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Mathematics of Voting

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Mathematics of Voting

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CONCEPT: Mathematical Reasoning
SKILLS: Solving problems, developing concepts
MATHEMATICS STANDARDS: Gr 7: MR 1.0, 2.0, 3.0
GRADES: 7–12
MATERIALS: Student Activity Sheets

BACKGROUND

Voting theory is a fascinating area of research involving mathematicians, political scientists, and economists. The American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics chose mathematics and voting as the theme for Mathematics Awareness Month 2008. There is more information on mathematics and voting at **www.mathaware.org/mam/08/**. It is a mathematical topic that is rich yet accessible to students, pertinent to their lives, especially during this election year, and has the potential to draw students who may not have a strong affinity for mathematics to become interested in mathematics.

The discussion questions on the Student Activity Sheets on pages 50–52 were adapted from the most recent Imagine Math Day at Harvey Mudd College (HMC), an outreach activity for secondary school students and their teachers hosted by the HMC Professional Development and Outreach Group. These Imagine Math Day activities are held every year in the spring, and are designed to engage students and teachers in rich, authentic mathematical discovery and inquiry.

These small-group discussion questions are designed to require no introduction to the subject of mathematical voting theory. The questions lead students to think about different systems of voting, their properties, and their advantages and disadvantages. Pose the sets of questions in the order presented on each Student Activity Sheet. You can modify them to accommodate time constraints.

The discussion questions about movies work best in groups of six to ten students in order to make a clear top choice unlikely. The movie names are parodies of movies that were playing in the spring of 2008; you will need to replace these with current movie titles. The questions are intentionally worded vaguely so as to provoke a lively discussion and to encourage students to come up with their own interpretations. For example, there are many different ways of characterizing when a voting system is "fair," and students may have their own interpretations as well.

The problem of constructing a "voting paradox" is discussed in much greater detail in Saari's *Chaotic Elections* book (2001). Younger students might use enumeration and trial and error to find a suitable paradox; students with more mathematical background might set up and use algebraic inequalities.

References

Brams, Steven J., and Peter C. Fishburn. *Approval Voting*, 2nd edition. New York: Springer-Verlag, 2007.

Nurmi, Hannu. *Voting Paradoxes and How to Deal with Them*. New York: Springer-Verlag, 1999.

Saari, Donald G. *Basic Geometry of Voting*. New York: Springer-Verlag, 1995.

________. *Chaotic Elections! A Mathematician Looks at Voting*. American Mathematical Society, 2001.

________. *Decisions and Elections: Explaining the Unexpected*. Cambridge University Press, 2001.

Taylor, Alan D. *Mathematics and Politics: Strategy, Voting, Power and Proof*. New York: Springer-Verlag, 1995.

________. *Choice and the Mathematics of Manipulation*. Cambridge University Press, 2005.

Student Activity Sheets, pages 50–52 . . .

Mathematics of Voting
Activity Sheet 1

by Darryl Yong

Movies

Suppose everyone at your table is getting together tonight to watch a movie and these are the four available choices.

Movie A: 3^3 Dresses

Movie B: The i

Movie C: Calvin and the Hip Monks

Movie D: $\frac{\text{cl}}{\text{field}}$

Unfortunately, the four movies are showing at different theaters and all of you are in the same vehicle so everyone has to watch the same movie. Let's explore different ways you could vote on which movie to watch.

1. Ask each person at your table to choose one of the four movies to watch. Tally the number of votes each movie receives. Is there a clear winner? Is there a clear loser?

   Group's votes: ____ votes for A ____ votes for B ____ votes for C ____ votes for D

2. In the space below, put a check mark next to the movies you are willing to watch. You can select as many movies as you like; you can even select all the movies or none of the movies. Then, count the number of check marks that each movie receives from the others at your table. Which movie receives the most number of check marks? Which movie receives the least number of votes?

   I'm willing to watch: _____ A _____ B _____ C _____ D

   Group's votes: ____ votes for A ____ votes for B ____ votes for C ____ votes for D

3. Now suppose that each person at the table has three votes and can assign them to the four movies any way he/she wants: Each person can assign all three points to one movie or give two points to one movie and one point to another, etc. After everyone at your table decides how to assign points, which movie receives the most number of points? Which movie receives the least number of points?

   My votes: ____ votes for A ____ votes for B ____ votes for C ____ votes for D

   Group's votes: ____ votes for A ____ votes for B ____ votes for C ____ votes for D

4. Rank each of the four movies according to your interest in watching it. Your first choice will be the movie you're most interested in watching, your fourth choice will be the movie you're least interested in watching. Collect this ranking information from everyone else at your table. (Use the back of this paper or some scratch paper.) Can you come up with a way to use this information to determine the movie that would make most people happy?

   My 1st choice: _____ My 2nd choice: _____ My 3rd choice: _____ My 4th choice: _____

5. Now imagine that the group has collectively decided which movie to watch. You drive to the theater and realize that the movie time information was incorrect and you've already missed half the movie. Now you have to choose one of the remaining three movies to watch. Can you find a good way to decide, using the information you already have? Share your ideas with the others at your table.

Mathematics of Voting
Activity Sheet 2

by Darryl Yong

Presidential Elections and TV Shows

1. Instead of our current system for presidential elections, what if we just roll a (certifiably fair) die to pick our next president? Why would this be a good or bad idea?
2. Instead of our current system for presidential elections, consider an alternative system in which every eligible voter in America votes and there is a lottery to pick one lucky voter. The candidate that this voter wrote on his or her ballot will be the next president. Do you think this is a good voting system? Why or why not?
3. Think of some reality TV shows that involve voting (for example, Gauntlet, American Idol, Dancing with the Stars, Survivor, Project Runway, Top Chef). How do the voting systems compare with any of the ones that have come up so far? Do you think the voting systems on the shows are fair? Why or why not? How would you improve the voting system either to make it more fair or to make the show more interesting?

A Conundrum of Olympic Proportions

During the 1994 Winter Olympic Games in Lillehammer, Norway, the final top three female figure skaters were Oksana Baiul (Ukraine), Nancy Kerrigan (United States), and Chen Lu (China). Here is how the judges ranked the three skaters after the free skate event.

	GBR	POL	CZE	UKR	CHN	USA	JPN	CAN	GER
Baiul	3	1	1	1	1	2	2	3	1
Kerrigan	2	2	2	2	2	1	1	1	2
Chen Lu	1	4	3	3	3	3	3	2	4

1. According to International Skating Union rules, the skater with the most number of first-place votes wins the gold medal. Who received the gold medal in 1994?
2. Another way to choose a winner is to allow each judge to vote for two skaters by giving one point to his or her top choice and another point to his or her second choice. The skater with the most number of points wins. What happens if we use this voting system instead?
3. Yet another way to choose a winner is to think of the rank that each judge assigns each skater as points. We can then sum up the number of points that each skater receives and the one with the lowest number of points is the winner. What happens if we use this voting system instead?
4. If Great Britain (GBR) had ranked Baiul or Kerrigan in 10th place, would it have made any difference for any of the three voting systems? Which voting system do you think is best, and why? Can you think of other changes in the judges' rankings that would produce different final standings?

Source of data: http://en.wikipedia.org/wiki/Figure_skating_at_the_1994_Winter_Olympics

Mathematics of Voting
Activity Sheet 3

by Darryl Yong

Voting Paradoxes

Suppose there is a group of people who want to elect a president from a pool of three candidates: A, B, and C. This fictitious group's preferences are summarized in the table below, which you will complete.

Number of people . . .	. . . with the preference
12	A ▶ B ▶ C
	A ▶ C ▶ B

We are writing A ▶ B as a shorthand for "A preferred over B." (By the way, how many different ways are there to arrange the letters A, B, and C?)

Fill in the table above to create a scenario in which condition (i) is true. Can you fill in the table above to create a scenario in which conditions (i) and (ii) are simultaneously true? What about a scenario in which conditions (i), (ii), and (iii) are simultaneously true?

i. Candidate A will win a plurality vote (the candidate with the most number of first-place votes wins).

ii. Candidate B will win if each person is allowed two votes and each person gives one vote to his or her top choice, and one vote to his or her second choice.

iii. Candidate C will win if a "Borda count" is used (each person's top choice receives 2 points, second choice receives 1 point, third choice receives 0 points, and the candidate with the most number of points wins).

> If you're feeling adventurous, try to find a voting paradox involving four candidates.