1-1-2011

A Space-Filling, Nonregular Tetrahedron

Margaret Cagle
Lawrence Gifted and Highly Gifted Magnet Middle School

Joyce Frost

Christine Latulippe
California State Polytechnic University - Pomona

Darryl H. Yong
Harvey Mudd College

Recommended Citation

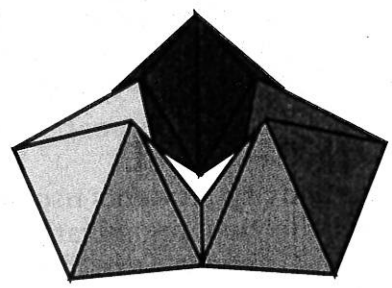
M. Cagle, J. Frost, C. Latulippe, Yong, D.H., "A Space-Filling, Nonregular Tetrahedron," ComMuniCator (California Mathematics Council), 35:4 (June 2011), 52-55.

tion? Give a convincing sentence or two for justifying your choice of a domain.

4. Is every whole number of marbles part of the range? Give a convincing sentence or two for justifying your answer.
5. Were there any data points that did not follow the trend? If there were, what might be the cause? How would a data point that does not follow the trend change the line drawn and the predictions made?
6. Would changing the distance between the chairs / tables affect the results? If so, how? Describe and carry out an experiment to test your theory.
7. Compare your results with other groups. What do you think might account for the differences?
8. If you did the experiment with tape and then without tape, was there a difference in your results? If there was a difference, what caused it; if not, why do you think that there was no change?

A Space-Filling, Nonregular Tetrahedron

by Margaret Cagle, Lawrence Gifted and Highly Gifted Magnet Middle School, peg@cagle.com; Joyce Frost, Mathematics Education Consultant, frostjoycee@gmail.com; Christine Latulippe, Cal Poly Pomona, cllatulippe@csupomona.edu; and Darryl Yong, Harvey Mudd College, dyong@hmc.edu



CONCEPTS: Geometry and Measurement
SKILLS: Measuring lengths and calculating areas of triangles; identifying two- and three-dimensional shapes and solids
MATHEMATICS STANDARDS: Gr 4: MG 3.6, 3.7; Gr 5: MG 1.1; Geom 9.0, 10.0, 15.0
GRADES: 4–12
MATERIALS: Scotch tape, packing tape, rulers, tetrahedron nets printed on cardstock (alternatively, nets can be created by folding paper and transferring the nets onto heavy paper such as manila folders), calculator, Student Activity Sheets (pages 54–55)

DESCRIPTION

This activity is an investigation of a special nonregular tetrahedron that can be arranged to fill space without leaving any internal gaps in the same way that certain planar figures tessellate the plane. These tetrahedra can be connected together with hinges to make fun and interesting puzzles. More background information can be found in the paper "An Amazing, Space-Filling, Non-Regular Tetrahedron" by Joyce Frost and Peg Cagle, published by the IAS / Park City Mathematics Institute (available at **mathforum.org/pcmi/hstp/resources/dodeca/**).

Students can construct the tetrahedron in a variety of ways. If time is limited, the instructor can print out tetrahedron nets (available at **tinyurl.com/tetrahedronnet**) and even have the nets cut out in advance so that students do not have to do any cutting. If there is more time, the students can construct the tetrahedron net through paper folding as described on the first web page above. This paper folding process supports a rich discussion of triangles and the Pythagorean theorem in a geometry class.

Making Tetrahedrons

Students should fold the net first before trying to tape the edges together. It is important to place tape *parallel to the edges* of the tetrahedron so as to leave the smallest possible untaped gap along each edge. While Scotch tape is sufficient for these edges, packing tape will make a more secure shape, but it will take more time to tape each edge. The easiest and most accurate way to tape the folded net together is to apply tape to one face first, then line up the other face carefully and adhere the tape to the other face.

The fun part of this activity occurs when students attach tetrahedra to each other to make interesting shapes. This is accomplished by taping tetrahedra together along common edges to create a hinge. It is important to apply tape to *both sides* to create a secure hinge. Packing tape is the best choice for creating these hinges—Scotch tape will wear out quickly.

All four faces of this special tetrahedron are congruent isosceles triangles and, with some geometric reasoning, one can deduce the ratios of the side lengths that cause this tetrahedron to be able to fill space. Students can create a rotating ring by hinging together eight tetrahedra along their longer edges. If they hinge 24 tetrahedra together along their longer edges, they will create a ring that can be folded into a rhombic dodecahedron (with no internal spaces). They can also create this rhombic dodecahedron by hinging the tetrahedra along their shorter sides, but the process is trickier.

Teachers can differentiate this activity to meet the needs and interests of a wide range of students. All students will enjoy playing with the completed tetrahedra puzzles. The Student Activity Sheets provide some sample questions that teachers can use to deepen geometric thinking and reasoning in Grades 4–6 and in a secondary-school geometry course. Students in Grades 4–6 should use Student Activity Sheet 1, while students in secondary school can complete both Student Activity Sheets 1 and 2.

EXTENSIONS

Teachers can extend this activity in a variety of ways. The lesson could lead students to discover Euler's Formula ($V - E + F = 2$) for convex polyhedra. Students might enjoy seeing what other shapes they can make by hinging tetrahedra along both shorter and longer edges. They can make a stellated dodecahedron by taping 48 tetrahedra along their shorter edges, as described in the paper referenced above.

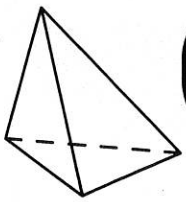
Authors' Note: The tetrahedron net printed at **tinyurl.com/tetrahedronnet** is designed to produce a tetrahedron with the longer edge exactly 13 centimeters to make the measurements and calculations easier on Student Activity Sheet 1. To ensure that the tetrahedron net prints correctly, please make sure that you set Page Scaling to None in the print dialog box on your PDF viewer program so that the net is printed with an effect scaling of 100%. One of the co-authors (Darryl Yong, dyong@hmc.edu) has a die that can be used to cut out the necessary net; send an e-mail if you would like to use this die.

Student Activity Sheets, pages 54–55. . .

Themes (cont. from page 3)

mance may be more fully realized when teachers function as teams. Collaboration among teachers involves the regular sharing of ideas, developing common goals, and determining ways to attain those goals.

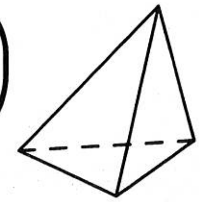
The *Communicator* Editorial Panel is seeking articles that share experiences that involve working in collaborative teams. We are also looking for articles that provide answers to some of the following questions: What does research say about teacher collaboration? How do we encourage teachers to work and plan together in teams? What part do academic coaches play in collaboration? What are some examples of successful collaborative work in action (for instance, Lesson Study or Professional Learning Communities)? How can schools foster continued collaboration in the current economic climate? What are some of the hurdles teachers face when working together and how can we overcome them?



A Space-Filling, Nonregular Tetrahedron

Activity Sheet 2

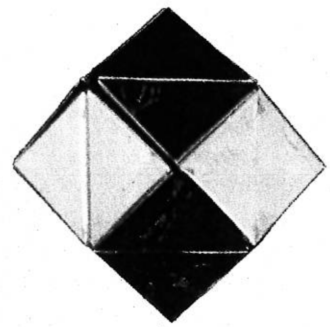
by Margaret Cagle, Joyce Frost, Christine Latulippe, Darryl Yong



In Activity Sheet 1 you found that the faces of this special tetrahedron are congruent isosceles triangles. The tetrahedron only has two different edge lengths. Because of the ratio of its edge lengths, this tetrahedron has the ability to fill space—that is, it can be arranged in such a way that it will fill a space without leaving any gaps. If you made the rhombic dodecahedron, you can see that there are no empty spaces between tetrahedra.

In this activity sheet you will further investigate the space-filling properties of this tetrahedron.

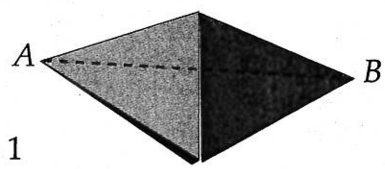
1. Twenty-four tetrahedra pack around a central point to create a rhombic dodecahedron. If you look at this rhombic dodecahedron in just the right way, it is possible to see the rhombic dodecahedron as a cube with a rectangular pyramid attached to each of its six faces. What is the edge length of that cube?



2. What is the ratio of the volume of that cube to the volume of the rhombic dodecahedron? How do you know?

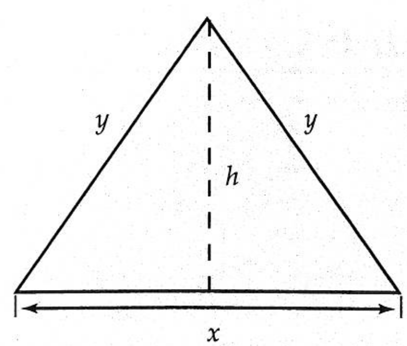
3. Place two tetrahedra together along their longer sides so they make a flat rhombus as shown in the picture at the right. Calculate the length of the diagonal AB. Measure the diagonal to verify your answer.

   (Hint: Turn the tetrahedra around until you can visualize AB as the hypotenuse of an isosceles triangle. Use Problems 1 and 2 to see why this isosceles triangle is also a right triangle.)



4. Look at one of the faces of the tetrahedron., Let x be the longer side length in the triangle. Use what you have learned so far about the tetrahedron to calculate the lengths h and y in terms of x. (Hint: What is the relationship between h and AB from Problem 3?)

   Calculate the area of the triangle. Compare your answer to the measurements you made in Activity Sheet 1, Problems 3 and 5.

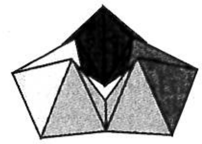


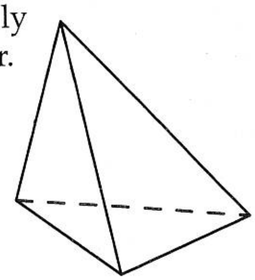
5. What is the volume of one tetrahedron in terms of x?

A Space-Filling, Nonregular Tetrahedron

Activity Sheet 1

by Margaret Cagle, Joyce Frost, Christine Latulippe, Darryl Yong





1. Follow your teacher's instructions to create a tetrahedron by carefully folding a tetrahedron net. Tape the edges of the tetrahedron together. (You can download a net at **tinyurl.com/tetrahedronnet**.)

2. How many faces does your tetrahedron have?
   How many edges does your tetrahedron have?
   How many vertices does your tetrahedron have?

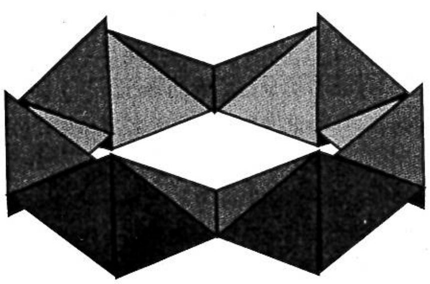
3. Use a ruler to measure the length of each edge of your tetrahedron to the nearest millimeter. Write the length of each edge on your tetrahedron.

4. What kind of triangle—scalene, isosceles, or equilateral—is each face of the tetrahedron?

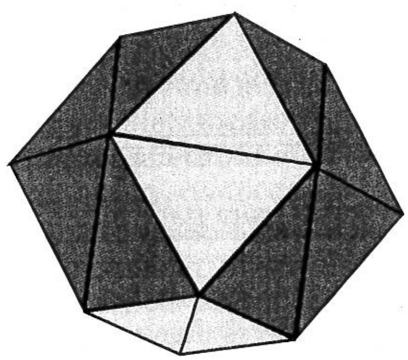
5. Use your ruler to measure a base and height of each triangle. What is the area of each triangular face? What is the total surface area of the tetrahedron?

6. Fold another tetrahedron. Carefully tape your two tetrahedra together along their **longest** sides to create a hinge. Tape both sides of the hinge to keep the two tetrahedra from falling apart.

7. Now that you and your fellow classmates have built several tetrahedra, you can put them together to make some interesting shapes. If you tape 8 tetrahedra together along their longest sides, you will get a rotating ring. If you tape 24 tetrahedra together along their longest sides, you will get a rotating ring that folds into a rhombic dodecahedron.



Tetrahedron Ring



Rhombic Dodecahedron

8. Look up the definition of a dodecahedron. How many vertices, faces and edges does it have? Explain why the picture shows a rhombic dodecahedron.