1-1-1994

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European Journal of Applied Mathematics / Volume 5 / Issue 03 / September 1994, pp 267 - 282
DOI: 10.1017/S0956792500001467, Published online: 26 September 2008

Euro. Jnl of Applied Mathematics (1994), *vol.* 5, *pp.* 267–282

Finite amplitude convection between stress-free boundaries; Ginzburg–Landau equations and modulation theory

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(Received 22 March 1993)

The stability theory for rolls in stress-free convection at finite Prandtl number is affected by coupling with low wavenumber two-dimensional mean-flow modes. In this work, a set of modified Ginzburg–Landau equations describing the onset of convection is derived which accounts for these additional modes. These equations can be used to extend the modulation equations of Zippelius & Siggia describing the breakup of rolls, bringing their stability theory into agreement with the results of Busse & Bolton.

1 Introduction

Classically, the analytically simple problem of Rayleigh–Benard convection between stress-free upper and lower boundaries has been studied. Early work (cf. Busse, 1978) only considered roll-like disturbances with zero vertical vorticity. This yielded an interval of stable wavenumbers of width $0(\sqrt{\epsilon})$, where ϵ is the reduced Rayleigh number. Zippelius & Siggia (1982, 1983) noted that large scale mean-flow modes with non-zero vertical vorticity couple to the convective motion, and suggested that they lead to an instability of roll patterns. Using modulation theory, they showed that this coupling reduced the interval of stable convective motion to a width $0(\epsilon)$ at moderate Prandtl number, due to skew-varicose type instabilities (Fig. 1). Their results suggested that for sufficiently small Prandtl number (σ), stable roll solutions disappeared altogether. They also showed that this coupling to the mean flow is present in the more physically realistic case of rigid boundaries; however, in this case the effects only become relevant for fully nonlinear convection (cf. Cross & Newell, 1984).

Busse & Bolton (1984) investigated the same problem using perturbation theory. Their results, which disagree with those of Zippelius & Siggia, showed that stable uniform roll solutions disappeared for $\sigma < \sigma_c = 0.543$ due to skew-varicose type instabilities.

In this paper, an attempt is made to resolve these conflicting results. A centre-manifold method is used to derive the governing modified Ginzburg–Landau equations for convection between stress-free boundaries near onset. In this expansion, the amplitudes of the convection motion, the mean-flow modes, and distances transverse and parallel to the rolls are scaled independently. It is shown that the long wavelength linear stability theory

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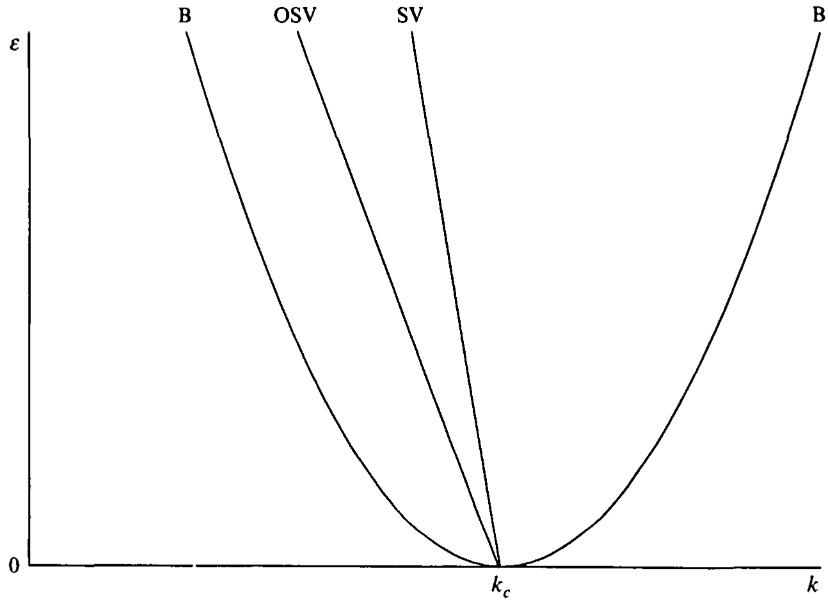

FIGURE 1. Stability boundaries in stress-free convection; wavenumber *versus* reduced Rayleigh number. the stability boundaries for uniform rolls in stress-free convection at Prandtl number $\sigma = 1$ are shown as a function of their wavenumber (k) and the reduced Rayleigh number (ϵ), following the work of Busse & Bolton (1984). When $\epsilon = 0$ a roll with wavenumber k_c becomes unstable. As ϵ is increased, the region of wavenumbers that are unstable increases quadratically (the boundary marked by the parabola labelled B). Inside this region, uniform roll solutions will grow and saturate at some finite amplitude. There finite amplitude solutions may be subject to secondary instabilities. The two instabilities which bound a region of stable rolls are the oscillatory skew-varicose (labelled OSV and leading to instability to the left of the boundary) and the skew varicose (labelled SV and leading to instability to the right of the boundary). the boundaries are linear in the limit of small ϵ, and consequently they bound a region of stable wavenumbers of width $0(\epsilon)$. For lower σ these boundaries may cross eliminating the range of stable rolls.

for a uniform roll solution decouples into a finite dimensional eigenvalue problem due to the symmetries of the system. In the limit of infinitely long wavelength, this eigenvalue problem has a double zero eigenvalue, with one zero corresponding to the translational invariance of the roll solution and the second corresponding to a mean-flow mode. It is the perturbation of this double zero eigenvalue that leads to the skew-varicose type instabilities of the problem. A major difficulty of perturbation problems of this type is knowing where to truncate the expansion. In this paper, the expansion is kept to a sufficiently high order to ensure a non-degenerate unfolding of the double zero eigenvalue of the long wavelength linear stability problem. This methodology can be used to extend the modulation theory of Zippelius & Siggia to higher order, bringing their roll stability theory into agreement with that of Busse & Bolton.

The layout of the paper is as follows. In the next section, a set of modified Ginzburg–Landau equations are derived from the Boussinesq equations; some of the details of this calculation are deferred to an Appendix. The relation between these equations and earlier models is indicated. In §3, a modulation theory is developed using the modified Ginzburg–Landau equations as opposed to more classical scaling arguments. The result is an extension of work done by Zippelius & Siggia (1993), which reconciles their roll stability results with those obtained by Busse & Bolton (1984) using amplitude expansions.

2 Derivation of the modified Ginzburg–Landau equations

The equations of motion for Boussinesq convection in a horizontally infinite layer heated from below with stress-free boundaries is the starting point for this derivation. It is convenient to write the differential equations in terms of a vector V made up of the temperature and velocity. In this formulation, due to Schluter *et al.* (1965), it is possible to write the linearization around the basic state as an operator that is self-adjoint with the appropriate inner-product. This will prove invaluable in the calculation done below. Here the nomenclature used is that of Cross (1980); note that there is no *ad hoc* scaling chosen for the amplitude or the horizontal length scales.

There are two parameters in the problem, the Prandtl number σ, which is the ratio of kinematic viscosity to thermal diffusivity, and the Rayleigh number R, which is a non-dimensional measure of the thermal forcing. When R is increased, there is a transition when the conductive state loses stability at $R_c = 27\pi^4/4$ (cf. Busse, 1978). The following expansion is valid in the limit of small reduced Rayleigh number

$$\epsilon = \frac{(R-R_c)}{R_c}$$

A horizontally infinite layer of unit depth is considered. The vertical direction z is bounded by $0 \leqslant z \leqslant 1$, with $\hat{z}$ being a unit vector pointing upwards. Similarly, the horizontal directions (x, y) are spanned by $(\hat{x}, \hat{y})$. The state of this system is specified by the velocity $\boldsymbol{v}$, the pressure P, and the deviation of the temperature from the conductive state, θ. The state of the system can be specified by a vector V

$$\boldsymbol{v} = (u, v, w), \quad \boldsymbol{u} = (u, v), \quad V = (\theta; \boldsymbol{u}, w) \tag{2.1}$$

with the pressure determined implicitly by incompressibility (2.2b) below.

The equations governing the advection and diffusion of momentum and heat, and the incompressibility of the liquid are

$$V_t = DV - \partial P + N(V, V), \tag{2.2a}$$

$$\partial \cdot V = 0. \tag{2.2b}$$

Here, the gradient operator ∂ is composed of a horizontal gradient operator ∇ and a vertical derivative ∂_z, so that

$$\partial = (0; \nabla, \partial_z), \quad \nabla = (\partial_x, \partial_y). \tag{2.3}$$

The advective nonlinear term is defined using this operator as

$$N(V, V') = -(V \cdot \partial)\, V'. \tag{2.4}$$

The linear matrix operator $D = D_0 + \delta D$ is defined by

$$D_0 = \begin{pmatrix} (\nabla^2 + \partial_z^2) & & R_c \\ 0 & \sigma(\nabla^2 + \partial_z^2) & 0 \\ \sigma & & \sigma(\nabla^2 + \partial_z^2) \end{pmatrix} \tag{2.5}$$

and δD is the matrix with $R-R_c$ in the upper right-hand corner as the only non-zero entry. To complete the formulation, stress-free, fixed-temperature boundary conditions are chosen

$$\theta = \partial_z u = \partial_z v = w = 0 \quad \text{at} \quad z = 0, 1. \tag{2.6}$$

To facilitate calculation, an inner product

$$\langle V, V' \rangle = [\sigma\theta^*\theta' + R_c\, \boldsymbol{v}^* \cdot \boldsymbol{v}']_m, \tag{2.7}$$

is introduced. Here * denotes complex conjugates and $[\ ]_m$ signifies an average over the fluid layer. The motivation for choosing this form of the equations of motion and defining the inner product is that the operator D_0 is self-adjoint subject to the constraint of equation (2.2b), i.e.

$$\langle V, D_0 V' \rangle = \langle V', D_0 V \rangle^*. \tag{2.8}$$

Now, given two eigenvectors V^i, V^j of the linear portion of equation (2.2a) at $R = R_c$,

$$D_0 V^i - \partial P = \lambda_i V^i, \tag{2.9}$$

Equation (2.8) implies the orthogonality condition

$$\langle V^i, V^j \rangle = c\delta_{ij}, \tag{2.10}$$

where c is a normalization constant and δ_{ij} is the Kronecker delta function. This allows the inner product (2.8) to be used as a projection operator; $\langle V^i, \cdot \rangle$ applied to equation (2.2a) picks out the portion of the equation projected on the eigenvector V^i.

The basic philosophy of most near-equilibrium amplitude expansions is to divide the eigenspace into two subspaces. The first is spanned by a set of 'fast' eigenvectors whose eigenvalues have negative real parts of order unity, and consequently motion in this subspace is damped on a fast time scale. The remaining subspace is spanned by a set of 'slow' eigenvectors whose spectrum is clustered close to zero, corresponding to slow growth or decay rates. The motion on the fast subspace can be adiabatically eliminated; it is rapidly driven to a quasi-equilibrium solution which evolves on the slow subspace. The result is a reduced set of equations describing the motion on this manifold (cf. Normand *et al.* 1977; Haken, 1978; Cross, 1980; Arneodo *et al.*, 1983; Spiegel, 1985a, b). In the case of ordinary differential equations and partial differential equations with discrete spectra, this process can be carried out using centre manifold theory (Guckenheimer & Holmes, 1983; Arnold, 1983). If the set of slow eigenvalues is sufficiently close to zero, the process of projecting out the fast subspace and confining the motion to an invariant manifold tangent to the slow subspace (called the centre manifold) is justifiable using the centre manifold theorem.

For this problem, the spectrum of eigenvalues is continuous. Fortunately, the form of the expansion is insensitive to where the division between fast and slow eigenvalues is made. The expansion is derived by direct analogy with centre manifold theory. First, the set of eigenvectors to be included in the slow subspace must be determined.

As R passes through R_c a set of eigenvalues passes through zero, signalling the onset of convective motion. The translation invariance of the layer forces the eigenvector to have a horizontal dependence of $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. The stress-free boundaries lead to a sinusoidal depth dependence, and allow us to solve analytically for the most unstable wavenumber at onset,

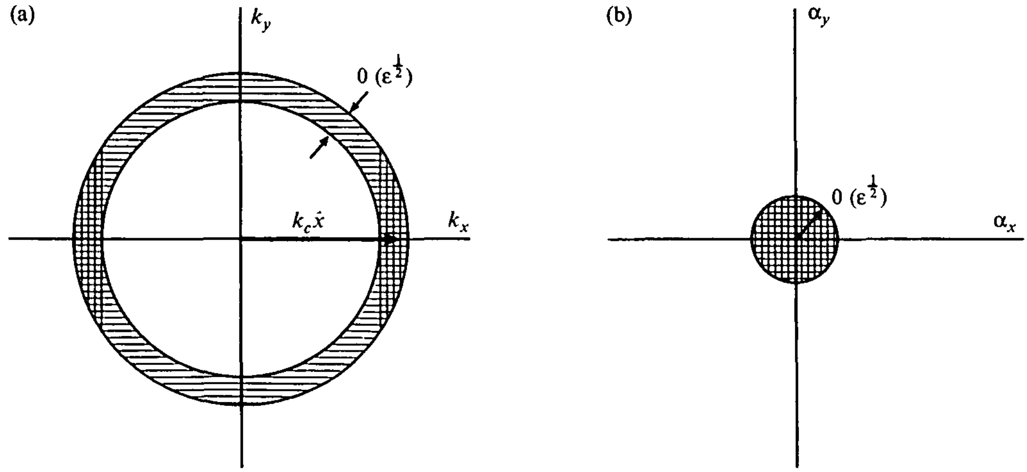

FIGURE 2. Wave vectors in the slow subspace. (a) The annulus in $\boldsymbol{k}$ space where V^k is in the slow subspace is shown. The annulus contains a band of wavevectors whose length is within $0(\sqrt{\epsilon})$ of k_c. This is restricted to the cross-hatched region near $\pm k_c \hat{x}$ for the modulation theory; (b) the disk in $\boldsymbol{\alpha}$ space where V^α is in the slow subspace. This has radius $0(\sqrt{\epsilon})$. For equations (2.18), the integrals are performed only over the vectors in the slow subspace.

$k_c = \pi/\sqrt{2}$. The eigenvectors for $k \approx k_c$ (here $k = |\boldsymbol{k}|$) can then be approximated by (cf. Schluter *et al.*, 1965)

$$V^k = \begin{bmatrix} (\pi^2+k^2)^2 \sin(\pi z) \\ ik\pi \cos(\pi z) \\ k^2 \sin(\pi z) \end{bmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + 0(\epsilon), \tag{2.11}$$

for $|k-k_c| = 0(\sqrt{\epsilon})$. The corresponding eigenvalue for V^k is

$$\lambda^k \approx \frac{1}{\tau}[\epsilon - \zeta^2(k-k_c)^2], \tag{2.12}$$

where

$$\tau = \frac{2(\sigma+1)}{3\pi^2\sigma}, \quad \zeta^2 = \frac{8}{3\pi^2}. \tag{2.13}$$

As λ^k is $0(\epsilon)$ for $|k-k_c| = 0(\sqrt{\epsilon})$, the corresponding eigenvectors must be contained in the slow subspace. For $|k-k_c| \gg \sqrt{\epsilon}$, $\mathrm{Re}(\lambda^k) \ll -\epsilon$, which is adopted as the condition for V^k to be in the fast subspace. Figure 2a shows the annulus in $\boldsymbol{k}$ space of eigenvectors contained in the slope subspace.

The linear stability theory for rolls is modified for finite Prandtl number by the presence of mean-flow modes with non-zero vertical vorticity (Zippelius & Siggia, 1982, 1983; Busse & Bolton, 1984). The eigenvectors representing these modes are

$$V^\alpha = \begin{bmatrix} 0 \\ i\hat{z}\times\boldsymbol{\alpha} \\ 0 \end{bmatrix} e^{i\boldsymbol{\alpha}\cdot\boldsymbol{x}}. \tag{2.14}$$

The associated eigenvalues are

$$\lambda^{\alpha} = -\sigma\alpha^2, \tag{2.15}$$

where $\alpha = |\boldsymbol{\alpha}|$. So for $\sigma\alpha^2 \approx 0(\epsilon)$ these modes must be included in the slow subspace. It is assumed that σ is of 0(1) so that the eigenvectors in the slow subspace are contained in a circle of radius $0(\sqrt{\epsilon})$ in $\boldsymbol{\alpha}$ space (Fig. 2b).

The vector V can now be expanded in terms of these two sets of eigenvectors

$$V \approx \int dk\, A^k(t)\, V^k + \int d\alpha\, B^{\alpha}(t)\, V^{\alpha}, \tag{2.16}$$

with the additional restriction that

$$A^{-k} = (A^k)^*, \quad B^{-\alpha} = (B^{\alpha})^*, \tag{2.17}$$

to ensure the reality of V. Here a time dependent amplitude has been associated with each vector in the slow subspace. Equation (2.16) is the lowest order approximation to the centre manifold.

Typically at this point in a centre manifold calculation, the lowest order nonlinear terms in the expansion are calculated, and the series is then truncated with the hope that the equations encapsulate the dynamical behaviour of the full system. This calculation is relegated to the Appendix due to its algebraic intricacy. The form of the final equations obtained is

$$a_1 A_t^k = a_2 A^k + \int dk'\, dk''\, a_3 A^{k'} A^{k''} A^{k-k'-k''} + \int d\alpha'\, a_4 B^{\alpha'} A^{k-\alpha'} \tag{2.18a}$$

$$b_1 B_t^{\alpha} = b_2 B^{\alpha} + \int dk' b_3 A^{k'} A^{\alpha-k'}, \tag{2.18b}$$

where the real constants a_i, b_i are computed in the Appendix, and the wave vectors $\boldsymbol{k}$, $\boldsymbol{k}'$, $\boldsymbol{k}''$ and $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ are restricted to the regions described in Fig. 2.

These 'modified Ginzburg–Landau' equations (Haken, 1978) should preserve the symmetries of the full equations. The transformations which leave equation (2.2) unchanged and the equivalent transformation for the system (2.18) are listed below.

Translation

$$\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{x}_0 \Rightarrow A^k \to A^k e^{i\boldsymbol{k}\cdot\boldsymbol{x}_0} B^{\alpha} \to B^k e^{i\boldsymbol{k}\cdot\boldsymbol{x}_0}, \tag{2.19a}$$

for arbitrary $\boldsymbol{x}_0$.

Rotation

$$\boldsymbol{x} \to M\boldsymbol{x} \Rightarrow \boldsymbol{k} \to M^{-1}\boldsymbol{k}, \quad \boldsymbol{\alpha} \to M^{-1}\alpha, \tag{2.19b}$$

where M is a two dimensional rotation matrix.

Reflection

$$(x, y) \to (-x, y) \Rightarrow (k_x, k_y) \to (-k_x, k_y), \quad (\alpha_x, \alpha_y) \to (-\alpha_x, \alpha_y), \tag{2.19c}$$

$$(x, y) \to (x, -y) \Rightarrow (k_x, k_y) \to (k_x, -k_y), \quad (\alpha_x, \alpha_y) \to (\alpha_x, -\alpha_y). \tag{2.19d}$$

In addition, there is the so-called 'Boussinesq symmetry', corresponding to an inversion of the layer and changing the sign of the temperature perturbation

$$z \to 1-z, \quad \theta \to -\theta, \quad w \to -w \Rightarrow A^k \to -A^k. \tag{2.19e}$$

The symmetries restrict the form of the amplitude equations obtained. Applying the two reflections (2.19c, d) to equation (2.18) and then using relation (2.17) and equating coefficients implies that all the a_i, b_i's are real. Also, (2.19e) implies that all the terms of even order in A^k vanish in (2.18a) and those of odd order vanish in (2.18b). As a check on this calculation, roll solutions were calculated and their stability computed from (2.18). This produced results identical with those derived by Busse & Bolton (1984) for the full Boussinesq equations.

If the value of B^α is set to zero, equation (2.18a) resembles that derived by Schluter *et al.* (1965) and Cross (1980); however, the value of a given in these papers is computed by assuming $|\boldsymbol{k}| = |\boldsymbol{k}'| = |\boldsymbol{k}''| = k_c$. This approximation leads to erroneous results for roll stability at finite Prandtl number. Busse (1986) has derived an amplitude equation for this problem similar to (2.18). His approach is modelled on the methods used in Busse & Bolton (1984), and as such the coefficients should correspond to a Taylor expansion around k_c of those tabulated here.

By assuming the solutions are rolls with nearly the same orientation over the whole layer, the system (2.18) can be reduced to a modulation type theory. These equations are derived in the next section.

3 Modulation theory for nearly roll-like solutions

In this section, the modulation equations for nearly parallel roll-like solutions are derived. Zippelius & Siggia (1982, 1983) attempted to analyse this problem using modulation theory and scaling arguments. Although they realized the role of mean-flow modes in the dynamics, their expansion is not of sufficiently high order to capture the quantitative details of the stability theory for rolls. By using an alternative derivation for the modulation equations for this problem starting from equations (2.18), a theory consistent with the results of Busse & Bolton (1984) is produced. This derivation has the advantage that it makes no *ad hoc* assumptions for the length scales on which the modulation acts.

By restricting the spectrum of (2.18) to small regions in Fourier space, the variation of the solution is restricted to large spatial scales. The mean-flow modes are naturally restricted to a region close to the origin. For this derivation, the convective eigenvectors are restricted to a region close to a single vector of length k_c, chosen to be $k_c \hat{x}$ without loss of generality (see Fig. 2a). Condition (2.17) defines implicitly the amplitudes in the region near $-k_c \hat{x}$. Outside these two regions, the amplitudes are taken to zero.

A modulation amplitude is defined by

$$S(\boldsymbol{x}, t) = \int d\boldsymbol{k}'\, A^{k'}(t)\, e^{i\boldsymbol{\Delta}\cdot\boldsymbol{x}}, \tag{3.1a}$$

where $\boldsymbol{\Delta} = \boldsymbol{k}' - k_c \hat{x}$. In addition, a streamfunction for the mean-flow modes is introduced via

$$Q(x,t) = \int d\alpha' \, B^{\alpha'}(t)\, e^{i\alpha' \cdot x}, \tag{3.1b}$$

where (2.19) ensures that Q is real. The velocity and temperature deviation can be reconstructed from $S(x,t)$ with the help of definitions (2.11) and (2.16); S is the slow modulation proportional to $e^{ik_c x}\{\sin(\pi z) \text{ or } \cos(\pi z)\}$.

To derive the equations governing these amplitudes, the coefficients in (2.18) are expanded in Taylor series around $k_c \hat{x}$ in terms of the perturbation wave vectors. Thus

$$\Delta = \Delta_x \hat{x} + \Delta_y \hat{y}, \tag{3.2a}$$

$$\alpha = \alpha_x \hat{x} + \alpha_y \hat{y}. \tag{3.2b}$$

A number of the terms vanish due to symmetry properties of the fluid layer and because the expansion is at the point of onset. The coefficients are expanded retaining the minimum number of terms needed to account for the independent small quantities ($\epsilon, A^k, B^\alpha, \alpha_x, \alpha_y, \Delta_x, \Delta_y$) and to break the accidental symmetries of a lower order truncation. This yields

$$\begin{aligned}\tau(1+f_1\Delta_x)A_t^k = {} & \left\{\epsilon(1+f_2\Delta_x) - \zeta^2\left(\frac{\Delta_y^4}{4k_c^2} + \Delta_x^2 + \frac{\Delta_x \Delta_y^2}{k_c}\right)\right\} A^k \\ & - \int dk'\, dk''(f_3 + f_4\Delta_x + f_5\Delta_x' + f_6\Delta_x'')A^{k'}A^{k''}A^{k-k'-k''} \\ & + \int d\alpha' \,[f_7\alpha_y' + f_8\Delta_x\alpha_y' + f_9\alpha_x'\alpha_y' + f_{10}\Delta_y\alpha_x']B^{\alpha'}A^{k-\alpha'} \end{aligned} \tag{3.3a}$$

$$\begin{aligned}\alpha^2 B_t^\alpha = {} & -\sigma\alpha^4 B^\alpha \\ & + \int dk'[g_1\alpha_x\alpha_y + g_2(4\Delta_x'\alpha_x\alpha_y + 4\Delta_y'(\alpha_x^2 - \alpha_y^2) - \alpha_y\alpha^2)]A^{k'}A^{\alpha-k'} \end{aligned} \tag{3.3b}$$

where the f_i's and g_i's are now solely functions of σ. It is argued below that a sufficient number of terms are retained to compute correctly the linear stability of a uniform roll solution.

The equations can be transformed into physical space by performing the Fourier transform described by equation (3.1) and using the convolution theorem. The perturbation wavevectors act as derivative operators.

The scalings used by Zippelius & Siggia (1983) are adopted. Define

$$X = \frac{\sqrt{\epsilon}\,x}{\zeta}, \quad Y = \frac{\epsilon^{\frac{1}{4}}y}{\zeta_y}, \quad T = \frac{\epsilon t}{\tau}, \tag{3.4a}$$

$$\tilde{S} = \frac{S}{S_0}, \quad \tilde{Q} = \frac{Q}{Q_0}, \tag{3.4b}$$

where

$$\zeta_y = \left[\frac{\zeta}{2k_c}\right]^{\frac{1}{2}}, \quad S_0 = \frac{\sqrt{12\epsilon}}{\pi}, \quad Q_0 = \frac{\epsilon^{\frac{3}{4}}\zeta_y}{\tau k_c}. \tag{3.4c}$$

This rescaling yields

$$\tilde{S}_T = \tilde{S}+(\partial_X - i\partial_Y^2)^2\tilde{S} - \tilde{S}|\tilde{S}|^2 + i\tilde{S}\tilde{Q}_Y + s_1 i\tilde{S}_X + s_2 i\tilde{S}_X|\tilde{S}|^2 + s_3 i\tilde{S}^2\tilde{S}_X^* + s_4\tilde{S}\tilde{Q}_{XY} + s_5(\tilde{S}_X\tilde{Q}_Y - \tilde{S}_Y\tilde{Q}_X), \tag{3.5a}$$

$$q_1(\partial_Y^2+\delta\partial_X^2)\tilde{Q}_T = (\partial_Y^2+\delta\partial_X^2)^2\tilde{Q} + q_2\partial_{XY}|\tilde{S}|^2 + q_3 i\partial_Y(\tilde{S}(\partial_Y^2+\delta\partial_X^2)\tilde{S}^* - \tilde{S}^*(\partial_Y^2+\delta\partial_X^2)\tilde{S}) - q_4 i\partial_X(\tilde{S}_X\tilde{S}_Y^* - \tilde{S}_Y\tilde{S}_X^*), \tag{3.5b}$$

$$\left.\begin{aligned} s_1 &= \frac{-\sqrt{3\epsilon}}{3}, \quad s_2 = \frac{\sqrt{3\epsilon}}{2}\left[\frac{1}{8}\left(\frac{5}{3\sigma}-1\right)\left(\frac{9}{8}+\frac{1}{\sigma}\right)+\frac{3}{2}\right],\\ s_3 &= \frac{\sqrt{3\epsilon}}{2}\left[\frac{1}{8}\left(\frac{5}{3\sigma}-1\right)\left(\frac{9}{8}+\frac{1}{\sigma}\right)-\frac{5}{2}\right],\\ s_4 &= \frac{\sqrt{3\epsilon}}{3}\left[\frac{1}{\sigma+1}-2\right], \quad s_5 = \frac{\sqrt{3\epsilon}}{2}, \end{aligned}\right\} \tag{3.6a}$$

$$\left.\begin{aligned} \delta &= \frac{\sqrt{3\epsilon}}{4}, \quad q_1 = \frac{\sqrt{3\epsilon}}{\sigma+1},\\ q_2 &= q_3 = 2\left[\frac{\sigma+1}{\sigma^2}\right], \quad q_4 = \left[\frac{\sqrt{3\epsilon}}{2}\right]q_2. \end{aligned}\right\} \tag{3.6b}$$

Because of the restriction to particular wave vectors, equations (3.5) have different symmetry properties from (2.18). Rotational symmetry is lost, but the isolation of $\boldsymbol{k}$ from $-\boldsymbol{k}$ in Fourier space introduces a phase invariance of (3.5)

$$\tilde{S} \rightarrow \tilde{S}e^{i\phi}, \tag{3.7a}$$

for arbitrary real ϕ. This means the modulation model is insensitive to the phase of the underlying rolls. This, coupled to the translation invariance of the problem, leads to the symmetry

$$\boldsymbol{X} \rightarrow \boldsymbol{X}+\boldsymbol{X}_0, \tag{3.7b}$$

for arbitrary $\boldsymbol{X}_0$. Reflection in X yields invariance under

$$X \rightarrow -X, \quad \tilde{S} \rightarrow \tilde{S}^*, \quad \tilde{Q} \rightarrow -\tilde{Q}, \tag{3.7c}$$

while reflection in Y yields

$$Y \rightarrow -Y, \quad \tilde{Q} \rightarrow -\tilde{Q}. \tag{3.7d}$$

The Boussinesq symmetry reduces to

$$\tilde{S} \rightarrow -\tilde{S}. \tag{3.7e}$$

A symmetry also is introduced because $\tilde{Q}$ corresponds to the streamfunction (or by considering (2.14) and (3.1b))

$$\tilde{Q} \rightarrow \tilde{Q}+\tilde{Q}_0, \tag{3.7f}$$

for arbitrary $\tilde{Q}_0$.

Although the rotational symmetry is broken, it must be preserved to the order of the expansion of the amplitude equation. One consequence of this is that a uniform roll solution with any orientation cannot drive the mean-flow modes. Equation (3.5b) must be satisfied by a solution of the form

$$\tilde{S} = R_0 \mathrm{e}^{i\gamma_0 \cdot X}, \quad \tilde{Q} = 0, \tag{3.7g}$$

for arbitrary R_0, γ_0.

By setting $\tilde{Q} \equiv 0$, the Newell–Whitehead–Segel amplitude equation is obtained at lowest order (Newell & Whitehead, 1969; Segel, 1969). The higher order terms can be compared with the expansion of Cross *et al.* (1983) for time independent interior solutions derived for a sidewall problem. The two agree after applying an operator of the form $(1+c\partial_X)$ to the right-hand side of equation (3.5a). All the terms present in Zippelius & Siggia's (1983) expansion are found here, but there are a number of additional terms of order $\sqrt{\epsilon}$. The addition of these terms reconciles the stability theory for rolls derived by modulation theory with that derived by Busse & Bolton (1984) via amplitude expansions.

The question of how to choose the scalings and to what order to expand the modulation equation is difficult and not fully resolved. Here a strong argument that the stability theory for a uniform roll can be computed correctly in the long-wave limit of perturbations from the present expansion will be presented.

At least three basic philosophies can be presented for determining the correct order to truncate the expansion. The first is to compare the results of the truncation with an independent calculation performed by a different method. It will be shown below that the result obtained for the theory of a uniform roll are in agreement with those of Busse & Bolton (1984) which, in turn, they have verified by numerical calculations of the full system. Although satisfying, this method of checking the present calculation denies it any predictive value of its own. Moreover, as discussed earlier, other results in the literature are at variance with these calculations.

A second method is to use the ideas of centre manifold theory (cf. Guckenheimer & Homes, 1983) to truncate the expansion. Equations (3.5) are, in fact, an expansion in (ϵ, $\tilde{Q}$, $\tilde{S}$, $i\partial_X$, $i\partial_Y$). These quantities can be considered as scaling independently as opposed to in some fixed ratio. The equations can be expanded to an order that breaks any accidental symmetries that occur in the equation, and until the terms neglected can be shown to be small in comparison to those retained. In fact, the equations were originally obtained by applying these ideas to expansion (3.3). In equation (3.5a), this yields the conclusion that the terms with coefficients s_1 through s_5 must be retained to break the accidental symmetry $X \to -X$. A similar argument leads to the retention of q_3 and q_4. This approach has certain advantages; the form of the expansion of (3.5) can be determined by taking the linear dispersion terms (as known from a linear analysis of the full equations) and then determining all the nonlinear terms that satisfy symmetries (3.7). However, the basis for these ideas can be shown to be quantitatively valid only for systems of ordinary differential equations, and cannot be rigorously applied here.

The methodology here will be to show that these equations correctly reproduce the stability theory for a uniform roll solution in the limit of long-wave perturbations. Note that for linear stability theory, only terms linear in $\tilde{Q}$ needs to be retained in equation (3.5). To compute the stability of a uniform roll solution from (3.5), a steady solution of the form

$$\tilde{S} = \tilde{S}_R e^{i\Gamma X}, \quad \tilde{Q} = 0, \tag{3.8a}$$

is considered. Substitution in (3.5) yields

$$|\tilde{S}_R|^2 = \frac{1 - s_1 \Gamma - \Gamma^2}{1 + (s_2 - s_3)\Gamma}. \tag{3.8b}$$

This corresponds to a roll solution with wavenumber

$$k_R = k_c + \frac{\sqrt{\epsilon}\Gamma}{\zeta}. \tag{3.9}$$

It is well known that the stability of a roll solution can be determined a perturbation localized to a pair of sidebands (e.g. Newell & Whitehead, 1969). By considering the perturbation as expanded as a set of Fourier modes, examining the coupling, and using the symmetries (3.7c, d), it can be shown that the linear perturbation theory decouples into acting on sets of perturbations of the form

$$\tilde{S} = e^{i\Gamma X}[\tilde{S}_R + A e^{\lambda T} \cos(\gamma \cdot X + \Phi) + B e^{\lambda T} i \sin(\gamma \cdot X + \Phi)] \tag{3.10a}$$

$$\tilde{Q} = C e^{\lambda T} \cos(\gamma \cdot X + \Phi) \tag{3.10b}$$

with

$$\gamma = (\gamma_X, \gamma_Y). \tag{3.10c}$$

Here A, B, C are real constants and Φ can be chosen arbitrarily, although it suffices to take the values $\Phi = 0, \pi$ to span the set of perturbations for a given γ. At this point, it will be assumed that

$$\Gamma \sim \sqrt{\epsilon}$$

corresponding to the expected skew-varicose type instabilities. Although this restriction is not necessary, it streamlines the presentation of the results. The perturbations will be examined in the long wavelength limit and it will be assumed that

$$\gamma_Y \sim \sqrt{\delta}\gamma_X \to 0.$$

This scaling corresponds to the transverse and parallel components of the perturbation occurring on the same lengthscales in physical space. Substituting this form in system (3.5) expanding and linearizing in A, B, C yields a 3×3 set of equations

$$\begin{pmatrix} -2-\lambda & (s_3 - s_1 - s_2 - 2\Gamma)\gamma_X & -s_4 \gamma_X \gamma_Y \\ -(s_1 + s_2 + s_3 + 2\Gamma)\gamma_X & -\lambda - \gamma_X^2 & -\gamma_Y \\ -2q_2 \gamma_X \gamma_Y & -2q_3(\gamma_Y^2 + \delta\gamma_X^2)\gamma_Y & (\gamma_Y^2 + \delta\gamma_X^2)^2 + \lambda q_1(\gamma_Y^2 + \delta\gamma_X^2) \end{pmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.11}$$

Note that when $\gamma_X = \gamma_Y = \lambda = 0$ every entry except the upper left-hand corner of the coefficient matrix vanishes. The matrix has a double zero eigenvalue in this case; these are generated by the translation symmetry (3.7b) in the X direction applied to the solution (3.8a) and by the streamfunction symmetry (3.7f). It is the perturbation of this double zero, corresponding to the coupling of the translational mode to large scale mean flows that drives the instability. The pair of zero eigenvalues, when perturbed, may become a complex pair corresponding to an oscillation or two real eigenvalues. The third eigenvalue is always -2 to leading order.

The coefficients in the matrix have all been expanded to lowest order in γ_X, γ_Y. The form of the eigenfunction expansion guarantees that each coefficient will be a series of terms of only even or odd total degree in γ_X, γ_Y. The leading term is present in all cases with one exception (examined below), and by comparison with the form of (3.5) it can be seen that the coefficients of these terms are expanded to leading order in ϵ. Below, when the eigenvalues of this matrix are found, it is expected that they will faithfully give the leading order approximation to the full system. Note that every term in (3.5) contributes to the expansion, with the exception of those with coefficients s_5 and q_4 which have the same scaling behaviour as terms that do contribute. Moreover, all the terms contribute to the stability boundaries derived below. Herein lies the explanation for the order of the truncation; a sufficient number of terms have been kept so that a leading order approximation to each coefficient of the equations (3.11) can be made; this should yield a non-degenerate stability calculation.

The second entry in the third row has no linear term; this absence can be explained by symmetry. The reflection symmetry (3.7d) implies that the nonlinear terms in (3.5b) must contain an odd number of derivatives with respect to Y, and consequently at least one derivative in this direction. A solution of the form

$$\tilde{S} = e^{i(\Gamma X+\alpha Y)} \sim e^{i\Gamma X}(1+i\alpha Y) \quad \text{as} \quad \alpha \to 0 \tag{3.12}$$

must vanish if substituted in (3.5b) due to the rotational symmetry (3.7g). If the linear perturbation (3.10a) is substituted into (3.5b), any coefficient linear in γ must be proportional to γ_Y due to the Y derivative present in the nonlinearity. This coefficient can be computed by substituting (3.10a) linearized in γ_Y into equation (3.5b)

$$\tilde{S} \sim e^{i\Gamma X}(1+iC\gamma_Y Y). \tag{3.13}$$

The form of (3.12) assures that (3.13) will vanish when substituted into (3.5b), and consequently the linear term in the second entry in the third row of (3.11) is absent.

Note that sending either γ_X or γ_Y to zero separately with the other fixed also yields a non-singular matrix, which should assure the validity of these limits also.

Expanding the characteristic polynomial of (3.11) yields the boundaries for the skew-varicose (SV) and the oscillatory skew-varicose (OSV) instabilities, which are of greatest interest when $\sigma \approx 1$. Define

$$\rho = \frac{\delta\gamma_X^2}{\gamma_Y^2}; \tag{3.14}$$

then the conditions for instability are

(SV) $$\Gamma > -\sqrt{3\epsilon}\left[\frac{7}{12}-\frac{1}{4\rho}\right], \tag{3.15a}$$

(OSV) $$\Gamma < -\sqrt{3\epsilon}\left[\frac{\sigma^2}{8(\sigma+1)^2}\left((\sigma+5)\rho+\frac{\sigma+1}{\rho}\right)+\frac{(\sigma+3)(3\sigma^2+2\sigma+2)}{12(\sigma+1)^2}\right]. \tag{3.15b}$$

If, for a given ρ, only (3.15a) is satisfied, there is a single positive real eigenvalue. If only (3.15b) is satisfied, there is a complex pair of eigenvalues with positive real part. If both conditions are satisfied, there are two positive eigenvalues. To find the boundary of the region of stability, the right-hand side of the (3.15a) must be maximized and the right hand side of (3.15b) must be minimized. This is accomplished by

(SV) $$\rho \to \infty, \tag{3.16a}$$

and

$$\text{(OSV)} \qquad \rho = \left[\frac{\sigma+1}{\sigma+5}\right]^{\frac{1}{2}}. \tag{3.16b}$$

These results and those for the other possible stability boundaries agree with those of Busse & Bolton (1984). Note that these two boundaries cross for $\sigma = \sigma_c = 0.543$, below which there are no stable roll solutions.

4 Conclusions

The modified Ginzburg–Landau formulation derived here allows the derivation of modulation equations for the onset of stress-free convection. These equations show how the modulation theory of Zippelius & Siggia can be extended so that the stability theory of a uniform roll solution agrees with the amplitude expansion results of Busse & Bolton.

By examining the stability theory in the long wavelength limit, it can be shown that the skew-varicose and the oscillatory skew-varicose instabilities are related to the coupling of two zero eigenvalues; one corresponds to the translational invariance of the uniform roll solution, and the second due to the uniform mean flow solution allowed by the stress-free boundary conditions. By relating this instability to the eigenvalues of a 3×3 matrix, it is possible to determine which terms in the modulation theory contribute to the roll stability problem in the long wavelength limit; this yields a concrete method for determining which terms to retain in the theory.

The present methodology has an advantage over the methods of Busse & Bolton in that it generates a set of modulation equation that can be integrated numerically to examine the nonlinear behaviour of the instability (Bernoff, 1986). A concrete method of unfolding the double zero eigenvalue in the long wavelength limit is described here.

Although stress-free boundaries may be physically unrealistic, they are a paradigm for understanding the coupling of convection rolls to mean flows. The importance of mean flows in rigid boundary convection at moderate Rayleigh numbers is clear from the work of Newell and his collaborators (Cross & Newell, 1983; Newell *et al.* 1990).

Acknowledgements

This work was done while preparing my PhD and was supported by a Marshall Scholarship. I would like to thank Mike Procter, Jim Swift and Nigel Weiss for their help and guidance. I would like to thank Bernie Matkowsky and Alan Newell for encouraging me to publish this calculation.

Appendix: Coefficients of the modified Ginzburg–Landau equations

In this section, the coefficients for equations (2.18) are derived. At lowest order, the centre manifold is approximated by

$$V = V_0 = \int dk\, A^k(t)\, V^k + \int d\alpha\, B^\alpha(t)\, V^\alpha, \tag{A 1}$$

where the regions of integration are defined in Fig. 2. To determine the equations governing the time dependence of A^k and B^α, V is substituted into equations (2.2) and the projection operators $\langle V^k, \cdot\rangle$ and $\langle V^\alpha, \cdot\rangle$ are applied to yield, respectively,

$$a_1 A_t^k = a_2 A^k + \int d\alpha' a_4 B^{\alpha'} A^{k-\alpha'}, \tag{A 2a}$$

$$b_1 B_t^\alpha = b_2 B^\alpha + \int dk' b_3 A^{k'} A^{\alpha-k'} + 0(B^\alpha B^{\alpha'}), \tag{A 2b}$$

where

$$a_1 = \tfrac{1}{2}(\pi^2+k^2)[\sigma(\pi^2+k^2)^3 + R_c k^2],$$

$$a_2 = \frac{\sigma}{2}k^2(\pi^2+k^2)^2\left[R - \frac{(\pi^2+k^2)^3}{k^2}\right],$$

$$a_4 = \hat{z}\cdot\alpha'\times k\left[\frac{\sigma}{2}(\pi^2+|\alpha'-k|^2)^2(\pi^2+k^2)^2 + \frac{R_c}{2}\{\pi^2(k^2-|\alpha'|^2)+k^2|\alpha'-k|^2\}\right], \tag{A 3a}$$

$$b_1 = \alpha^2, \quad b_2 = -\sigma\alpha^4, \quad b_3 = \frac{\pi^2}{4}(\hat{z}\cdot k'\times\alpha)[\alpha\cdot(2k'-\alpha)]. \tag{A 3b}$$

A factor of R_c has been divided out of (A 2b). The quadratic terms in B^α arises in equation (A 2b) due to the self-interaction of the mean-flow modes. For B^α small, this term is always smaller than the linear B^α term, and consequently plays no role in the dominant balance in this equation. Note that there are no quadratic A^k terms in (A 2a); this is a direct consequence of (2.19e).

It remains to calculate the cubic interaction term in (2.18a). The only quadratic interaction unaccounted for is that of V^k with itself. It is convenient to divide this interaction term by its depth dependence

$$N(V^k, V^k) = N_0(V^k, V^{k'}) + N_2(V^k, V^{k'}), \tag{A 4}$$

where

$$N_0(V^k, V^{k'}) = \begin{bmatrix} 0 \\ c_1(k,k') \\ 0 \end{bmatrix}, \tag{A 5a}$$

$$N_2(V^k, V^{k'}) = \begin{bmatrix} c_2(k,k')\sin(2\pi z) \\ c_3(k,k')\cos(2\pi z) \\ c_4(k,k')\sin(2\pi z) \end{bmatrix}. \tag{A 5b}$$

Here N_0 has constant depth dependence, and for small $k+k'$ is contained in the slow subspace. It gives rise to the b_3 term in (A 2b). Outside the slow subspace, this term generates a contribution to the cubic term easily computed by extending (A 2b), which is always smaller than that computed below. The N_2 term in (A 4) is part of the fast subspace. This term yields a first order correction to the centre manifold for V

$$V = V_0 + V_1, \tag{A 6}$$

where

$$D_0 V_1 - \partial P_1 = -\int dk'\, dk''\, A^{k'} A^{k''} N_2(V^{k'}, V^{k''}). \tag{A 7}$$

This yields an additional contribution to the right-hand side of (A 2a) bringing it into the same form as (2.18a)

$$+\int dk'\, dk''\, a_3 A^{k'} A^{k''} A^{k-k'-k''}. \tag{A 8}$$

Here

$$a_3 = \langle V^k, N(V_0, V_1) + N(V_1, V_0)\rangle$$

$$= \frac{\pi^2}{8} k' \cdot (k' - k'') \tag{A 9}$$

$$[K \cdot (k+K)\{R_c[2\pi^2(k\cdot\Sigma) + k^2(k'+k'')\cdot\Sigma] + \sigma\theta_0(\pi^2+k^2)^2\}$$
$$+\Sigma\cdot(k+K)\{R_c[\pi^2(k\cdot K) - k^2K^2] - \sigma(\pi^2+K^2)^2(\pi^2+k^2)^2\}],$$

where

$$K = k - k' - k'', \quad k^2 = |k|^2, \quad K^2 = |K|^2,$$

$$\Sigma = \frac{P_0 k' + (P_0 - \pi) k''}{2\pi\sigma H},$$

$$H = 4\pi^2 + |k' + k''|^2,$$

$$\theta_0 = \frac{2\pi R_c P_0 + \sigma H(\pi^2 + |k''|^2)^2 + R_c |k''|^2}{\sigma[R_c - H^2]},$$

$$P_0 = \frac{\pi k''\cdot(k''-k')(R_c - H^2) - 2\pi R_c |k''|^2 - 2\pi\sigma H(\pi^2 + |k''|^2)^2}{H^3 - R_c|k'+k''|^2}. \tag{A 10}$$

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