H, so this is equal to

$$\{(a_0x_0 + a_1x_1 : b_0x_0 + b_1x_1 : c_0x_0 + c_1x_1 : d_0x_0 + d_1x_1) | (x_0 : x_1) \in \mathbb{P}^1\}/\sim$$

where we require $(x_0 : x_1) \in \mathbb{P}^1$ since this only means that at least one of the coordinates is nonzero. (This is necessary to ensure that the corresponding linear combination has at least one nonzero coordinate.) This gives us the desired one-to-one correspondence between lines in $\mathbb{P}^3$ and 2-dimensional planes in K^4. $\square$

We remark that in general, there is a similar correspondence between $k-1$-dimensional planes in $\mathbb{P}^{n-1}$ and k-dimensional planes in K^n, though for our purposes we will mostly refer to the case where $k = 2, n = 4$.

Proposition 20. *A subset $L \subseteq \mathbb{P}^3$ is a line if and only if it is the intersection of two distinct hyperplanes. (A hyperplane in $\mathbb{P}^3$ is the zero locus of a single linear homogeneous polynomial and is thus 2-dimensional.)*

Proof. By the last proposition and the above remark (with $k = 2, n = 4$), we may rephrase this proposition as: A subset $L' \subseteq K^4$ is a 2-dimensional plane if and only if it is the intersection of two distinct 3-dimensional planes. This is an elementary fact of linear algebra, which we cite to prove the proposition. $\square$

We now return to the case of $\mathbb{P}^2$, where a line may be viewed as a linear parameterization or the zero set of a single homogeneous linear polynomial. We use the second characterization for the following proposition.

Proposition 21. *Let $L = V(ax + by + cz)$ and $L' = V(a'x + b'y + c'z)$ be two distinct lines in $\mathbb{P}^2$. Then, L and L' intersect in exactly one point.*

Proof. We observe that a point $(x : y : z) \in L \cap L'$ if and only if it is a solution to both linear equations, that is, if

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since L and L' are distinct lines, the vectors (a, b, c) and (a', b', c') are linearly independent. Hence, the matrix above has rank two and a null space of dimension 1. Suppose the null space is spanned by (x_0, y_0, z_0). Then, all elements of the null space are of the form $(\lambda x_0, \lambda y_0, \lambda z_0)$. If $\lambda = 0$, then the vector does not represent a point in $\mathbb{P}^2$ and can be ignored. All such vectors with $\lambda \neq 0$ are equivalent under projective equivalence and account for exactly one point of $\mathbb{P}^2$. Therefore, the lines L and L' meet in that one point. □

A natural question one might ask is how to extend the notion of projective variety to the tropical setting. We choose to follow Richter-Gebert et al. (2005) and use the following definition.

Definition 22. Let I be an ideal in $K[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by homogeneous polynomials. Then a *tropical projective variety* is of the form

$$\mathcal{T}(I) = \overline{val(V(I))}/\mathbb{R}(1, \ldots, 1)$$

where modding out by $\mathbb{R}(1, \ldots, 1)$ means by tropical scalar multiplication, i.e., the usual addition by the vector $\lambda(1, 1, 1, 1, 1, 1)$ for $\lambda \in \mathbb{R}$. For example, $(1, 2, 3, 4) \sim (5, 6, 7, 8)$. A tropical projective variety is contained in $\mathbb{TP}^n = \mathbb{R}^{n+1}/\mathbb{R}(1, \ldots, 1)$.

Chapter 4

Defining the Structure of a Set of Lines

In this thesis, we attempt to decribe a particular set of lines as the union of two disjoint conics, but in order to answer our main question, we must first answer another: what does it mean for a set of lines to have a particular structure? This motivates the definition of the Grassmannian, first for affine space, and later for projective space.

4.1 The Classical Grassmannian

Definition 23. The *Grassmannian* $G(k,n)$ is the set of all k-dimensional planes through the origin in K^n.

By itself, this definition is not very useful, but we gain a lot by considering $G(k,n)$ as an embedded object in projective space. But in order to do this, we must first define the wedge product.

Definition 24. Let V be a finite-dimensional vector space. For $v, w \in V$ and $a \in K$, define the *wedge product* $\wedge$ as follows:

(a) $v \wedge v = 0$

(b) $v \wedge w = -w \wedge v$

(c) $(av + w) \wedge u = av \wedge u + w \wedge u$

Products such as these may be added, scaled by constants, or wedged together following the above rules. If an α is the sum of products of the form $v_1 \wedge \ldots \wedge v_k$, where k is fixed, then we say α is a k-form. The set of all k-forms is a vector space, denoted $\Gamma^k(V)$. If we are given an ordered indexed set of vectors $\{v_1, \ldots, v_m\}$, then we may write as shorthand $v_I = v_{i_1} \wedge \ldots \wedge v_{i_k}$, where $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

Proposition 25. *The vector space $\Gamma^k(V)$ has dimension $\binom{n}{k}$ and a basis given by $\{e_I\}$, with I ranging over all subsets of size k of $\{1, \ldots, n\}$ and $\{e_i\}$ being the standard basis.*

Corollary 26. *$\Gamma^k(V)$ is isomorphic to $\mathbb{R}^{\binom{n}{k}}$ as a vector space.*

We note that by the above corollary, we have an induced definition of $\mathbb{P}(\Gamma^k(V)) \cong \mathbb{P}^{\binom{n}{k}-1}$. Additionally, it is standard to order the standard basis lexicographically by subsets I. In the case of $k = 2, n = 4$, the ordering is:

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$$

For a more rigorous development of the wedge product, see Roman (2008).

We are now ready to define the Plücker embedding.

Definition 27. Let W be a k-dimensional plane in K^n spanned by vectors $w_1, \ldots, w_k$. Then we define a map $\Phi : G(k, n) \to \mathbb{P}(\Gamma^k(V))$ by

$$W \mapsto w_1 \wedge \ldots \wedge w_k$$

The map Φ is known as the *Plücker embedding*.

It may be shown that the image of the Grassmannian under this map is a variety. For example, $\Phi(G(2, 4)) = V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})$, where the p_{ij} are the coordinates of $\mathbb{P}^5$. Harris (1995)

Example 28. Let H be the plane in $\mathbb{R}^4$ spanned by the vectors $(1, -2, 3, 0)$ and $(0, 0, 4, -5)$. Then,

$$\begin{aligned}(1, -2, 3, 0) \wedge (0, 0, 4, -5) &= (e_1 - 2e_2 + e_3) \wedge (4e_3 - 5e_4) \\ &= 4e_1 \wedge e_3 - 5e_1 \wedge e_4 - 8e_2 \wedge e_3 + 10e_2 \wedge e_4 + 12e_3 \wedge e_3 - 15e_3 \wedge e_4 \\ &= 4e_1 \wedge e_3 - 5e_1 \wedge e_4 - 8e_2 \wedge e_3 + 10e_2 \wedge e_4 - 15e_3 \wedge e_4,\end{aligned}$$

which corresponds to the point $(0 : 4 : -5 : -8 : 10 : 15) \in \mathbb{P}^5$.

What this means is that any collection of k-planes has a natural embedding in a space we already know and understand. It makes sense for use to say that a set of lines has the structure of a conic; it is a conic in $\mathbb{P}^{\binom{n}{k}-1}$.

We now turn our attention to our real goal, the projective Grassmannian.

Definition 29. The *projective Grassmannian* $\mathbb{G}(k-1, n-1)$ is the set of $(k-1)$-dimensional planes in $\mathbb{P}^{n-1}$.

At first glance, it may seem strange that we defined the projective Grassmannian in terms of $k-1$ and $n-1$. However, we now recall the equivalence established in the previous section between $k-1$-planes in $\mathbb{P}^{n-1}$ and k-planes in K^n. This demonstrates that $\mathbb{G}(k-1, n-1) \cong G(k, n)$. Therefore, our definition of the Plücker embedding extends to the projective Grassmannian.

4.2 The Tropical Grassmannian

For this thesis, we must also define the tropical Grassmannian, which carries over much of the structure from the classical Grassmannian. Speyer and Sturmfels (2004) defines the tropical Grassmannian of lines in $\mathbb{TP}^3$ as the tropical projective variety $\mathcal{T}(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})$, which is the corner locus of $x_{12}x_{34} \oplus x_{13}x_{24} \oplus x_{14}x_{23}$. Speyer and Sturmfels show that there is a bijective correspondence between points in this tropical Grassmannian and lines in $\mathbb{TP}^3$, which we describe below.

Like all other tropicalized varieties, lines in $\mathbb{TP}^3$ may be constructed as the tropicalization of classical lines. However, we provide that following description due to Richter-Gebert et al. (2005).

A tropical line is given by its 6 coordinates in the tropical Grassmannian, $(a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34})$. There are 3 cases, due to the description of the Grassmannian as a corner locus. Either

$$a_{14} + a_{23} = a_{13} + a_{24} \leq a_{12} + a_{34},$$

$$a_{14} + a_{23} = a_{12} + a_{34} \leq a_{13} + a_{24}, \text{or}$$

$$a_{13} + a_{24} = a_{12} + a_{34} \leq a_{14} + a_{23}.$$

In the first case, the line is the union of a line segment with endpoints $(a_{34} - a_{13}, a_{34} - a_{23}, a_{24} - a_{23}, 0)$ and $(a_{24} - a_{12}, a_{14} - a_{12}, a_{24} - a_{23}, 0)$ and four rays, e_1 and e_2 emanating from the first endpoint, and e_3 and $-e_1 - e_2 - e_3$ from the second endpoint. We choose representatives in $\mathbb{TP}^3$ so that the line segment may be viewed in $\mathbb{R}^3$. Note that the line segment is parallel to $e_1 + e_2$.

The other two cases are similar and may be derived by swapping axes in the first case.

Chapter 5

Classical Solution

In this section we prove the classical result through techniques that will assist its potential applications towards proving the tropical analogue. But first, we provide a characterization of quadric surfaces and prove a couple of necessary lemmas.

Let $f(x_0, \ldots, x_3) = \Sigma_{0 \le i \le j \le 3} c_{ij} x_i x_j$ be a homogeneous quadratic polynomial. Then, we may write $Q(x_0, \ldots, x_3) = X^T A X$, where $X = (x_0, \ldots, x_3)$ is a column vector and

$$A = \begin{pmatrix} c_{00} & \frac{1}{2}c_{01} & \frac{1}{2}c_{02} & \frac{1}{2}c_{03} \\ \frac{1}{2}c_{01} & c_{11} & \frac{1}{2}c_{12} & \frac{1}{2}c_{13} \\ \frac{1}{2}c_{02} & \frac{1}{2}c_{12} & c_{22} & \frac{1}{2}c_{23} \\ \frac{1}{2}c_{03} & \frac{1}{2}c_{13} & \frac{1}{2}c_{23} & c_{33} \end{pmatrix}.$$

A matrix may be either singular (degenerate) or invertible (nondegenerate), so we extend this definition to quadratic polynomials by saying that a quadratic is (non)degenerate if and only if the above matrix is (non)degenerate.

Lemma 30. *Let $Q \subset \mathbb{P}^3$ be the nonempty zero set of a single nondegenerate quadratic polynomial $f(x_0, \ldots, x_3)$. Then, there exists a change of coordinates $(x_0, \ldots, x_3) \mapsto (z_0, \ldots, z_3)$ such that $Q = V(z_0 z_3 - z_1 z_2)$.*

Proof. We observe that the matrix A corresponds not only to a quadratic form, but to a symmetric bilinear form B, where $B(X, Y) = X^T A Y$.

Fix nonzero $e_1 \in K^4$ such that $f(e_1) = 0$. Such an e_1 exists because we take Q to be nonempty. Since f is nondegenerate, there exists $f_2 \in K^4$ such that $B(e_1, f_2) \neq 0$. By scaling f_2, assume that $B(e_1, f_2) = \frac{1}{\sqrt{2}}$. It follows that f_2 is not a mulitple of e_1 since $B(e_1, \lambda e_1) = \lambda B(e_1, e_1) = \lambda f(e_1) = 0$.

Let $e_2 = \frac{1}{\sqrt{2}} f_2 - \frac{1}{2} B(f_2, f_2) e_1$ so that

$$\begin{aligned} B(e_2, e_2) &= B(\frac{1}{\sqrt{2}} f_2, \frac{1}{\sqrt{2}} f_2) - 2B(\frac{1}{\sqrt{2}} f_2, \frac{1}{2} B(f_2, f_2) e_1) + B(\frac{1}{2} B(f_2, f_2) e_1, \frac{1}{2} B(f_2, f_2) e_1) \\ &= \frac{1}{2} B(f_2, f_2) - \frac{1}{\sqrt{2}} B(f_2, f_2) B(f_2, e_1) + \frac{1}{4} B(f_2, f_2)^2 B(e_1, e_1) \\ &= \frac{1}{2} B(f_2, f_2) - \frac{1}{2} B(f_2, f_2) \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} B(e_1, e_2) &= B(e_1, \frac{1}{\sqrt{2}} f_2) - B(e_1, \frac{1}{2} B(f_2, f_2) e_1) \\ &= \frac{1}{\sqrt{2}} B(e_!, f_2) - \frac{1}{2} B(f_2, f_2) B(e_1, e_1) \\ &= \frac{1}{2}. \end{aligned}$$

We may repeat this process in the orthogonal complement (with respect to B) of $span\{e_1, e_2\}$ to obtain e_3, e_4 such that $B(e_3, e_3) = B(e_4, e_4) = 0$ and $B(e_3, e_4) = -\frac{1}{2}$ (with the negative from a simple negation of e_3).

Observe that $f(x_0 e_1 + x_1 e_2 + x_2 e_3 + x_3 e_4) = x_0 x_1 - x_2 x_3$. These basis vectors may be renumbered to show the desired result. □

Theorem 31. *Let K be algebraically closed. Let Q be a smooth (non-degenerate) quadric surface in $\mathbb{P}^3$. Then, the set of lines contained in Q is two disjoint conics in $\mathbb{G}(2,4) \subseteq \mathbb{P}^5$. The explicit description of these lines is given in the proof.*

Proof. We may assume by Lemma 30, without loss of generality, that Q is the zero set of the quadratic $z_0 z_3 - z_1 z_2 = 0$. It can be seen that Q is the image of the injective map.

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

$$((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1)$$

In fact, this is a special case of the Segre embedding. We may see immediately that two sets of lines are contained in Q, both families parametrized by $\mathbb{P}^1$.

$$\phi((a_0 : a_1) \times \mathbb{P}^1) = \{(a_0 y_0 : a_0 y_1 : a_1 y_0 : a_1 y_1) | (y_0 : y_1) \in \mathbb{P}^1\}$$

$$\phi(\mathbb{P}^1 \times (b_0 : b_1)) = \{(x_0b_0 : x_0b_1 : x_1b_0 : x_1b_1) | (x_o : x_1) \in \mathbb{P}^1\}$$

We claim these are all the lines contained in Q. To see this, look at the inverse map of ϕ, given by

$$\phi^{-1}((z_0 : z_1 : z_2 : z_3)) = \begin{cases} ((z_0 : z_2), (z_0 : z_1)) & : z_0 \neq 0 \\ ((z_1 : z_3), (z_0 : z_3)) & : z_1 \neq 0 \\ ((z_0 : z_2), (z_2 : z_3)) & : z_2 \neq 0 \\ ((z_1 : z_3), (z_2 : z_3)) & : z_3 \neq 0. \end{cases}$$

This means that ϕ^{-1} is linear, so if we have a line $L \subseteq Q$, then $\phi^{-1}(L)$ is a line in $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, we need only consider lines in $\mathbb{P}^1 \times \mathbb{P}^1$. Such lines may be parametrized

$$(x : y) \mapsto ((a_0x + b_0y : a_1x + b_1y), (c_0x + d_0y : c_1x + d_1y))$$

In order for the image of a line in $\mathbb{P}^1 \times \mathbb{P}^1$ under ϕ to be a line in $\mathbb{P}^3$, we must have either $a_0 = a_1 = b_0 = b_1 = 0$ or $c_0 = c_1 = d_0 = d_1 = 0$, else we would have quadratic terms. But, these lines are precisely the lines we have already described, that is, one of the coordinates $\mathbb{P}^1$ is fixed. Therefore, there are these two families of lines and no other lines in Q.

We now consider the image of these families under the Plücker embedding $G(2,4) \to \mathbb{P}^5$. We begin with lines of the form $(a_0y_0 : a_0y_1 : a_1y_0 : a_1y_1)$. Observe that this line in $\mathbb{P}^3$ may be viewed as a plane through the origin in K^4 spanned by vectors $(a_0, 0, a_1, 0)$ and $(0, a_0, 0, a_1)$.

By definition of the Plücker embedding, this corresponds to the point $(a_0^2 : 0 : a_0a_1 : -a_0a_1 : 0 : a_1^2) \in \mathbb{P}^5$. Since $(a_0 : a_1)$ varies over $\mathbb{P}^1$, this is the parametrization of a conic. We may repeat this process for the second family of lines and obtain the conic determined by parametrization $(b_0, b_1) \mapsto (0 : b_0^2 : b_0b_1 : b_ob_1 : b_1^2 : 0)$.

□

Chapter 6

Application to the Tropical Case

Now we consider the applications of this theorem to the tropical analogue. We begin by demonstrating the existence of two families of 2-dimensional planes in a general quadric surface in $(K^*)^4$ defined by a homogeneous polynomial.

Let $Q = V(f)$ be such a quadric surface in $(K^*)^4$, i.e., $f \in K[x_0, \ldots, x_3] \subseteq K[x_0^{\pm 1}, ..., x_3^{\pm 1}]$ is a homogeneous quadratic polynomial. Then, we let Q' be the zero set of f in $\mathbb{P}^3$. By the classical solution proven in the last section, we have the existence of a set of lines contained in Q', which we denote $L = \{l_\alpha\}$. Let $l_\alpha = V(f_\alpha, g_\alpha)$. Then, by the reverse containment of varieties and ideals, we have

$$(f_\alpha, g_\alpha) \supseteq (f)$$

as subsets of $K[x_0, \ldots, x_3]$. That is, $f = af_\alpha + bg_\alpha$ for some $a, b \in K[x_0, \ldots, x_3] \subseteq K[x_0^{\pm 1}, ..., x_3^{\pm 1}]$. So, the above equation also applies when we consider the ideals as generated in $K[x_0^{\pm 1}, ..., x_3^{\pm 1}]$. As a result, $V(f_\alpha, g_\alpha) \subseteq V(f)$ when viewed as subsets of $(K^*)^4$.

To see the applications to the tropical setting, recall that the definition of a tropical projective variety we are using is the set

$$T(I) = \overline{val(V(I))}/\mathbb{R}(1, ...1)$$

where $I \subseteq K[x_0^{\pm 1}, ..., x_3^{\pm 1}]$ is a homogeneous ideal. If I is generated by two linear forms, then we say $T(I)$ is a line in $\mathbb{TP}^3$. If J is a homogeneous ideal generated by a single quadratic form, then we say $T(J)$ is a quadric surface in $\mathbb{TP}^3$.

Now, if I and J are as described above, then $V(I)$ is a 2-dimensional plane in $(K^*)^4$, $V(J)$ is a quadric surface in $(K^*)^4$, and by the above definition of a tropical variety, $V(I) \subseteq V(J)$ implies $T(I) \subseteq T(J)$ (since $J \subseteq I$). Therefore, by the result of the previous section and the beginning of this section, we already have two families of tropical lines contained in a general tropical quadric.

It is fairly easy to see that these families are conics in $\mathbb{TP}^5$. The explicit form we have as of now is fairly complicated, so we will first describe the general idea by which one takes a family of lines in $\mathbb{TP}^3$ and turns it into a tropical variety in $\mathbb{TP}^5$, then include explicit computations at the end of this section.

Let $I = \langle f = a_1x_1 + a_1x_2 + a_3x_3 + a_4x_4, g = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 \rangle$. According to Richter-Gebert et al. (2005), the coefficients in a tropical basis for I must satisfy the Grassmann-Plücker relation, which defines the Grassmannian in the textbook. Thus, finding a tropical basis for I determines its image in $G(2,4)$. More specifically, I has tropical basis of the form

$$U = \{p_{12}(t)x_2 + p_{13}(t)x_3 + p_{14}(t)x_4, -p_{12}(t)x_1 + p_{23}(t)x_3 + p_{24}(t)x_4,$$

$$-p_{13}(t)x_1 - p_{23}(t)x_2 + p_{34}(t)x_4, -p_{14}(t)x_1 - p_{24}(t)x_2 - p_{34}(t)x_3\}$$

Define a circuit of an ideal to be a minimal set of indices for linear forms in that ideal. Maclagan and Sturmfels (In progress) says that a set of linear forms in I whose supports (set of indices) cover all circuits exactly once is a tropical basis for I. This means that starting with the 2 generators of I, we can find this tropical basis by taking the set of $a_ig - b_if$. This gives us $p_{ij} = a_ib_j - a_jb_i$. So, the point in $\mathbb{TP}^5$ is really just $(a_1, a_2, a_3, a_4) \wedge (b_1, b_2, b_3, b_4)$, but all this reasoning gives us the necessary justifications for doing this.

It's somewhat inaccurate to consider a point in $\mathbb{TP}^5$ in the sense that this point will have all rational coordinates and $G(2,4)$ is the closure of all these points, but when we consider families of lines, this will be much more natural.

For instance, if we consider the a_i and b_j to be homogeneous linear functions of variables, then each p_{ij} is a quadratic form and the tropicalization of the family is a conic.

In the classical quadric $Q = V(z_0z_3 - z_1z_2)$, we have demonstrated the existence of two families of lines. We perform computations with just one of them, since the other case is similar. So consider the line

$$(x_0 : x_1) \mapsto (x_0b_0 : x_0b_1 : x_1b_0 : x_1b_1).$$

This line may be written as $L_b = V(b_1 z_0 - b_0 z_1, b_1 z_2 - b_0 z_3)$. To obtain an arbitrary quadric surface, in the sense that it will give us an arbitrary tropical quadric, we perform a classical change of coordinates on projective space, meaning we determine new coordinates $(x_1 : x_2 : x_3 : x_4)$ such that $(z_0, \ldots, z_3)^T = M(x_1, \ldots, x_4)^T$ for an invertible matrix $M = (m_{ij})$. We change the first index from 0 to 1 so the indices better correspond to those in M. We say that the defining equation of Q written in terms of x_i is arbitrary. Through this transformation, we may see that our family of lines has the following description in x-coordinates.

$$\begin{gathered} L_b = V[(b_1 m_{11} - b_0 m_{21})x_1 + (b_1 m_{12} - b_0 m_{22})x_2 + \\ (b_1 m_{13} - b_0 m_{23})x_3 + (b_1 m_{14} - b_0 m_{24})x_4, \\ (b_1 m_{31} - b_0 m_{41})x_1 + (b_1 m_{32} - b_0 m_{42})x_2 + \\ (b_1 m_{33} - b_0 m_{43})x_3 + (b_1 m_{34} - b_0 m_{44})x_4] \end{gathered}$$

Under the Plücker embedding, we obtain the point $(p_{12} : \ldots : p_{34})$, where

$$\begin{gathered} p_{ij} = (m_{1j}m_{3i} - m_{1i}m_{3j})b_1^2 - (m_{2j}m_{3i} - m_{2i}m_{3j} + m_{1j}m_{4i} - m_{1i}m_{4j})b_0 b_1 + \\ (m_{2j}m_{4i} - m_{2i}m_{4j})b_0^2 \end{gathered}$$

To understand the significance of this equation, observe that the above terms involving entries of M may be rewritten in terms of its 2×2 minors. Fixing columns i and j, we denote the 2×2 minor of M involving rows k and l as M_{kl}. We also flip the sign of all the p_{ij} through projective equivalence to obtain:

$$p_{ij} = M_{13}b_1^2 - (M_{23} + M_{14})b_0 b_1 + M_{24}b_0^2.$$

Note that the other ruling of lines is given by the tropicalization of:

$$p_{ij} = M_{12}b_1^2 + (M_{23} - M_{14})b_0 b_1 + M_{34}b_0^2.$$

We now re-emphasize the main point that $(b_0 : b_1)$ parameterizes the family of lines and each p_{ij} is a homogeneous quadratic polynomial in b_0, b_1, so the tropicalization in $\mathbb{TP}^5$ is a conic. Thus, we have shown the existence of two families of tropical lines in our tropical quadric, whose image in the tropical Grassmannian is the union of two conics. The rest of this thesis is dedicated to determining if those two conics are disjoint.

Chapter 7

Subdivisions

In this chapter, we introduce the notion of the subdivision of the Newton polytope of a tropical polynomial. This allows us to define smoothness of a tropical surface, which we will see for quadric surfaces can have a huge impact on whether or not the tropicalized rulings are distinct. We then give an algorithm which allows us to eliminate many instances of nonsmooth surfaces.

7.1 Definitions and Duality

Definition 32. Let $S \subset \mathbb{R}^n$ be a finite set. Then the *convex hull* of S is the minimal convex set containing all the points of S. This minimal set is unique because the intersection of convex sets is convex. If S is finite, then its convex hull is a *polytope*. A *polyhedron* is the intersection of finitely many closed half-spaces of R^n. One may show that polytopes are bounded polyhedra. Ziegler (1995)

Definition 33. A *face* of a polyhedron P is a set of the form $\{x \in P : w\dot{x} \leq w\dot{y} \text{ for all } y \in P\}$ for some $w \in \mathbb{R}^n$. That is, a face is the minimum of a linear functional on P. A face of a three-dimensional polyhedron may refer to either a vertex, an edge, a usual two-dimensional face, or the entire polyhedron.

Definition 34. A *polyhedral complex* is a collection of polyhedra such that:

1. If P is in the collection, then so is any face of P.

2. If P and Q are both in the collection, then $P \cap Q$ is a face of both P and Q or the empty set.

The *support* of a polyhedral complex is the union of its polyhedra.

Theorem 35. *If f is a tropical polynomial, then $trop(f)$ is the support of a polyhedral complex.*

For a proof, see Maclagan and Sturmfels (In progress).

Definition 36. Let $f(x_1, ..., x_n) = \oplus_{u \in S \subset \mathbb{N}^n} c_u x^u$ be a tropical polynomial, where S is a finite set and if $u = (u_1, ... u_n)$, then x^u denotes $x_1^{u_1} x_2^{u_2} ... x_n^{u_n}$. Then the *Newton polytope* of f is the convex hull of $-S$.

The *subdivision* of a Newton polytope of a tropical polynomial f is the lower part of the convex hull of $\{(-u, c_u) : u \in S\} \subset \mathbb{R}^n$. Itenberg et al. (2009)

We remark that this definition differs slightly from the one given in Itenberg et al. (2009) due to our choice of the minimum operation (as opposed to maximum) in the tropical semiring. This alteration is used to preserve the following duality theorem

Theorem 37. *There is a bijective correspondence $\mathcal{B}$ between the elements of $trop(f)$ (as a polyhedral complex) and the subdivision of its Newton polytope such that if σ is an i-dimensional element of $trop(f)$, then $\mathcal{B}(\sigma)$ is an $(n-i)$-dimensional element of the subdivision of the Newton polytope. Additionally, the linear spans of σ and $\mathcal{B}(\sigma)$ are orthogonal and $\mathcal{B}$ preserves incidence. Itenberg et al. (2009)*

To illustrate the meaning and usefulness of this duality theorem, we give two examples of tropical conics in the plane and their Newton polytopes. Also, note that this implies that there is an edge between two vertices of the subdivision of the Newton polytope if and only if both terms are simultaneously the minimum of all terms of f, a fact that will be useful in the next section of this chapter.

Example 38. Let $g(x, y) = \min\{2x, 2y + 2, 4, y - 2, x - 3, x + y - 4\}$. Then the corner locus of g and the subdivision of its Newton polytope are shown below.

Example 39. Let $h(x, y) = \min\{2x + 3, x + y + 2, y + 4, 1\}$. Its corner locus and Newton polytope subdivision are shown below.

Definition 40. A tropical hypersurface in $\mathbb{R}^n$ is *smooth* if:

1) its Newton polytope is the convex hull of

$$\{(-d, 0, ..., 0), (0, -d, 0, ..., 0), ..., (0, ..., 0, -d)\}$$

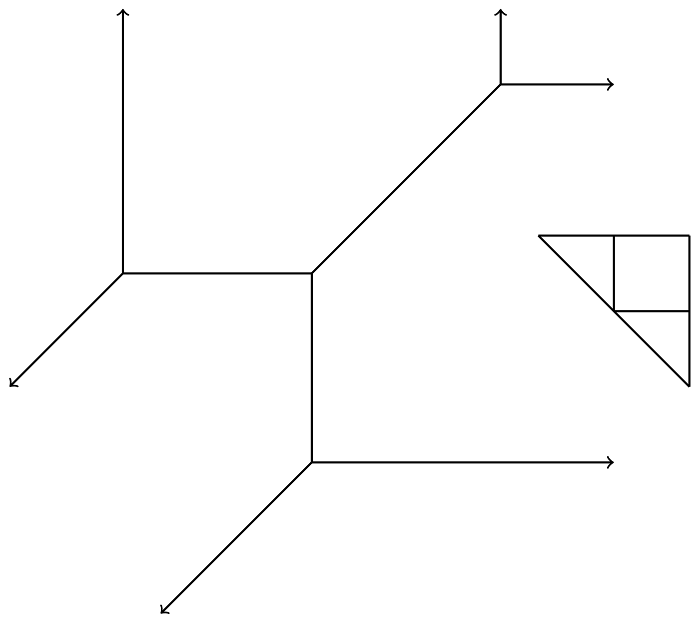

Figure 7.1 Example 38.

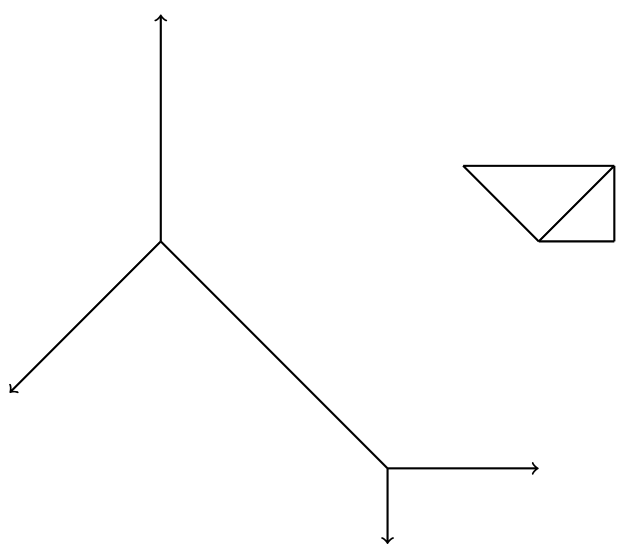

Figure 7.2 Example 39.

, where d is the degree of f and

2) the subdivision of its Newton polytope is maximal, that is, if all n-dimensional polytopes in the complex are simplexes with volume $\frac{1}{n!}$.

The conic shown in Example 38 above is smooth, while the one in Example 39 is not, failing on both counts.

7.2 Algorithm for Smooth Quadrics

For this section, consider the tropical quadric $f(x,y,z,w) = ax^2 \oplus bxy \oplus cxz \oplus dxw \oplus ey^2 \oplus fyz \oplus gyw \oplus hz^2 \oplus izw \oplus jw^2$. Let Q denote the tropical hypersurface defined by f in $\mathbb{TP}^3$. We observe that in projective coordinates, we may take $w = 0$, so that Q may also be considered the corner locus of $f(x,y,z) = ax^2 \oplus bxy \oplus cxz \oplus dx \oplus ey^2 \oplus fyz \oplus gy \oplus hz^2 \oplus iz \oplus j$.

We claim that Q is a smooth tropical surface (the Newtonian subdivision corresponding to f is maximal) only if the following conditions hold:

i) Each of the following equations is satisfied

$$a+e > 2b, a+h > 2c, a+j > 2d, e+h > 2f, e+j > 2g, h+j > 2i$$

ii) None of the following pairs of numbers is equal

$$\{b+h, c+f\}, \{a+f, b+c\}, \{c+e, b+f\}, \{e+i, f+g\}, \{g+h, f+i\}, \{f+j, g+i\}$$

$$\{a+i, c+d\}, \{d+h, c+i\}, \{c+j, d+i\}, \{a+g, b+d\}, \{d+e, b+g\}, \{b+j, d+g\}$$

iii) The set $\{b+i, c+g, d+f\}$ has a distinct minimum.

If any of these conditions fail, then Q is not smooth.

7.3 Proof of Algorithm

In the following proof, let 2Δ denote the simplex with vertices $(0,0,0)$, $(-2,0,0)$, $(0,-2,0)$, and $(0,0,-2)$.

Suppose Q is smooth, so the subdivision of its Newton polytope is maximal. The condition i) simply says that each of the terms in the polynomial contribute to the function defined by it, that each term determines the unique minimum for some (x,y,z,w). For instance, $ax^2 \oplus bxy \oplus ey^2 = \min\{2x+a, x+y+b, 2y+e\}$. For the $x+y+b$ term to contribute, we require that it be strictly less than the average of the other two terms, which leads to the inequality $a+e > 2b$. The other 5 inequalities follow similarly.

Next, each face of the polytope must also have a maximal subdivision. Consider the face F of 2Δ that is the convex hull of $(0,0,0)$, $(-2,0,0)$ and

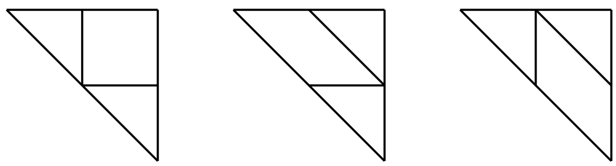

Figure 7.3 The three ways a maximal subdivision can 'go wrong.'

$(0, -2, 0)$. If F does not have a maximal subdivision, then it must contain a polygon that contains one of the three area 1 paralellograms shown in the figure below, meaning that 4 distinct terms give rise to $f(p)$ at some point p. (This fact may be verified by checking all possible polygons of area ≥ 1 and eliminating those which involve 3 colinear vertices and thus cannot correspond to a dual tropical element given condition i).) This is why we consider the 3 pairs $\{b+j, d+g\}, \{a+g, b+d\}$, and $\{d+e, b+g\}$. Now suppose for instance, that $b+j = d+g$. Then, choose x_0, y_0 such that

$$j = d + x_0 = g + y_0 = b + x_0 + y_0$$

Thus, the four terms of f j, dx, gy, and bxy are all equal for such x_0 and y_0. Choose z_0 sufficiently large such that we need only consider the other terms ax^2 and ey^2 in determining $f(x_0, y_0, z_0)$. By condition i), $a + j > 2d$. By substitution, $a + d + x_0 > 2d$, so $a + 2x_0 > d + x_0$. Similarly, $e + 2y_0 > g + y_0$, so the four equal terms are all equal to $f(x_0, y_0, z_0)$ at (x_0, y_0, z_0). Thus, $b + j = d + g$ implies nonsmoothness, so smoothness implies $b + j \neq d + g$. The other 11 terms in condition ii) follow by considering the other two polygons shown in the figure and the other faces of the simplex.

A maximal subdivision of 2Δ contains 25 edges, 24 of which have already been determined and one edge that intersects the interior of the simplex. Maclagan and Sturmfels (In progress) This edge may connect $(-1, -1, 0)$ to $(0, 0, -1)$, $(-1, 0, -1)$ to $(0, -1, 0)$ or $(0, -1, -1)$ to $(-1, 0, 0)$. Suppose, without loss of generality, that we have the first case, and let (x_0, y_0, z_0) be a point in which bxy and iz are the distinct minimum of the ten terms of f. Then, $b + x_0 + y_0 = i + z_0 < c + x_0 + z_0, g + y_0, d + x_0, f + y_0 + z_0$. In particular,

$$b + i + x_0 + y_0 + z_0 < c + g + x_0 + y_0 + z_0, d + f + x_0 + y_0 + z_0$$

and $b + i$ is the distinct minimum of $\{b + i, c + g, d + f\}$. The other two edges result in the other two terms being the distinct minimum of the set.

Remark 41. We believe that this algorithm may be stated as an 'if and only if,' but as we will see, this was not necessary to restrict the quadric surfaces we consider to those whose tropicalized rulings are distinct.

Chapter 8

Tropical Parametrization

As we saw in Chapter 6, our problem involves two conics in the tropical Grassmannian. Much of the existing theory on tropical varieties involves the intersection of tropical hypersurfaces (for instance, the use of a tropical basis). Thinking back to the classical setting, a conic in $\mathbb{P}^5$ is the intersection of many hypersurfaces, yet may be parametrized in a single variable. In this section, we develop a method to extend this simple description to tropical curves which are tropicalizations of parametrized curves.

8.1 Line Example

Consider the line $L = V(x + y - 1) \subseteq K^2$. By the Fundamental Theorem of Tropical Geometry, L tropicalizes to $\mathcal{T}(L)$, the corner locus of the tropical polynomial:

$$x \oplus y \oplus 0,$$

which is depicted in Figure 8.1

Since L is a line, it may not only be written as the zero locus of a linear polynomial, but also as the image of a parameter, i.e., $L = \{(a, 1-a) | a \in K\}$

Thus, $\mathcal{T}(L) = cl(\{(val(a), val(1-a)) | a \in K\})$. We could parameterize $\mathcal{T}(L)$ over K, but here it will be more useful to conceive a 'parametrization' over $\mathbb{Q}$ with $val(a)$ as the parameter.

If we think of a as a Puiseux series $a(t)$, then it is easy to see that when $val(a) < 0$, a and $1-a$ have the same first term. Thus, $val(1-a) = val(a)$. Similarly, when $val(a) > 0$, 1 and $1-a$ have the same first term and $val(1-a) = val(1) = 0$. If we only require that $val(a) = 0$, then we may choose $a = 1 + t^k$, where $k \in \mathbb{Q}$ and $k \geq 0$, so that $val(1-a) = val(t^k) = k$.

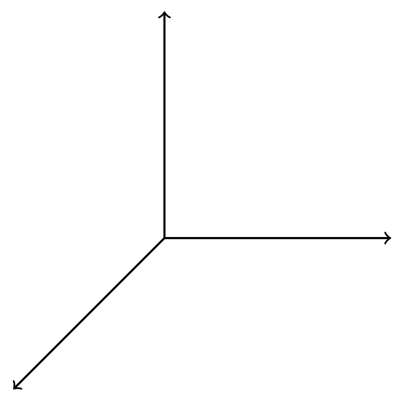

Figure 8.1 The tropical graph of $x \oplus y \oplus 0$

Looking at Figure 8.1, we can see this phenomenon play out graphically. $\mathcal{T}(L)$ is 'almost' a piecewise linear function of x. It is a piecewise linear function for $x \neq 0$, but at $x = 0$, it has a range of $[0, \infty)$. We formalize this notion for more general curves in the next section.

8.2 General Curves in the Plane

In order to succinctly describe points where the image "goes off to infinity," we introduce the following definition.

Definition 42. Let $f(x)$ be a tropical polynomial of one variable. Then, if $x_0 \in \mathcal{T}(f)$, then x_0 is a *breakpoint* of f.

This allows us to state the proposition central to this section.

Proposition 43. *Let $f(x) \in K[x]$ and $g(x,y) = y - f(x) \in K[x,y]$. Then, $\mathcal{T}(g)$ is the union of the graph of $trop(f)$ and the collection of $\{x_i\} \times [f(x_i), \infty)$, where the x_i are the breakpoints of $trop(f)$.*

Proof. Let $trop(f)(x) = a_n x^n \oplus ... \oplus a_1 x \oplus a_0$. Then, $\mathcal{T}(g)$ is the corner locus of $trop(g) = y \oplus a_n x^n \oplus ... \oplus a_1 x \oplus a_0$.

($\subseteq$) Suppose $(x_0, y_0) \in \mathcal{T}(g)$. Then, two terms of $trop(g)$ obtain their minimum at (x_0, y_0). If those terms are y and $a_i x^i$, then $y_0 = a_i x_0^i = trop(f)(x_0)$. If those terms are $a_i x^i$ and $a_j x^j$, then $a_i x_0^i = a_j x_0^j$, meaning x_0 is a breakpoint of $trop(f)$ and $y_0 \geq ix_0 + a_i = jx_0 + a_j = trop(f)(x_0)$, meaning (x_0, y_0) is on one of the rays described in the proposition statement.

($\supseteq$) Suppose x_0 is a breakpoint of $trop(f)$ and (x_0, y_0) is on the ray $\{x_0\} \times [trop(f)(x_0), \infty)$. Then, for some $i \neq j$, $a_i x_0^i = a_j x_0^j$ and $y_0 \geq ix_0 + a_i = jx_0 + a_j = trop(f)(x_0)$, so $(x_0, y_0) \in \mathcal{T}(g)$. If $y_0 = trop(f)(x_0)$, then $trop(g)$ obtains its minimum at both the terms y and whichever term of $trop(f)$ is the minimum at x_0, so $(x_0, y_0) \in \mathcal{T}(g)$. $\square$

Corollary 44. *Let f and g be as stated in the proposition above and let $b \in K$. Then the possible values of $val(f(b))$ are:*

- *A dense subset of $[trop(f)(b), \infty)$ if $val(b)$ is a breakpoint of $trop(f)$*
- *$trop(f)(b)$ otherwise*

Proof. In the classical setting, $V(g) = \{(a, f(a)) | a \in K\}$. By the Fundamental Theorem of Tropical Geometry, $\mathcal{T}(g) = cl(\{(val(a), val(f(a)) | a \in K\})$. Thus, in the xy-plane, $val(a) = x$ and $val(f(a)) = y$. But, by the proposition above, $\mathcal{T}(g)$ is also the union of the graph of $trop(f)$ with rays towards ∞ described as $\{x_i\} \times [f(x_i), \infty)$ at the breakpoints x_i. Equating the two descriptions proves the corollary. $\square$

Remark 45. This corollary could likely be proven with purely algrebraic techniques, but we provide a geometric proof for visualization purposes.

8.3 Higher Dimensions

We now consider one-dimensional curves in higher dimensions. Suppose $\gamma : \mathbb{P}^1 \to \mathbb{P}^n$ by

$$(x_0 : x_1) \mapsto (g_0(x_0 : x_1) : ... : g_n(x_0 : x_1))$$

is a parametrization of a curve C in $\mathbb{P}^n$, where the g_i are homogeneous polynomials of the same degree. We use the parametrization technique developed in the previous section to describe its tropicalization, which we denote $trop(C)$. (Not to be confused with the tropicalization of a polynomial, of course.)

First, we break $\mathbb{P}^1$ into elements that may be written as $(x : 1)$ and the single element $(1 : 0)$ so we may consider C as parameterized by a single variable plus an additional point. We denote these two parts as C' and p, so $C = C' \cup \{p\}$. Note that it is trivial to determine $val(p)$.

Let $f_i(x) = g_i(x : 1)$ so C' is the image of the following parametrization $\gamma' : K \to \mathbb{P}^n$ by

$$x \mapsto (f_0(x) : ... : f_n(x)).$$

Thus, $trop(C') = cl(\{(val(f_0(x)) : ... : val(f_n(x)))|x \in K\})$. If $val(x)$ is given, then the previous section tells us the possible corresponding points of C', with one caveat. According to the Structure Theorem for Tropical Varieties Maclagan and Sturmfels (In progress), the tropicalization of a 1-dimensional classical variety is a polyhedral complex of dimension 1. This means if $trop(f_i)$ and $trop(f_j)$ ($i \neq j$) share a breakpoint $val(x^*)$ and $(val(f_0(x^*)) : ... : val(f_n(x^*))) \in C'$, then only one of $val(f_i(x^*))$ and $val(f_j(x^*))$ may exceed $trop(f_i(x^*))$ or $trop(f_j(x^*))$, respectively.

We remark that this technique only maps out a curve in $\mathbb{R}^{n+1}$. Since we only considered points of the form $(x : 1) \in \mathbb{P}^1$, this curve does not necessarily contain the entire equivalence class of points in $\mathbb{TP}^n$. However, it does give us a single point for each equivalence class.

Chapter 9

Computer Generation of Examples

9.1 Integer Method

In performing the arithmetic of tropicalizations, particularly that of valuations, one generally must only consider the first term of a Puiseux series $a(t)$. So instead of using $a(t) = t^{-3} + 2it^{-1} + \dots$, we may wish to use $a(t) = t^{-3}$ instead. However, if we are forced to compute $val(a(t) - b(t))$, then it is possible that the first terms of both series will cancel. If this is the case, then we must consider more terms of the Puiseux series. But if not, then $val(a(t) + b(t)) = \min\{val(a), val(b)\}$ and the tropicalization of classical objects may be computed solely in terms of the valuations of coefficients.

In general, the assumption that $val(a(t) + b(t)) = \min\{val(a), val(b)\}$ is not valid, of course, but for the sake of constructing examples, it is nice to know when it is. There are two ways of dealing with this problem, as described below.

First, we may restrict ourselves to cases in which we never take the valuation of the sum of terms with different valuations. That is, we choose coefficients carefully enough that whenever we compute $val(a(t) + b(t))$, $val(a) \neq val(b)$ so that $val(a(t) + b(t)) = \min\{val(a), val(b)\}$. In this case, we may assume that all Puiseux series coefficients are single terms.

Example 46. Relating to our original problem, in which we use a classical change of coordinates matrix M, we would like to encode M in a computer as a matrix of integers.

Suppose

$$M = \begin{pmatrix} t^{-2}+t^{1}+... & t^{2}+t^{6}+... & 1+t^{5}+... & t^{-10}+t^{-9}+... \\ t^{-3}+t^{5}+... & t^{-2}+1+... & t^{3}+t^{6}+... & 1+t+... \\ t^{8}+t^{9}+... & t+t^{2}+... & t^{-4}+t^{-2}+... & t^{-3}+t^{-2}+... \\ t^{-9}+t^{4}+... & t^{-1}+1+... & t^{5}+t^{7}+... & t^{3}+t^{6}+... \end{pmatrix}$$

It can be verified (by hand or though the programs described later in this chapter) that in computation of the tropical quadric derived from this matrix, and in computation of the lines contained in the corresponding tropical surface, we never add or subtract terms of equal valuation. Therefore, we would obtain the exact same tropicalizations were we to use

$$M' = \begin{pmatrix} t^{-2} & t^{2} & 1 & t^{-10} \\ t^{-3} & t^{-2} & t^{3} & 1 \\ t^{8} & t & t^{-4} & t^{-3} \\ t^{-9} & t^{-1} & t^{5} & t^{3} \end{pmatrix}$$

For the purposes of computation, we replace each entry with its valuation to obtain

$$M'' = \begin{pmatrix} -2 & 2 & 0 & -10 \\ -3 & -2 & 3 & 0 \\ 8 & 1 & -4 & -3 \\ -9 & -1 & 5 & 3 \end{pmatrix}$$

Second, we note that above, we used a canonical choice of a single-termed $a(t)$ such that $val(a) = c$, that is, t^c. However, we could have just as easily used $2t^c$, or it^c, or $1000t^c$. When using the canonical choices, cancellation happens easily, but when we choose other single term coefficients with the same valuation, cancellation does not occur so easily. The test used in the computer program is strict, in the sense that it only works with entries t^k, but this is not always necessary when computing examples by hand.

9.2 Programming in Python

Given the tropical version of parametrization developed earlier, the computations involved in determining whether two tropical conics in $\mathbb{TP}^5$ intersect are trivial, though rather tedious. A major task in this thesis then, was to develop a computer program that could look at many tropical quadric surfaces and determine if the tropical conics associated to them intersected.

The first two functions are shorter and found in the appendix with all helper functions.

smoothtest: This function takes in the defining polynomial of a tropical quadric surface in $\mathbb{TP}^3$ and determines if the surface is smooth. If so, then it returns the combinatorial type of the surface. This is done by listing the edges of the corresponding Newtonian subdivision. This assumes that the algorithm from Chapter 7 is an 'if and only if,' but at least narrows our attention to surfaces that are almost smooth, if not completely. That is, we have not strictly shown that these surfaces are smooth, but all smooth surfaces will pass our test. As we will see, this is enough to restrict our attention to surfaces which, in our simulations, were found to exhibit distinct rulings.

findmatrix: finds a matrix M with integer coefficients such that the corresponding tropical quadric surface is smooth (using smoothtest) and such that the lines may be computed without worry of first term cancellation as described in the previous section.

findconics: given a matrix M, finds the tropical conics in $\mathbb{TP}^5$ associated to the tropicalized rulings of the classical surface.

conicintersect: This is the most important function, since it determines if two given conics intersect or not. It works as follows. For each of the two conics, it extracts the corresponding breakpoints of the quadratics in each coordinate and compiles them into a single list. If a point p is on the conic, then it must be the image corresponding to a parameter that is a breakpoint or is in an interval between breakpoints. (Once we are past all of the breakpoints, then each of the coordinates changes at the same rate, meaning we get the same point projectively.) Thus, each intersection may be characterized as either 'breakpoint-breakpoint,' 'breakpoint-interval,' or 'interval-interval.' This function matches up all possibilities and attempts to find an intersection in each case.

When the potential intersection is interval-interval, this is trivial, since it simply amounts to solving a system of linear equations. However, when there is a breakpoint involved, it must first solve the system for all coordinates in which the quadratics do not have the breakpoint in question, then test the remaining coordinates. Working through each possible combination, it returns 'yes' if the conics intersect and 'no' if they do not.

Running conicintersect on 157,000 examples of matrices of integer entries in $[-1000, 1000]$ that determine smooth quadric surfaces, none of them were found to have tropical conics which intersected. While we do not offer a proof, this certainly supports the hypothesis that all smooth quadric surfaces have distinct rulings.

Chapter 10

Examples

In this chapter, we explore two examples of quadric surfaces, one which is not smooth, the other which has 25 edges in the subdivision of its Newtonian polytope. The first demonstrates significant intersection of the tropical conics in the Grassmannian, while the second has disjoint conics.

10.1 Overlapping Conics

Let M be the change of coordinates matrix in $K^{4\times 4}$

$$\begin{pmatrix} t^{-30} & t^{25} & t^{-5} & t^{47} \\ t^{-20} & t^{6} & t^{-33} & t^{34} \\ t & t^{31} & t^{-9} & t^{34} \\ t^{-20} & t^{-10} & t^{-9} & t^{-22} \end{pmatrix}$$

which takes $z_0z_3 - z_1z_2$ to the polynomial $g(x_0, x_1, x_2, x_3)$. Then, if f is the tropicalization of g, $f = -50x^2 \oplus -40xy \oplus -39xz \oplus -52x \oplus 15y^2 \oplus -15yz \oplus 3y \oplus -42x^2 \oplus -27z \oplus 25$.

Recall that the first conic in the tropical Grassmannian is given by the coordinates:

$$a_{ij} = val(M_{13}b_1^2 - (M_{23} + M_{14})b_0b_1 + M_{24}b_0^2)$$

for $1 \leq i < j \leq 4$, with $(b_0 : b_1)$ ranging over $\mathbb{P}^1$. Recall that by M_{kl}, we mean the 2×2 minor of M involving rows k and l and columns i and j. In our example, we have

$$a_{12} = val((t - t^{26})b_1^2 + (t^{11} - t^7 + t^{-40} - t^5)b_0b_1 + (t^{-30} - t^{-14})b_0^2).$$

By the parametrization technique outlined in Chapter 8, we may first consider a_{12} written as

$$a_{12} = val((t - t^{26})b^2 + (t^{11} - t^7 + t^{-40} - t^5)b + (t^{-30} - t^{-14})).$$

Furthermore, if $B = val(b)$, then

$$a_{12} \begin{cases} = 2B+1 & : B < -41 \\ \geq -81 & : B = -41 \\ = B - 40 & : -41 < B < 10 \\ \geq -30 & : B = 10 \\ = -30 & : B > 10 \end{cases}$$

That is, $a_{12} = \min\{2B+1, B-40, -30\}$ when B is not a breakpoint of the tropical polynomial (-41 or 10). Continuing this process for the other 5 coordinates, we find the breakpoints $\{-56, -41, -31, -28, -19, -14, 0, 10\}$. Note that we are able to determine the entire behavior of the a_{ij} from their values at their breakpoints since the slope of $trop(p_{ij})$ is 2 when B is less than the first breakpoint, 1 in between the two breakpoints and 0 when B is greater than the second breakpoint. Thus, we can summarize the behavior of the a_{ij} in the following chart, where the horizontal axis corresponds to B, the vertical axis to the coordinate a_{ij}, and the table values are the possible outputs. A '+' symbolizes a breakpoint, so 6^+, for instance, means $a_{ij} \in [6, \infty)$.

B	-56	-41	-31	-28	-19	-14	0	10
a_{12}	-111	-81^+	-71	-68	-59	-54	-40	-30^+
a_{13}	-151	-121	-101	-95	-77	-67^+	-53^+	-53
a_{14}	-108^+	-93	-83	-80	-71	-66	-52	-42^+
a_{23}	-96	-66	-46^+	-43^+	-43	-43	-43	-43
a_{24}	-53^+	-38	-28	-25	-16^+	-16	-16	-16
a_{34}	-83^+	-68	-58	-55^+	-55	-55	-55	-55

For the other ruling, the corresponding coordinates are given by

$$a'_{ij} = val(M_{12}b_1^2 - (-M_{23} + M_{14})b_0 b_1 + M_{34}b_0^2)$$

which gives us the following chart:

B	-56	-50	-41	-16	$-\mathbf{7}$	$-\mathbf{4}$	6	10	24	31
a'_{12}	-136	-124	-106	-56^+	$-\mathbf{47}$	$-\mathbf{44}$	-34	-30	-16	-9^+
a'_{13}	-175	-163	-145	-95	$-\mathbf{77}$	$-\mathbf{71}$	-51	-43^+	-29^+	-29
a'_{14}	-108^+	-102	-93	-68	$-\mathbf{59}$	$-\mathbf{56}$	-46	-42	-28	-21^+
a'_{23}	-120	-108	-90	-40	$-\mathbf{22}^+$	$-\mathbf{19}^+$	-19	-19	-19	-19
a'_{24}	-59	-47^+	-38	-13	$-\mathbf{4}$	$-\mathbf{1}$	9^+	9	9	9
a'_{34}	-98	-86	-68^+	-43	$-\mathbf{34}$	$-\mathbf{31}^+$	-31	-31	-31	-31

One property of parametrizations, both tropical and classical, is that if two parametrized curves intersect, it need not be when their parameters are equal. The images of these parametrizations are invariant under appropriate transformation of the parameter. Additionally, we are working in tropical projective space, so the images are also invariant under addition by scalar multiples of the vector $(1,1,1,1,1,1)$. In order to demonstrate the overlap of the two conics, we reprint the first chart with the parameter translated by 24 and the images translated by $(24,24,24,24,24,24)$. The overlapped sections are bolded.

B	-32	-17	$-\mathbf{7}$	$-\mathbf{4}$	5	10	24	34
a_{12}	-87	-57^+	$-\mathbf{47}$	$-\mathbf{44}$	-35	-30	-16	-6^+
a_{13}	-127	-97	$-\mathbf{77}$	$-\mathbf{71}$	-59	-43^+	-29^+	-29
a_{14}	-84^+	-69	$-\mathbf{59}$	$-\mathbf{56}$	-47	-42	-28	-18^+
a_{23}	-72	-42	$-\mathbf{22}^+$	$-\mathbf{19}^+$	-19	-19	-19	-19
a_{24}	-29^+	-14	$-\mathbf{4}$	$-\mathbf{1}$	8^+	8	8	8
a_{34}	-59^+	-44	$-\mathbf{34}$	$-\mathbf{31}^+$	-31	-31	-31	-31

As the charts above show, the two tropical conics intersect, and not just at a single point, but at a continuum of solutions. In fact, they intersect for all $B \in [-16,5]$.

10.2 Disjoint Conics

Now let M be the change of coordinates matrix

$$\begin{pmatrix} 2t^{10} & t^{-5} & t^{5} & t^{-9} \\ t^{6} & t^{-9} & t^{7} & t^{-1} \\ t^{-4} & t^{6} & t^{-7} & t^{8} \\ t^{-7} & t^{4} & t^{7} & t^{9} \end{pmatrix}$$

which takes $z_0z_3 - z_1z_2$ to the polynomial $g(x_0, x_1, x_2, x_3)$. Then, if f is the tropicalization of g, $f = 2x^2 \oplus -13xy \oplus -2xz \oplus -16x \oplus -3y^2 \oplus -16yz \oplus -5y \oplus 0x^2 \oplus -8z \oplus 0$.

We remark that the 2 in the upper left corner is present to prevent the cancellation of terms that would otherwise occur in computing the lines in $trop(f)$. Also, the 0's are necessary in the expression of f because 0 is not the additive identity of the tropical semiring.

Repeating the same process as before, we get the two following charts for the conics.

B	-4	-3	-2	0	2	5	8	14
a_{12}	-17^+	-16^+	-16	$\mathbf{-16}$	$\mathbf{-16}$	$\mathbf{-16}$	$\mathbf{-16}$	$\mathbf{-16}$
a_{13}	-7	-5^+	-4	-2	0^+	0	0	0
a_{14}	-21	-19^+	-18	-16	-14	-11	-8^+	-8
a_{23}	-20^+	-19	-18	-16	-14	-11	-8^+	-8
a_{24}	-11	-9	-7^+	-5	-3	0^+	0	0
a_{34}	$\mathbf{-24}$	$\mathbf{-22}$	$\mathbf{-20}$	$\mathbf{-16}$	-12	-6	0^+	6^+

B	-14	-13	-12	-6	2	10	12	13	17
a_{12}	-27^+	$-26+$	-25	-19	-11	-3	-1^+	-1	-1
a_{13}	-17	-15^+	-14^+	-14	$\mathbf{-14}$	$\mathbf{-14}$	$\mathbf{-14}$	$\mathbf{-14}$	$\mathbf{-14}$
a_{14}	-31	-29^+	-28	-22	-14	-6	-4	-3	1^+
a_{23}	-32	-30	-28^+	-22	-14	-6	-4	-3^+	-3
a_{24}	$\mathbf{-46}$	$\mathbf{-44}$	$\mathbf{-42}$	$\mathbf{-30}$	$\mathbf{-14}$	2	6	8^+	12^+
a_{34}	-30	-28	-26	-14^+	-6	2^+	2	2	2

We now present a simple argument for why these two conics do not intersect. This type of argument may not necessarily be applied to all pairs of non-intersecting conics, but we have chosen this example to make the argument as simple as possible.

Suppose the conics did intersect at some point p. Then, the lowest coordinate of p must be in the same position no matter which representative of p we choose, as equivalence is up to addition by scalar multiples of the vector $(1,1,1,1,1,1)$. In both charts above, we have bolded the lowest term at each breakpoint. Note that these are necessarily the lowest terms since none of the bolded boxes have a '+' in them. The first conic only has the a_{12} and a_{34} terms as the lowest, whereas the second conic only has the a_{13} and a_{24} terms as the lowest. Thus, the two conics cannot intersect where this is the case. There is one exception we must check, the boundary point where the lowest term changes from a_{34} to a_{12} (and a_{24} to a_{13}). This occurs at $B = 0$ for the first conic and $B = 2$ for the second, so we have included both columns on our charts despite neither being a breakpoint. A quick inspection of the two columns shows that they are not equal and we confirm the conics do not intersect.

Chapter 11

Future Plans

While there is certainly strong evidence for the claim that smooth tropical quadric surfaces hvae distinct rulings, a proof remains unfound. One question we ask is if there is a connection between combinatorial type (the subdivision of the Newton polytope) and whether or not the tropicalized rulings are distinct. For instance, if one quadric surface were to have distinct rulings, would all other surfaces of the same combinatorial type have distinct rulings as well? If so, then a proof could be found using computer experimentation as done in this thesis. However, the programs used here were only able to output 168 of the 192 possible maximal subdivisions Maclagan and Sturmfels (In progress) of 2Δ. This must be due to the restrictions we made in assuming a lack of term cancellation in classical calculations, but not necessarily expected. Using purely random coefficients, we were able to find surfaces of all 192 types, so we wonder if those 24 missing types have nondistinct rulings.

We also acknowledge the failure of tropicalized rulings to be distinct in the case of some nonsmooth quadric surfaces. We wonder if there is a proof strategy that would prove the contrapositive of our hypothesis, that if two lines through a point tropicalize to the same line, then the tropicalization of that point must correspond to an element of the subdivision which is larger than would be allowed in a maximal subdivision.

Lastly, we would like to consider the presence of other lines on the quadric surface, that may not arise as the tropicalizations of classical lines on the classical surface. In general tropical lifting problems such as this are difficult and an answer to this question would be necessary to complete the answer to the posed problem.

Appendix A

Code

Here is the commented code used to find examples and test the smoothness of tropical quadrics:

```
import random
import math


def dismin(L):

    # determines if a list has a distinct minimum, returns 1 if yes, 0 if not
    # This will be useful in determining if a tropical quadric is smooth.
    m=min(L)
    count=0
    for i in range(len(L)): #counts how many times the minimum shows up
        if L[i]==m:
            count+=1

    if count==1:
        return 1
    else:
        return 0

def makematrix(N):

    # makes a random matrix with integer coefficients that correspond to
    # valuations of Puiseux series.

    m11=float(random.randint(-N,N))
    m12=float(random.randint(-N,N))
    m13=float(random.randint(-N,N))
    m14=float(random.randint(-N,N))
    m21=float(random.randint(-N,N))
    m22=float(random.randint(-N,N))
    m23=float(random.randint(-N,N))
    m24=float(random.randint(-N,N))
    m31=float(random.randint(-N,N))
    m32=float(random.randint(-N,N))
    m33=float(random.randint(-N,N))
    m34=float(random.randint(-N,N))
    m41=float(random.randint(-N,N))
    m42=float(random.randint(-N,N))
    m43=float(random.randint(-N,N))
    m44=float(random.randint(-N,N))

    return [[m11,m12,m13,m14],[m21,m22,m23,m24],[m31,m32,m33,m34],[m41,m42,m43,m44]]\\


def m2c(M):
```

```
# takes matrix M (list of 16 integers) and turns it into
# tropical quadric coefficients by treating M as a change of
# basis matrix applied to the standard classical quadric:
# z0z3-z1z2=0, then tropicalizing the result

m11=M[0][0]
m12=M[1][0]
m13=M[2][0]
m14=M[3][0]
m21=M[0][1]
m22=M[1][1]
m23=M[2][1]
m24=M[3][1]
m31=M[0][2]
m32=M[1][2]
m33=M[2][2]
m34=M[3][2]
m41=M[0][3]
m42=M[1][3]
m43=M[2][3]
m44=M[3][3]

c1=min([m11+m41,m21+m31])
# In the classical setting, (m11m41-m21m31) is the coefficient
# on $x^2$

c2=min([m11+m42,m21+m32,m31+m22,m41+m12])
c3=min([m11+m43,m21+m33,m31+m23,m41+m13])
c4=min([m11+m44,m21+m34,m31+m24,m41+m14])
c5=min([m12+m42,m22+m32])
c6=min([m12+m43,m22+m33,m32+m23,m42+m13])
c7=min([m12+m44,m22+m34,m32+m24,m42+m14])
c8=min([m13+m43,m23+m33])
c9=min([m13+m44,m23+m34,m33+m24,m43+m14])
c10=min([m14+m44,m24+m34])

if dismin([m11+m41,m21+m31])==0:
    # ensures that we are allowed
    # to let val(a+b)=min(val(a),val(b))

    # putting all zeroes makes the rest fail
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m11+m42,m21+m32,m31+m22,m41+m12])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m11+m43,m21+m33,m31+m23,m41+m13])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m11+m44,m21+m34,m31+m24,m41+m14])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m12+m42,m22+m32])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m12+m43,m22+m33,m32+m23,m42+m13])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m12+m44,m22+m34,m32+m24,m42+m14])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m13+m43,m23+m33])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m13+m44,m23+m34,m33+m24,m43+m14])==0:
    return [0,0,0,0,0,0,0,0,0,0]

if dismin([m14+m44,m24+m34])==0:
    return [0,0,0,0,0,0,0,0,0,0]


return [c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,'split']+M
# returns the coefficients with the original matrix at the end, so
# we can comose functions and still see what the matrix is.
```

```
def smoothtest(C):

    a=C[0]
    # Note that this ignores any list elements past the 10th,
    # such as the elements of the matrix that were attached at the end of
    # m2c.

    b=C[1]
    c=C[2]
    d=C[3]
    e=C[4]
    f=C[5]
    g=C[6]
    h=C[7]
    i=C[8]
    j=C[9]

    combtype=[]
    # list of connections made in the construction of the
    # subdivision antiw: refers to the face of the simplex (Newton
    # polytope) opposite the vertex corresponding to the $w^2$ term.

    if b+h<c+f: #the following steps reflect condition ii) of the  algorithm
        combtype+=['bh']

    if b+h>c+f:
        combtype+=['cf'] # adds this connection to the list if it is satisfied

    if b+h==c+f:
        return 'fail'

    if b+c<a+f:
        combtype+=['bc']

    if b+c>a+f:
        combtype+=['af']

    if b+c==a+f:
        return 'fail'

    if b+f<c+e:
        combtype+=['bf']

    if b+f>c+e:
        combtype+=['ce']

    if b+f==c+e:
        return 'fail'

    # antix

    if e+i<f+g:
        combtype+=['ei']

    if e+i>f+g:
        combtype+=['fg']

    if e+i==f+g:
        return 'fail'

    if f+i<h+g:
        combtype+=['fi']

    if f+i>h+g:
        combtype+=['gh']

    if f+i==h+g:
        return 'fail'

    if g+i<f+j:
        combtype+=['gi']

    if g+i>f+j:
        combtype+=['fj']

    if g+i==f+j:
```

```
    return 'fail'

# antiy

if a+i<c+d:
    combtype+=['ai']

if a+i>c+d:
    combtype+=['cd']

if a+i==c+d:
    return 'fail'

if c+i<d+h:
    combtype+=['ci']

if c+i>d+h:
    combtype+=['dh']

if c+i==d+h:
    return 'fail'

if d+i<c+j:
    combtype+=['di']

if d+i>c+j:
    combtype+=['cj']

if d+i==c+j:
    return 'fail'

# antiz

if a+g<b+d:
    combtype+=['ag']

if a+g>b+d:
    combtype+=['bd']

if a+g==b+d:
    return 'fail'

if b+g<d+e:
    combtype+=['bg']

if b+g>e+d:
    combtype+=['de']

if b+g==e+d:
    return 'fail'

if d+g<b+j:
    combtype+=['dg']

if d+g>b+j:
    combtype+=['bj']

if d+g==b+j:
    return 'fail'


# distinct minimum for "3D" terms

if dismin([b+i,c+g,d+f])==0: #condition iii)
    return 'fail'
else:
    if b+i==min([b+i,c+g,d+f]):
        combtype+=['bi']

    if c+g==min([b+i,c+g,d+f]):
        combtype+=['cg']

    if d+f==min([b+i,c+g,d+f]):
        combtype+=['df']

# basic part of smooth test
```

```
    if a+e<=2*b: #condition i)
        return 'fail'

    if a+h<=2*c:
        return 'fail'

    if a+j<=2*d:
        return 'fail'

    if e+h<=2*f:
        return 'fail'

    if e+j<=2*g:
        return 'fail'

    if h+j<=2*i:
        return 'fail'


    return combtype+['split']+C
    # returns the combinatorial type of the subdivision, along with
    # everything entered into the function


def nicelinetest(M):

    # determines if we are allowed to assume
    # val(a+b)=min(val(a),val(b)) in computation of the tropical lines
    # in the surface defined by the tropical quadric resulting from
    # the matrix M

    for i in range(3):
        for j in range(i+1,4):
            if dismin([M[0][i]+M[1][j],M[0][j]+M[1][i]])==0:
                return 'fail'

            if dismin([M[2][i]+M[1][j],M[2][j]+M[1][i],M[0][i]+M[3][j],M[0][j]+M[3][i]])==0:
                return 'fail'

            if dismin([M[2][i]+M[3][j],M[2][j]+M[3][i]])==0:
                return 'fail'

            if dismin([M[0][i]+M[2][j],M[0][j]+M[2][i]])==0:
                return 'fail'

            if dismin([M[1][i]+M[2][j],M[1][j]+M[2][i],M[0][i]+M[3][j],M[0][j]+M[3][i]])==0:
                return 'fail'

            if dismin([M[1][i]+M[3][j],M[1][j]+M[3][i]])==0:
                return 'fail'

    return 'pass'



    def findmatrix(N):
        # keeps computing random matrices until one of them leads to a
        # smooth tropical quadric surface, then returns all relevant
        # information regarding that matrix.

        word='fail'
        while word=='fail':
            word=smoothtest(m2c(makematrix(N)))
            return word
```

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