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Logarithmic Spirals and Projective Geometry in M.C. Escher's "Path of Life III"

Cover Page Footnote

Logarithmic Spirals and Projective Geometry
in M.C. Escher's *Path of Life III*

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Abstract

M.C. Escher's use of dilation symmetry in *Path of Life III* gives rise to a pattern of logarithmic spirals and an oddly ambiguous sense of depth.

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1. Introduction

The geometry of M.C. Escher's works has been studied extensively [2, 4, 6, 7, 8]. This article undertakes an in-depth study of the 1966 woodcut *Path of Life III* – on display at `http://www.mcescher.com/Gallery/recogn-bmp/LW445.jpg`[1] – expanding upon its brief mention in [6]. We find that its symmetry is relatively simple, that its highlighted spirals are logarithmic, and that the illusion of depth created by the varying size of its tiles is surprisingly complex. If we ignore Escher's artistic modifications at the boundary of the image, the woodcut can be described as a conformal mapping of a regular tiling.

2. Symmetry

Path of Life III is an image of black and white fish and birds. Tiny white creatures appear at the center of the circle, fly along brilliant red-gold

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[1]Accessed January 27, 2012.

"paths" toward the viewer, then fade to black to swim (and shrink) back to the center along a mirror image of the spiral.

As noted in *Visions of Symmetry* [6, p. 316], Escher classified tiles by shape, disregarding distortions of and decorations within that shape. If we are blind to all but the outline of the birds and fish of *Path of Life III*, we see a center of 4-fold rotational symmetry at the point where four wings or fins touch and where lines of mirror symmetry down the backs of the birds and fish cross. (See Figure 1.) Thus, Escher identified the symmetries of this tiling as those of the group p4g, denoted 4*2 in Conway and Thurston's notation [1].[2]
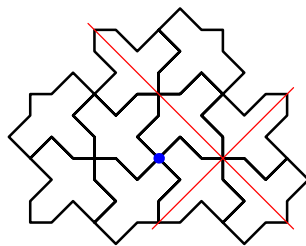

Figure 1: The birds and fish of *Path of Life III*; flat, colorless and featureless. Two lines of mirror symmetry are indicated in red; a center of 4-fold gyrational symmetry is highlighted in blue.

If we consider the color of the fish and birds when calculating their symmetries, we find that the 4-fold gyrational symmetry becomes a 2-fold one. The lines of fish and birds now have two distinct types of meeting points. Roughly speaking, this tiling has only half the symmetries of the uncolored one; the new group is denoted 2*22 or equivalently cmm.

Once eyes are added to differentiate fish from birds (we refer to the creatures with long, gilled heads as "fish" and the ones with long, flared tails as "birds"), only the translational symmetries of the tiling remain; the symme-

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[2]Thurston's notation describes the symmetry group of a surface in terms of its quotient space – the result of gluing together all the look-alike points on the surface. In this case, the 4-fold gyrational symmetry corresponds to a cone point on the quotient surface. This cone point is denoted by the leading 4 in Thurston's notation, corresponding to the degree of the gyrational symmetry it arose from. The * indicates that this quotient surface has a boundary, as would be formed by folding the surface along a line of mirror symmetry. The final 2 describes the angle of the single corner on that boundary, which is the result of two lines of mirror symmetry meeting at right angles.
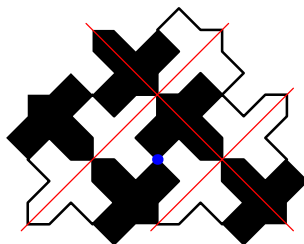
Figure 2: The birds and fish of *Path of Life III*, flat and featureless. Note the difference between the two points of intersection of red lines.

try group becomes ∘ or p1.

Escher's notes include many images of this sort [6], but his finished works are rarely this simple. In the case of *Path of Life III* he modifies the tiling to fill a circle. The birds and fish shrink as they approach the center of the circle, thereby compressing an infinite expanse of tiling into a finite area. The final figure has a 6-fold gyrational symmetry about its center and the interior of the figure has a dilation symmetry about the center. By taking measurements of the image, we find that the dilation factor is approximately 2.5. In other words, if the woodcut were to be magnified 5-fold (or 2.5-fold) the center of the figure should appear unchanged.

3. Paths of Life

The red-gold spirals running through the piece represent the "path of life" of its title. How can we describe those paths mathematically?

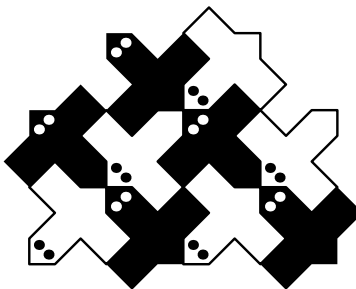Schattschneider suggests that Escher employed spiral symmetry in creat-

Figure 3: The birds and fish of *Path of Life III* laid out flat. Adding eyes breaks the rotational and mirror symmetries of the previous figures.

ing this woodcut [6, p. 248]. Starting from a small white fish, the next white fish along a path is approximately 2.5 times larger than and $\pi/3$ radians clockwise from that smaller fish. This leads us to conjecture that a typical path is described in polar coordinates by:

$$r = \left(\frac{5}{2}\right)^{-\frac{3}{\pi}\theta}.$$

This is the equation of the logarithmic spiral shown in Figure 4.

Figure 4: Logarithmic or equiangular spiral.

Superimposing twelve of these spirals as in Figure 5 yields an image comparable to the paths of Escher's woodcut.
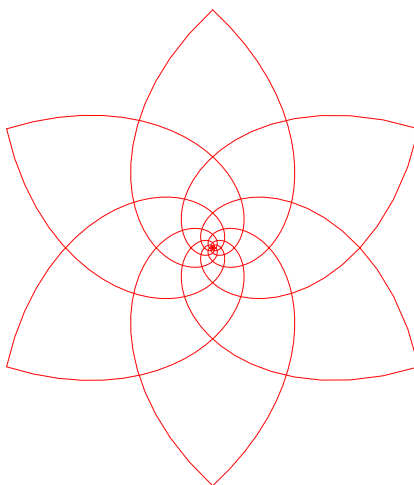

Figure 5: Twelve logarithmic spirals.

Differentiating to find a tangent vector to this curve, we find that it intersects any line through the origin at an angle of:

$$\alpha = \frac{\pi}{2} - \arctan\left(\frac{3}{\pi}\ln\left(\frac{5}{2}\right)\right) \text{ radians,}$$

or approximately 49 degrees. Hence, two curves should intersect at an angle of $2\alpha \approx 98$ degrees.

4. Comparison to Perspective Drawing

The lines of black and white creatures in Figure 3 meet at right angles; by changing the angle of intersection and drawing creatures decreasing in size, Escher has created an illusion of depth in this woodcut. Sugihara applied projective geometry to the study of Escher's optical illusions in [8]. Can *Path of Life III* be interpreted as a perspective projection?

Looking at the tiling in Figure 3, one might conjecture that this image can be understood as a perspective view of a regular (all tiles congruent) tiling of the inside of a cylinder. However, as Schattschneider has remarked [6, p. 248], the fish do not decrease in size the way the posts of a picket fence in a one point perspective drawing do. If we take three equidistant posts of such a fence and join the top of the closer post to the base of the further one and vice-versa, the diagonals thus constructed meet at the center of the middle post, as seen in Figure 6.
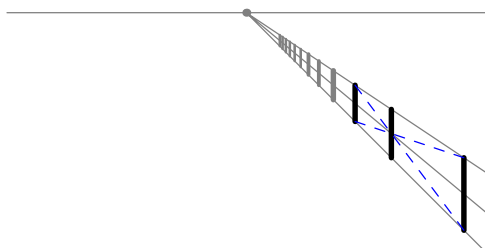

Figure 6: Perspective drawing of a fence.

Using the space between crossings of paths in *Path of Life III* as our "posts" we see that the fish decrease in size more rapidly than the fenceposts of Figure 6 do. The two diagonals meet closer to the center of the woodcut than they would if the image were a perspective projection of a regularly tiled cylinder.

How much does Escher's drawing differ from a perspective projection? Figure 7 is a very simple perspective drawing of fenceposts receding into the distance; a typical post has height $1/k$ and is distance $1/k$ from the leftmost corner of the drawing. Comparing the sequence of fencepost heights 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ..., $\frac{1}{n}$ to the sequence of creature sizes 1, $\frac{2}{5}$, $\frac{4}{25}$, $\frac{8}{125}$, ..., $\left(\frac{2}{5}\right)^n$ we see an extreme difference.

We've established that the image cannot be interpreted as a perspective projection of a regularly tiled cylinder. Could it be the projection of some
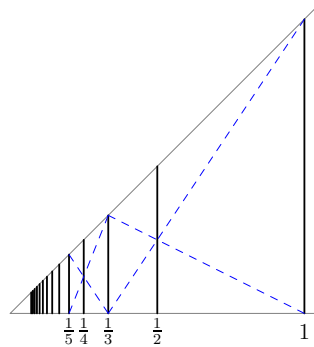
Figure 7: Simplified perspective drawing of a fence.

other regularly tiled surface?

Figure 8 shows a side view of a series of sight lines projected onto a plane. The distances between the intersections of the sight lines with the plane are 1, $\frac{2}{5}$, $\frac{4}{25}$, ..., $\left(\frac{2}{5}\right)^n$. In order for the projections of same-sized fish and birds onto this plane to appear as they do in the woodcut, the surface they cover would have to be trumpet-shaped, as indicated by the series of colored segments in the figure. Because this surface would be wider near the viewer and narrow into the distance, there is no way to cover it with congruent tiles so that its projection matches *Path of Life III*. A distant ring on that trumpet would have a smaller circumference than twelve fish in a ring close to the viewer, so the birds and fish on the surface would be elongated as they recede into the distance.
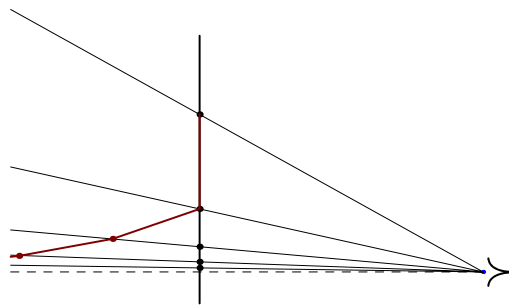

Figure 8: Perspective projection onto a plane.

5. Conformal Geometry

While flatness is not a property that *Path of Life III* would seem to enjoy — and hence curvature is present that cannot be accounted for in a flat perspective — one might imagine that Escher's depictions of the bird and fish figures which comprise its tiling are nevertheless faithful to angles if not curvatures. In other words, the birds and fish which comprise the ideal congruent tiling underlying the woodcut have the same wing, gill, and fin angles as those in the woodcut itself.

To investigate this possibility, we conjecture that there exists a tiling of the plane by congruent fish and birds to which *Path of Life III*, away from its boundary, is related by a conformal transformation: one which preserves angles but need not preserve length, area, or curvature. Indeed, we cannot hope to preserve area if we are to transform a congruent tiling into one whose figures decrease in area as they converge on the center point.

De Smit and Lenstra [3] showed in 2003 that Escher's 1956 lithograph *Prentententoonstelling (Print Gallery)*, in which an observer views his own city in a print that deftly wraps around the image to include himself, is related to its underlying "flat" scene via a conformal transformation. Their work relied upon the observation that an idealized version of *Prentententoonstelling* would possess two global symmetries, being preserved both by a full rotation of 2π radians about its center and by a rotated scaling (dilation) toward its center.

In the same way, the interior of *Path of Life III* possesses two global symmetries: rotation by $\frac{\pi}{3}$ radians and scaling toward its center by a factor of A. We have estimated $A \approx \frac{5}{2}$, that is, each bird along a spiral path is $\frac{5}{2}$ times as large as the bird which precedes it.

Each of these symmetries describes a conformal transformation. It is most convenient to express conformal transformations as complex-valued functions of a complex variable: $w = f(z)$. Indeed, conformality is a generic property of complex functions; see, for instance, Chapter 4 of [5] for details.

We will write i for the conventional square root of -1 and $w = x + iy$ for an arbitrary point on the complex plane in which we embed the woodcut. The two above-mentioned symmetries of *Path of Life III* may be expressed by the two scalar-multiplication functions $R(w)$ and $S(w)$, where

$$R(w) = \frac{1 + i\sqrt{3}}{2} w \quad \text{and} \quad S(w) = A \cdot w,$$

respectively corresponding to rotation of the complex plane by $\frac{\pi}{3}$ radians about its origin and scaling of the complex plane about the origin by a (most likely real) factor of A.

As illustrated in Figure 3, the underlying tiling in *Path of Life III* is preserved by a two-dimensional lattice of translations. Using $z = x + iy$ to represent an arbitrary point on the tiling in the complex plane, we may say that the tiling is preserved by two addition functions $H(z)$ and $V(z)$, where

$$H(z) = z + \omega_1 \quad \text{and} \quad V(z) = z + \omega_2,$$

where ω_1 and ω_2 are independent complex numbers describing the "width" and "height" of the tiling's repeating unit.

The symmetries of Figure 3 correspond to addition in the complex plane, while those of Escher's woodcut correspond to multiplication. To translate between addition and multiplication we need only employ an exponential function. You may recall that $e^m \cdot e^n = e^{m+n}$, i.e., addition of inputs to an exponential function is equivalent to multiplication of the corresponding outputs.

6. Lifted Image and Logarithmic Spirals

The meaning of a complex number as an exponent being not immediately obvious, the complex exponential function $w = e^z$ is typically defined by reference to a notorious formula of Leonhard Euler.[3] If $z = x + iy$, then

$$e^z = e^x \, e^{iy} = e^x \left(\cos y + i \, \sin y\right).$$

It is not difficult to show that the complex exponential function exchanges addition with multiplication in the same way as its real counterpart: the relation $e^m \cdot e^n = e^{m+n}$ holds, even when m and n are complex numbers.

Importantly, this means that any image in the complex plane which is preserved by addition will be transformed by the complex exponential function to an image which is preserved by multiplication. The converse will also hold, once a suitable inverse transformation is chosen. In choosing an inverse and applying it to this image we are creating a "lift" of *Path of Life III*.

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[3]Editor's note: See Carl Behrens' essay in this issue regarding this formula.
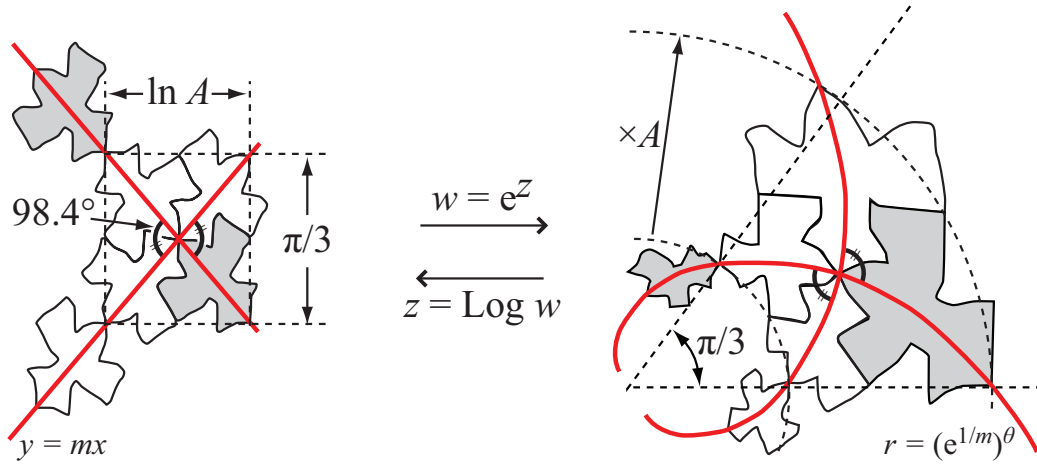
Figure 9: The complex exponential function exchanges addition and multiplication symmetries. The angle between paths in the lift, along with the height of the repeating unit in the tiling, may be used to compute the scale factor A of the original image.

One such lift is the so-called *principal natural logarithm*: we define the principal natural logarithm of the complex number $w = x + iy$ by the formula

$$\text{Log } w = \ln \sqrt{x^2 + y^2} + i \arctan \frac{y}{x}.$$

This provides a recipe for turning *Path of Life III* into a congruent tiling of the plane: if Escher's woodcut truly possesses multiplicative (rotation and scale) symmetries, its image under the principal natural logarithm will possess additive symmetries and hence be a congruent tiling.

Applying the principal natural logarithm transformation to *Path of Life III* using Filter Forge image processing software does appear to yield such a tiling. The lift shown on the left in Figure 10 (see next page) is an image whose repeating unit shows astonishing agreement with a congruent tiling of the plane by birds and fish.

The spiral paths of *Path of Life III* now appear as straight line paths in this lifted image. Using image editing software, the obtuse angle between these paths was measured to be approximately 98.4 degrees. Furthermore, since the transformation $w = e^z$ is conformal (angle-preserving), the angle of intersection of the spirals of *Path of Life III* will also be approximately 98.4 degrees, refining prior measurements.
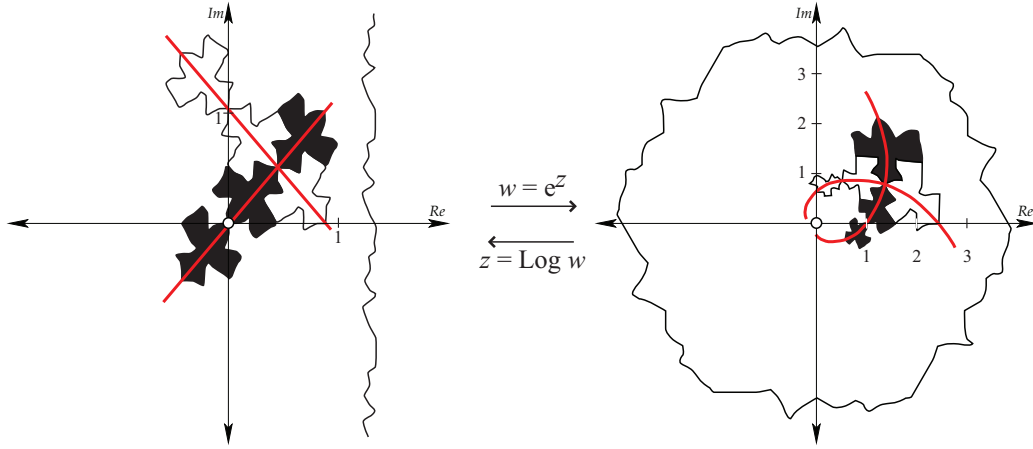
Figure 10: The "lifted" image of *Path of Life III* exhibits a congruent tiling. The scale on the axes reflects the identity $e^0 = 1$, revealed by the tip of the black bird's nose.

Moreover, assuming the lifted paths to be straight lines guarantees that their images (in the woodcut) are exactly logarithmic spirals. To wit, a straight line through the origin with slope m can be described by the equation $z(t) = (1 + im)t$. The exponential function will transform this line into the image

$$w(t) = e^{z(t)} = e^{(1+im)t} = e^t \, e^{imt}.$$

Along this curve the distance r of each point from the origin and the angle θ made by the same point will satisfy

$$r = |w(t)| = e^t \quad \text{and} \quad \theta = \arg w(t) = mt.$$

Thus each point on the image satisfies $r = e^{\theta/m}$, or in other words $r = \left(e^{1/m}\right)^\theta$. This equation describes a logarithmic spiral with pitch $e^{1/m}$. Since the lifted image's red paths show strong agreement with straight lines, their images in the woodcut *Path of Life III* are in strong agreement with logarithmic spirals.[4]

In particular, if the angle between red paths is taken to be 98.4 degrees, then the slope of the upward red lines in the lift is $m = \tan \frac{98.4°}{2} \approx 1.159$ and the pitch of the spirals in *Path of Life III* corresponding to the upward red

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[4]Earlier we estimated that the spirals had equation $\left(\frac{5}{2}\right)^{-3\theta/\pi}$, which leads us to expect $e^{1/m} \approx 0.4169$.

lines in the lift is $e^{1/m} \approx e^{0.8632} \approx 2.371$. The downward red lines having opposite slope will result in their corresponding spirals having the (anticipated) reciprocal pitch $e^{-0.8632} \approx 0.422$.

One final observation will permit us to measure the scale factor of the original image as well. Since it possesses the 6-fold rotation symmetry $R(w) = \frac{1+i\sqrt{3}}{2}w$, its lift will possess the translation symmetry

$$V(z) = z + \mathrm{Log}\, \frac{1+i\sqrt{3}}{2} = z + i\frac{\pi}{3}.$$

This translation is vertical, since $i\frac{\pi}{3}$ is purely imaginary. Thus the height of the repeating unit – the vertical distance that separates each pair of congruent fish and birds – must be exactly $\frac{\pi}{3}$. This vertical distance is (as it must be) exactly the angle of the rotational symmetry of Escher's woodcut about its center.

If the original image possesses the scale symmetry $S(w) = A \cdot w$ for a real number A, then its lift via the principal natural logarithm must possess the translation symmetry

$$H(z) = z + \mathrm{Log}\, A = z + \ln A.$$

This translation is horizontal, since $\ln A$ is a real number. Thus the width of the repeating unit in the lift will equal the natural logarithm of the scale factor A.

Combined with the measured angle of the red line paths, we can use right-triangle trigonometry to measure the horizontal width of the repeating unit:

$$\tan 49.2° = \frac{\pi/3}{\ln A} \quad \text{so} \quad A = e^{\frac{\pi}{3 \cdot \tan 49.2°}} \approx 2.469.$$

That is, away from its boundary, *Path of Life III* is invariant under a scaling about its origin by a factor of approximately 2.469.

These values – the angle of approximately 98.4 degrees between paths, the scale factor of $A \approx 2.469$, and the logarithmic spiral $r \approx 2.371^\theta$ – refine the predicted values of 98 degrees for the angle, $\frac{5}{2} = 2.5$ for the scale factor, and $(5/2)^{3/\pi} \approx 2.399$ for the pitch of the spirals.

7. The Illusion of Depth

Discarding the assumption that the tiles on the surface depicted in this picture are all the same size, we find that we can interpret *Path of Life III*

as a parallel projection of a conical surface, as a perspective projection of a hemispherical surface, or as a simple planar image.

The red-gold paths meet at an angle of approximately $2\alpha = 98.4$ degrees, but the tiling creates the illusion that they meet at right angles. Viewing the woodcut, we may interpret this as a right angle foreshortened by a perspective projection. Assuming that the image is the result of a parallel projection – i.e. that the viewer is expected to stand far from the picture when viewing it – we can analyze a tetrahedron whose faces include a 45-45-90 triangle and its projection onto a β-β-2α isosceles triangle in a plane parallel to the plane of the image ($\beta = 90 - \alpha$ degrees). We find that the angle of 2α between red-gold paths in Escher's work can be interpreted as the projection of a right angle tilted approximately 30 degrees away from the viewing plane. This suggests that *Path of Life III* depicts birds and fish swimming up out of and then down into a shallow cone with a cone angle of approximately 120 degrees.
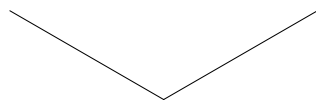

Figure 11: An angle of 120 degrees.

However, the anticipated cone point is not clearly visible in the picture. One possible explanation of this is that the level of detail in the image encourages one to view it close up; the viewer does not experience the image as a parallel projection. Instead, the fact that the birds and fish are similar to each other despite their varying size encourages the viewer to believe that her or his lines of sight always intersect the surface at the same angle. This in turn produces the illusion that the surface is spherical rather than cone-shaped.

We conclude that the illusion of depth in *Path of Life III* is ambiguous. We can perceive the image as flat or as a projection of a cone-shaped or a spherical surface.

8. Boundary Conditions

At the center of the woodcut, white birds follow white fish or black birds follow black fish, nose to tail. At the boundary of the image, a bird with a white tail and shaded head follows another bird with a shaded tail and

black head. These two birds have a slightly different shape; instead of being between two same-colored fish, one in front and one in back they lie between a same-colored fish and a shaded bird. Escher has smoothed the 98-degree angle at which they would have met into a curved line. The deformation of these birds is most evident in the outer wing of the white bird and the angle at which the black bird's outer wing meets its body.

Between these two birds, the paths of life followed by white and black creatures do not cross and continue outward, but instead curve toward each other and join smoothly. Given a point on one of the spiral paths shown in Figure 5, there is a unique parabola joining it smoothly to its mirror image. Visual inspection suggests that this is the curve Escher chose to draw, with the parabolic arcs beginning near the midpoints of the two shaded birds.

Escher's *Sun and Moon* woodcut [6, p. 265] shows rays of light emanating from the center of the piece. In *Path of Life III*, rays of light (or darkness?) transform white figures into black. These rays do not emanate from a single center; instead, each pair of shaded birds shares a separate nearby "source" for the rays. Perhaps this was necessary to establish contrast between the birds on the boundary, or perhaps this feature had some deeper significance to Escher.

9. Conclusion

In *Path of Life III* Escher follows his usual practice of starting with a simple, symmetric tiling and embellishing it to subtly change its symmetry. He then uses dilation (a conformal transformation) to pack an infinite expanse of this tiling into the interior of a circle. This gives rise to the logarithmic spiral paths of the woodcut's name and to an ambiguous illusion of depth.

The woodcut does not depict the perspective projection of any regularly tiled surface. The image may be interpreted as flat or as the projection of a spherical or conical surface.

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