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# Quotient Sets and Diophantine Equations

**Stephan Ramon Garcia, Vincent Selhorst-Jones, Daniel E. Poore, and Noah Simon**

**Abstract.** Quotient sets $\mathbb{U}/\mathbb{U} = \{u/u' : u, u' \in \mathbb{U}\}$ have been considered several times before in the MONTHLY [**4**], [**5**], [**9**]. We consider more general quotient sets $\mathbb{U}/\mathbb{V}$ and we apply our results to certain simultaneous Diophantine equations with side constraints.

In the following, we let $\mathbb{R}^+ = (0, \infty)$ denote the set of positive real numbers and $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers. A subset $\mathbb{S}$ of $\mathbb{R}^+$ is said to be *dense* in $\mathbb{R}^+$ if for each $\xi \in \mathbb{R}^+$ and every $\epsilon > 0$, there exists some $s \in \mathbb{S}$ so that $|\xi - s| < \epsilon$. This condition is equivalent to asserting that each nonempty subinterval of $\mathbb{R}^+$, no matter how small, contains infinitely many elements of $\mathbb{S}$.

For $\mathbb{U}, \mathbb{V} \subseteq \mathbb{N}$ we define the corresponding *set of quotients* $\mathbb{U}/\mathbb{V}$ by

$$\mathbb{U}/\mathbb{V} := \left\{ \frac{u}{v} : u \in \mathbb{U}, v \in \mathbb{V} \right\}.$$

Since $\mathbb{U}/\mathbb{V}$ is a subset of $\mathbb{R}^+$, we shall occasionally use the phrase "$\mathbb{U}/\mathbb{V}$ is dense" to mean that "$\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$."

Quotient sets of the form $\mathbb{U}/\mathbb{U}$ have been extensively discussed, for their own sake, in the MONTHLY over the years [**4**], [**5**], [**9**]. We consider here more general quotient sets $\mathbb{U}/\mathbb{V}$ and apply our results to certain systems of Diophantine equations.

Before proceeding, we require some additional notation. For $\mathbb{U} \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, we let $m\mathbb{U} := \{mu : u \in \mathbb{U}\}$ and $\mathbb{U} + m := \{u + m : u \in \mathbb{U}\}$. The proof of the following lemma is straightforward and is therefore omitted.

**Lemma 1.** *Let* $\mathbb{U}, \mathbb{V} \subseteq \mathbb{N}$.

(i) $\mathbb{U}/\mathbb{V}$ *is dense if and only if* $\mathbb{V}/\mathbb{U}$ *is dense.*
(ii) *If* $\mathbb{U}/\mathbb{V}$ *is dense and* $m, n \in \mathbb{N}$, *then* $m\mathbb{U}/n\mathbb{V}$ *is dense.*
(iii) *If* $\mathbb{U}/\mathbb{V}$ *is dense and* $m, n \in \mathbb{N}$, *then* $(\mathbb{U} + m)/(\mathbb{V} + n)$ *is dense.*

The main link between quotient sets and Diophantine equations is provided by the following elementary lemma:

**Lemma 2.** *If* $\mathbb{U}, \mathbb{V} \subseteq \mathbb{N}$, $a, b, c, d \in \mathbb{N}$, *and* $ad - bc = 1$, *then the system*

$$\begin{aligned} ax + by &= u, \\ cx + dy &= v, \end{aligned} \tag{1}$$

*has a solution* $(x, y, u, v) \in \mathbb{N} \times \mathbb{N} \times \mathbb{U} \times \mathbb{V}$ *if and only if there exists* $(u, v) \in \mathbb{U} \times \mathbb{V}$ *such that* $\frac{b}{d} < \frac{u}{v} < \frac{a}{c}$. *In particular, if* $\mathbb{U}/\mathbb{V}$ *is dense in* $\mathbb{R}^+$ *then for infinitely many pairs* $(u, v) \in \mathbb{U} \times \mathbb{V}$ *the system* (1) *has a solution* $(x, y) \in \mathbb{N}^2$.

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*Proof.* Solving for $x, y$ in (1) we find that $x = du - bv$ and $y = av - cu$. Both of these integers are strictly positive if and only if $\frac{b}{d} < \frac{u}{v} < \frac{a}{c}$. ∎

**Example 3.** The system

$$21x^{13} + 17y^{13} = 61^j - 3,$$
$$58x^{13} + 47y^{13} = 61^k + 4,$$

has no solutions $x, y, j, k \in \mathbb{N}$. Indeed, there do not exist $j, k \in \mathbb{N}$ such that

$$0.361702 \approx \frac{17}{47} < \frac{61^j - 3}{61^k + 4} < \frac{21}{58} \approx 0.362069.$$

The exponent 13 is irrelevant—there are no solutions to the given system even if $x, y$ appear only to the first power.

For $\mathbb{U} \subseteq \mathbb{N}$ the function $\pi_{\mathbb{U}}(x) := \#\{u \in \mathbb{U} : u \leq x\}$ is called the *counting function of* $\mathbb{U}$. In the following theorem, observe that the structure of $\mathbb{V}$ plays almost no role whatsoever.

**Proposition 4.** *Let* $\mathbb{U}$ *and* $\mathbb{V}$ *be infinite subsets of* $\mathbb{N}$. *If*

$$\lim_{n\to\infty} \frac{\pi_{\mathbb{U}}(n\alpha)}{\pi_{\mathbb{U}}(n\beta)} < 1$$

*holds whenever* $0 < \alpha < \beta$, *then* $\mathbb{U}/\mathbb{V}$ *is dense in* $\mathbb{R}^+$.

*Proof.* Let $0 < \alpha < \beta$ and observe that

$$\lim_{n\to\infty} [\pi_{\mathbb{U}}(n\beta) - \pi_{\mathbb{U}}(n\alpha)] = \lim_{n\to\infty} \pi_{\mathbb{U}}(n\beta) \cdot \lim_{n\to\infty} \left(1 - \frac{\pi_{\mathbb{U}}(n\alpha)}{\pi_{\mathbb{U}}(n\beta)}\right) = \infty$$

since $\mathbb{U}$ is infinite. Since $\mathbb{V}$ is infinite, there is a $v \in \mathbb{V}$ such that $\pi_{\mathbb{U}}(v\beta) - \pi_{\mathbb{U}}(v\alpha) \geq 2$. Thus there exists a $u \in \mathbb{U}$ such that $v\alpha < u < v\beta$ whence $\alpha < \frac{u}{v} < \beta$. ∎

Let $\mathbb{P}$ denote the set of primes, $\mathbb{P}_{m,r}$ the set of primes congruent to $r$ modulo $m$, and $\pi_{m,r}$ the counting function of $\mathbb{P}_{m,r}$. It is somewhat of a folk theorem that $\mathbb{P}/\mathbb{P}$ is dense in $\mathbb{R}^+$ (see [**1**, Ex. 218], [**2**, Ex. 4.19], [**5**, Thm. 4], [**7**, Ex. 7, p. 107], [**8**, Thm. 4], [**9**, Cor. 2]). In fact, this result dates back at least to Sierpiński (1959), who himself attributed the result to Schinzel [**6**]. Using Proposition 4, we can greatly generalize this observation.

**Corollary 5.** *If* $\gcd(m, r) = 1$ *and* $\mathbb{V} \subseteq \mathbb{N}$ *is infinite, then* $\mathbb{P}_{m,r}/\mathbb{V}$ *is dense.*

*Proof.* We let $0 < \alpha < \beta$ and check the criterion from Proposition 4. The Prime Number Theorem for Arithmetic Progressions [**2**, Thm. 4.7.4] asserts that

$$\lim_{x\to\infty} \frac{\pi_{m,r}(x)}{x/\log x} = \frac{1}{\phi(m)},$$

where $\phi$ denotes Euler's $\phi$-function. Thus

$$\lim_{n\to\infty} \frac{\pi_{m,r}(n\alpha)}{\pi_{m,r}(n\beta)} = \lim_{n\to\infty} \frac{n\alpha \log(n\beta)}{n\beta \log(n\alpha)} = \frac{\alpha}{\beta} < 1.$$ ∎

**Example 6.** The set of all rational numbers of the form $\frac{p}{(2q-1)!}$ where $p, q$ are primes ending in 123,456,789 and 987,654,321, respectively, is dense in $\mathbb{R}^+$. Indeed, let $\mathbb{U} = \mathbb{P}_{10^9,123456789}$ and $\mathbb{V} = \mathbb{P}_{10^9,987654321}$ and apply Corollary 5.

**Example 7.** By Corollary 5 the set $\mathbb{P}_{97,83}/\mathbb{P}_{103,59}$ is dense in $\mathbb{R}^+$. By Lemma 2 it follows that there are infinitely many $x, y \in \mathbb{N}$ and $p, q \in \mathbb{P}$ such that

$$\begin{aligned} 21x + 17y &= p, \qquad & p &\equiv 83 \pmod{97}, \\ 58x + 47y &= q, \qquad & q &\equiv 59 \pmod{103}. \end{aligned}$$

Some solutions with small $x, y$ are given in the following table. As predicted by

| $x$ | $y$ | $p$ | $q$ | $p/q$ |
|---|---|---|---|---|
| 100 | 509 | 10,753 | 29,723 | 0.361774 |
| 696 | 1,553 | 41,017 | 113,359 | 0.361833 |
| 1,442 | 1,921 | 62,939 | 173,923 | 0.361879 |
| 1,848 | 1,157 | 58,477 | 161,563 | 0.361945 |

Lemma 2, each quotient $p/q$ belongs to the interval $(\frac{17}{47}, \frac{21}{58}) \approx (0.361702, 0.362068)$.

**Example 8.** Applying Corollary 5 with $\mathbb{U} = \mathbb{P}_{1000,999}$ and $\mathbb{V} = \{(2q-1)! : q \in \mathbb{P}_{1000,123}\}$, we see that there are infinitely many solutions to the system

$$\begin{aligned} 9650x + 967y &= p, \\ 11097x + 1112y &= (2q-1)!, \end{aligned}$$

where $x, y \in \mathbb{N}$, $p$ is a prime ending in 999, and $q$ is a prime ending in 123.

Before presenting our next result, let us briefly recall Kronecker's Approximation Theorem [**3**, Thm. 440], which asserts that if $\beta > 0$ is irrational, $\alpha \in \mathbb{R}$, and $\delta > 0$, then there exist $n, m \in \mathbb{N}$ such that $|n\beta - \alpha - m| < \delta$. The next result generalizes [**4**, Thm. 2].

**Proposition 9.** *If $u, v > 1$ are distinct natural numbers such that the sets $\mathbb{U} = \{u^n : n \in \mathbb{N}\}$ and $\mathbb{V} = \{v^n : n \in \mathbb{N}\}$ are disjoint, then $\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$.*

*Proof.* By Lemma 1, we may assume that $v < u$. Let $\xi \in \mathbb{R}^+$, $\epsilon > 0$, and note that $\beta = \log_v u > 0$ is irrational. By the continuity of the function $f(x) = v^x$ at $\log_v \xi$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\log_v x - \log_v \xi| < \delta \quad \Rightarrow \quad |x - \xi| < \epsilon. \tag{2}$$

Kronecker's Theorem with $\beta = \log_v u$ and $\alpha = \log_v \xi$ now yields $n, m \in \mathbb{N}$ so that

$$\left| \log_v \left( \frac{u^n}{v^m} \right) - \log_v \xi \right| = |n \log_v u - \log_v \xi - m| < \delta.$$

In light of (2), it follows that $|u^n/v^m - \xi| < \epsilon$ whence $\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$. ■

**Example 10.** The system

$$21x + 17y = 7 \cdot 29^j + 19,$$
$$58x + 47y = 11 \cdot 52^k + 13,$$

has infinitely many solutions $(x, y, j, k) \in \mathbb{N}^4$. Indeed, by Proposition 9 the set $\{29^j/52^k : j, k \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$. Now appeal to Lemma 1.

In the following, for $\mathbb{U} \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we let $\mathbb{U}^{(n)} := \{u^n : u \in \mathbb{U}\}$ denote the set of $n$th powers of elements of $\mathbb{U}$.

**Proposition 11.** *Let $\mathbb{U}, \mathbb{V} \subseteq \mathbb{N}$ and $n \in \mathbb{N}$. If $\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$, then $\mathbb{U}^{(n)}/\mathbb{V}^{(n)}$ is dense in $\mathbb{R}^+$.*

*Proof.* If $\xi > 0$, then there exist sequences $u_j$ in $\mathbb{U}$ and $v_j$ in $\mathbb{V}$ such that $u_j/v_j$ tends to $\sqrt[n]{\xi}$. Since the map $x \mapsto x^n$ is continuous, it follows that $u_j^n/v_j^n$ tends to $\xi$ whence $\mathbb{U}^{(n)}/\mathbb{V}^{(n)}$ is dense in $\mathbb{R}^+$. ■

**Example 12.** By Lemma 2, the system

$$21x + 17y = 7p^{11} + 6, \qquad p \equiv 83 \pmod{97},$$
$$58x + 47y = 13q^{11} + 15, \qquad q \equiv 59 \pmod{103}.$$

has infinitely many solutions $x, y \in \mathbb{N}$ and $p, q \in \mathbb{P}$. Indeed, $\mathbb{P}_{97,83}/\mathbb{P}_{103,59}$ is dense in $\mathbb{R}^+$ by Example 7. Now apply Proposition 11 and Lemma 1.

Intuitively, one expects that if $\mathbb{U}/\mathbb{V}$ is dense and if $\mathbb{X}$ and $\mathbb{Y}$ are in some sense "close" to $\mathbb{U}$ and $\mathbb{V}$, respectively, then $\mathbb{X}/\mathbb{Y}$ should also be dense. One way to make this precise is the following.

**Proposition 13.** *Let $\mathbb{U}, \mathbb{V}, \mathbb{X}, \mathbb{Y} \subseteq \mathbb{N}$, suppose that $\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$, and consider $\mathbb{U}, \mathbb{V}, \mathbb{X}, \mathbb{Y}$ as strictly increasing sequences in $\mathbb{N}$. If $\mathbb{X}$ and $\mathbb{Y}$ have subsequences $x_{n_j}$ and $y_{n_k}$, respectively, such that*

$$\lim_{j \to \infty} \frac{x_{n_j}}{u_j} = \lim_{k \to \infty} \frac{y_{n_k}}{v_k} = 1, \tag{3}$$

*then $\mathbb{X}/\mathbb{Y}$ is dense in $\mathbb{R}^+$.*

*Proof.* If $\xi > 0$, then there exist subsequences $u_{j_m}$ in $\mathbb{U}$ and $v_{k_m}$ in $\mathbb{V}$ such that $u_{j_m}/v_{k_m}$ tends to $\xi$. Since

$$\lim_{m \to \infty} \frac{x_{n_{j_m}}}{y_{n_{k_m}}} = \lim_{m \to \infty} \left(\frac{x_{n_{j_m}}}{u_{j_m}}\right)\left(\frac{u_{j_m}}{v_{k_m}}\right)\left(\frac{v_{k_m}}{y_{n_{k_m}}}\right) = \xi$$

by (3), it follows that $\mathbb{X}/\mathbb{Y}$ is dense in $\mathbb{R}^+$. ■

**Corollary 14.** *Let $\mathbb{U}, \mathbb{V} \subseteq \mathbb{N}$. If $\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$ and $f, g \in \mathbb{Z}[x]$ both have positive leading coefficients and the same degree, then the set $\{f(u)/g(v) : u \in \mathbb{U}, v \in \mathbb{V}\}$ is dense in $\mathbb{R}^+$.*

*Proof.* Suppose that $f$ and $g$ are both of degree $n$ and let $f_n, g_n > 0$ denote their leading coefficients, respectively. By Proposition 11 the set $\mathbb{U}^{(n)}/\mathbb{V}^{(n)}$ is dense in $\mathbb{R}^+$ and hence so is $\{(f_n u^n)/(g_n v^n) : u \in \mathbb{U}, v \in \mathbb{V}\}$ by Lemma 1. Let $\mathbb{X} = \{f(u_j) : j \in \mathbb{N}\}$ and $\mathbb{Y} = \{g(v_k) : k \in \mathbb{N}\}$ and note that

$$\lim_{j\to\infty} \frac{f(u_j)}{f_n u_j^n} = 1 = \lim_{k\to\infty} \frac{g(v_k)}{g_n v_k^n},$$

whence $\mathbb{X}/\mathbb{Y}$ is dense in $\mathbb{R}^+$ by Proposition 13. ■

**Example 15.** The system

$$\begin{aligned} 21x + 17y &= 7p^3 + 2p^2 + 8p + 1, \qquad & p &\equiv 83 \pmod{97}, \\ 58x + 47y &= 13q^3 + 5q^2 + 6q + 2, \qquad & q &\equiv 59 \pmod{103}, \end{aligned}$$

has infinitely many solutions $x, y, p, q \in \mathbb{N}^2 \times \mathbb{P}^2$. Indeed, by Example 7 and Corollary 14, it follows that $\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$ where $\mathbb{U} = \{7p^3 + 2p^2 + 8p + 1 : p \in \mathbb{P}_{97,83}\}$ and $\mathbb{V} = \{13q^3 + 5q^2 + 6q + 2 : q \in \mathbb{P}_{103,59}\}$.

The following example shows how even more imposing systems can be devised using a little matrix arithmetic and basic calculus.

**Example 16.** We claim that the system

$$\begin{aligned} 21x + 17y &= 16p^4 + 5p^2q + 13p^2 + 24pq^2 + 9pq + 7q^4 + 7, \\ 58x + 47y &= 44p^4 + 15p^2q + 36p^2 + 66pq^2 + 25pq + 21q^4 + 20, \end{aligned} \tag{4}$$

has infinitely many solutions $(x, y, p, q) \in \mathbb{N} \times \mathbb{N} \times \mathbb{P}_{97,83} \times \mathbb{P}_{103,59}$. We first rewrite this system as

$$\begin{pmatrix} 21 & 17 \\ 58 & 47 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \begin{pmatrix} f(p,q) \\ g(p,q) \end{pmatrix}, \tag{5}$$

where

$$\begin{aligned} f(p,q) &= 4p^4 + 3p^2 + 6pq^2 + 2pq + 1, \\ g(p,q) &= 7q^4 + 5p^2q + pq + p^2 + 3. \end{aligned}$$

Since

$$\begin{pmatrix} 21 & 17 \\ 58 & 47 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}, \tag{6}$$

we obtain from (5) the equivalent system

$$\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(p,q) \\ g(p,q) \end{pmatrix}. \tag{7}$$

This leads us to consider the set

$$\left\{ \frac{f(p,q)}{g(p,q)} : p \in \mathbb{P}_{97,83}, q \in \mathbb{P}_{103,59} \right\}. \tag{8}$$

By Proposition 11, for each $\xi > 0$ there exist increasing sequences $p_n$ in $\mathbb{P}_{97,83}$ and $q_n$ in $\mathbb{P}_{103,59}$ such that $p_n^4/q_n^4$ tends to $\frac{7}{4}\xi$. Thus the sequence $\frac{p_n}{q_n}$ is bounded and hence

$$\lim_{n\to\infty} \frac{f(p_n, q_n)}{g(p_n, q_n)} = \lim_{n\to\infty} \frac{(\frac{1}{q_n^4})(4p_n^4 + 3p_n^2 + 6p_n q_n^2 + 2p_n q_n + 1)}{(\frac{1}{q_n^4})(7q_n^4 + 5p_n^2 q_n + p_n q_n + p_n^2 + 3)} = \xi.$$

Therefore the set (8) is dense in $\mathbb{R}^+$. By Lemma 2, we conclude that the system (7), and hence (4), has infinitely many solutions $(x, y, p, q) \in \mathbb{N} \times \mathbb{N} \times \mathbb{P}_{83,97} \times \mathbb{P}_{59,103}$.

We turn now from Diophantine equations and instead focus on quotient sets themselves. The general problem of determining whether $\mathbb{U}/\mathbb{V}$ is dense in $\mathbb{R}^+$ appears to be intractable, except in a few special cases such as those considered above. It may come as some surprise, however, that even if $\mathbb{U} = \mathbb{V}$ the situation is far from clear. Let us say that $\mathbb{U} \subseteq \mathbb{N}$ is *fractionally dense* if $\mathbb{U}/\mathbb{U}$ is dense in $\mathbb{R}^+$.

**Example 17.** A few sets which are fractionally dense are the set $\mathbb{P}$ of primes (Corollary 5), infinite arithmetic progressions (Lemma 1), and the set $\mathbb{N}^{(2)} = \{n^2 : n \in \mathbb{N}\}$ of perfect squares (Proposition 11). A few sets which are not fractionally dense are $\{2^n : n \in \mathbb{N}\}$, $\{n! : n \in \mathbb{N}\}$, and the set $\mathbb{F} = \{1, 2, 3, 5, 8, 13, 21, \ldots\}$ of all nonzero Fibonacci numbers (the $n$th term in this sequence is closely approximated by $5^{-1/2}\phi^n$ where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the Golden Ratio [**3**, X.10.14]).

**Example 18.** There are uncountably many subsets of $\mathbb{N}$ that are fractionally dense and uncountably many that are not. Indeed, let $\mathbb{U} = 3\mathbb{N} = \{3, 6, 9, \ldots\}$ and note that for any sequence $(\epsilon_n)_{n=1}^{\infty}$ of $+1$'s and $-1$'s, the set $\mathbb{U}' = \{3 + \epsilon_1, 6 + \epsilon_2, 9 + \epsilon_3, \ldots\}$ is fractionally dense (Proposition 13). Since there are uncountably many sign sequences, there are uncountably many fractionally dense subsets of $\mathbb{N}$. Now repeat the argument with the set $\mathbb{U} = \{2^2, 2^3, 2^4, \ldots\}$, which is not fractionally dense.

Let us recall that the reciprocal sum

$$\sum_{u\in\mathbb{U}} \frac{1}{u} \tag{9}$$

provides a qualitative measure of the "thickness" of a subset $\mathbb{U} \subseteq \mathbb{N}$. For instance, Euler showed that (9) diverges for $\mathbb{U} = \mathbb{P}$ (see [**10**] for a number of elementary proofs) and Brün proved that (9) converges if $\mathbb{U}$ is the set of all twin primes. Thus in this sense, the set of twin primes is "sparser" than the set of primes.

**Example 19.** Let $\mathbb{U} \subset \mathbb{N}$ denote the set of natural numbers beginning with the digit 1 in base-10 notation. Although the series

$$\sum_{u\in\mathbb{U}} \frac{1}{u} = 1 + \underbrace{\frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{19}}_{> \frac{10}{20} = \frac{1}{2}} + \underbrace{\frac{1}{100} + \frac{1}{101} + \cdots \frac{1}{199}}_{> \frac{100}{200} = \frac{1}{2}} + \cdots$$

diverges, $\mathbb{U}$ is not fractionally dense since $\mathbb{U}/\mathbb{U} \cap (4, \frac{9}{2}) = \varnothing$. In other words, $\mathbb{U}$ is a "thick" subset of $\mathbb{N}$ which fails to be fractionally dense (see Table 1).

We conclude this note with the following versatile construction which can be used to disprove virtually any naïve conjecture about fractional density (see also [**4**, Thm. 1]).

**Table 1.** Reciprocal sums do not say anything about fractional density.

| The set $\mathbb{U}$ is | $\sum_{u\in\mathbb{U}} \frac{1}{u}$ | $\mathbb{U}$ fractionally dense? |
|---|---|---|
| $\{2^n : n \in \mathbb{N}\}$ | converges | no |
| see Example 19 | diverges | no |
| see Example 20 | converges | yes |
| $2\mathbb{N} = \{2n : n \in \mathbb{N}\}$ | diverges | yes |

**Example 20.** Let $\mathbb{Q} \cap (0, 1) = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots\}$ be an enumeration of the rational numbers, written in lowest terms, in the open interval $(0, 1)$. Select a sequence of natural numbers $m_1, m_2, m_3, \ldots$ inductively so that

$$\underbrace{m_1a_1 < m_1b_1}_{\text{quotient is } \frac{a_1}{b_1}} < \underbrace{m_2a_2 < m_2b_2}_{\text{quotient is } \frac{a_2}{b_2}} < \underbrace{m_3a_3 < m_3b_3}_{\text{quotient is } \frac{a_3}{b_3}} < \cdots . \tag{10}$$

Letting $\mathbb{U} = \{u_1, u_2, \ldots\}$ denote the elements of this sequence, it is clear that $\mathbb{U}/\mathbb{U}$ is dense in $\mathbb{R}^+$. However, the $m_i$ may be chosen so that the gaps between successive pairs in (10) grow arbitrarily fast. In particular, we can select the $m_i$ so that $\sum_{u\in\mathbb{U}} \frac{1}{u}$ converges. Also observe that if the $m_i$ are selected so that $2m_ia_i \leq m_{i+1}a_{i+1}$, then $\mathbb{U}$ is fractionally dense but $\{u_1, u_3, u_5, \ldots\}$ and $\{u_2, u_4, u_6, \ldots\}$ are not.

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