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UNITARY EQUIVALENCE TO A COMPLEX SYMMETRIC MATRIX: LOW DIMENSIONS

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ABSTRACT. A matrix $T \in \mathbf{M}_n(\mathbb{C})$ is *UECSM* if it is unitarily equivalent to a complex symmetric (i.e., self-transpose) matrix. We develop several techniques for studying this property in dimensions three and four. Among other things, we completely characterize 4×4 nilpotent matrices which are UECSM and we settle an open problem which has lingered in the 3×3 case. We conclude with a discussion concerning a crucial difference which makes dimension three so different from dimensions four and above

1. INTRODUCTION

Following [27], we say that a matrix $T \in \mathbf{M}_n(\mathbb{C})$ is *UECSM* if it is unitarily equivalent to a complex symmetric (i.e., self-transpose) matrix. Here we use the term *unitarily equivalent* in the sense of operator theory: we say that two matrices A and B are unitarily equivalent if $A = UBU^*$ for some unitary matrix U. We denote this relationship by $A \cong B$. In contrast, the term *unitarily similar* is frequently used in the matrix-theory literature.

Since every square complex matrix is *similar* to a complex symmetric matrix [18, Thm. 4.4.9] (see also [11, Ex. 4] and [8, Thm. 2.3]), determining whether a given matrix is UECSM is sometimes difficult, although several numerical methods [1, 10, 27] have recently emerged. To illustrate the subtlety of this problem, we remark that exactly one of the following matrices is UECSM (see Section 3)

$$\begin{pmatrix} 0 & 2 & 9 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 9 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 9 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 9 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Let us briefly discuss our results. First, we adapt, from the three-dimensional to the four-dimensional setting, a highly successful method developed in [9] based upon the Pearcy-Sibirskiĭ trace criteria [20, 25] (Section 2). This work depends crucially upon a recent breakthrough of Djokivić [7] in the study of Poincaré series. As a concrete example, we use our new criteria to completely characterize 4×4 nilpotent matrices which are UECSM (Section 3). Following a somewhat different thread, we settle in the affirmative a conjecture which has lingered in the 3×3 case for the last few years (Section 4). Moreover, we also provide a theoretical

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explanation for the failure of this conjecture in dimensions four and above (Section 5). In particular, we are able to construct the counterexample [1, Ex. 5] from scratch, as opposed to resorting to a brute-force random search. We conclude this note with a discussion concerning a crucial difference which makes dimension three so different from dimensions four and above (Section 6).

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2. Trace Criteria

In 1968, Sibirskiĭ [25] refined a striking result of Pearcy [20] and proved that $A, B \in \mathbf{M}_3(\mathbb{C})$ are unitarily equivalent if and only if $\Phi(A) = \Phi(B)$ where $\Phi : \mathbf{M}_3(\mathbb{C}) \to \mathbb{C}^7$ is the function defined by

$$\Phi(X) = (\operatorname{tr} X,\ \operatorname{tr} X^2,\ \operatorname{tr} X^3,\ \operatorname{tr} X^*X,\ \operatorname{tr} X^*X^2,\ \operatorname{tr} X^{*2}X^2,\ \operatorname{tr} X^*X^2X^{*2}X). \tag{2}$$

Pearcy's original 1962 result included the words X^*XX^*X and $X^*X^2X^*X$, which were later shown by Sibirksiĭ to be redundant.

Recently, the first and third authors proved that for $n \leq 7$, a matrix $T \in \mathbf{M}_n(\mathbb{C})$ is UECSM if and only if $T \cong T^t$ and, moreover, that this result fails for $n \geq 8$ [13]. Consequently, $T \in \mathbf{M}_3(\mathbb{C})$ is UECSM if and only if $\Phi(T) = \Phi(T^t)$. Fortunately, the first six traces in (2) are automatically equal for $X = T$ and $X = T^t$, whence T is UECSM if and only if $\operatorname{tr} X^*X^2X^{*2}X$ yields the same value for $X = T$ and $X = T^t$. Using standard properties of the trace, one sees that this is equivalent to

$$\operatorname{tr}[T^*T(T^*T - TT^*)TT^*] = 0. \tag{3}$$

In other words, $T \in \mathbf{M}_3(\mathbb{C})$ is UECSM if and only if (3) holds.

A simple extension of the Pearcy-Sibirskiĭ theorem to the 4×4 setting appeared hopeless for many years until Djoković [7, Thm. 4.4] recently proved that $A, B \in \mathbf{M}_4(\mathbb{C})$ are unitarily equivalent if and only if $\operatorname{tr} w_i(A, A^*) = \operatorname{tr} w_i(B, B^*)$ for $i = 1, 2, \ldots, 20$, where the words $w_i(x, y)$ are defined by

(1) x	(6) x^4	(11) x^2yx^2y	(16) x^3y^3xy
(2) x^2	(7) x^3y	(12) x^2y^2xy	(17) y^3x^3yx
(3) xy	(8) x^2y^2	(13) y^2x^2yx	(18) x^3yx^2yxy
(4) x^3	(9) $xyxy$	(14) x^3y^2xy	(19) $x^2y^2xyx^2y$
(5) x^2y	(10) x^3y^2	(15) $x^3y^2x^2y$	(20) $x^3y^3x^2y^2$.

In light of the fact that $T \in \mathbf{M}_4(\mathbb{C})$ is UECSM if and only if $T \cong T^t$, it follows that T is UECSM if and only if $\operatorname{tr} w_i(T, T^*) = \operatorname{tr} w_i(T^t, \overline{T})$ for $i = 1, 2, \ldots, 20$. Since a matrix and its transpose have the same trace, the preceding is equivalent to

$$\operatorname{tr} w_i(T, T^*) = \operatorname{tr} \widetilde{w}_i(T, T^*), \tag{4}$$

where $\widetilde{w}_i(x, y)$ is the reverse of $w_i(x, y)$ (e.g., $\widetilde{xy^2} = y^2x$). Fortunately, the desired condition (4) holds automatically for $i = 1, 2, \ldots, 11$. For instance,

$$\operatorname{tr} \widetilde{w}_{11}(T, T^*) = \operatorname{tr} T^*T^2T^*T^2 = \operatorname{tr} T^2T^*T^2T^* = \operatorname{tr} w_{11}(T, T^*).$$

Thus T is UECSM if and only if (4) holds for the nine values $i = 12, 13, \ldots, 20$. However, we can do even better for we claim that (4) holds for $i = 12$ if and only if (4) holds for $i = 13$:

$$\begin{aligned}
\operatorname{tr} w_{12}(T,T^*) = \operatorname{tr} \widetilde{w}_{12}(T,T^*) \quad &\Leftrightarrow \quad \operatorname{tr} T^2T^{*2}TT^* = \operatorname{tr} T^*TT^{*2}T^2 \\
&\Leftrightarrow \quad \operatorname{tr} TT^*T^2T^{*2} = \operatorname{tr} T^{*2}T^2T^*T \\
&\Leftrightarrow \quad \operatorname{tr} \widetilde{w}_{13}(T,T^*) = \operatorname{tr} w_{13}(T,T^*).
\end{aligned}$$

Similarly, (4) holds for $i = 16$ if and only if (4) holds for $i = 17$:

$$\begin{aligned}
\operatorname{tr} w_{16}(T,T^*) = \operatorname{tr} \widetilde{w}_{16}(T,T^*) \quad &\Leftrightarrow \quad \operatorname{tr} T^3T^{*3}TT^* = \operatorname{tr} T^*TT^{*3}T^3 \\
&\Leftrightarrow \quad \operatorname{tr} TT^*T^3T^{*3} = \operatorname{tr} T^{*3}T^3T^*T \\
&\Leftrightarrow \quad \operatorname{tr} \widetilde{w}_{17}(T,T^*) = \operatorname{tr} w_{17}(T,T^*).
\end{aligned}$$

Thus we need only consider the indices $i = 12, 14, 15, 16, 18, 19, 20$. Now observe that for $i = 20$ the desired condition $\operatorname{tr} w_{20}(T,T^*) = \operatorname{tr} \widetilde{w}_{20}(T,T^*)$ is equivalent to

$$\begin{aligned}
\operatorname{tr}(T^3T^{*3}T^2T^{*2} - T^{*2}T^2T^{*3}T^3) = 0 \ &\Leftrightarrow \ \operatorname{tr}(T^3T^{*3}T^2T^{*2} - T^2T^{*3}T^3T^{*2}) = 0 \\
&\Leftrightarrow \ \operatorname{tr}[T^2(TT^{*3} - T^{*3}T)T^2T^{*2}] = 0.
\end{aligned}$$

Similar computations for $i = 12, 14, 15, 16, 18, 19$ yield the following theorem:

Theorem 1. *A matrix $T \in \mathbf{M}_4(\mathbb{C})$ is UECSM if and only if the traces of the following seven matrices vanish:*

(1) $T(TT^{*2} - T^{*2}T)TT^*$,
(2) $T(T^2T^{*2} - T^{*2}T^2)TT^*$,
(3) $T^2(TT^{*2} - T^{*2}T)T^2T^*$,
(4) $T(T^2T^{*3} - T^{*3}T^2)TT^*$,
(5) $T[(T^2T^*)^2 - (T^*T^2)^2]TT^*$,
(6) $T^2T^*(T^*T - TT^*)T^*T^2T^*$,
(7) $T^2(TT^{*3} - T^{*3}T)T^2T^{*2}$.

For the sake of convenience, we adopt the following notation. For $i = 1, 2, \ldots, 7$, let $\Psi_i(T)$ denote the trace of the ith matrix listed in Theorem 1 and define a function $\Psi : \mathbf{M}_4(\mathbb{C}) \to \mathbb{C}^7$ by setting $\Psi(T) = (\Psi_1(T), \Psi_2(T), \ldots, \Psi_7(T))$. In light of Theorem 1, we see that $T \in \mathbf{M}_4(\mathbb{C})$ is UECSM if and only if $\Psi(T) = 0$.

Example 1. In [10] it is observed that neither of the matrices

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

are susceptible to testing with `UECSMTest` [27], `ModulusTest` [10], or `StrongAngleTest` [1], although ad-hoc arguments can be employed. Since $\Psi(T_1) = 0$ and $\Psi(T_2) = (-12, 0, 0, 0, 0, 0, 0)$, we conclude that T_1 is UECSM and that T_2 is not.

3. Canonical forms: 4×4 nilpotent UECSMs

As an application example of Theorem 1 we completely characterize those 4×4 nilpotent matrices which are UECSM. This is an illuminating exercise for several reasons. First of all, characterizing objects up to unitary equivalence is typically a difficult task and previous work has mostly been confined to the 3×3 case (e.g., [9, Thms. 5.1, 5.2], [27, Sect. 4]). Second, we encounter many families which can be independently proven to be UECSM based upon purely theoretical considerations (i.e., providing independent confirmation of our results). Finally, we discover several interesting classes of matrices which are UECSM but which do not fall into any previously known class.

In light of Schur's Theorem on unitary triangularization, we restrict our attention to matrices of the form

$$T = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Noting that T^3 has at most one nonzero entry, we first consider the fourth and seventh conditions in Theorem 1 since we expect these traces to be simple when expanded symbolically. Indeed, a computation reveals that

$$\Psi_4(T) = |a|^2|d|^2|f|^2(|a|^2 + |b|^2 - |e|^2 - |f|^2), \tag{6}$$

$$\Psi_7(T) = |a|^2|d|^4|f|^2(|a|^2 - |f|^2). \tag{7}$$

Since we require $\Psi(T) = 0$, we examine several special cases.

3.1. **The case $d = 0$.** A few routine computations tell us that $\Psi_i(T) = 0$ for $i = 2, 3, 4, 5, 7$. For $i = 1$ and $i = 6$ we have

$$\Psi_1(T) = |ae + bf|^2(|a|^2 + |b|^2 - |e|^2 - |f|^2),$$
$$\Psi_6(T) = \overline{c}(ae + bf)\Psi_1(T),$$

whence $\Psi(T) = 0$ if and only if either

$$ae + bf = 0 \tag{8}$$

or

$$|a|^2 + |b|^2 = |e|^2 + |f|^2. \tag{9}$$

The condition (8) has a simple interpretation, for if $d = 0$, then $T^2 = 0$ if and only if (8) holds. Now recall that a matrix which is nilpotent of order two is UECSM [14, Cor. 4].

On the other hand, condition (9) does not have an obvious theoretical interpretation. We remark that the third matrix in (1) is obtained by setting $a = 2$, $b = 9$, $c = 1$, $d = 0$, $e = 6$, $f = 7$ and noting that $2^2 + 9^2 = 85 = 6^2 + 7^2$. The remaining three matrices in (1) are not UECSM since their entries do not satisfy (9).

3.2. **The case $a = 0$.** In this case, let us write

$$T = \left(\begin{array}{c|c|c|c} 0 & 0 & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Another calculation shows that $\Psi_i(T) = 0$ for $i = 2, 3, 4, 5, 7$ and that

$$\Psi_1(T) = |f|^2\Big[(\underbrace{|b|^2 + |d|^2}_{\|v_3\|^2})(\underbrace{|b|^2 + |d|^2 - |c|^2 - |e|^2 - |f|^2}_{\|v_3\|^2 - \|v_4\|^2}) + |\underbrace{b\overline{c} + d\overline{e}}_{\langle v_3, v_4\rangle}|^2 \Big], \tag{10}$$

$$\Psi_6(T) = f(\underbrace{b\overline{c} + d\overline{e}}_{\langle v_3, v_4\rangle})\Psi_1(T), \tag{11}$$

where v_1, v_2, v_3, v_4 denote the columns of T. Depending upon whether $f = 0$ or not, there are two cases to consider.

(1) If $f = 0$, then $\Psi(T) = 0$ whence T is UECSM. This agrees with theory, since in this case

$$T = \left(\begin{array}{cc|cc} 0 & 0 & b & c \\ 0 & 0 & d & e \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

is nilpotent of order two and hence UECSM by [14, Cor. 4].

(2) If $f \neq 0$, then according to (10) and (11) there are several possibilities.

(a) If $|b|^2 + |d|^2 = 0$, then $b = d = 0$ and $\Psi(T) = 0$ whence T is UECSM. This agrees with the fact that every rank-one matrix is UECSM [14, Cor. 5].

(b) If $(|b|^2 + |d|^2)(|b|^2 + |d|^2 - |c|^2 - |e|^2 - |f|^2) + |b\overline{c} + d\overline{e}|^2 = 0$, then T is UECSM. In particular, observe that if v_3 and v_4 are orthogonal vectors with the same norm, then $\Psi(T) = 0$. This agrees with the observation that every partial isometry on $\mathbb{C}^4$ is UECSM [16, Cor. 2]. Otherwise we obtain matrices which are UECSM but which do not lie in any previously understood class.

3.3. **The case** $f = 0$**.** In this case we have

$$T^t \cong \begin{pmatrix} 0 & 0 & e & c \\ 0 & 0 & d & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We therefore have the same results as Subsection 3.2, after exchanging the roles of a and f, and b and e, respectively.

3.4. **The case** $a, d, f \neq 0$**.** If $a, d, f \neq 0$, then it follows from (6) and (7) that the conditions $|a| = |f|$ and $|b| = |e|$ are necessary for T to be UECSM. In fact, we claim that these conditions are also sufficient. Indeed, if $|a| = |f|$ and $|b| = |e|$, then upon conjugating T by a diagonal unitary matrix we see that

$$T \cong \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is unitarily equivalent to its transpose via the symmetric unitary matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus T is UECSM whenever $a, d, f \neq 0$, $|a| = |f|$, and $|b| = |e|$.

The following theorem summarizes our findings:

Theorem 2. *The matrix*

$$T = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is UECSM if and only if at least one of the following occurs:

(1) $d = 0$ *and* $ae + bf = 0$,

(2) $d = 0$ *and* $|a|^2 + |b|^2 = |e|^2 + |f|^2$,

(3) $a = 0$ *and* $f = 0$,

(4) $a = 0$ *and* $(|b|^2 + |d|^2)(|b|^2 + |d|^2 - |c|^2 - |e|^2 - |f|^2) + |b\overline{c} + d\overline{e}|^2 = 0$,

(5) $f = 0$ *and* $(|d|^2 + |e|^2)(|d|^2 + |e|^2 - |a|^2 - |b|^2 - |c|^2) + |c\overline{e} + b\overline{d}|^2 = 0$,

(6) $|a| = |f|$ *and* $|b| = |e|$.

4. An angle criterion in three dimensions

Suppose that $T \in \mathbf{M}_n(\mathbb{C})$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding normalized eigenvectors $x_1, x_2, \ldots, x_n$. Let $y_1, y_2, \ldots, y_n$ denote normalized eigenvectors of T^* corresponding to the eigenvalues $\overline{\lambda_i}$. Observe that y_j is characterized up to a scalar multiple by the fact that $\langle x_i, y_j \rangle = 0$ when $i \neq j$. Under these circumstances, it is known that the condition

$$|\langle x_i, x_j \rangle| = |\langle y_i, y_j \rangle| \tag{12}$$

for $1 \leq i < j \leq n$ is necessary for T to be UECSM [1, Thm. 1] (in fact, the first use of such a procedure in this context dates back to [11, Ex. 7]).

Although it was initially unclear whether (12) is sufficient for T to be UECSM, L. Balayan and the first author eventually showed that there exist matrices 4×4 and larger which satisfy (12) but which are not UECSM. These counterexamples will be discussed further in Section 5. On the other hand, based upon extensive numerical evidence they also conjectured that (12) *is* sufficient in the 3×3 case [1, Sec. 6]. Theorem 3 below settles this conjecture in the affirmative.

Strangely enough, the proof relies critically upon complex function theory and the emerging theory of truncated Toeplitz operators. Interest in truncated Toeplitz operators has blossomed over the last several years [2, 3, 6, 5, 12, 21, 22, 23, 24, 26], sparked by a seminal paper of D. Sarason [21]. In [9], W.T. Ross and the first two authors established that if $T \in \mathbf{M}_3(\mathbb{C})$ has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding normalized eigenvectors x_1, x_2, x_3 satisfying $\langle x_i, x_j \rangle \neq 0$ for $1 \leq i, j \leq 3$, then the following are equivalent:

(1) T is unitarily equivalent to a complex symmetric matrix,

(2) T is unitarily equivalent to an analytic truncated Toeplitz operator,

(3) The condition

$$\det X^*X = (1 - |\langle x_1, x_2 \rangle|^2)(1 - |\langle x_2, x_3 \rangle|^2)(1 - |\langle x_3, x_1 \rangle|^2) \tag{13}$$

holds, where $X = (x_1|x_2|x_3)$ is the matrix having x_1, x_2, x_3 as its columns.

In particular, a direct proof that (3) $\Rightarrow$ (1), independent of the theory of truncated Toeplitz operators, has not yet been discovered.

Theorem 3. *Suppose that $T \in \mathbf{M}_3(\mathbb{C})$ has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding unit eigenvectors x_1, x_2, x_3. Let y_1, y_2, y_3 denote unit eigenvectors of T^* corresponding to the eigenvalues $\overline{\lambda_1}, \overline{\lambda_2}, \overline{\lambda_3}$. Under these circumstances, the condition* (12) *is necessary and sufficient for T to be UECSM.*

Proof. Since the necessity of (12) is well-known [1, Thm. 1], we focus here on sufficiency. We first show that it suffices to consider the case where $\langle x_i, x_j \rangle \neq 0$ for $1 \leq i, j \leq 3$.

Suppose that T has a pair of eigenvectors which are orthogonal. Upon scaling, translating by a multiple of the identity, and applying Schur's Theorem on unitary triangularization, we may further assume that

$$T \cong \begin{pmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & \lambda \end{pmatrix}$$

where $\lambda \neq 0, 1$. Since T satisfies (12), the eigenspaces of T^* corresponding to the eigenvalues 1 and $\overline{\lambda}$ must be orthogonal. A routine calculation shows that $(\overline{a}, 1, 0)$ and $(\overline{b}, 0, \overline{\lambda})$ are eigenvectors of T^* with eigenvalues 1 and $\overline{\lambda}$, respectively, and so we must have $a = 0$ or $b = 0$. It is straightforward to check that T satisfies (3) in either case, and thus T is UECSM (one could also observe that both cases lead to the conclusion that T is unitarily equivalent to the direct sum of a 2×2 and a 1×1 matrix whence T is UECSM by any of [1, Cor. 3], [4, Cor. 3.3], [11, Ex. 6], [13], [14, Cor. 1], [19, p. 477], [27, Cor. 3], or [10, Ex. 2]).

Assuming now that $\langle x_i, x_j \rangle \neq 0$ for $1 \leq i, j \leq 3$, we intend to use the fact that (13) implies that T is UECSM. Let $X = (x_1|x_2|x_3)$ and $Y = (y_1|y_2|y_3)$ denote the 3×3 matrices having the vectors x_1, x_2, x_3 and y_1, y_2, y_3 as columns, respectively. In particular, note that

$$Y^* X = \begin{pmatrix} \langle x_1, y_1 \rangle & 0 & 0 \\ 0 & \langle x_2, y_2 \rangle & 0 \\ 0 & 0 & \langle x_3, y_3 \rangle \end{pmatrix}. \tag{14}$$

We now claim that

$$\det X^* X = |\langle x_1, y_1 \rangle|^2 (1 - |\langle x_2, x_3 \rangle|^2). \tag{15}$$

Since y_1 is a unit vector orthogonal to x_2 and x_3, we may write

$$x_1 = \langle x_1, y_1 \rangle y_1 + x'$$

for some x' in $\operatorname{span}\{x_2, x_3\}$. Let Λ be the multilinear function given by

$$\Lambda(w_1, w_2, w_3) = \det X^* W,$$

where $W = (w_1|w_2|w_3)$ is the matrix whose columns are the w_i. We then have

$$\begin{aligned} \det X^* X &= \Lambda(x_1, x_2, x_3) \\ &= \langle x_1, y_1 \rangle \Lambda(y_1, x_2, x_3) + \Lambda(x', x_2, x_3) \end{aligned}$$

Since x' belongs to $\operatorname{span}\{x_2, x_3\}$, the second term vanishes and we have

$$\det X^* X = \langle x_1, y_1 \rangle \det \begin{pmatrix} \langle y_1, x_1 \rangle & \langle x_2, x_1 \rangle & \langle x_3, x_1 \rangle \\ 0 & 1 & \langle x_3, x_2 \rangle \\ 0 & \langle x_2, x_3 \rangle & 1 \end{pmatrix},$$

from which the desired condition (15) is immediate.

Similarly we obtain

$$\det X^*X = |\langle x_2, y_2\rangle|^2(1 - |\langle x_3, x_1\rangle|^2), \tag{16}$$

$$\det X^*X = |\langle x_3, y_3\rangle|^2(1 - |\langle x_1, x_2\rangle|^2), \tag{17}$$

by relabeling the indices and using the same argument. Moreover, we can also perform these computations with Y^*Y in place of X^*X, which provides

$$\det Y^*Y = |\langle x_1, y_1\rangle|^2(1 - |\langle y_2, y_3\rangle|^2). \tag{18}$$

Thus if T satisfies (12), then it follows from (15) and (18) that

$$|\det X|^2 = \det X^*X = \det Y^*Y = |\det Y|^2,$$

whence $|\det X| = |\det Y|$. Multiplying (15), (16), and (17) together and appealing to (14), we obtain

$$(\det X^*X)^3 = |\det Y^*X|^2(1 - |\langle x_1, x_2\rangle|^2)(1 - |\langle x_2, x_3\rangle|^2)(1 - |\langle x_3, x_1\rangle|^2).$$

However, since $\det X^*X = |\det X|^2 = |\det Y||\det X| = |\det Y^*X|$ it follows from the preceding that

$$\det X^*X = (1 - |\langle x_1, x_2\rangle|^2)(1 - |\langle x_2, x_3\rangle|^2)(1 - |\langle x_3, x_1\rangle|^2).$$

As we have discussed above, this establishes that T is UECSM. □

5. The angle criterion in dimensions $n \geq 4$

Following the notation and conventions established in Section 4, we assume that the matrix T in $\mathbf{M}_n(\mathbb{C})$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding normalized eigenvectors $x_1, x_2, \ldots, x_n$. Similarly, we select normalized eigenvectors of T^* corresponding to the eigenvalues $\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n}$ and denote them $y_1, y_2, \ldots, y_n$. Recall from the preceding discussion that the condition

$$|\langle x_i, x_j\rangle| = |\langle y_i, y_j\rangle| \tag{12}$$

for $1 \leq i < j \leq n$ is necessary and sufficient for T to be UECSM if $n \leq 3$, but insufficient if $n \geq 4$. Indeed, there exist matrices 4×4 or larger which are *not* UECSM but which nevertheless satisfy (12). The first known example was discovered by L. Balayan using a random search of matrices having integer entries [1, Ex. 5]. In this section, we provide a solid theoretical explanation for the existence of such counterexamples and we illustrate this process by constructing L. Balayan's original counterexample from scratch.

Unlike (12), the related condition

$$\langle x_i, x_j\rangle\langle x_j, x_k\rangle\langle x_k, x_i\rangle = \overline{\langle y_i, y_j\rangle\langle y_j, y_k\rangle\langle y_k, y_i\rangle}, \tag{19}$$

for $1 \leq i \leq j \leq k \leq n$, is equivalent to asserting that T is UECSM [1, Thm. 2]. Following [1], we refer to (19) as the *Strong Angle Test* (SAT) and (12) as the *Weak Angle Test* (WAT). Observe that the WAT can be obtained from the SAT by setting $k = j$ in (19). In particular, we remark that a matrix which passes the SAT automatically passes the WAT, although the converse does not hold.

Curiously, the counterexample discussed above satisfies the related condition

$$\langle x_i, x_j\rangle\langle x_j, x_k\rangle\langle x_k, x_i\rangle = \langle y_i, y_j\rangle\langle y_j, y_k\rangle\langle y_k, y_i\rangle, \tag{20}$$

for $1 \leq i \leq j \leq k \leq n$ [1, Ex. 5]. We say that a matrix which satisfies (20) passes the *Linear Strong Angle Test* (LSAT). Our aim in this section is to describe a method for producing matrices which pass the LSAT (20) and hence the WAT (12), but not the SAT (19).

Theorem 4. *A matrix T in $\mathbf{M}_n(\mathbb{C})$ which has distinct eigenvalues satisfies the Linear Strong Angle Test* (20) *if and only if T is unitarily equivalent to a matrix of the form QDQ^{-1} where D is diagonal and Q belongs to $\mathbf{SU}(k, n-k)$ for some $1 \leq k \leq n$.*

Here $\mathbf{SU}(k, n-k)$ refers to the group of complex matrices having determinant 1 and which preserve the Hermitian form

$$\langle v, w \rangle_k := \sum_{j=1}^{k} v_j \overline{w_j} \; - \sum_{j=k+1}^{n} v_j \overline{w_j}.$$

In particular, we observe that a matrix Q belongs to $\mathbf{SU}(k, n-k)$ if and only if

$$Q^* A Q = A, \tag{21}$$

where

$$A := I_k \oplus -I_{n-k}.$$

In contrast, a matrix passes the Strong Angle Test (19) if and only if it is unitarily equivalent to a matrix of the form QDQ^{-1} where D is diagonal and Q belongs to $\mathbf{O}(n)$, the complex orthogonal group of order n [18, Thm. 4.4.13] (see also [15, Sect. 5]).

In order to prove Theorem 4, we require the following lemma.

Lemma 1. *Maintaining the notation and conventions established above, if a matrix T in $\mathbf{M}_n(\mathbb{C})$ satisfies the Linear Strong Angle Test* (20)*, then there is a selfadjoint unitary matrix U and unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $Ux_i = \alpha_i y_i$ for $1 \leq i \leq n$.*

Proof of Lemma 1. The proof is similar to that of [1, Thm. 2]. We first work under the assumption that $\langle x_i, x_j \rangle \neq 0$ for all i, j. Setting $k = i$ in (20) reveals that $|\langle y_i, y_j \rangle| = |\langle x_i, x_j \rangle|$ so that the constants

$$\gamma_i := \frac{\langle y_1, y_i \rangle}{\langle x_1, x_i \rangle}$$

each have unit modulus. Since T satisfies the LSAT, we next observe that

$$\gamma_i \overline{\gamma_j} = \frac{\langle y_1, y_i \rangle \langle y_j, y_1 \rangle}{\langle x_1, x_i \rangle \langle x_j, x_1 \rangle} = \frac{\langle x_i, x_j \rangle}{\langle y_i, y_j \rangle}. \tag{22}$$

Let R be the $n \times n$ matrix which satisfies $Rx_i = \gamma_i y_i$ for $1 \leq i \leq n$. By (22), we see that

$$\langle Rx_i, Rx_j \rangle = \gamma_i \overline{\gamma_j} \langle y_i, y_j \rangle = \langle x_i, x_j \rangle$$

from which it follows that R is unitary. We now briefly sketch how to modify this construction if $\langle x_i, x_j \rangle = 0$ for some pair (i, j). The details are largely technical and can be found in the proof of [1, Thm. 2], *mutatis mutandis.*

Consider the partially-defined, selfadjoint matrix $(\beta_{ij})_{i,j=1}^n$ whose (obviously unimodular) entries are given by

$$\beta_{ij} = \frac{\langle y_i, y_j \rangle}{\langle x_i, x_j \rangle},$$

for those $1 \leq i, j \leq n$ for which this expression is well-defined. Since T satisfies the LSAT (20), it follows that $\beta_{ij}\beta_{jk} = \beta_{ik}$ holds whenever all of the quantities involved are well-defined. It turns out that one can inductively fill in the undefined entries of the matrix $(\beta_{ij})_{i,j=1}^n$ so that each entry β_{ij} is unimodular (i.e., $\beta_{ij} = \overline{\beta_{ji}}$) and such that the multiplicative property $\beta_{ij}\beta_{jk} = \beta_{ik}$ holds whenever $1 \leq i, j, k \leq n$. One then constructs the unitary matrix R by setting $\gamma_i = \beta_{1i}$ and letting $Rx_i = \gamma_i y_i$ as before. We refer the reader to the proof of [1, Thm. 2] for further details.

Now let $X = (x_1|x_2|\cdots|x_n)$ denote the $n \times n$ matrix whose columns are the eigenvectors $x_1, x_2, \ldots, x_n$ of T and let $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the diagonal matrix whose entries are the corresponding eigenvalues of T. In particular, we note that the matrix X diagonalizes T in the sense that

$$T = XDX^{-1}. \tag{23}$$

We next remark that

$$R^*T^*R = XD^*X^{-1}, \tag{24}$$

since both matrices agree on the basis $x_1, x_2, \ldots, x_n$. Taking adjoints in (23) we find that

$$T^* = (X^*)^{-1}D^*X^*,$$

from which it follows that the ith column of $(X^*)^{-1}$ is an eigenvector of T^* corresponding to the eigenvalue $\overline{\lambda_i}$. One can therefore check that

$$RTR^* = (X^*)^{-1}DX^* \tag{25}$$

by noting that $(X^*)^{-1}DX^*$ has the same eigenvectors as T^* and evaluating both sides of (25) on the basis $y_1, y_2, \ldots, y_n$. Taking adjoints in (25) yields

$$RT^*R^* = XD^*X^{-1}.$$

Comparing the preceding with (24) we find that

$$RT^*R^* = R^*T^*R.$$

In other words, T^* commutes with the unitary matrix R^2.

If T is irreducible (i.e., has no proper, nontrivial reducing subspaces), then $R^2 = \omega I$ for some constant ω of unit modulus. Letting $\alpha_i = \omega^{-1/2}\gamma_i$ (either branch of the square root is acceptable), we find that the matrix $U = \omega^{-1/2}R$ is selfadjoint, unitary, and satisfies $Ux_i = \alpha_i y_i$ for $1 \leq i \leq n$. To conclude the proof in the general case, one simply applies the preceding reasoning on each maximal proper reducing subspace of T to obtain the desired matrix U. □

Proof of Theorem 4. ($\Leftarrow$) First assume that T is unitarily equivalent to QDQ^{-1} where D diagonal and Q belongs to $\mathbf{SU}(k, n-k)$. Since the condition (20) of the LSAT is invariant under unitary transformations, we may assume that $T = QDQ^{-1}$. Recalling that $A = I_k \oplus (-I_{n-k})$ is diagonal and using (21), we have

$$\begin{aligned} T^* &= (Q^{-1})^*D^*Q^* \\ &= (AQA)D^*(AQ^{-1}A) \\ &= AQ(AD^*A)Q^{-1}A \\ &= (AQ)D^*(AQ)^{-1}. \end{aligned}$$

Writing $Q = (q_1|q_2|\cdots|q_n)$ in column-by-column format, we obtain normalized eigenvectors

$$x_i = \frac{q_i}{\|q_i\|}, \qquad y_i = Ax_i,$$

of T and T^*, respectively. Since A is unitary it follows that $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle$ whence T satisfies the Linear Strong Angle Test (20).

($\Rightarrow$) Suppose that T satisfies the LSAT. By Lemma 1, there is a selfadjoint unitary matrix U and unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $Ux_i = \alpha_i y_i$ for $1 \leq i \leq n$. As in the proof of Lemma 1, let $X = (x_1|x_2|\cdots|x_n)$ and let $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ so that

$$T = XDX^{-1}. \tag{26}$$

As in (25), we have

$$T = U(X^*)^{-1}DX^*U, \tag{27}$$

as both sides agree on the basis $x_1, x_2, \ldots, x_n$ (recall that $(X^*)^{-1}DX^*$ has the same eigenvectors as T^*). In light of (26) and (27), we conclude that the matrix X^*UX commutes with D. Since the diagonal entries of D are distinct and U is selfadjoint, we conclude that X^*UX is a diagonal matrix having only real entries. Without loss of generality, we may assume that the vectors $x_1, x_2, \ldots, x_n$ are ordered so that the first k diagonal entries of X^*UX are positive and the last $n-k$ are negative (note that X^*UX is invertible since both U and X are invertible).

Now let $w_i = |\delta_i|^{-\frac{1}{2}} x_i$, where δ_i is the ith diagonal entry of X^*UX. With $W = (w_1|w_2|\cdots|w_n)$, we have

$$W^*UW = A. \tag{28}$$

Since U is selfadjoint and unitary, we may appeal to both Sylvester's Law of Inertia [18, Thm. 4.5.8] and the Spectral Theorem to find a unitary matrix Z such that

$$U = Z^*AZ. \tag{29}$$

Plugging (29) into (28) we find that

$$(ZW)^*A(ZW) = A, \tag{30}$$

which tells us that the matrix $Q = ZW$ belongs to $\mathbf{SU}(k, n-k)$. Since the columns w_i of W are nonzero multiples of the corresponding columns x_i of X, it follows that

$$\begin{aligned} T &= WDW^{-1} \\ &= Z^*\left[(ZW)D(ZW)^{-1}\right]Z \\ &= Z^*(QDQ^{-1})Z. \end{aligned}$$

Thus T is unitarily equivalent to a matrix of the form QDQ^{-1} where Q belongs to $\mathbf{SU}(k, n-k)$. This completes the proof of Theorem 4. $\square$

We can now use Theorem 4 to construct matrices that satisfy the Weak Angle Test but not the Strong Angle Test. We begin by constructing a matrix T that satisfies the Linear Strong Angle Test. From the theorem, we know that this can be done by constructing a matrix Q in $\mathbf{SU}(k, n-k)$ and setting $T = QDQ^{-1}$ for any diagonal matrix D with distinct entries. Putting $k = j$ in the LSAT shows that T will satisfy the WAT, but T may satisfy the SAT as well.

Comparing (19) and (20), we can see that T will satisfy the SAT if and only if

$$\langle x_i, x_j\rangle\langle x_j, x_k\rangle\langle x_k, x_i\rangle \in \mathbb{R} \tag{31}$$

for $1 \leq i \leq j \leq k \leq n$. In practice, this condition is rarely satisfied. We illustrate the process with the following example.

Example 2. To construct a matrix that satisfies the WAT but not the SAT, we first need to construct an element of $\mathbf{SU}(k, n-k)$. We will do this for $n = 4$, as we know from Theorem 3 that examples of matrices that satisfy WAT but not SAT do not exist for smaller choices of n. We will use $k = 2$, which turns out to be necessary when $n = 4$ (see Theorem 5).

An element of $\mathbf{SU}(2,2)$ can be produced by applying an indefinite analogue of the Gram-Schmidt process to a collection of four elements of $\mathbb{C}^4$ [17, Sec. 3.1]. Using this method on a matrix with small random entries in $\mathbb{Z}[i]$ produced

$$Q = \begin{pmatrix} 1+\frac{i}{2} & 0 & -\frac{1}{2\sqrt{6}}(1-i) & \frac{i}{\sqrt{6}} \\ -\frac{i}{2} & 2i & \frac{1}{2\sqrt{6}}(7+5i) & -\frac{i}{\sqrt{6}} \\ -\frac{1}{2}(1-i) & 1-i & -\frac{1}{\sqrt{6}}(1+4i) & -\sqrt{\frac{2}{3}} \\ 0 & -i & -\sqrt{\frac{2}{3}}(1+i) & \sqrt{\frac{2}{3}} \end{pmatrix}.$$

Let D be the diagonal matrix with diagonal $(-1, 0, 1, 2)$ and let $T = QDQ^{-1}$. Explicitly, we have

$$T = \frac{1}{6}\begin{pmatrix} -10 & 4-6i & -3-11i & 2i \\ 4+6i & -22 & -15+17i & -12-2i \\ 3-11i & 15+17i & 28 & 2+6i \\ 2i & 12-2i & 2-6i & 16 \end{pmatrix}. \tag{32}$$

By Theorem 4, we know that T passes the LSAT and hence the WAT. On the other hand, if q_i denotes the ith column of Q, then we have

$$\langle q_1, q_2\rangle\langle q_2, q_3\rangle\langle q_3, q_1\rangle = \frac{1}{3}(100 - 8i) \notin \mathbb{R}.$$

Hence T does not satisfy the SAT and is therefore not UECSM. Although there was no guarantee that the matrix T obtained in this manner would not satisfy the SAT, in practice this does not appear to occur frequently.

Example 3. In this example, we consider the matrix

$$T = \begin{pmatrix} 5 & 0 & -1 & 3 \\ 2 & 4 & 1 & 2 \\ 2 & -2 & 6 & -2 \\ 0 & -2 & 1 & 4 \end{pmatrix}, \tag{33}$$

which was the first known example of a matrix which passes the Weak Angle Test (12) yet fails to be UECSM [1, Ex. 5]. This matrix was originally obtained by a brute force search, but now Theorem 4 puts it into a broader context and explains *why* matrices such as (33) exist. Indeed, the computations carried out in [1, Ex. 5] confirm that T passes the WAT and the LSAT, but fails the SAT. By following the proof of Theorem 4, it is possible to explicitly compute a Q in $\mathbf{SU}(2,2)$ and a diagonal matrix D such that T is unitarily equivalent to QDQ^{-1}.

6. Contrasting dimensions three and four

We conclude this note with some remarks concerning certain phenomena which distinguish dimension three from dimensions four and above. In the following, we maintain the notation and conventions established in the preceding two sections.

As we have seen, Theorem 4 provides a method for constructing matrices which pass the Weak Angle Test (WAT) and which *may* fail to be UECSM. On the other hand, Theorem 3 asserts that passing the WAT *is* sufficient for a matrix to be UECSM if $n = 3$. Therefore something peculiar must occur in dimension three which prevents the method of Theorem 4 from ever actually producing examples such as the matrices (32) from Example 2 and (33) from Example 3. The following theorem helps explain this curious dichotomy.

Theorem 5. *If T in $\mathbf{M}_n(\mathbb{C})$ has distinct eigenvalues and is unitarily equivalent to a matrix of the form QDQ^{-1} where D is diagonal and Q belongs to $\mathbf{SU}(n-1,1)$, then T is UECSM.*

Proof. Without loss of generality, we may assume that $T = QDQ^{-1}$ where D is diagonal and Q belongs to $\mathbf{SU}(n-1,1)$. We may also assume that D has real entries, as it follows from the Strong Angle Test that T being UECSM is independent of the actual eigen*values* of T. By Theorem 4, it follows that T satisfies the Linear Strong Angle Test (20). Returning to the proof of Theorem 4, we note that (27) asserts that

$$T = U(X^*)^{-1}DX^*U$$

where U is a selfadjoint unitary matrix and $D = D^*$ by assumption. However, this simply means that

$$T = UT^*U. \tag{34}$$

As shown in the proof of Theorem 4, the fact that Q belongs to $\mathbf{SU}(n-1,1)$ implies that U is unitarily equivalent to $A = \operatorname{diag}(1, 1, \ldots, 1, -1)$. Indeed, plugging $Q = ZW$ into (30) and using (29), we find that $W^*UW = A$ (i.e., U is $*$-congruent to A). The desired result follows upon appealing to Sylvester's Law of Inertia [18, Thm. 4.5.8].

After performing a unitary change of coordinates in (34), we may assume that $U = \operatorname{diag}(1, 1, \ldots, 1, -1)$ so that T has the form

$$\begin{pmatrix} T_{1,1} & T_{1,2} \\ -T_{1,2}^* & T_{2,2} \end{pmatrix}$$

where $T_{1,1}$ is $(n-1)\times(n-1)$ and selfadjoint and $T_{2,2}$ is 1×1 and real. Conjugating T by an appropriate block-diagonal unitary matrix we may further assume that $T_{1,1}$ is diagonal. Conjugating again by a diagonal unitary matrix, we may also arrange for the $(n-1)\times 1$ matrix $T_{1,2}$ to be purely imaginary. In other words, T is UECSM. □

Corollary 1. *If T is 3×3 and satisfies the Linear Strong Angle Test (20), then T is UECSM.*

Proof. By Theorem 4, T is unitarily equivalent to a matrix of the form QDQ^{-1} where Q belongs to either $\mathbf{SU}(3,0)$ or $\mathbf{SU}(2,1)$ (the cases $\mathbf{SU}(1,2)$ and $\mathbf{SU}(0,3)$ being identical to these first two). If Q belongs to $\mathbf{SU}(3,0)$, then Q is unitary whence T is unitarily equivalent to the diagonal matrix D. On the other hand, if Q belongs to $\mathbf{SU}(2,1)$, then we may appeal to Theorem 5 to conclude that T is UECSM. □

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