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Dagan Karp  
*Harvey Mudd College*

Chiu-Chu Melissa Liu  
*Northwestern University*

Marcos Mariño  
*CERN*

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## The local Gromov–Witten invariants of configurations of rational curves

DAGAN KARP  
CHIU-CHU MELISSA LIU  
MARCOS MARIÑO

We compute the local Gromov–Witten invariants of certain configurations of rational curves in a Calabi–Yau threefold. These configurations are connected subcurves of the “minimal trivalent configuration”, which is a particular tree of  $\mathbb{P}^1$ 's with specified formal neighborhood. We show that these local invariants are equal to certain global or ordinary Gromov–Witten invariants of a blowup of  $\mathbb{P}^3$  at points, and we compute these ordinary invariants using the geometry of the Cremona transform. We also realize the configurations in question as formal toric schemes and compute their formal Gromov–Witten invariants using the mathematical and physical theories of the topological vertex. In particular, we provide further evidence equating the vertex amplitudes derived from physical and mathematical theories of the topological vertex.

[14N35](#); [53D45](#)

### 1 Introduction

Let  $Z$  be a closed subvariety of a smooth projective threefold  $X$  such that  $X$  is a local Calabi–Yau threefold near  $Z$ . In some cases, the contribution to the Gromov–Witten invariants of  $X$  by maps to  $Z$  can be isolated and defines local Gromov–Witten invariants of  $Z$  in  $X$ . Information obtained from the study of local Gromov–Witten theory can be used to gain insight into Gromov–Witten theory in general. This has led to a great amount of interest in the subject.

The study of the local invariants of curves in a Calabi–Yau threefold has a particularly rich history. Their study goes back to the famous Aspinwall–Morrison formula for the local invariants of a single  $\mathbb{P}^1$  smoothly embedded in a Calabi–Yau threefold with normal bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ ; this result is studied by Aspinwall and Morrison [3], Cox and Katz [9], Faber and Pandharipande [10], Kontsevich [18], Lian, Liu and Yau [23], Manin [27], Pandharipande [31], and Voisin [32]. The local invariants of nonsingular curves of any genus have been completely determined by Bryan and Pandharipande [6; 7; 8]. In [5], Bryan, Katz and Leung computed local invariants of

certain rational curves with nodal singularities, and in particular, contractible *ADE* configurations of rational curves. The local invariants of the closed topological vertex, which is a configuration of three  $\mathbb{P}^1$ 's meeting in a single triple point, were computed by Bryan and the first author [4].

In this paper, we will compute local invariants of certain configurations of rational curves. The configurations considered in this paper are all connected subtrees of the *minimal trivalent configuration*, which is a configuration of three chains of  $\mathbb{P}^1$ 's meeting in a triple point (see Figure 1 below). A precise description of the formal neighborhood will be given in Section 3.

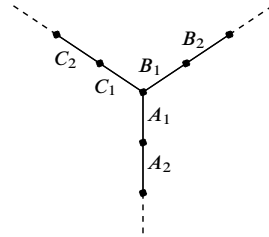


Figure 1: The minimal trivalent configuration  $Y^N = \bigcup_{i=1}^N A_i \cup B_i \cup C_i$ . The normal bundles of  $A_1, B_1, C_1$  are isomorphic to  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ ; the normal bundle of any other irreducible component is isomorphic to  $\mathcal{O} \oplus \mathcal{O}(-2)$ .

### 1.1 Local Gromov–Witten invariants

Let  $Z \subset X$  be a closed subvariety of a smooth projective Calabi–Yau threefold. Let  $\overline{\mathcal{M}}_g(X, \mathbf{d})$  denote the stack of genus  $g$  stable maps to  $X$  representing  $\mathbf{d} \in H_2(X, \mathbb{Z})$ . It is a Deligne–Mumford stack with a perfect obstruction theory of virtual dimension zero which defines a virtual fundamental zero-cycle  $[\overline{\mathcal{M}}_g(X, \mathbf{d})]^{\text{vir}}$ .

Whenever the substack  $\overline{\mathcal{M}}_g(Z)$  consisting of stable maps whose image lies in  $Z$  is a union of path connected components of  $\overline{\mathcal{M}}_g(X, \mathbf{d})$ , it inherits a degree-zero virtual class. The genus- $g$  *local Gromov–Witten invariant* of  $Z$  in  $X$  is defined to be the degree of this virtual class, and is denoted by  $N_{\mathbf{d}}^g(Z \subset X)$ . We write  $N_{\mathbf{d}}^g(Z)$  when the formal neighborhood is understood.

We will consider genus  $g$ , degree  $\mathbf{d}$  local Gromov–Witten invariants  $N_{\mathbf{d}}^g(Y^N)$ , where

$$\mathbf{d} = \sum_{j=1}^N (d_{1,j}[A_j] + d_{2,j}[B_j] + d_{3,j}[C_j]) \in H_2(Y^N; \mathbb{Z}).$$

For simplicity, we write  $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$  where  $\mathbf{d}_i = (d_{i,1}, \dots, d_{i,N})$ . In this paper, we always assume  $\mathbf{d}$  is *effective* in the sense that  $d_{i,j} \geq 0$ . We will show that the local invariants  $N_{\mathbf{d}}^g(Y^N)$  are well defined in the following cases:

- (i) (The minimal trivalent configuration)  $d_{1,1} = d_{2,1} = d_{3,1} = 1$ .
- (ii) (A chain of rational curves)  $d_{1,1} > 0, d_{2,j} = d_{3,j} = 0$  for  $1 \leq j \leq N$ .

We will see in Section 3 that the formal neighborhood of  $Y^N$  has a cyclic symmetry, so one can cyclically permute  $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$  in Case (ii). We show that in the above cases the local invariants  $N_{\mathbf{d}}^g(Y^N)$  are equal to certain global or ordinary Gromov–Witten invariants of a blowup of  $\mathbb{P}^3$  at points (Section 4), and we compute these ordinary invariants using the geometry of the Cremona transform (Section 2). To state our results, define constants  $C_g$  by

$$(1) \quad \sum_{g=0}^{\infty} C_g t^{2g} = \left( \frac{t/2}{\sin(t/2)} \right)^2 = \sum_{g=0}^{\infty} \frac{|B_{2g}(2g-1)|}{(2g)!} t^{2g}.$$

**Theorem 1** (The minimal trivalent configuration) *Suppose that*

$$d_{1,1} = d_{2,1} = d_{3,1} = 1.$$

*Then*

$$N_{\mathbf{d}}^g(Y^N) = \begin{cases} C_g & \text{if } 1 = d_{i,1} \geq \dots \geq d_{i,N} \geq 0 \text{ for } i = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2** (A chain of rational curves) *Suppose that*

$$\mathbf{d}_1 = (d_1, \dots, d_N), \quad \mathbf{d}_2 = \mathbf{d}_3 = (0, \dots, 0),$$

*where*  $d_1 > 0$ . *Then*

$$N_{\mathbf{d}}^g(Y^N) = \begin{cases} C_g d^{2g-3} & \text{if } d_1 = d_2 = \dots = d_k = d > 0 \text{ and} \\ & d_{k+1} = d_{k+2} = \dots = d_N = 0 \text{ for some } 1 \leq k \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Our results are new and add to the list of configurations of rational curves for which the local Gromov–Witten invariants are known.

The configuration in Theorem 2 is an  $A_N$  curve. It is interesting to compare Theorem 2 with the result for a generic contractible  $A_N$  curve  $E = E_1 \cup \dots \cup E_N$  from Bryan–Katz–Leung [5, Proposition 2.10]:

**Fact 1** (A generic contractible  $A_N$  curve [5]) Assume  $d_i > 0$  for  $i = 1, \dots, N$ . Let  $N_g(d_1, \dots, d_N)$  denote genus  $g$  local Gromov–Witten invariants of  $E$  in the class  $\sum_{j=1}^N d_j[E_j]$ . Then

$$N_g(d_1, \dots, d_N) = \begin{cases} C_g d^{2g-3} & d_1 = \dots = d_N = d > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $Y^1$  is the closed topological vertex. By the results in Faber–Pandharipande [10] and Bryan–Karp [4],  $N_{d_1, d_2, d_3}^g(Y^1)$  is defined in the following cases:

- (iii) (Super-rigid  $\mathbb{P}^1$ )  $d_1 > 0, d_2 = d_3 = 0$  (and its cyclic permutation).
- (iv) (The closed topological vertex)  $d_1, d_2, d_3 > 0$ .

**Fact 2** (Super-rigid  $\mathbb{P}^1$  [10]) Suppose that  $d > 0$ . Then

$$N_{d,0,0}^g(Y^1) = N_{0,d,0}^g(Y^1) = N_{0,0,d}^g(Y^1) = C_g d^{2g-3}.$$

**Fact 3** (The closed topological vertex [4]) Suppose that  $d_1, d_2, d_3 > 0$ . Then

$$N_{d_1, d_2, d_3}^g(Y^1) = \begin{cases} C_g d^{2g-3} & d_1 = d_2 = d_3 = d > 0, \\ 0 & \text{otherwise.} \end{cases}$$

## 1.2 Formal Gromov–Witten invariants

The minimal trivalent configuration  $Y^N$  together with its formal neighborhood is a nonsingular formal toric Calabi–Yau (FTCY) scheme  $\hat{Y}^N$ . The formal Gromov–Witten invariants  $\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N)$  of  $\hat{Y}^N$  are defined for all nonzero effective classes (see Section 5.1 and Bryan–Pandharipande [8, Section 2.1]). Moreover,

$$\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N) = N_{\mathbf{d}}^g(Y^N)$$

in all the above cases (i)–(iv). Introduce formal variables  $\lambda, t_{i,j}$  and define

$$\tilde{Z}_N(\lambda; \mathbf{t}) = \exp\left(\sum_{g \geq 0} \sum_{\mathbf{d}} \lambda^{2g-2} \tilde{N}_{g, \mathbf{d}}(\hat{Y}^N) e^{-\mathbf{d} \cdot \mathbf{t}}\right)$$

where  $\mathbf{d}$  runs over all nonzero effective classes, and

$$\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3), \quad \mathbf{t}_i = (t_{i,1}, \dots, t_{i,N}), \quad \mathbf{d} \cdot \mathbf{t} = \sum_{i=1}^3 \sum_{j=1}^N d_{i,j} t_{i,j}.$$

We call  $\tilde{Z}_N(\lambda; \mathbf{t})$  the *partition function* of formal Gromov–Witten invariants of  $\hat{Y}^N$ . It is the generating function of *disconnected* formal Gromov–Witten invariants of  $\hat{Y}^N$ .

In Section 5, we will compute  $\tilde{Z}_N(\lambda; \mathbf{t})$  by the mathematical theory of the topological vertex (see Li–Liu–Liu–Zhou [22]) and get the following expression (Proposition 17):

$$(2) \quad \tilde{Z}_N(\lambda; \mathbf{t}) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n[n]^2} \sum_{i=1}^3 \sum_{2 \leq k_1 \leq k_2 \leq N} e^{-n(t_{i,k_1} + \dots + t_{i,k_2})}\right) \\ \cdot \sum_{\vec{\mu}} \tilde{\mathcal{W}}_{\vec{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_{i,1}} s_{(\mu^i)'}(u^i(q, \mathbf{t}_i)).$$

where  $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$  is a triple of partitions,  $q = e^{\sqrt{-1}\lambda}$ ,  $[n] = q^{n/2} - q^{-n/2}$ . The precise definitions of  $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$  and  $s_{(\mu^i)'}(u^i(q, \mathbf{t}_i))$  will be given in Section 1.3. In particular, we will show that

$$(3) \quad \tilde{Z}_1(\lambda; \mathbf{t}) = \sum_{\vec{\mu}} \tilde{\mathcal{W}}_{\vec{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_i} \mathcal{W}_{(\mu^i)'}(q) = \exp\left(\sum_{n=1}^{\infty} \frac{Q_n(\mathbf{t})}{-n[n]^2}\right)$$

where  $\mathbf{t} = (t_1, t_2, t_3)$ ,  $\mathcal{W}_{\mu}(q)$  is defined by (9) in Section 1.3, and

$$(4) \quad Q_n(\mathbf{t}) = e^{-nt_1} + e^{-nt_2} + e^{-nt_3} - e^{-n(t_1+t_2)} - e^{-n(t_2+t_3)} - e^{-n(t_3+t_1)} + e^{-n(t_1+t_2+t_3)}.$$

In Section 6, we will compute  $\tilde{Z}_N(\lambda; \mathbf{t})$  by the physical theory of the topological vertex (see Aganagic–Klemm–Mariño–Vafa [2]) and get the following expression (Proposition 20):

$$(5) \quad Z_N(\lambda; \mathbf{t}) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n[n]^2} \sum_{i=1}^3 \sum_{2 \leq k_1 \leq k_2 \leq N} e^{-n(t_{i,k_1} + \dots + t_{i,k_2})}\right) \\ \cdot \sum_{\vec{\mu}} \mathcal{W}_{\vec{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_{i,1}} s_{(\mu^i)'}(u^i(q, \mathbf{t}_i))$$

where  $\mathcal{W}_{\vec{\mu}}(q)$  is defined by (8) in Section 1.3. In particular, we will show that (Proposition 19):

$$(6) \quad Z_1(\lambda; \mathbf{t}) = \sum_{\vec{\mu}} \mathcal{W}_{\vec{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_i} \mathcal{W}_{(\mu^i)'}(q) = \exp\left(\sum_{n=1}^{\infty} \frac{Q_n(\mathbf{t})}{-n[n]^2}\right)$$

The equivalence of the physical and mathematical theories of the topological vertex boils down to the following combinatorial identity:

$$(7) \quad \mathcal{W}_{\mu^1, \mu^2, \mu^3}(q) = \tilde{\mathcal{W}}_{\mu^1, \mu^2, \mu^3}(q).$$

It is known that (7) holds when one of the three partitions is empty (see the work of Li, C-C M Liu, K Liu and Zhou [24; 22]). When none of the partitions is empty, Klemm has checked all the cases where  $|\mu^i| \leq 6$  by computer. Up to now, a mathematical proof of (7) in full generality is not available. Equations (3) and (6) imply the following result.

### Theorem 3

$$\sum_{\vec{\mu}} \mathcal{W}_{\vec{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_i} \mathcal{W}_{(\mu^i)^t}(q) = \sum_{\vec{\mu}} \tilde{\mathcal{W}}_{\vec{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_i} \mathcal{W}_{(\mu^i)^t}(q).$$

Theorem 3 provides further evidence of (7) equating the vertex amplitudes derived from physical and mathematical theories of the topological vertex.

### 1.3 The topological vertex

In [2], Aganagic, Klemm, Mariño, and Vafa proposed that Gromov–Witten invariants of any toric Calabi–Yau threefold can be expressed in terms of certain relative invariants of its  $\mathbb{C}^3$  charts, called *the topological vertex*. They suggested that these local relative invariants should count holomorphic maps from bordered Riemann surfaces to  $\mathbb{C}^3$  where the boundary circles are mapped to three explicitly specified Lagrangian submanifolds  $L_1, L_2, L_3$ . The topological vertex depends on three partitions  $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ , where  $\mu^i$  corresponds to the winding numbers (the homology classes of boundary circles) in  $L_i$ . There is a symmetry on  $\mathbb{C}^3$  cyclically permuting  $L_1, L_2, L_3$ , so one expects the topological vertex to be symmetric under a cyclic permutation of the three partitions  $\mu_1, \mu_2, \mu_3$ .

In [2] the topological vertex was computed by using the conjectural relation between open Gromov–Witten invariants on toric Calabi–Yau threefolds and Chern–Simons invariants of knots and links. It has the following form:

$$(8) \quad \mathcal{W}_{\vec{\mu}}(q) = q^{\kappa_{\mu^2/2} + \kappa_{\mu^3/2}} \sum_{\rho, \rho^1, \rho^3} c_{\rho \rho^1}^{\mu^1} c_{\rho \rho^3}^{(\mu^3)^t} \frac{\mathcal{W}_{(\mu^2)^t \rho^1}(q) \mathcal{W}_{\mu^2 \rho^3}(q)}{\mathcal{W}_{\mu^2}(q)}.$$

In (8),  $\mu^t$  denotes the partition transposed to  $\mu$ . The expression (8) involves various quantities that we now define.  $\kappa_\mu$  is given by

$$\kappa_\mu = \sum_i \mu_i(\mu_i - 2i + 1).$$

The coefficients  $c_{\mu\nu}^\rho$  are Littlewood–Richardson coefficients. They can be defined in terms of Schur functions as follows

$$s_\mu s_\nu = \sum_\rho c_{\mu\nu}^\rho s_\rho.$$

Here, Schur functions are regarded as a basis for the ring  $\Lambda$  of symmetric polynomials in an infinite number of variables. The quantity  $\mathcal{W}_\mu(q)$  can be also defined in terms of Schur functions as follows:

$$(9) \quad \mathcal{W}_\mu(q) = s_\mu(x_i = q^{-i+\frac{1}{2}}).$$

One can show that

$$(10) \quad \mathcal{W}_{\mu^t}(q) = q^{-\kappa_\mu/2} \mathcal{W}_\mu(q).$$

We also define, in analogy to skew Schur functions,

$$(11) \quad \mathcal{W}_{\mu/\nu}(q) = \sum_\lambda c_{\nu\lambda}^\mu \mathcal{W}_\lambda(q).$$

Finally,  $\mathcal{W}_{\mu\nu}(q)$  is defined by

$$(12) \quad \mathcal{W}_{\mu\nu}(q) = q^{\kappa_\mu/2+\kappa_\nu/2} \sum_\lambda \mathcal{W}_{\mu^t/\lambda}(q) \mathcal{W}_{\nu^t/\lambda}(q).$$

This expression for  $\mathcal{W}_{\mu\nu}(q)$  is different from the one used originally in [2]. The fact that both agree follows from cyclicity of the vertex, and it has been proved in detail by Zhou [33]. The expression for the vertex in terms of Schur functions is given in Okounkov–Reshetikhin–Vafa [30] where the cyclicity of the vertex is also proved.

In Li–Liu–Liu–Zhou [22] the topological vertex was interpreted and defined as local relative invariants of a configuration  $C_1 \cup C_2 \cup C_3$  of three  $\mathbb{P}^1$ 's meeting at a point  $p_0$  in a relative Calabi–Yau threefold  $(Z, D_1, D_2, D_3)$ , where  $K_Z + D_1 + D_2 + D_3 \cong \mathcal{O}_Z$ ,  $C_i$  intersects  $D_i$  at a point  $p_i \neq p_0$ , and  $C_i \cap D_j$  is empty for  $i \neq j$ . The partition  $\mu^i$  corresponds to the ramification pattern over  $p_i$ . It is shown in [22] that Gromov–Witten invariants of any toric Calabi–Yau threefold (or more generally, formal Gromov–Witten invariants of formal toric Calabi–Yau threefolds) can be expressed in terms of local relative invariants as described above, and the gluing rules coincide with those stated



in Aganagic–Klemm–Mariño–Vafa [2]. The following expression of the vertex was derived in [22]:

$$(13) \quad \tilde{\mathcal{W}}_{\tilde{\mu}}(q) = q^{-(\kappa_{\mu^1} - 2\kappa_{\mu^2} - \frac{1}{2}\kappa_{\mu^3})/2} \sum_{v^+, v^1, v^3, \eta^1, \eta^3} c_{(v^1)^t, \mu^2}^{v^+} c_{(\eta^1)^t, v^1}^{\mu^1} c_{\eta^3(v^3)^t}^{\mu^3} \cdot q^{(-2\kappa_{v^+} - \frac{\kappa_{v^3}}{2})/2} \mathcal{W}_{v^+v^3}(q) \sum_{\sigma} \frac{1}{z_{\sigma}} \chi_{\eta^1}(\sigma) \chi_{\eta^3}(2\sigma).$$

Here  $2\sigma = (2\sigma_1 \geq 2\sigma_2 \geq \dots)$  if  $\sigma = (\sigma_1 \geq \sigma_2 \geq \dots)$ . Recall that

$$z_{\sigma} = \prod_{i \geq 1} i^{m_i} \cdot m_i!$$

where  $m_i = m_i(\sigma)$  is the number of parts of the partition  $\sigma$  equal to  $i$  (see Macdonald [26, p.17]).

It is expected that the two different enumerative interpretations in [2] and in [22] of the vertex give rise to equivalent counting problems, in the spirit of the following simple example: counting ramified covers of a disc by bordered Riemann surfaces with prescribed winding numbers is equivalent to counting ramified covers of a sphere by closed Riemann surfaces with prescribed ramification pattern over  $\infty$ .

Finally, we introduce some notation which will arise in computations in Section 6. For any positive integer  $n$ , define

$$(14) \quad u_n^i(q, \mathbf{t}_i) = \frac{1}{[n]} \left( 1 + \sum_{k=2}^N e^{-n(t_{i,2} + \dots + t_{i,k})} \right).$$

Given a partition  $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_{\ell} > 0)$ , define

$$(15) \quad u_{\mu}^i(q, \mathbf{t}_i) = \prod_{j=1}^{\ell} u_{\mu_j}^i(q, \mathbf{t}_i).$$

and

$$(16) \quad s_{\mu}(u^i(q, \mathbf{t}_i)) = \sum_{|v|=|\mu|} \frac{\chi_{\mu}(v)}{z_v} u_v^i(q, \mathbf{t}_i).$$

In particular, when  $N = 1$ , we have

$$u_n^i = \frac{1}{[n]} = \sum_{i>0} q^{-i+1/2}$$

So

$$(17) \quad s_\mu(u^i(q, \mathbf{t}_i)) = s_\mu(x_i = q^{-i+\frac{1}{2}}) = \mathcal{W}_\mu(q).$$

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## 2 Cremona

In this section we prove Theorems 1 and 2 using the geometry of the Cremona transform. We assume that the formal neighborhood  $Y^N \subset X$  is as constructed in Section 3. We also assume that the local invariants of  $Y^N$  are equal to certain ordinary invariants of  $X$ , which we prove in Section 4.

### 2.1 The blowup of $\mathbb{C}\mathbb{P}^3$ at points

We briefly review the properties of the blowup of  $\mathbb{P}^3$  at points used here for completeness and to set notation. This material can be found in much greater detail in, for instance, Griffiths–Harris [13].

Let  $X \rightarrow \mathbb{P}^3$  be the blowup of  $\mathbb{P}^3$  along  $M$  distinct points  $\{p_1, \dots, p_M\}$ . We describe the homology of  $X$ . All (co)homology is taken with integer coefficients. Note that we may identify homology and cohomology as rings via Poincaré duality, where cup product is dual to intersection product.

Let  $H$  be the total transform of a hyperplane in  $\mathbb{P}^3$ , and let  $E_i$  be the exceptional divisor over  $p_i$ . Then  $H_4(X, \mathbb{Z})$  has a basis

$$H_4(X) = \langle H, E_1, \dots, E_M \rangle.$$

Furthermore, let  $h \in H_2(X)$  be the class of a line in  $H$ , and let  $e_i$  be the class of a line in  $E_i$ . The collection of all such classes form a basis of  $H_2(X)$ .

$$H_2(X) = \langle h, e_1, \dots, e_M \rangle$$

The intersection ring structure is given as follows. Let  $pt \in H_0(X)$  denote the class of a point. Two general hyperplanes meet in a line, so  $H \cdot H = h$ . A general hyperplane

and line intersect in a point, so  $H \cdot h = pt$ . Also, a general hyperplane is far from the center of a blowup, so all other products involving  $H$  or  $h$  vanish. The restriction of  $\mathcal{O}_X(E_i)$  to  $E_i \cong \mathbb{P}^2$  is the dual of the bundle  $\mathcal{O}_{\mathbb{P}^2}(1)$ , so  $E_i \cdot E_i$  is represented by minus a hyperplane in  $E_i$ , i.e.  $E_i \cdot E_i = -e_i$ , and  $E_i^3 = (-1)^{3-1} pt = pt$  (see Fulton [11]). Furthermore, the centers of the blowups are far away from each other, so all other intersections vanish. In summary, the following are the only non-zero intersection products.

$$\begin{array}{ll} H \cdot H = h & H \cdot h = pt \\ E_i \cdot E_i = -e_i & E_i \cdot e_i = -pt \end{array}$$

Also, we point out that the canonical bundle  $K_X$  is easy to describe in this basis:

$$K_X = -4H + 2 \sum_{i=1}^M E_i$$

Finally, we introduce a notational convenience for the Gromov–Witten invariants of  $\mathbb{P}^3$  blown up at points in a Calabi–Yau class. Any curve class is of the form

$$\beta = dh - \sum_{i=1}^M a_i e_i$$

for some integers  $d, a_i$  where  $d$  is non-negative. Thus  $K_X \cdot \beta = 0$  if and only if  $2d = \sum_{i=1}^M a_i$ . In that case, the virtual dimension of  $\overline{\mathcal{M}}_g(X, \beta)$  is zero, and

$$\langle \rangle_{g, \beta}^X = \int_{[\overline{\mathcal{M}}_g(X, \beta)]^{\text{vir}}} 1$$

is determined by the discrete data  $\{d, a_1, \dots, a_M\}$ . Then, we may use the shorthand notation

$$\langle \rangle_{g, \beta}^X = \langle d; a_1, \dots, a_M \rangle_g^X.$$

For example,

$$\langle \rangle_{g, 5h - e_1 - e_2 - 2e_3 - 3e_5 - 3e_6}^X = \langle 5; 1, 1, 2, 0, 3, 3 \rangle_g^X.$$

Furthermore, the Gromov–Witten invariants of  $X$  do not depend on ordering of the points  $p_i$ , and thus for any permutation  $\sigma$  of  $M$  points,

$$\langle d; a_1, \dots, a_M \rangle_g^X = \langle d; a_{\sigma(1)}, \dots, a_{\sigma(M)} \rangle_g^X.$$

## 2.2 Properties of the invariants of the blowup of $\mathbb{P}^3$ at points

First, we use the fact, shown in Bryan–Karp [4], that the Gromov–Witten invariants of the blowup of  $\mathbb{P}^3$  along points have a symmetry which arises from the geometry of the Cremona transformation.

**Theorem 4** (Bryan–Karp [4]) *Let  $\beta = dh - \sum_{i=1}^M a_i e_i$  with  $2d = \sum_{i=1}^M a_i$  and assume that  $a_i \neq 0$  for some  $i > 4$ . Then we have the following equality of Gromov–Witten invariants:*

$$\langle \rangle_{g,\beta}^X = \langle \rangle_{g,\beta'}^X$$

where  $\beta' = d'h - \sum_{i=1}^M a'_i e_i$  has coefficients given by

$$d' = 3d - 2(a_1 + a_2 + a_3 + a_4)$$

$$a'_1 = d - (a_2 + a_3 + a_4)$$

$$a'_2 = d - (a_1 + a_3 + a_4)$$

$$a'_3 = d - (a_1 + a_2 + a_4)$$

$$a'_4 = d - (a_1 + a_2 + a_3)$$

$$a'_5 = a_5$$

$\vdots$

$$a'_M = a_M.$$

We also use the following vanishing lemma, and a few of its corollaries.

**Lemma 5** *Let  $X$  be the blowup of  $\mathbb{P}^3$  at  $M$  distinct generic points  $\{x_1, \dots, x_M\}$ , and  $\beta = dh - \sum_{i=1}^M a_i e_i$  with  $2d = \sum_{i=1}^M a_i$ , and assume that  $d > 0$  and  $a_i < 0$  for some  $i$ . Then*

$$\overline{\mathcal{M}}_g(X, \beta) = \emptyset.$$

**Corollary 6** *For any  $M$  points  $\{x_1, \dots, x_M\}$  and  $X$  and  $\beta$  as above the corresponding invariant vanishes;*

$$\langle \rangle_{g,\beta}^X = 0.$$

This follows immediately from the deformation invariance of Gromov–Witten invariants and [Lemma 5](#).

**Proof** In genus zero, [Lemma 5](#) follows from a vanishing theorem of Gathmann [[12](#), Section 3]. In order to prove [Lemma 5](#), for arbitrary genus, it suffices to show that the result holds for a specific choice of points, as if the moduli space is empty for a specific choice, then it is empty for the generic choice. By choosing some of the points to be coplanar, and the rest to also be coplanar on a second plane, the result follows. For further details, see Karp [[17](#)].  $\square$

**Corollary 7** Let  $X$  be the blowup of  $\mathbb{P}^3$  along  $M$  points and define  $\beta = dh - \sum_{i=1}^M a_i e_i$  where  $2d = \sum_{i=1}^M a_i$  and  $d > 0$ . Also define

$$X' \xrightarrow{\pi} X$$

to be the blowup of  $X$  at a generic point  $p$ , so that  $X'$  is deformation equivalent to the blowup of  $\mathbb{P}^3$  at  $M + 1$  distinct points. Let  $\{h', e'_1, \dots, e'_{M+1}\}$  be a basis of  $H_2(X')$ , and let  $\beta' = dh' - \sum_{i=1}^M a_i e'_i$ . Then

$$\langle d; a_1, \dots, a_M, 0 \rangle_g^{X'} = \langle d; a_1, \dots, a_M \rangle_g^X$$

**Proof** This result follows from the more general results of Hu [[14](#)]. An independent proof using [Lemma 5](#) can be found in Karp [[17](#)].  $\square$

### 2.3 Proof of [Theorem 1](#)

Let the blowup space  $X^{N+1}$  and the minimal trivalent configuration  $Y^N$  be as constructed in [Section 3](#) on page [128](#). By [Proposition 8](#) on page [133](#) we have

$$N_{\mathbf{d}}^g(Y^N) = \langle \rangle_{g, \mathbf{d}}^{X^{N+1}}.$$

Assume that the invariant is non-zero:

$$\begin{aligned} \langle \rangle_{g, \mathbf{d}}^{X^{N+1}} = & \langle 3; 1, 1 - d_{1,2}, \dots, d_{1,N-1} - d_{1,N}, d_{1,N}, \\ & 1, 1 - d_{2,2}, \dots, d_{2,N-1} - d_{2,N}, d_{2,N}, \\ & 1, 1 - d_{3,2}, \dots, d_{3,N-1} - d_{3,N}, d_{3,N} \rangle_g^{X^{N+1}} \\ & \neq 0 \end{aligned}$$

Then, by [Corollary 6](#), the coefficient of each  $e_i, f_i, g_i$  is non-negative. Thus, for  $i = 1, 2, 3$ ,

$$(18) \quad 1 \geq d_{i,2} \geq \dots \geq d_{i,N} \geq 0.$$

Therefore we compute

$$\begin{aligned} \langle \rangle_{g, \mathbf{d}}^{X^{N+1}} &= \langle 3; 1, 0, \dots, 0, 1, \\ &\quad 1, 0, \dots, 0, 1, \\ &\quad 1, 0, \dots, 0, 1 \rangle_g^{X^{N+1}} \\ &= \langle 3; 1, 1, 1, 1, 1, 1 \rangle_g^{X^2}, \end{aligned}$$

where the last equality follows from [Corollary 7](#). So when [\(18\)](#) holds, we have

$$N_{\mathbf{d}}^g(Y^N) = N_{1,1,1}^g(Y^1) = C_g.$$

The last equality follows from [Fact 3](#) (see [Bryan–Karp \[4\]](#)). □

## 2.4 Proof of [Theorem 2](#)

Let the blowup space  $\tilde{X}^{N+1}$  and the chain of rational curves  $Y_A^N$  be as constructed in [Section 3](#) on page [130](#). By [Proposition 10](#) on page [137](#) we have

$$N_{\mathbf{d}}^g(Y^N) = N_{\mathbf{d}_1}^g(Y_A^N) = \langle \rangle_{g, \mathbf{d}}^{\tilde{X}^{N+1}}$$

where

$$\mathbf{d}_1 = (d_1, \dots, d_N), \quad \mathbf{d}_2 = \mathbf{d}_3 = (0, \dots, 0).$$

Assume that the invariant is non-zero:

$$\langle \rangle_{g, \mathbf{d}}^{\tilde{X}^{N+1}} = \langle d_1; d_1, d_1 - d_2, \dots, d_{N-1} - d_N, d_N \rangle_g^{\tilde{X}^{N+1}} \neq 0.$$

By [Corollary 6](#) the multiplicities are decreasing:

$$d_1 \geq d_2 \geq \dots \geq d_N \geq 0$$

Therefore, as  $d_1 > 0$ , there exists some  $1 \leq j \leq N$  such that

$$d_1 \geq \dots \geq d_j > 0, \quad d_{j+1} = \dots = d_N = 0.$$

Then, using [Corollary 7](#), we compute

$$\begin{aligned} N_{\mathbf{d}}^g(Y^N) &= \langle d_1; d_1, d_1 - d_2, \dots, d_{j-1} - d_j, 0, \dots, 0 \rangle_g^{\tilde{X}^{N+1}} \\ &= \langle d_1; d_1, d_1 - d_2, \dots, d_{j-1} - d_j, d_j \rangle_g^{\tilde{X}^{j+1}}. \end{aligned}$$

Note that for any  $1 \leq i \leq j + 1$  we may reorder

$$\begin{aligned} \langle d_1; d_1, d_1 - d_2, \dots, d_{j-1} - d_j, d_j \rangle_{\tilde{X}^{j+1}} &= \\ & \langle d_1; d_1, d_1, d_i - d_{i+1}, 0, 0, d_1 - d_2, \dots \\ & \dots, d_{i-2} - d_{i-1}, d_{i+1} - d_{i+2}, \dots, d_{j-1} - d_j, d_j \rangle_{\tilde{X}^{j+1}} \end{aligned}$$

Applying Cremona invariance (Theorem 4) we compute

$$\begin{aligned} \langle \rangle_{\tilde{X}^{N+1}} &= \langle d_1 - 2(d_i - d_{i+1}); d_1 - (d_i - d_{i+1}), 0, d_{i+1} - d_i, d_{i+1} - d_i, \\ & d_1 - d_2, \dots, d_{j-1} - d_j, d_j \rangle_{\tilde{X}^{j+3}}. \end{aligned}$$

Then, by Corollary 6,  $d_{i+1} \geq d_i$ . Since this inequality holds for every  $1 \leq i \leq j$  we have  $d_1 \leq \dots \leq d_j$ . Therefore

$$d_1 = \dots = d_j = d.$$

Thus we have

$$\begin{aligned} \langle \rangle_{\tilde{X}^{N+1}} &= \langle d; d, 0, \dots, 0, d \rangle_{\tilde{X}^{j+1}} \\ &= \langle d; d, d \rangle_{\tilde{X}^2} \\ &= N_{d,0,0}^g(Y^1) \\ &= C_g d^{2g-3} \end{aligned}$$

The last equality follows from Faber–Pandharipande [10]. □

### 3 Construction

We construct these configurations as subvarieties of a locally Calabi–Yau space  $X^{N+1}$ , which is obtained via a sequence of toric blowups of  $\mathbb{P}^3$ :

$$X^{N+1} \xrightarrow{\pi_{N+1}} X^N \xrightarrow{\pi_N} \dots \xrightarrow{\pi_2} X^1 \xrightarrow{\pi_1} X^0 = \mathbb{P}^3$$

In fact,  $X^{i+1}$  will be the blowup of  $X^i$  along three points. Our rational curves will be labeled by  $A_i, B_i, C_i$ , where  $1 \leq i \leq N$ , reflecting the nature of the configuration. Curves in intermediary spaces will have super-scripts, and their corresponding proper transforms in  $X$  will not.

The standard torus  $\mathbb{T} = (\mathbb{C}^\times)^3$  action on  $\mathbb{P}^3$  is given by

$$(t_1, t_2, t_3) \cdot (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : t_1 x_1 : t_2 x_2 : t_3 x_3).$$

There are four  $\mathbb{T}$ -fixed points in  $X^0 := \mathbb{P}^3$ ; we label them  $p_0 = (1:0:0:0)$ ,  $q_0 = (0:1:0:0)$ ,  $r_0 = (0:0:1:0)$  and  $s_0 = (0:0:0:1)$ . Let  $A^0$ ,  $B^0$  and  $C^0$  denote the (unique,  $\mathbb{T}$ -invariant) line in  $X^0$  through the two points  $\{p_0, s_0\}$ ,  $\{q_0, s_0\}$  and  $\{r_0, s_0\}$ , respectively.

Define

$$X^1 \xrightarrow{\pi_1} X^0$$

to be the blowup of  $X^0$  at the three points  $\{p_0, q_0, r_0\}$ , and let  $A^1, B^1, C^1 \subset X^1$  be the proper transforms of  $A^0, B^0$  and  $C^0$ . The exceptional divisor in  $X^1$  over  $p_0$  intersects  $A^1$  in a unique fixed point; call it  $p_1 \in X^1$ . Similarly, the exceptional divisor in  $X^1$  also intersects each of  $B^1$  and  $C^1$  in unique fixed points; call them  $q_1$  and  $r_1$ .

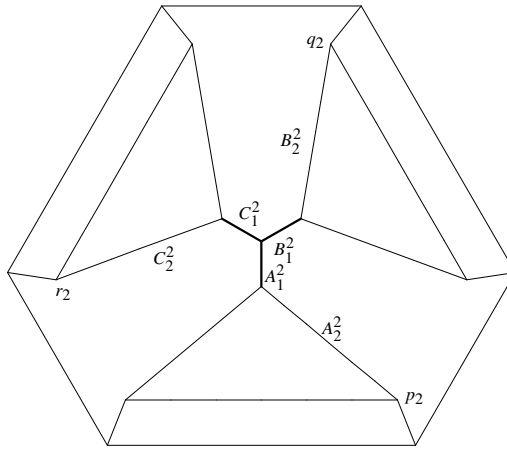


Figure 2: The  $\mathbb{T}$ -invariant curves in  $X^2$

Now define

$$X^2 \xrightarrow{\pi_2} X^1$$

to be the blowup of  $X^1$  at the three points  $\{p_1, q_1, r_1\}$ , and let  $A_1^2, B_1^2, C_1^2 \subset X^2$  be the proper transforms of  $A^1, B^1, C^1$ . The exceptional divisor over  $p_1$  contains two  $\mathbb{T}$ -fixed points disjoint from  $A_1^2$ . Choose one of them, and call it  $p_2$ ; this choice is arbitrary. Similarly, there are two fixed points in the exceptional divisors above  $q_1, r_1$  disjoint from  $B_1^2, C_1^2$ . Choose one in each pair identical to the choice of  $p_2$  and call them  $q_2$  and  $r_2$  (identical makes sense here as the configuration of curves in Figure 2 is rotationally symmetric). This choice is indicated in Figure 2. Let  $A_2^2$  denote the (unique,  $\mathbb{T}$ -invariant) line intersecting  $A_1^2$  and  $p_2$ . Define  $B_2^2, C_2^2$  analogously.



Clearly  $X^2$  is deformation equivalent to a blowup of  $\mathbb{P}^3$  at six distinct points. The  $\mathbb{T}$ -invariant curves in  $X^2$  are depicted in Figure 2, where each edge corresponds to a  $\mathbb{T}$ -invariant curve in  $X^2$ , and each vertex corresponds to a fixed point.

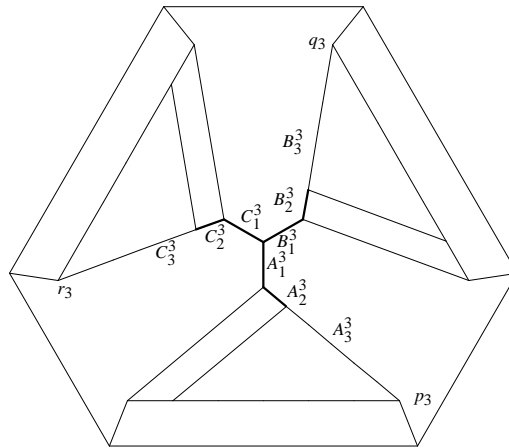


Figure 3: The  $\mathbb{T}$ -invariant curves in  $X^3$

We now define a sequence of blowups beginning with  $X^2$ . Fix an integer  $N \geq 2$ . For each  $1 < i \leq N$ , define

$$X^{i+1} \xrightarrow{\pi_{i+1}} X^i$$

to be the blowup of  $X^i$  along the three points  $p_i, q_i, r_i$ . Let  $A_j^{i+1} \subset X^{i+1}$  denote the proper transform of  $A_j^i$  for each  $1 \leq j \leq i$ . The exceptional divisor in  $X^{i+1}$  above  $p_i$  contains two  $\mathbb{T}$ -fixed points, choose one of them and call it  $p_{i+1}$ . Similarly choose  $q_{i+1}, r_{i+1}$ , and define  $A_{i+1}^{i+1} \subset X^{i+1}$  to be the line intersecting  $A_i^{i+1}$  and  $p_{i+1}$ , with  $B_{i+1}^{i+1}, C_{i+1}^{i+1}$  defined similarly. The  $\mathbb{T}$ -invariant curves in  $X^3$  are shown in Figure 3.

Finally, we define the *minimal trivalent configuration*  $Y^N \subset X^{N+1}$  by

$$Y^N = \bigcup_{1 \leq j \leq N} A_j \cup B_j \cup C_j,$$

where

$$A_j = A_j^{N+1}, \quad B_j = B_j^{N+1}, \quad C_j = C_j^{N+1}.$$

The configuration  $Y^N$  is shown in Figure 4, along with all other  $\mathbb{T}$ -invariant curves in  $X^{N+1}$ . It contains a chain of rational curves:

$$Y_A^N = A_1 \cup \dots \cup A_N.$$

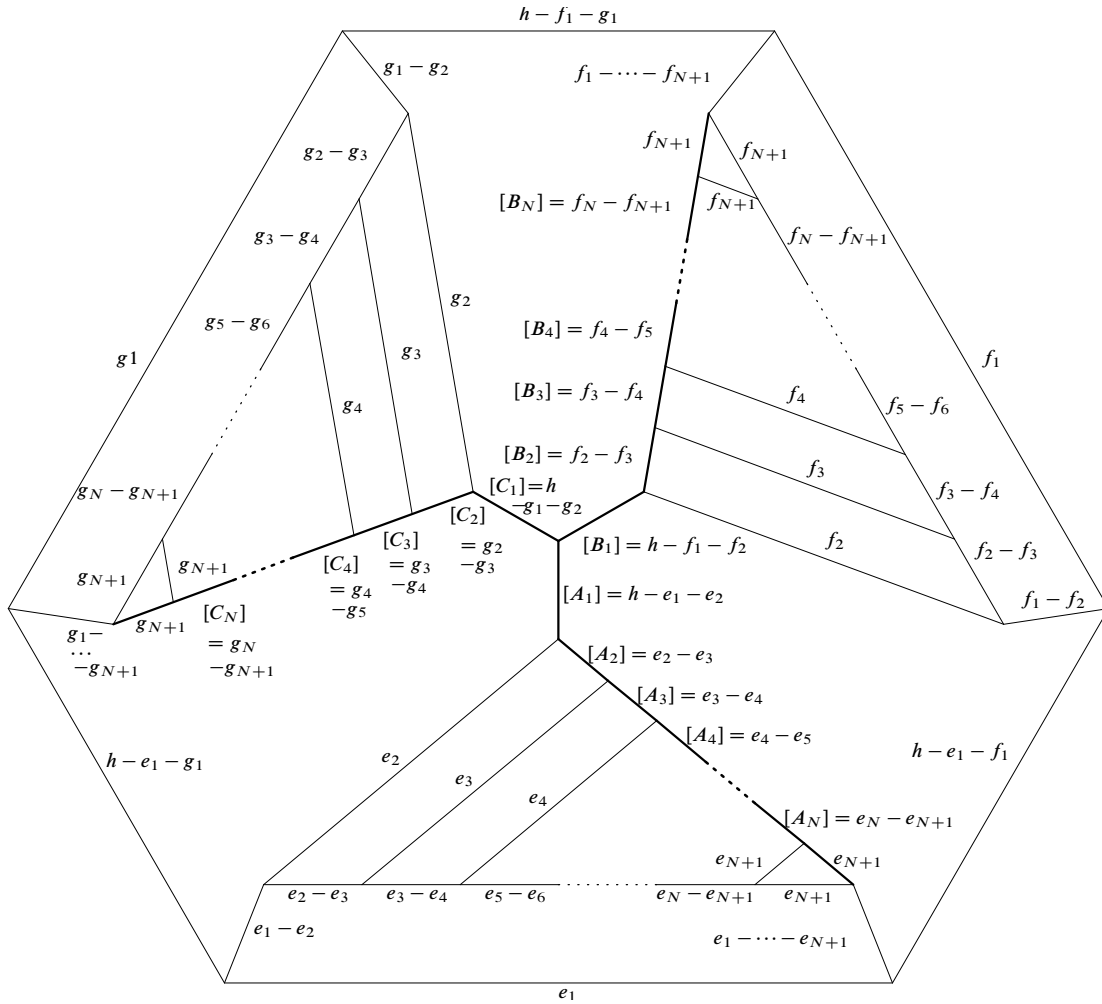


Figure 4: The  $\mathbb{T}$ -invariant curves in  $X^{N+1}$

### 3.1 Homology

We now compute  $H_*(X^{N+1}, \mathbb{Z})$  and identify the class of the configuration  $[Y^N] \in H_2(X^{N+1}, \mathbb{Z})$ . All (co)homology will be taken with integer coefficients. We denote divisors by upper case letters, and curve classes with the lower case. In addition, we decorate homology classes in intermediary spaces with a tilde, and their total transforms in  $X^N$  are undecorated.

Let  $\tilde{E}_1, \tilde{F}_1, \tilde{G}_1 \in H_4(X^1)$  denote the exceptional divisors in  $X^1 \rightarrow X^0$  over the points  $p_0, q_0$  and  $r_0$ , and let  $E_1, F_1, G_1 \in H_4(X)$  denote their total transforms. Continuing,

for each  $1 \leq i \leq N + 1$ , let  $\tilde{E}_i, \tilde{F}_i, \tilde{G}_i \in H_4(X^i)$  denote the exceptional divisors over the points  $p_{i-1}, q_{i-1}, r_{i-1}$  and let  $E_i, F_i, G_i \in H_4(X)$  denote their total transforms. Finally, let  $H$  denote the total transform of the hyperplane in  $X^0 = \mathbb{P}^3$ . The collection of all such classes  $\{H, E_i, F_i, G_i\}$ , where  $1 \leq i \leq N + 1$ , spans  $H_4(X^{N+1})$ .

Similarly, for each  $1 \leq i \leq N + 1$ , let  $\tilde{e}_i, \tilde{f}_i, \tilde{g}_i \in H_2(X^{i+1})$  denote the class of a line in  $\tilde{E}_i, \tilde{F}_i, \tilde{G}_i$  and let  $e_i, f_i, g_i \in H_2(X)$  denote their total transforms. In addition, let  $h \in H_2(X^{N+1})$  denote the class of a line in  $H$ . Then  $H_2(X^{N+1})$  has a basis given by  $\{h, e_i, f_i, g_i\}$ .

The intersection product ring structure is given as follows. Note that  $X^{N+1}$  is deformation equivalent to the blowup of  $\mathbb{P}^3$  at  $3N$  distinct points. Therefore, these

$$\begin{array}{ll} H \cdot H = h & H \cdot h = pt \\ E_i \cdot E_i = -e_i & E_i \cdot e_i = -pt \\ F_i \cdot F_i = -f_i & F_i \cdot f_i = -pt \\ G_i \cdot G_i = -g_i & G_i \cdot g_i = -pt \end{array}$$

are all of the nonzero intersection products in  $H_*(X^{N+1})$ .

In this basis, the classes of the components of  $Y^N$  are given as follows.

$$\begin{aligned} [A_i] &= \begin{cases} h - e_1 - e_2 & \text{if } i = 1 \\ e_i - e_{i+1} & \text{otherwise} \end{cases} \\ [B_i] &= \begin{cases} h - f_1 - f_2 & \text{if } i = 1 \\ f_i - f_{i+1} & \text{otherwise} \end{cases} \\ [C_i] &= \begin{cases} h - g_1 - g_2 & \text{if } i = 1 \\ g_i - g_{i+1} & \text{otherwise} \end{cases} \end{aligned}$$

To see this, recall that  $A_1$  is the proper transform of a line through two points which are centers of a blowup, and that  $A_i$ , for  $i > 1$ , is the proper transform of a line in an exceptional divisor containing a center of a blowup.  $B_i$  and  $C_i$  are similar.

### 4 Local to global

In this section, we will show that the local invariants  $N_{\mathbf{d}}^g(Y)$  are equal to the ordinary invariants  $\langle \rangle_{g, \mathbf{d}}^{X^{N+1}}$  in case  $Y$  is either the minimal trivalent configuration  $Y^N$  or the chain of rational curves  $Y_A^N$  defined in Section 3.

### 4.1 The minimal trivalent configuration

**Proposition 8** Let  $f: \Sigma \rightarrow X^{N+1}$  represent a point in  $\overline{\mathcal{M}}_g(X^{N+1}, \mathbf{d})$ , where

$$1 = d_{i,1} \geq \dots \geq d_{i,N} \geq 0.$$

Then the image of  $f$  is contained in the minimal trivalent configuration

$$Y^N = \bigcup_{1 \leq j \leq N} A_j \cup B_j \cup C_j.$$

**Proof** We use the toric nature of the construction. Assume that there exists a stable map

$$[f: \Sigma \rightarrow X^{N+1}] \in \overline{\mathcal{M}}_g(X^{N+1}, \mathbf{d})$$

such that  $\text{Im}(f) \not\subset Y^N$ . Then there exists a point  $p \in \text{Im}(f)$  such that  $p \notin Y^N$ .

Recall that  $\mathbb{T}$ -invariant subvarieties of a toric variety are given precisely by orbit closures of one-parameter subgroups of  $\mathbb{T}$ . So in particular the limit of  $p$  under the action of a one-parameter subgroup is a  $\mathbb{T}$ -fixed point. Moreover, since  $p \notin Y^N$ , there exists a one-parameter subgroup  $\psi: \mathbb{C}^\times \rightarrow \mathbb{T}$  such that

$$\lim_{t \rightarrow 0} \psi(t) \cdot p = q$$

where  $q$  is  $\mathbb{T}$ -fixed and  $q \notin Y^N$ .

The limit of  $\psi$  acting on  $[f]$  is a stable map  $f'$  such that  $q \in \text{Im}(f')$ . It follows that  $q$  is in the image of all stable maps in the orbit closure of  $[f']$ . Thus, there must exist a stable map  $[f'': \Sigma \rightarrow X^{N+1}] \in \overline{\mathcal{M}}_g(X^{N+1}, \mathbf{d})$  such that  $\text{Im}(f'')$  is  $\mathbb{T}$ -invariant and  $\text{Im}(f'') \not\subset Y^N$ .

We show that this leads to a contradiction. Let  $F$  denote the union of the  $\mathbb{T}$ -invariant curves in  $X^{N+1}$ ; it is shown above in Figure 4. We study the possible components of  $F$  contained in the image of  $f''$ .

Note that the push forward of the class of  $\Sigma$  is given by

$$\begin{aligned} f''_*[\Sigma] = & 3h - e_1 - \sum_{j=1}^{N-1} (d_{1,j} - d_{1,j+1})e_{j+1} - d_{1,N}e_{N+1} \\ & - f_1 - \sum_{j=1}^{N-1} (d_{2,j} - d_{2,j+1})f_{j+1} - d_{2,N}f_{N+1} \\ & - g_1 - \sum_{j=1}^{N-1} (d_{3,j} - d_{3,j+1})g_{j+1} - d_{3,N}g_{N+1}. \end{aligned}$$

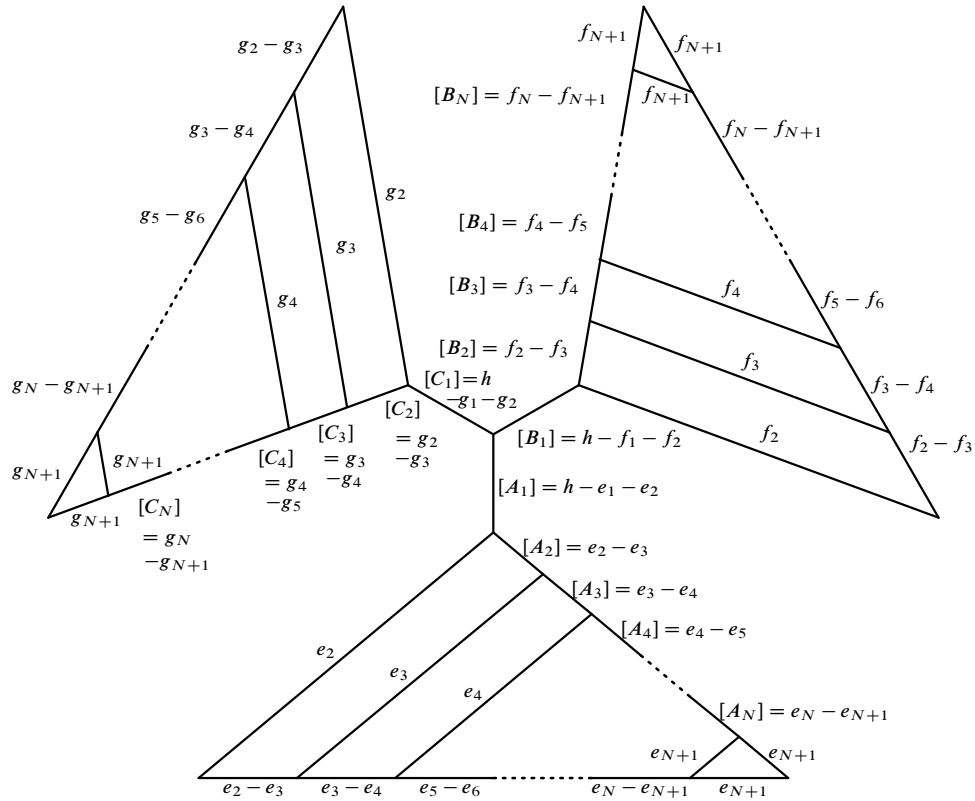


Figure 5: The possible curves in  $\text{Im}(f'')$

Suppose that  $A_1 \cup B_1 \cup C_1 \subset \text{Im}(f'')$ . Then  $f''_*[\Sigma]$  contains (at least)  $3h$ . Note that  $[F]$  has no  $-h$  terms. Therefore  $\text{Im}(f'')$  does not contain any of the curves  $h - e_1 - f_1, h - e_1 - g_1, h - f_1 - g_1$ . And furthermore each of  $A_1, B_1$  and  $C_1$  must have multiplicity one.

There are no remaining terms that contain  $-e_1, -f_1$  or  $-g_1$ . Also, since the image of  $f''$  contains precisely one of  $A_1, B_1, C_1$ , we conclude that the multiplicity of terms contain positive  $e_1, f_1, g_1$  must be zero. Thus,  $\text{Im}(f'')$  is contained in the configuration shown in Figure 5.

Now, note that in  $\mathbf{d}$  the sum of the multiplicities of the  $e_i$ 's is  $-2$ . This is true of the curve  $A_1$  as well. Therefore the total multiplicity of all other  $e$  terms must vanish. But all other  $e$  terms are of the form  $e_i - e_{i+1}$  or  $e_j$ . Since the former contribute nothing to the total multiplicity, we conclude that there are no  $e_j$  terms in the image of  $f''$ . Therefore  $\text{Im}(f'')$  must be contained in the configuration shown in Figure 6.

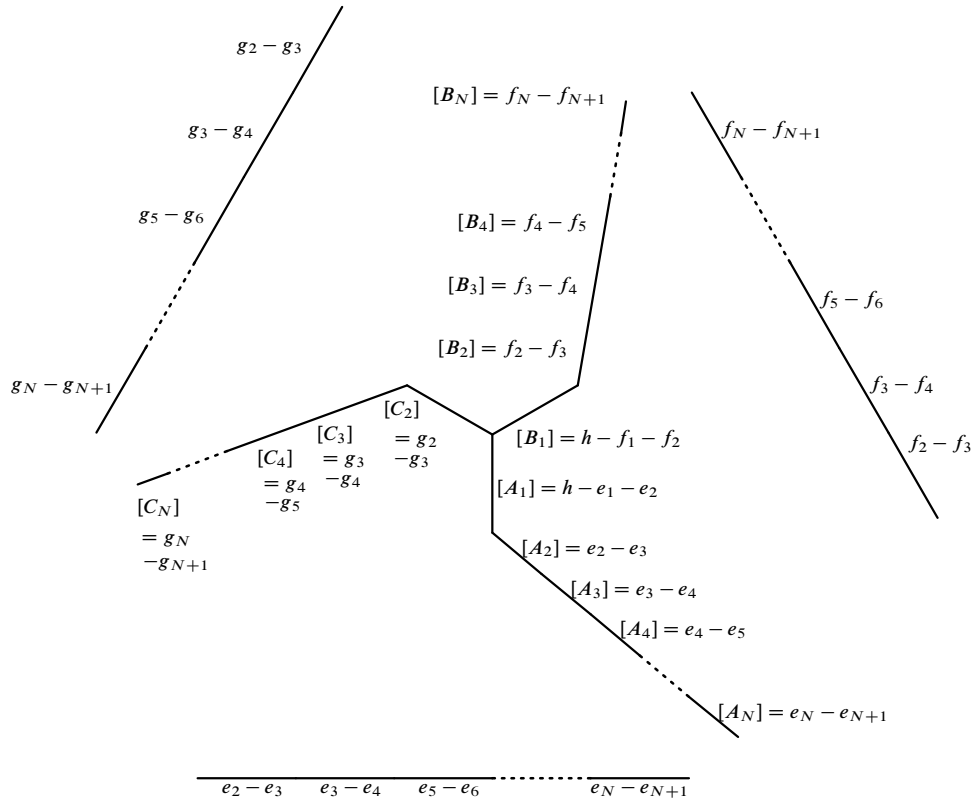


Figure 6: The remaining possible curves in  $\text{Im}(f'')$

But  $\text{Im}(f'')$  is connected, and contains  $h$  terms. Therefore it can not contain nor be contained in any of the three outer parts of Figure 6. Therefore  $\text{Im}(f'') \subset Y^N$ . This contradicts our assumption, and therefore at least one of  $A_1, B_1, C_1$  is not in  $\text{Im}(f'')$ .

Without loss of generality, suppose  $A_1 \notin \text{Im}(f'')$ . Let  $d_{e,f}, d_{e,g}, d_{f,g}$  denote the degree of  $f''$  on the components  $h - e_1 - f_1, h - e_1 - g_1, h - f_1 - g_1$  respectively. Since  $A_1$  is not contained in the image of  $f''$ , we must have

$$0 < d_{e,f} + d_{e,g} \leq 3$$

as these are the only multiplicities of  $-e_1$  terms, and there are no terms containing  $-h$ .

Furthermore, in order for  $\text{Im}(f'')$  to simultaneously be connected and contain  $-e_i$  terms for  $i > 1$ , it must be the case that  $\text{Im}(f'')$  contains two of

$$\{e_1, e_1 - e_2, e_1 - \dots - e_{N+1}\}.$$

Thus

$$d_{e,f} + d_{e,g} = 3, \quad d_{f,g} = 0$$

and  $B_1, C_1 \notin \text{Im}(f'')$ . This forces  $\text{Im}(f'')$  to be contained in the configuration shown in Figure 7.

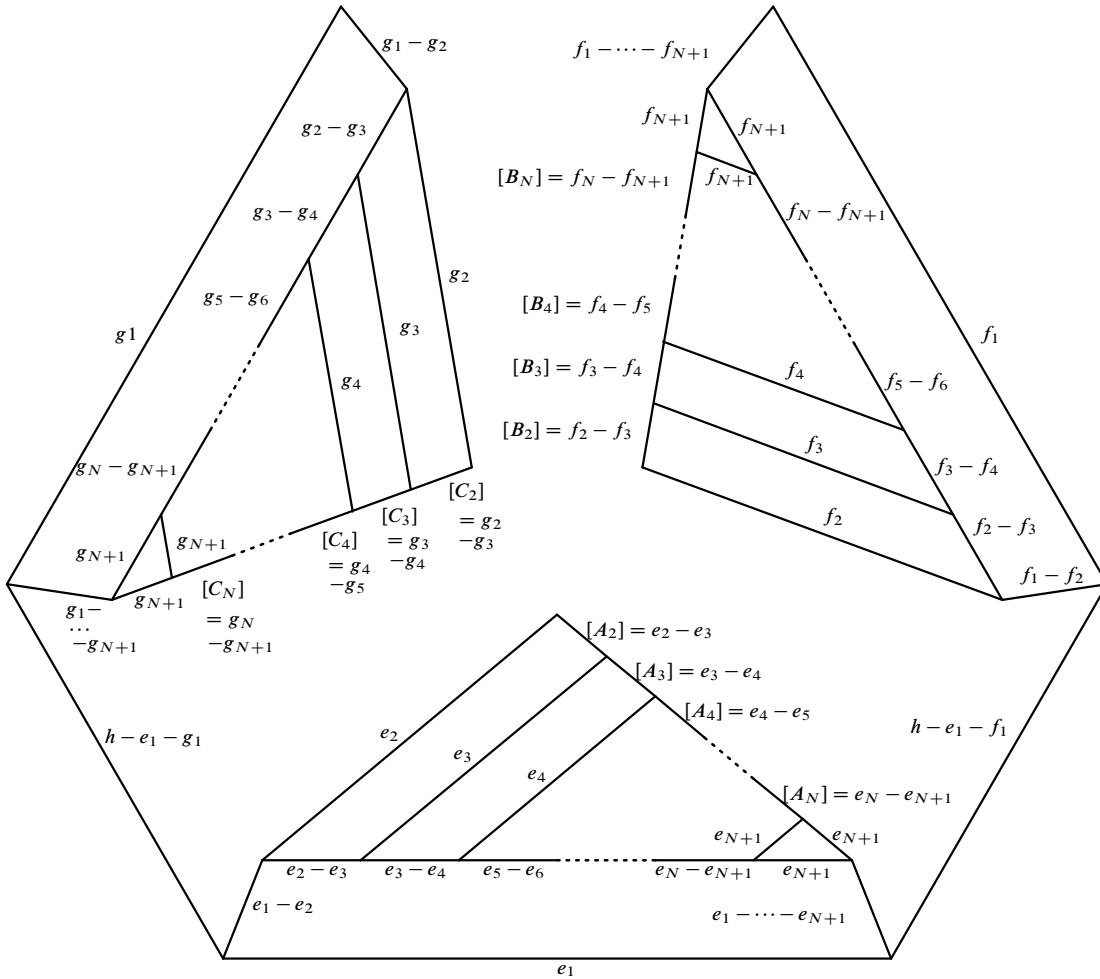


Figure 7: The other possibility for curves in  $\text{Im}(f'')$

Again we have that  $\text{Im}(f'')$  is connected and contains  $-f_i, -g_j$  for some  $i, j > 1$ . Therefore  $\text{Im}(f'')$  contains at least one of  $f_1 - f_2, f_1 - \dots - f_{N+1}$  and also at least one of  $g_1 - g_2, g_1 - \dots - g_{N+1}$ . But the multiplicity of  $f_1$  and  $g_1$  in  $\mathbf{d}$  is  $-1$ . Therefore

$$d_{e,f}, d_{e,g} \geq 2.$$

This contradiction shows that our assumption  $A_1 \not\subset \text{Im}(f'')$  is incorrect. Therefore  $A_1 \subset \text{Im}(f'')$ . An identical argument also shows that  $B_1, C_1 \subset \text{Im}(f'')$ . However we showed above that  $A_1, B_1, C_1 \not\subset \text{Im}(f'')$ .

This contradiction shows that our original assumption is incorrect. Therefore there does not exist a point  $p \in \text{Im}(f'')$  such that  $p \notin Y^N$ . Thus  $\text{Im}(f'') \subset Y^N$ , and the result holds.  $\square$

**Remark 9** Note that this argument does not hold for general  $a_1, b_1, c_1$ . For instance, it is a fun exercise to show that there is more than one  $\mathbb{T}$ -invariant configuration of curves in  $X$  in the following classes.

$$\begin{aligned}\beta_1 &= 2(h - e_1 - e_2) + (e_2 - e_3) \\ &\quad + 2(h - f_1 - f_2) + (f_2 - f_3) \\ &\quad + 2(h - g_1 - g_2) + (g_2 - g_3) \\ \beta_2 &= 4(h - e_1 - e_2) + (e_2 - e_3) + 2(h - f_1 - f_2) + 2(h - g_1 - g_2) + (g_2 - g_3) \\ \beta_3 &= 4(h - e_1 - e_2) + 4(e_2 - e_3) \\ &\quad + 4(h - f_1 - f_2) + 4(f_2 - f_3) \\ &\quad + 4(h - g_1 - g_2) + 4(g_2 - g_3).\end{aligned}$$

## 4.2 A chain of rational curves

**Proposition 10** Let  $f: C \rightarrow X^{N+1}$  represent a point in  $\overline{\mathcal{M}}_g(X^{N+1}, \mathbf{d})$ , where

$$d_{1,1} > 0, \quad d_{2,j} = d_{3,j} = 0, \quad j = 1, \dots, N.$$

Then the image of  $f$  is contained in the chain of rational curves

$$Y_A^N = A_1 \cup \dots \cup A_N$$

defined in [Section 3](#).

Since  $Y_A^N$  does not contain any of the curves  $B_i, C_i$ , the blowups with centers  $p_i$  and  $q_i$  in the construction of  $X^{N+1}$  are extraneous. In order to simplify the argument in this case, consider the space

$$\dots \xrightarrow{\pi_{N+2}} \tilde{X}^{N+1} \xrightarrow{\pi_{N+1}} \tilde{X}^N \xrightarrow{\pi_N} \dots \xrightarrow{\pi_1} \mathbb{P}^3,$$

where the construction of  $\tilde{X}^{N+1}$  follows that of  $X^{N+1}$ , without the extraneous blowups. So  $\tilde{X}^{i+1} \rightarrow \tilde{X}^i$  is the blowup of  $\tilde{X}^i$  along the point  $p_i$ , where  $p_i$  is defined in [Section 3](#). Thus,  $\tilde{X}^{N+1}$  is deformation equivalent to the blowup of  $\mathbb{P}^3$  at  $N + 1$



points. Since  $Y_A^N$  does not contain the curves  $B_i, C_i$ , clearly the formal neighborhood of  $Y_A^N$  in  $\tilde{X}^{N+1}$  agrees with the construction in  $X^{N+1}$ .

We continue to let  $E_i$  be the total transform of the exceptional divisor over  $p_i$ , and  $e_i$  be the class of a line in  $E_i$ . Furthermore, we continue to let  $H$  denote the pullback of the class of a hyperplane in  $\mathbb{P}^3$ , and  $h$  be the class of a line in  $H$ . Then,  $\{H, E_i\}$  is a basis for  $H_4(\tilde{X}^{N+1})$  and  $\{h, e_i\}$  is a basis for  $H_2(\tilde{X}^{N+1})$ . The non-zero intersection pairings are given as follows.

$$\begin{array}{l} H \cdot H = h \quad H \cdot h = pt \\ E_i \cdot E_i = -e_i \quad E_i \cdot e_i = -pt \end{array}$$

The  $\mathbb{T}$ -invariant curves in  $\tilde{X}^{N+1}$  are shown together with their homology classes in Figure 8.

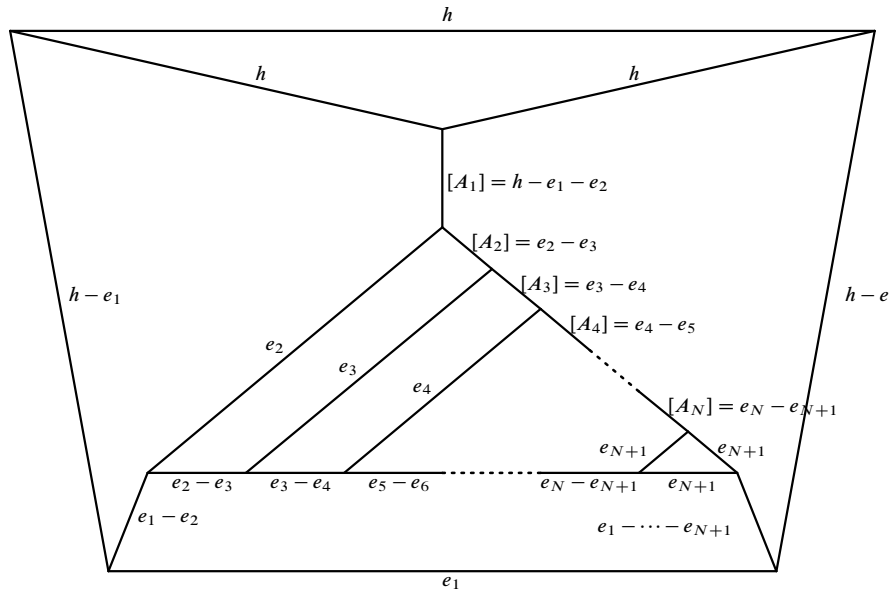


Figure 8: The  $\mathbb{T}$ -invariant curves in  $\tilde{X}^{N+1}$

**Proof of Proposition 10** As shown in above, we may use the toric nature of  $\tilde{X}^{N+1}$  to construct a stable map  $[f'': \Sigma \rightarrow \tilde{X}^{N+1}] \in \overline{\mathcal{M}}_g(\tilde{X}^{N+1}, \mathbf{d})$  such that  $\text{Im}(f'')$  is  $\mathbb{T}$ -invariant, but  $\text{Im}(f'') \not\subset Y_A^N$ . We show that this leads to a contradiction.

We study the class  $f''_*[\Sigma] = \mathbf{d}$ . Note that the multiplicity of the  $-e_1$  term is the same as that of  $h$ . Furthermore, each  $-e_1$  occurs along with  $h$ , and there are no  $-h$  terms.

Therefore  $\text{Im}(f'')$  can not contain any terms containing positive  $e_1$ , nor can it contain any of the curves in class  $h$ . Thus, the image of  $f''$  is contained in the configuration of curves shown in Figure 9.

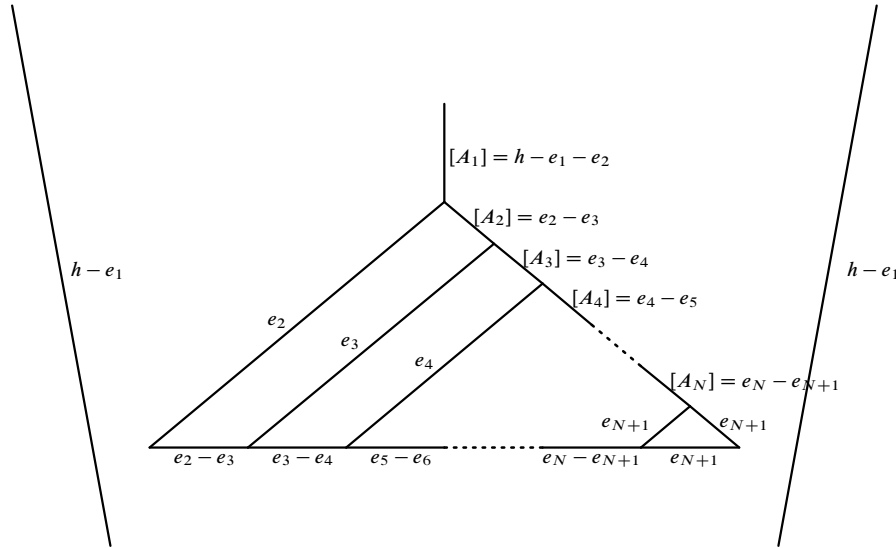


Figure 9: The possible curves in  $\text{Im}(f'')$

Since  $a_1 > 0$ , it must be that  $f''_*[\Sigma]$  contains at least one  $e_i$  term with non-zero multiplicity for  $i > 1$ . Also,  $\text{Im}(f'')$  is connected and so we conclude that the image of  $f$  must not contain either of the curves of class  $h - e_1$  in Figure 9.

Now, note that the total multiplicity of the  $e$  terms is  $-2a_1$ , and that the curve  $A_1$  must also have this property. Therefore the sum of all other  $e$  terms must be zero. Since the other  $e$  terms are of the form  $e_i - e_{i+1}$  or  $e_j$ , we conclude that  $\text{Im}(f'')$  does not contain any of the curves  $e_j$ . Thus  $\text{Im}(f'')$  is contained in the configuration depicted in Figure 10.

However, since  $\tilde{X}^{N+1}$  is connected and contains  $h$ , we conclude that  $\text{Im}(f'') \subset Y_A^N$ . This contradiction shows that our original assumption is incorrect, and that the result holds.  $\square$

### 5 Mathematical theory of the topological vertex

Let

$$(19) \quad X^{N+1} \xrightarrow{\pi_{N+1}} X^N \xrightarrow{\pi_N} \dots \xrightarrow{\pi_2} X^1 \xrightarrow{\pi_1} X^0 = \mathbb{P}^3$$

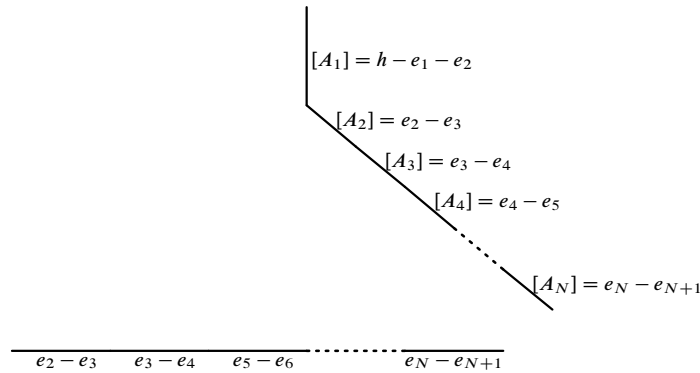


Figure 10: The remaining possible curves in  $\text{Im}(f'')$

be the toric blowups constructed in Section 3. Let  $Y^N \subset X^{N+1}$  be the minimal trivalent configuration, and let  $\widehat{Y}^N$  be the formal completion of  $X^{N+1}$  along  $Y^N$ . Then  $\widehat{Y}^N$  is a nonsingular formal scheme, and  $\overline{\mathcal{M}}_g(\widehat{Y}^N, \mathbf{d})$  is a separated formal Deligne–Mumford stack with a perfect obstruction theory of virtual dimension zero. It has a virtual fundamental class when it is proper, which is not true in general.

### 5.1 Formal Gromov–Witten invariants of $\widehat{Y}^N$

In (19)  $\mathbb{T} = (\mathbb{C}^\times)^3$  acts on  $X^j$  and the projections  $\pi_j$  are  $\mathbb{T}$ -equivariant, so  $\widehat{Y}^N$  is a formal scheme together with a  $\mathbb{T}$ -action. The point  $s_0 = A_1 \cap B_1 \cap C_1$  is fixed by the  $\mathbb{T}$ -action, so  $\mathbb{T}$  acts on  $T_{s_0}X^0$  and  $\Lambda^3 T_{s_0}X^0$ . Let  $\mathbb{S}$  be the rank 2 subtorus of  $\mathbb{T}$  which acts trivially on  $\Lambda^3 T_{s_0}X^0$ . The union of one dimensional orbit closures of the  $\mathbb{T}$ -action on  $X^j$  is a configuration of rational curves, which corresponds to a graph (see Figure 11).

The  $\mathbb{S}$ -action on  $X^j$  can be read off from the slopes of the edges of the graph associated to  $X^j$ . More precisely, let  $\Lambda_{\mathbb{S}} = \text{Hom}(\mathbb{S}, \mathbb{C}^\times)$  be the group of irreducible characters of  $\mathbb{S}$ . If we fix an identification  $\mathbb{S} \cong (\mathbb{C}^\times)^2$  then an element in  $\Lambda_{\mathbb{S}}$  is of the form  $s_1^p s_2^q$  where  $(s_1, s_2)$  are coordinates on  $(\mathbb{C}^\times)^2$  and  $p, q \in \mathbb{Z}$ . The line segment associated to  $C \cong \mathbb{P}^1$  is tangent to  $(p, q) \in \mathbb{Z} \oplus \mathbb{Z}$  if the irreducible characters of the  $\mathbb{S}$ -actions on  $T_x C$  and  $T_y C$  are  $s_1^p s_2^q$  and  $s_1^{-p} s_2^{-q}$  (see Figure 12). Similarly, the  $\mathbb{S}$ -action on  $\widehat{Y}^N$  can be read off from Figure 13 in Section 5.5.

Let  $u_1, u_2$  be a basis of  $H_{\mathbb{S}}^2(pt, \mathbb{Z})$  so that  $H_{\mathbb{S}}^2(pt, \mathbb{Z}) = \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$ . For any nonzero effective class  $\mathbf{d}$  ( $d_{i,j} \geq 0$ ), define

$$\widetilde{N}_{\mathbf{d}}^g(\widehat{Y}^N) = \int_{[\overline{\mathcal{M}}_g(\widehat{Y}^N, \mathbf{d})^{\mathbb{S}}]_{\text{vir}}} \frac{1}{e_{\mathbb{S}}(N^{\text{vir}})}.$$

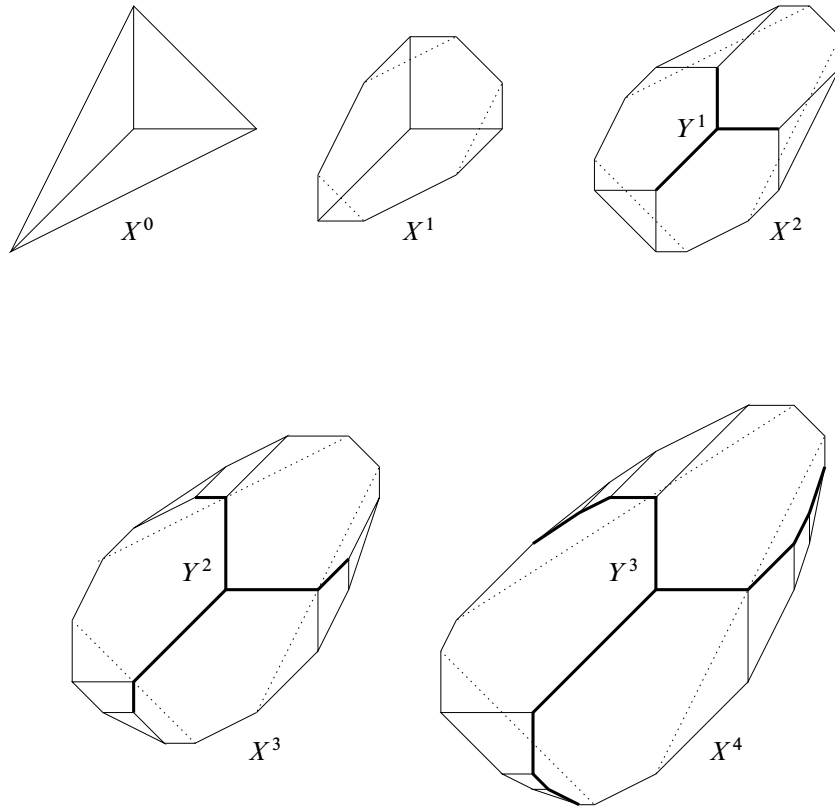


Figure 11: Configurations of  $\mathbb{T}$ -invariant curves

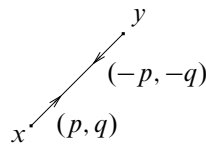


Figure 12: The  $\mathbb{S}$ -action can be read off from the slope

A priori  $\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N)$  is a rational function in  $u_1, u_2$  with  $\mathbb{Q}$  coefficients, homogeneous of degree 0. By results in Li–Liu–Liu–Zhou [22],  $\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N) \in \mathbb{Q}$  is a constant function independent of  $u_1, u_2$ . We call  $\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N)$  *formal Gromov–Witten invariants*. For the

cases (i)–(iv) described in Section 1,

$$\begin{aligned}
 N_{\mathbf{d}}^g(Y^N) &= \langle \rangle_{g, \mathbf{d}}^{X^{N+1}} \\
 &= \int_{[\overline{\mathcal{M}}_g(X^{N+1}, \mathbf{d})]^{\text{vir}}} 1 \\
 &= \int_{[\overline{\mathcal{M}}_g(X^{N+1}, \mathbf{d})^{\mathbb{S}}]^{\text{vir}}} \frac{1}{e_{\mathbb{S}}(N^{\text{vir}})} \\
 &= \int_{[\overline{\mathcal{M}}_g(\widehat{Y}^N, \mathbf{d})^{\mathbb{S}}]^{\text{vir}}} \frac{1}{e_{\mathbb{S}}(N^{\text{vir}})} \\
 &= \widetilde{N}_{\mathbf{d}}^g(\widehat{Y}^N).
 \end{aligned}$$

As in Section 1, introduce formal variables  $\lambda, t_{i,j}$ , and define a generating function

$$(20) \quad F_N(\lambda; \mathbf{t}) = \sum_{g \geq 0} \sum_{\mathbf{d}} \lambda^{2g-2} \widetilde{N}_{\mathbf{d}}^g(\widehat{Y}^N) e^{-\mathbf{d} \cdot \mathbf{t}}$$

where

$$\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3), \quad \mathbf{t}_i = (t_{i,1}, \dots, t_{i,N}), \quad \mathbf{d} \cdot \mathbf{t} = \sum_{i=1}^3 \sum_{j=1}^N d_{i,j} t_{i,j}.$$

The partition function of the formal Gromov–Witten invariants of  $\widehat{Y}^N$  is defined to be

$$\widetilde{Z}_N(\lambda; \mathbf{t}) = \exp(F_N(\lambda; \mathbf{t})).$$

By connectedness and cyclic symmetry, we only need to compute  $\widetilde{N}_{\mathbf{d}}^g(\widehat{Y}^N)$  in the following cases (see Figure 13):

- (D1)  $\mathbf{d} = (\mathbf{d}_1, \mathbf{0}, \mathbf{0})$ ,  $d_{1,j} > 0$  for  $j \leq k$  and  $d_{1,j} = 0$  for  $j > k$ , where  $1 \leq k \leq N$ .
- (D2)  $\mathbf{d} = (\mathbf{d}_1, \mathbf{0}, \mathbf{0})$ ,  $d_{1,j} > 0$  for  $k_1 \leq j \leq k_2$  and  $d_{1,j} = 0$  otherwise, where  $2 \leq k_1 \leq k_2 \leq N$ .
- (D3)  $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{0})$ ,  $d_{1,m} > 0$  for  $m \leq j$  and  $d_{1,m} = 0$  for  $m > j$ ,  $d_{2,m} > 0$  for  $m \leq k$  and  $d_{2,m} = 0$  for  $m > k$ , where  $1 \leq j, k \leq N$ .
- (D4)  $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ ,  $d_{i,j} > 0$  for  $j \leq k_i$  and  $d_{i,j} = 0$  for  $j > k_i$ , where  $1 \leq k_1, k_2, k_3 \leq N$ .

Any other  $\widetilde{N}_{\mathbf{d}}^g(\widehat{Y}^N)$  is either manifestly zero (because  $\overline{\mathcal{M}}_g(\widehat{Y}^N, \mathbf{d})$  is empty) or is equal to one of the above case. Let

$$F_N^1(\lambda; \mathbf{t}_1), \quad F_N^2(\lambda; \mathbf{t}_1), \quad F_N^3(\lambda; \mathbf{t}_1, \mathbf{t}_2), \quad F_N^4(\lambda; \mathbf{t})$$

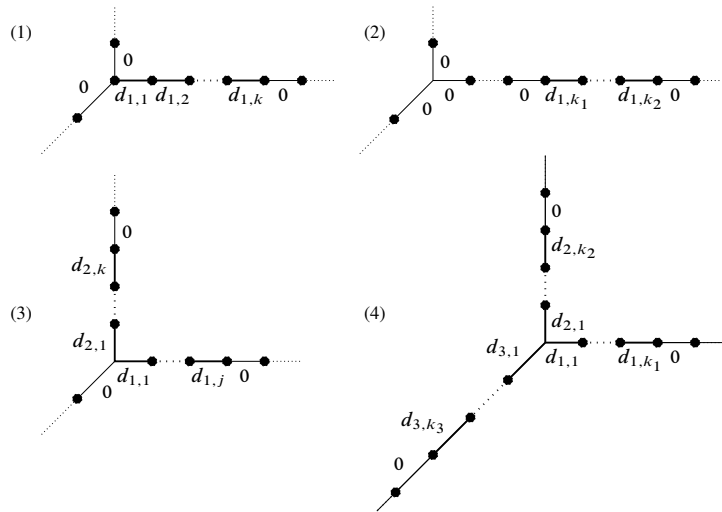


Figure 13: Four cases

denote the contribution to  $F_N(\lambda; \mathbf{t})$  from (D1), (D2), (D3), (D4), respectively. Then

$$\begin{aligned}
 F_N(\lambda; \mathbf{t}) &= \sum_{i=1}^3 F_N^1(\lambda; \mathbf{t}_i) + \sum_{i=1}^3 F_N^2(\lambda; \mathbf{t}_i) \\
 &\quad + F_N^3(\lambda; \mathbf{t}_1, \mathbf{t}_2) + F_N^3(\lambda; \mathbf{t}_2, \mathbf{t}_3) + F_N^3(\lambda; \mathbf{t}_3, \mathbf{t}_1) + F_N^4(\lambda; \mathbf{t}).
 \end{aligned}$$

### 5.2 Summary of results

Let  $C_g$  be defined by

$$\sum_{g=0}^{\infty} C_g t^{2g} = \left( \frac{t/2}{\sin(t/2)} \right)^2$$

as before. Then

$$\sum_{g \geq 0} C_g n^{2g-3} \lambda^{2g-2} = \frac{-1}{n[n]^2}$$

where

$$[n] = q^{n/2} - q^{-n/2}, \quad q = e^{\sqrt{-1}\lambda}.$$

In [Section 5.5](#), we will compute  $\tilde{N}_d^g(\hat{Y}^N)$  by the mathematical theory of the topological vertex and obtain the following results. We will do the computations by the physical theory of the topological vertex in [Section 6](#).

**Proposition 11** Suppose that

$$\mathbf{d}_1 = (d_1, \dots, d_N), \quad \mathbf{d}_2 = \mathbf{d}_3 = (0, \dots, 0).$$

where  $d_1 > 0$ . Then

$$\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N) = \begin{cases} C_g n^{2g-3} & d_1 = d_2 = \dots = d_k = n > 0 \text{ and} \\ & d_{k+1} = d_{k+2} = \dots = d_N = 0 \text{ for some } 1 \leq k \leq N \\ 0 & \text{otherwise} \end{cases}$$

which is equivalent to

$$(21) \quad F_N^1(\lambda; \mathbf{t}_1) = \sum_{n>0} \frac{-1}{n[n]^2} \sum_{1 \leq k \leq N} e^{-n(t_{1,1} + \dots + t_{1,k})}$$

Proposition 11 is equivalent to Theorem 2.

**Proposition 12** Suppose that

$$\mathbf{d}_1 = (d_1, \dots, d_N), \quad \mathbf{d}_2 = \mathbf{d}_3 = (0, \dots, 0).$$

where  $d_1 = 0$ . Then

$$\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N) = \begin{cases} -C_g n^{2g-3} & d_j = n > 0 \text{ for } k_1 \leq j \leq k_2 \text{ and} \\ & d_j = 0 \text{ otherwise, where } 2 \leq k_1 \leq k_2 \leq N \\ 0 & \text{otherwise} \end{cases}$$

which is equivalent to

$$(22) \quad F_N^2(\lambda; \mathbf{t}_1) = \sum_{n>0} \frac{1}{n[n]^2} \sum_{2 \leq k_1 \leq k_2 \leq N} e^{-n(t_{1,k_1} + \dots + t_{1,k_2})}$$

Proposition 12 corresponds to a chain of  $(0, -2)$  rational curves.

**Proposition 13** Suppose that

$$\mathbf{d}_1 = (n, 0, \dots, 0), \quad d_{2,1} > 0, \quad \mathbf{d}_3 = (0, \dots, 0)$$

where  $n > 0$ . Then

$$\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N) = \begin{cases} -C_g n^{2g-3} & d_{2,1} = d_{2,2} = \dots = d_{2,k} = n \text{ and} \\ & d_{k+1} = d_{k+2} = \dots = d_N = 0 \text{ for some } 1 \leq k \leq N \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 14** Suppose that  $d_{i,1} > 0$ . Then  $\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N) = 0$  unless

$$d_{i,1} \geq d_{i,2} \geq \dots \geq d_{i,N}.$$

**Proposition 15** Suppose that

$$d_{i,j} = \begin{cases} d_i > 0 & j \leq k_i \\ 0 & j > k_i \end{cases}$$

where  $i = 1, 2, 3$  and  $1 \leq k_i \leq N$ . Then

$$(a) \quad \tilde{N}_{\mathbf{d}_1, \mathbf{d}_2, \mathbf{0}}^g(\widehat{Y}^N) = \begin{cases} -C_g n^{2g-3} & d_1 = d_2 = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(b) \quad \tilde{N}_{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3}^g(\widehat{Y}^N) = \begin{cases} C_g n^{2g-3} & d_1 = d_2 = d_3 = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 14 and Proposition 15 are consistent with Theorem 1.

Let  $N = 1$  in Proposition 11 and Proposition 15, we get

**Corollary 16**

$$\tilde{Z}_1(\lambda; \mathbf{t}) = \exp\left(\sum_{n=1}^{\infty} \frac{Q_n(\mathbf{t})}{-n[n]^2}\right)$$

where  $\mathbf{t} = (t_1, t_2, t_3)$  and

$$Q_n(\mathbf{t}) = e^{-nt_1} + e^{-nt_2} + e^{-nt_3} - e^{-n(t_1+t_2)} - e^{-n(t_2+t_3)} - e^{-n(t_3+t_1)} + e^{-n(t_1+t_2+t_3)}.$$

Finally, we will derive the following expression of  $\tilde{Z}_N(\lambda; \mathbf{t})$ , where the notation is the same as that in Section 1:

**Proposition 17**

$$\tilde{Z}_N(\lambda; \mathbf{t}) = \exp\left(\sum_{i=1}^3 F_N^2(\lambda; \mathbf{t}_i)\right) \sum_{\tilde{\mu}} \tilde{\mathcal{W}}_{\tilde{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i|t_{i,1}} s_{(\mu^i)'}(u^i(q, \mathbf{t}_i)).$$

where

$$F_N^2(\lambda; \mathbf{t}_i) = \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \sum_{2 \leq k_1 \leq k_2 \leq N} e^{-n(t_{i,k_1} + \dots + t_{i,k_2})}$$

In particular, when  $N = 1$  we have

$$F_1^2(\lambda; \mathbf{t}_i) = 0, \quad s_{(\mu^i)'}(u^i(q, \mathbf{t}_i)) = \mathcal{W}_{(\mu^i)'}(q),$$

which gives the following corollary.



**Corollary 18**

$$\tilde{Z}_1(\lambda; \mathbf{t}) = \sum_{\vec{\mu}} \tilde{\mathcal{W}}_{\vec{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i|t_i} \mathcal{W}_{(\mu^i)t}(q).$$

where  $\mathbf{t} = (t_1, t_2, t_3)$ .

Equation (6) in Section 1 follows from Corollary 16 and Corollary 18.

**5.3 Three-partition Hodge integrals**

Three-partition Hodge integrals arise when we calculate

$$\tilde{N}_{\mathbf{d}}^g(\hat{Y}^N) = \int_{[\overline{\mathcal{M}}_{g, \mathbf{d}}(\hat{Y}^N, \mathbf{d})^{\text{vir}}]} \frac{1}{e_{\mathbb{S}}(N^{\text{vir}})}$$

by virtual localization (see Li–Liu–Liu–Zhou [22, Section 7] for such calculations). We recall their definition in this subsection.

Let  $w_1, w_2, w_3$  be formal variables, where  $w_3 = -w_1 - w_2$ . Let  $w_4 = w_1$ . Write  $\mathbf{w} = (w_1, w_2, w_3)$ . For  $\vec{\mu} = (\mu^1, \mu^2, \mu^3) \neq (\emptyset, \emptyset, \emptyset) = \vec{\emptyset}$ , define

$$d_{\vec{\mu}}^1 = 0, \quad d_{\vec{\mu}}^2 = \ell(\mu^1), \quad d_{\vec{\mu}}^3 = \ell(\mu^1) + \ell(\mu^2), \quad \ell(\vec{\mu}) = \sum_{i=1}^3 \ell(\mu^i).$$

The *three-partition Hodge integrals* are defined by

$$(23) \quad G_{g, \vec{\mu}}(\mathbf{w}) = \frac{(-\sqrt{-1})^{\ell(\vec{\mu})}}{|\text{Aut}(\vec{\mu})|} \prod_{i=1}^3 \prod_{j=1}^{\ell(\mu^i)} \frac{\prod_{a=1}^{\mu_j^i - 1} (\mu_j^i w_{i+1} + a w_i)}{(\mu_j^i - 1)! w_i^{\mu_j^i - 1}} \cdot \int_{\overline{\mathcal{M}}_{g, \ell(\vec{\mu})}} \prod_{i=1}^3 \frac{\Lambda_g^\vee(w_i) w_i^{\ell(\vec{\mu}) - 1}}{\prod_{j=1}^{\ell(\mu^i)} (w_i (w_i - \mu_j^i \psi_{d_{\vec{\mu}}^i + j}))}$$

where

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \dots + (-1)^g \lambda_g.$$

Note that  $G_{g, \vec{\mu}}(w_1, w_2, w_3)$  has a pole along  $w_i = 0$  if  $\mu^i \neq \emptyset$ . The following cyclic symmetry is clear from the definition:

$$(24) \quad \begin{aligned} G_{g, \mu^1, \mu^2, \mu^3}(w_1, w_2, w_3) &= G_{g, \mu^2, \mu^3, \mu^1}(w_2, w_3, w_1) \\ &= G_{g, \mu^3, \mu^1, \mu^2}(w_3, w_1, w_2) \end{aligned}$$

Note that

$$\sqrt{-1}^{\ell(\bar{\mu})} G_{g, \bar{\mu}}(\mathbf{w}) \in \mathbb{Q}(w_1, w_2, w_3)$$

is homogeneous of degree 0, so

$$G_{g, \bar{\mu}}(w_1, w_2, -w_1 - w_2) = G_{g, \bar{\mu}}\left(1, \frac{w_2}{w_1}, -1 - \frac{w_2}{w_1}\right).$$

Introduce variables  $\lambda$ ,  $p^i = (p_1^i, p_2^i, \dots)$ ,  $i = 1, 2, 3$ . Given a partition  $\mu$ , define

$$p_\mu^i = p_1^i \cdots p_{\ell(\mu)}^i$$

for  $i = 1, 2, 3$ . In particular,  $p_{\emptyset}^i = 1$ . Write

$$\mathbf{p} = (p^1, p^2, p^3), \quad \mathbf{p}_{\bar{\mu}} = p_{\mu^1}^1 p_{\mu^2}^2 p_{\mu^3}^3.$$

Define generating functions

$$\begin{aligned} G_{\bar{\mu}}(\lambda; \mathbf{w}) &= \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\bar{\mu})} G_{g, \bar{\mu}}(\mathbf{w}), \\ G(\lambda; \mathbf{p}; \mathbf{w}) &= \sum_{\bar{\mu} \neq \bar{\emptyset}} G_{\bar{\mu}}(\lambda; \mathbf{w}) \mathbf{p}_{\bar{\mu}}, \\ G^\bullet(\lambda; \mathbf{p}; \mathbf{w}) &= \exp(G(\lambda; \mathbf{p}; \mathbf{w})) = 1 + \sum_{\bar{\mu} \neq \bar{\emptyset}} G_{\bar{\mu}}^\bullet(\lambda; \mathbf{w}) \mathbf{p}_{\bar{\mu}}. \end{aligned}$$

In particular,

$$G_{g, \mu, \emptyset, \emptyset}(1, 0, -1) = \begin{cases} -\sqrt{-1} d^{2g-2} b_g, & \mu = (d), \\ 0, & \ell(\mu) > 1. \end{cases}$$

where

$$b_g = \begin{cases} 1, & g = 0, \\ \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2} \lambda_g & g > 0. \end{cases}$$

It was proved by Faber and Pandharipande [10] that

$$(25) \quad \sum_{g=0}^{\infty} b_g t^{2g} = \frac{t/2}{\sin(t/2)}.$$

So

$$(26) \quad G_{(n), \emptyset, \emptyset}(\lambda; 1, 0, -1) = \frac{-\sqrt{-1}}{2n \sin(\lambda n/2)} = \frac{1}{n[n]}.$$

Similarly, we have

$$(27) \quad G_{(n),\emptyset,\emptyset}(\lambda; 1, -1, 0) = \frac{(-1)^{n-1}}{n[n]}.$$

The following formula of three-partition Hodge integrals was derived in Li–Liu–Liu–Zhou [22]:

$$(28) \quad G_{\vec{\mu}}^{\bullet}(\lambda; \mathbf{w}) = \sum_{|v^i|=|\mu^i|} \prod_{i=1}^3 \left( q^{\frac{1}{2}\kappa_v i} \frac{w_i+1}{w_i} \frac{\chi_{v^i}(\mu^i)}{z_{\mu^i}} \right) \tilde{\mathcal{W}}_{\vec{v}}(q),$$

In particular,

$$(29) \quad G_{\mu,\emptyset,\emptyset}^{\bullet}(\lambda; 1, \tau, -\tau-1) = \sum_{|v|=|\mu|} \frac{\chi_v(\mu)}{z_{\mu}} q^{\frac{1}{2}\kappa_v \tau} \mathcal{W}_v(q).$$

$$(30) \quad G_{\mu^1,\mu^2,\emptyset}^{\bullet}(\lambda; 1, \tau, -\tau-1) = \sum_{|v^i|=|\mu^i|} \frac{\chi_{v^1}(\mu^1)}{z_{\mu^1}} \frac{\chi_{v^2}(\mu^2)}{z_{\mu^2}} q^{\frac{1}{2}(\kappa_{v^1} \tau - \kappa_{v^2} \tau^{-1})} \mathcal{W}_{v^1(v^2)^t}(q).$$

Equation (29) is equivalent to the formula of one-partition Hodge integrals conjectured in Mariño–Vafa [28], which was proved in Liu–Liu–Zhou [25] and Okounkov–Pandharipande [29]. Equation (30) is equivalent to the formula of two-partition Hodge integrals proved in Liu–Liu–Zhou [24].

#### 5.4 Relative formal GW invariants of the topological vertex

Symplectic relative Gromov–Witten theory was developed by Li and Ruan [19], and Ionel and Parker [15; 16]. The mathematical theory of the topological vertex in [22] is based on Jun Li’s algebraic relative Gromov–Witten theory [20; 21].

Given a triple of partitions  $\vec{\mu} = (\mu^1, \mu^2, \mu^3) \neq \vec{\emptyset}$  and a triple of integers  $\mathbf{n} = (n_1, n_2, n_3)$ , let

$$F_{\chi, \vec{\mu}}^{\bullet}(\mathbf{n}) \in \mathbb{Q}$$

be the disconnected relative formal GW invariants of the topological vertex defined in [22]. Introduce variables  $\lambda, p_j^i$  as in Section 5.3, and define generating functions

$$\begin{aligned}
 F_{\bar{\mu}}^{\bullet}(\lambda; \mathbf{n}) &= \sum_{\chi} \lambda^{-\chi + \ell(\bar{\mu})} F_{\chi, \bar{\mu}}^{\bullet}(\mathbf{n}) \\
 F^{\bullet}(\lambda; \mathbf{p}; \mathbf{n}) &= 1 + \sum_{\bar{\mu} \neq \emptyset} F_{\bar{\mu}}^{\bullet}(\lambda; \mathbf{n}) \mathbf{p}_{\bar{\mu}} \\
 F(\lambda; \mathbf{p}; \mathbf{n}) &= \log(F^{\bullet}(\lambda; \mathbf{p}; \mathbf{n})) = \sum_{\bar{\mu} \neq \emptyset} F_{\bar{\mu}}(\lambda; \mathbf{n}) \mathbf{p}_{\bar{\mu}} \\
 F_{\bar{\mu}}(\lambda; \mathbf{n}) &= \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\bar{\mu})} F_{g, \bar{\mu}}(\mathbf{n}).
 \end{aligned}$$

By virtual localization,  $F_{g, \bar{\mu}}(\mathbf{n})$  can be expressed in terms of three-partition Hodge integrals and double Hurwitz numbers. We have

$$(31) \quad F_{\bar{\mu}}^{\bullet}(\lambda; \mathbf{n}) = (-1)^{\sum_{i=1}^3 (n_i-1)|\mu^i|} (-\sqrt{-1})^{\ell(\bar{\mu})} \sum_{|v^i|=|\mu^i|} G_{\bar{v}}^{\bullet}(\lambda; \mathbf{w}) \prod_{i=1}^3 z_{v^i} \Phi_{v^i, \mu^i}^{\bullet} \left( \sqrt{-1} \left( n_i - \frac{w_{i+1}}{w_i} \right) \right)$$

where

$$(32) \quad \Phi_{v, \mu}^{\bullet}(\lambda) = \sum_{\chi} H_{\chi, v, \mu}^{\bullet} \frac{\lambda^{-\chi + \ell(v) + \ell(\mu)}}{(\chi + \ell(v) + \ell(\mu))!} = \sum_{\eta} e^{\kappa_{\eta} \lambda / 2} \frac{\chi_{\eta}(v)}{z_v} \frac{\chi_{\eta}(\mu)}{z_{\mu}}.$$

is a generation function of disconnected double Hurwitz numbers  $H_{\chi, \mu, v}^{\bullet}$ . Equations (28), (31), and (32) imply

$$(33) \quad F_{\bar{\mu}}^{\bullet}(\lambda; \mathbf{n}) = \sum_{|v^i|=|\mu^i|} \prod_{i=1}^3 \left( q^{\frac{1}{2} \kappa_{v^i} n_i} \frac{\chi_{v^i}(\mu^i)}{z_{\mu^i}} \right) \tilde{\mathcal{W}}_{\bar{v}}(q),$$

Note that the  $\mathbf{w}$  dependence on the right hand side of (31) cancels and the right hand side of (33) is independent of  $\mathbf{w}$ . Since  $\Phi_{v\mu}^{\bullet}(0) = \delta_{v\mu} / z_{\mu}$ , we have

$$F_{\bar{\mu}}^{\bullet} \left( \frac{w_2}{w_1}, \frac{w_3}{w_2}, \frac{w_1}{w_3} \right) = (-1)^{\sum_{i=1}^3 (n_i-1)|\mu^i|} (-\sqrt{-1})^{\ell(\bar{\mu})} \sum_{|v^i|=|\mu^i|} G_{\bar{v}}^{\bullet}(\lambda; w_1, w_2, w_3).$$

Also  $F_{\mu}^{\bullet}(\lambda; \mathbf{n})$  is independent of  $n_i$  if  $\mu^i$  is empty.

$$(34) \quad F_{g,\mu,\emptyset,\emptyset}(0, n_2, n_3) = (-1)^{|\mu|} (-\sqrt{-1})^{\ell(\mu)} G_{g,\mu,\emptyset,\emptyset}(1, 0, -1)$$

$$(35) \quad F_{g,\mu,v,\emptyset}(-1, 0, n_3) = (-1)^{|v|} (-\sqrt{-1})^{\ell(\mu)+\ell(v)} G_{g,\mu,v,\emptyset}(1, -1, 0)$$

From (26), (27), (34), (35) we conclude that if  $\mu \neq \emptyset$  then

$$(36) \quad F_{\mu,\emptyset,\emptyset}(\lambda; 0, n_2, n_3) = \begin{cases} \frac{(-1)^{n-1} \sqrt{-1}}{n[n]} & \mu = (n) \\ 0 & \ell(\mu) > 1. \end{cases}$$

$$(37) \quad F_{\mu,\emptyset,\emptyset}(\lambda; -1, n_2, n_3) = \begin{cases} \frac{(-1)^n \sqrt{-1}}{n[n]} & \mu = (n) \\ 0 & \ell(\mu) > 1. \end{cases}$$

If  $\mu \neq \emptyset, v \neq \emptyset$  then (see Liu–Liu–Zhou [24, p7] for details)

$$(38) \quad F_{\mu,v,\emptyset}(\lambda; -1, 0, n_3) = \begin{cases} \frac{1}{n} & \mu = v = (n) \\ 0 & \text{otherwise.} \end{cases}$$

We also have

$$F_{(1),(1),\emptyset}(\lambda; 0, 0, 0) = -1, \quad F_{(1),(1),(1)}(\lambda; 0, 0, 0) = -\sqrt{-1}[1].$$

### 5.5 Computations

The  $\mathbb{S}$ -action on  $\widehat{Y}^N$  can be read off from Figure 13 as explained in the first two paragraphs of Section 5.1.

We now degenerate each  $\mathbb{P}^1$  into two  $\mathbb{P}^1$ 's intersecting at a node. The total space of the normal bundle  $\mathcal{O}(n) \oplus \mathcal{O}(-n-2)$  degenerates to  $\mathcal{O}(a) \oplus \mathcal{O}(-a-1)$  and  $\mathcal{O}(b) \oplus \mathcal{O}(-b-1)$  with  $a+b=n$ . For each node we introduce a pair of framing vectors to encode the  $\mathbb{S}$ -action (see Figure 15). We refer to [22, Section 4] for details.

The framing here corresponds to the framing of Lagrangian submanifolds in the article by Aganagic, Klemm, Mariño and Vafa [2] and the framing of knots and links in Chern–Simons theory. In Figure 14, all the  $\mathbb{P}^1$ 's have normal bundles  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  or  $\mathcal{O} \oplus \mathcal{O}(-2)$ ; in Figure 15, all the  $\mathbb{P}^1$ 's have normal bundles  $\mathcal{O} \oplus \mathcal{O}(-1)$ .

Using the connected version of the gluing formula [22, Theorem 7.5], we have

$$F_N^4(\lambda; \mathbf{t}) = \sum_{|\mu^i| > 0} F_{\tilde{\mu}}(\lambda; 0, 0, 0) \prod_{i=1}^3 \prod_{j=1}^{\ell(\mu^i)} \sum_{k=1}^N e^{-\mu_j^i(t_{i,1} + \dots + t_{i,k})} \cdot \left( \mu_j^i F_{(\mu_j^i), (\mu_j^i), \emptyset}(-1, 0, 0) \right)^{k-1} \mu_j^i F_{(\mu_j^i), \emptyset, \emptyset}(\lambda; 0, 0, 0)$$

$$= \sum_{|\mu^i|>0} F_{\vec{\mu}}(\lambda; 0, 0, 0) \prod_{i=1}^3 \prod_{j=1}^{\ell(\mu^i)} \left( \mu_j^i \frac{(-1)^{\mu_j^i-1} \sqrt{-1}}{\mu_j^i [\mu_j^i]} \sum_{k=1}^N e^{-\mu_j^i (t_{i,1} + \dots + t_{i,k})} \right)$$

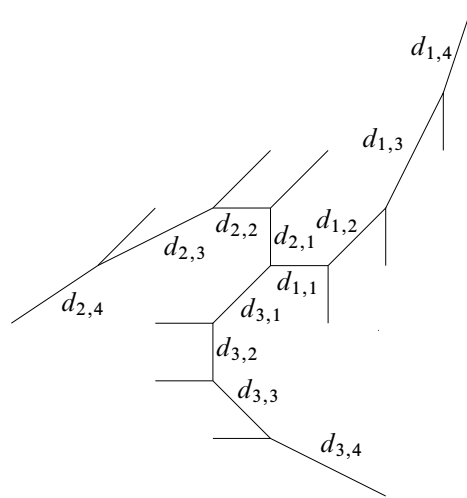


Figure 14: Graph of  $\hat{Y}_N$

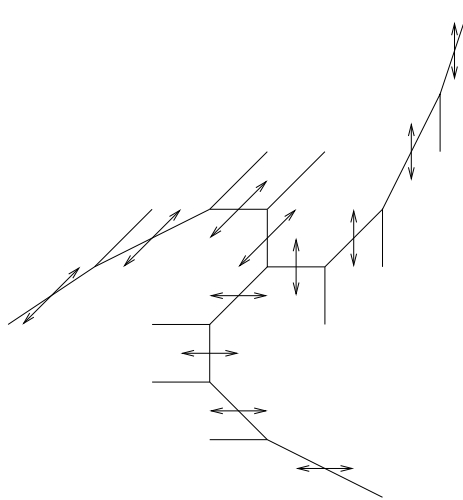


Figure 15: Degeneration of  $\hat{Y}_N$

So

$$(39) \quad F_N^4(\lambda; \mathbf{t}) = \sum_{|\mu^i| > 0} F_{\vec{\mu}}(\lambda; 0, 0, 0) \prod_{i=1}^3 (-1)^{|\mu^i|} (-\sqrt{-1})^{\ell(\mu^i)} \prod_{j=1}^{\ell(\mu^i)} \left( \frac{1}{[\mu_j^i]} \sum_{k=1}^N e^{-\mu_j^i (t_{i,1} + \dots + t_{i,k})} \right)$$

In particular, using  $F_{(1),(1),(1)}(\lambda; 0, 0, 0) = -\sqrt{-1}[1]$  and (39), we can recover [Theorem 1](#). Equation (39) is equivalent to

$$(40) \quad F_N^4(\lambda; \mathbf{t}) = \sum_{|\mu^i| > 0} \tilde{F}_{\vec{\mu}}(\lambda; 0, 0, 0) \prod_{i=1}^3 (-1)^{\ell(\mu^i)} e^{-|\mu^i| t_{i,1}} u_{\mu^i}^i(q; \mathbf{t}_i)$$

where  $u_{\mu}^i(q; \mathbf{t}_i)$ 's are defined as in [Section 1.3](#), and

$$\tilde{F}_{\vec{\mu}}(\lambda; 0, 0, 0) = (-1)^{\sum_{i=1}^3 |\mu^i|} \sqrt{-1}^{\ell(\vec{\mu})} F_{\vec{\mu}}(\lambda; 0, 0, 0).$$

Similarly,

$$(41) \quad F_N^3(\lambda; \mathbf{t}_1, \mathbf{t}_2) = \sum_{|\mu^1| > 0, |\mu^2| > 0} \tilde{F}_{\mu^1, \mu^2, \emptyset}(\lambda; 0, 0, 0) \prod_{i=1}^2 (-1)^{\ell(\mu^i)} e^{-|\mu^i| t_{i,1}} u_{\mu^i}^i(q; \mathbf{t}_i).$$

$$F_N^1(\lambda; \mathbf{t}_1) = \sum_{\mu \neq \emptyset} \tilde{F}_{\mu, \emptyset, \emptyset}(\lambda; 0, 0, 0) (-1)^{\ell(\mu)} e^{-|\mu| t_{1,1}} u_{\mu}^1(q; \mathbf{t}_1)$$

$$= \sum_{\mu \neq \emptyset} F_{\mu, \emptyset, \emptyset}(\lambda; 0, 0, 0) (-1)^{|\mu|} (-\sqrt{-1})^{\ell(\mu)} e^{-|\mu| t_{1,1}} u_{\mu}^1(q; \mathbf{t}_1)$$

$$= \sum_{n > 0} \frac{(-1)^{n-1} \sqrt{-1}}{n[n]} (-1)^n (-\sqrt{-1}) e^{-n t_{1,1}} u_n^1(q; \mathbf{t}_1).$$

So

$$(42) \quad F_N^1(\lambda; \mathbf{t}_1) = \sum_{n > 0} \frac{-1}{n[n]^2} \sum_{k=1}^N e^{-n(t_{1,1} + \dots + t_{1,k})}.$$

This proves [Proposition 11](#).

$$\begin{aligned}
 F_N^2(\lambda; \mathbf{t}_1) &= \sum_{n>0} F_{(n),\emptyset,\emptyset}(\lambda; -1, 0, 0) \frac{(-1)^{n-1} \sqrt{-1}}{[n]} \sum_{2 \leq k_1 \leq k_2 \leq N} e^{-n(t_{1,k_1} + \dots + t_{1,k_2})} \\
 &= \sum_{n>0} \frac{(-1)^n \sqrt{-1}}{n[n]} \cdot \frac{(-1)^{n-1} \sqrt{-1}}{[n]} \sum_{2 \leq k_1 \leq k_2 \leq N} e^{-n(t_{1,k_1} + \dots + t_{1,k_2})}
 \end{aligned}$$

So

$$(43) \quad F_N^2(\lambda; \mathbf{t}_1) = \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \sum_{2 \leq k_1 \leq k_2 \leq N} e^{-n(t_{1,k_1} + \dots + t_{1,k_2})}$$

This proves Proposition 12.

From (40), (41), and (42), it is clear that if  $d_{i,1} > 0$ , then  $\tilde{N}_d^g(\hat{Y}^N) = 0$  unless

$$d_{i,1} \geq d_{i,2} \geq \dots \geq d_{i,k_i} > 0.$$

This proves Proposition 14.

Let  $F_N^5(\lambda; t_{1,1}, \mathbf{t}_2)$  be the contribution to  $F_N^3(\lambda; \mathbf{t}_1, \mathbf{t}_2)$  from the case in Proposition 13. To compute  $F_N^5(\lambda; t_{1,1}, \mathbf{t}_2)$ , we consider the degeneration in Figure 16.

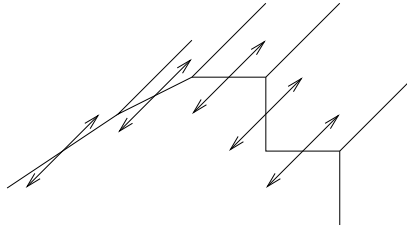


Figure 16: Another degeneration

We have

$$\begin{aligned}
 F_N^5(\lambda; t_{1,1}, \mathbf{t}_2) &= \sum_{n>0} F_{(n),\emptyset,\emptyset}(-1, 0, 0) n F_{(n),(n),\emptyset}(-1, 0, 0) \\
 &\quad \cdot \left( \sum_{k=1}^N (n F_{(n),(n),\emptyset}(-1, 0, 0))^{k-1} n F_{(n),\emptyset,\emptyset}(0, 0, 0) e^{-nt_{1,1} - n(t_{2,1} + \dots + t_{2,k})} \right) \\
 &= \sum_{n>0} \frac{(-1)^n \sqrt{-1}}{n[n]} \cdot \frac{(-1)^{n-1} \sqrt{-1}}{[n]} \left( \sum_{k=1}^N e^{-nt_{1,1} - n(t_{2,1} + \dots + t_{2,k})} \right)
 \end{aligned}$$



So

$$(44) \quad F_N^5(\lambda; t_{1,1}, \mathbf{t}_2) = \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \sum_{k=1}^N e^{-nt_{1,1} - n(t_{2,1} + \dots + t_{2,k})}$$

Proposition 13 follows from (44).

We now prove Proposition 15. Suppose that  $d_{i,j} = d_i > 0$  for  $j \leq k_i$  and  $d_{i,j} = 0$  for  $j > k_i$ , where  $i = 1, 2, 3$  and  $1 \leq k_i \leq N$ . From (41) and (40) it is easy to see that

$$\begin{aligned} \tilde{N}_{\mathbf{d}_1, \mathbf{d}_2, 0}^g(\hat{Y}^N) &= \tilde{N}_{d_1, d_2, 0}^g(\hat{Y}^1), \\ \tilde{N}_{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3}^g(\hat{Y}^N) &= \tilde{N}_{d_1, d_2, d_3}^g(\hat{Y}^1). \end{aligned}$$

Recall that

$$\tilde{N}_{d_1, d_2, d_3}^g(\hat{Y}^1) = N_{d_1, d_2, d_3}^g(Y^1)$$

where  $N_{d_1, d_2, d_3}^g(Y^1)$  are given by Fact 3. Proposition 15 follows from Proposition 13 and Fact 3.

Finally, we prove Proposition 17. Let

$$F'_N(\lambda; \mathbf{t}) = F_N(\lambda; \mathbf{t}) - \sum_{i=1}^3 F_N^2(\lambda; \mathbf{t}_i).$$

Then

$$F'_N(\lambda; \mathbf{t}) = \sum_{\tilde{\mu} \neq \emptyset} \tilde{F}_{\tilde{\mu}}(\lambda; 0, 0, 0) \prod_{i=1}^3 (-1)^{\ell(\mu^i)} e^{-|\mu^i| t_{i,1}} u_{\mu^i}^i(q; \mathbf{t}^i).$$

It was proved in [22] that

$$(45) \quad \tilde{F}_{\tilde{\mu}}^{\bullet}(\lambda; 0, 0, 0) = \sum_{|v^i|=|\mu^i|} \tilde{\mathcal{W}}_{\tilde{v}}(q) \prod_{i=1}^3 \frac{\chi_{\mu^i}(v^i)}{z_{\mu^i}}$$

So

$$\begin{aligned} \exp(F'_N(\lambda; \mathbf{t})) &= \sum_{\tilde{v}} \tilde{\mathcal{W}}_{(\tilde{v}^1)^t, (\tilde{v}^2)^t, (\tilde{v}^3)^t}(q) \prod_{i=1}^3 (-1)^{|\tilde{v}^i|} e^{-|\tilde{v}^i| t_{i,1}} s_{\tilde{v}^i}(u^i(q, \mathbf{t}_i)) \\ &= \sum_{\tilde{\mu}} \tilde{\mathcal{W}}_{\tilde{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_{i,1}} s_{(\mu^i)^t}(u^i(q, \mathbf{t}_i)) \end{aligned}$$

where  $s_\mu(u^i(q, \mathbf{t}_i))$ 's are defined as in Section 1.3. We conclude that

$$\exp(F_N(\lambda; \mathbf{t})) = \exp\left(\sum_{i=1}^3 F_N^2(\lambda; \mathbf{t}_i)\right) \sum_{\tilde{\mu}} \tilde{\mathcal{W}}_{\tilde{\mu}}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| t_{i,1} s_{(\mu^i)'}(u^i(q, \mathbf{t}_i))}.$$

This completes the proof of Proposition 17.

## 6 Physical theory of the topological vertex

In this section, we compute the local Gromov–Witten invariants considered in this paper by using the physical theory of the topological vertex. Typical computations in this theory involve formal sums over Young tableaux, and in some cases, like the one considered here, it is more convenient to use the operator formalism on Fock spaces. After a short overview of this formalism, we will use it to compute the partition functions for local Gromov–Witten invariants.

### 6.1 Operator formalism

We introduce:

$$|p_\mu\rangle = \prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i} |0\rangle, \quad |s_\mu\rangle = \sum_{|v|=|\mu|} \frac{\chi_\mu(v)}{z_v} |p_v\rangle, \quad |s_{\lambda/\mu}\rangle = \sum_v c_{\mu v}^\lambda |s_v\rangle$$

where  $\alpha_n$  satisfy the commutation relations

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0},$$

and for  $n > 0$ ,  $\alpha_n |0\rangle = 0$ . The dual vector space is obtained by acting with the operators  $\alpha_n$  on the state  $\langle 0|$ , and the pairing is defined by  $\langle 0|0\rangle = 1$ . One then finds,

$$\langle p_\mu | p_\nu \rangle = z_\mu \delta_{\mu,\nu}, \quad \langle s_\mu | s_\nu \rangle = \delta_{\mu,\nu}.$$

The coherent state  $|t\rangle$  is defined as

$$|t\rangle = \exp\left(\sum_{n=1}^{\infty} \frac{t_n}{n} \alpha_{-n}\right) |0\rangle = \sum_{\mu} \frac{t_\mu}{z_\mu} |p_\mu\rangle$$

where

$$t_\mu = t_{\mu_1} \cdots t_{\mu_{\ell(\mu)}}$$

and one has

$$\langle s|t \rangle = \exp\left(\sum_{n=1}^{\infty} \frac{s_n t_n}{n}\right).$$

The elements  $|s_\mu\rangle$ , where  $\mu$  is a partition, span a vector space  $\mathcal{H}$  that can be identified with the ring of symmetric functions  $\Lambda$  in an infinite number of variables, and therefore it inherits a ring structure from  $\Lambda$ . This identification can be made by considering the map

$$|s_\mu\rangle \rightarrow \langle t|s_\mu\rangle \equiv s_\mu(t),$$

where  $s_\mu(t)$  gives the Schur function after identifying  $t_n = p_n = x_1^n + x_2^n + \dots$ .

Given a coherent state  $|t\rangle$ , it is useful to define the coherent states

$$|t^\omega\rangle, \quad |t^\xi\rangle, \quad |t^\xi\rangle$$

where

$$t_n^\omega = (-1)^{n+1} t_n, \quad t_n^\xi = -t_n, \quad t_n^\xi = (-1)^n t_n.$$

Using  $\chi_{\mu^t}(v) = (-1)^{|\nu|+\ell(\nu)} \chi_\mu(v)$  it is easy to show that

$$(46) \quad s_\mu(t^\omega) = s_{\mu^t}(t), \quad s_\mu(t^\xi) = (-1)^{|\mu|} s_\mu(t).$$

The ring of symmetric polynomials is endowed with a coproduct structure (see for example Macdonald [26, Ex. 25 of I.5])

$$\Delta: \Lambda \rightarrow \Lambda \otimes \Lambda$$

which is a ring homomorphism, and is defined by

$$\Delta(s_\lambda) = \sum_{\mu} s_{\lambda/\mu} \otimes s_\mu.$$

The  $n$ th power sums  $p_n$  are primitive elements of  $\Lambda$  under this coproduct, and one has

$$\Delta p_n = p_n \otimes 1 + 1 \otimes p_n.$$

We then have a inherited coproduct  $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ , and it is easy to see that it acts as follows on coherent states:

$$\Delta(|t\rangle) = |t\rangle \otimes |t\rangle.$$

This gives the following identity, which will be useful in proving [Proposition 19](#):

$$(47) \quad \sum_{\mu, \nu} s_\mu(t) |s_{\mu/\nu}\rangle \langle s_\nu| = |t\rangle \langle t|.$$

We need now explicit expressions for  $\mathcal{W}_\mu(q)$  and  $\mathcal{W}_{\mu\nu}(q)$  in the operator formalism. Using (9), one immediately finds

$$(48) \quad \mathcal{W}_\mu(q) = s_\mu(\beta), \quad \beta_n = \frac{1}{[n]} = \frac{1}{q^{n/2} - q^{-n/2}},$$

therefore

$$(49) \quad \sum_{\mu} \mathcal{W}_\mu(q) |s_\mu\rangle = |\beta\rangle,$$

where  $|\beta\rangle$  is a coherent state with  $\beta_n$  given in (48). One can then write

$$\mathcal{W}_\mu(q) = \langle s_\mu | \beta \rangle.$$

We introduce the operator  $q^{\pm\kappa/2}$  defined by

$$q^{\pm\kappa/2} |s_\mu\rangle = q^{\pm\kappa\mu/2} |s_\mu\rangle.$$

We also define an operator  $W$  as

$$\mathcal{W}_{\mu\nu}(q) = \langle s_\mu | W | s_\nu \rangle.$$

In the proof of Proposition 19, we will need an explicit expression for

$$\langle t | q^{-\kappa/2} W q^{-\kappa/2} | \bar{t} \rangle,$$

where  $|t\rangle, |\bar{t}\rangle$  are coherent states. Using (12) one finds

$$\begin{aligned} \langle t | q^{-\kappa/2} W q^{-\kappa/2} | \bar{t} \rangle &= \sum_{\mu, \nu} q^{-\kappa\mu/2 - \kappa\nu/2} \mathcal{W}_{\mu, \nu}(q) s_\mu(t) s_\nu(\bar{t}) \\ &= \sum_{\mu, \nu, \sigma} s_\mu(t) \mathcal{W}_{\mu^t/\sigma}(q) \mathcal{W}_{\nu^t/\sigma}(q) s_\nu(\bar{t}) \\ &= \sum_{\mu, \nu, \sigma, \tau} s_\mu(t) \langle \beta | s_{\mu/\sigma} \rangle \langle s_\sigma | s_\tau \rangle \langle s_\nu/\tau | \beta \rangle s_\nu(\bar{t}) \\ &= \langle \beta | t^\omega \rangle \langle t^\omega | \bar{t}^\omega \rangle \langle \bar{t}^\omega | \beta \rangle, \end{aligned}$$

where in the last step we have used (47) twice. The last quantity is expressed solely in terms of products of coherent states, and we finally find

$$(50) \quad \langle t | q^{-\kappa/2} W q^{-\kappa/2} | \bar{t} \rangle = \exp\left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (t_n + \bar{t}_n)}{n[n]} + \frac{t_n \bar{t}_n}{n} \right).$$

This result was previously obtained, in slightly different form, by Aganagic, Dijkgraaf, Klemm, Mariño and Vafa [1], and Zhou [34]. It is also possible to compute

$\langle t|q^{-\kappa/2}Wq^{-\kappa/2}|s_\mu\rangle$  by following the same steps. One finds

$$\langle t|q^{-\kappa/2}Wq^{-\kappa/2}|s_\mu\rangle = \langle \beta|t^\omega\rangle \sum_{\nu} \mathcal{W}_{\mu^t/\nu}(q)s_\nu(t^\omega).$$

The sum over the representation  $\nu$  can be performed explicitly by using the following formula (see Macdonald [26]):

$$\sum_{\nu} s_{\mu/\nu}(x)s_\nu(y) = s_\mu(x, y),$$

where  $x, y$  are variables of the Schur polynomials. We then find

$$(51) \quad \langle t|q^{-\kappa/2}Wq^{-\kappa/2}|s_\mu\rangle = \langle \beta|t^\omega\rangle s_\mu(u_n), \quad u_n = \frac{(-1)^{n+1}}{[n]} + t_n.$$

### 6.2 The closed topological vertex

The physical theory of the topological vertex [2] gives the following expression for  $\tilde{Z}_1(\lambda; \mathbf{t})$ :

$$\begin{aligned} Z_1(\lambda; \mathbf{t}) &= \sum_{\mu^1, \mu^2, \mu^3} \mathcal{W}_{(\mu^3)^t, (\mu^2)^t, (\mu^1)^t}(q) \mathcal{W}_{\mu^1}(q) \mathcal{W}_{\mu^2}(q) \mathcal{W}_{\mu^3}(q) e^{-\sum_{i=1}^3 |\mu^i| t_i} (-1)^{\sum_{i=1}^3 |\mu^i|} = \\ & \sum_{\mu^1, \mu^2, \mu^3} \mathcal{W}_{\bar{\mu}}(q) \mathcal{W}_{(\mu^1)^t}(q) \mathcal{W}_{(\mu^2)^t}(q) \mathcal{W}_{(\mu^3)^t}(q) e^{-|\mu^1| t_1 - |\mu^2| t_2 - |\mu^3| t_3} (-1)^{|\mu^1| + |\mu^2| + |\mu^3|} \end{aligned}$$

where  $\mathbf{t} = (t_1, t_2, t_3)$ , and in the last step we have used

$$\mathcal{W}_{(\mu^3)^t, (\mu^2)^t, (\mu^1)^t}(q) = q^{-\sum_{i=1}^3 \kappa_{\mu^i/2}} \mathcal{W}_{\mu^1, \mu^2, \mu^3}(q), \quad \mathcal{W}_\mu(q) = q^{\kappa_\mu/2} \mathcal{W}_{\mu^t}(q).$$

In this subsection, we will prove (6) in Section 1:

**Proposition 19**

$$Z_1(\lambda; \mathbf{t}) = \exp\left(\sum_{n=1}^{\infty} \frac{Q_n(\mathbf{t})}{-n[n]^2}\right)$$

where

$$Q_n(\mathbf{t}) = e^{-nt_1} + e^{-nt_2} + e^{-nt_3} - e^{-n(t_1+t_2)} - e^{-n(t_2+t_3)} - e^{-n(t_3+t_1)} + e^{-n(t_1+t_2+t_3)}.$$

**Proof**

$$\begin{aligned}
 Z_1(\lambda; \mathbf{t}) = & \sum_{\substack{\mu^1, \mu^2, \mu^3, \\ \rho, \rho^1, \rho^3, \rho'}} c_{\rho\rho^1}^{\mu^1} \mathcal{W}_{(\mu^1)t} (q) \langle s_{(\mu^2)t} | W | s_{\rho^1} \rangle \langle s_{\rho} | \mathcal{W}_{(\mu^3)t} (q) c_{\rho'\rho^3}^{\mu^3} | s_{\rho'} \rangle \langle s_{\rho^3} | W | s_{\mu^2} \rangle \\
 & \cdot e^{-|\mu^1|t_1} (-1)^{|\mu^1|} \cdot e^{-|\mu^2|t_2} (-1)^{|\mu^2|} \cdot e^{-|\mu^3|t_3} (-1)^{|\mu^3|} = \\
 (52) \quad & \sum_{\mu^2} (-1)^{|\mu^2|} e^{-|\mu^2|t_2} \langle s_{(\mu^2)t} | W \left( \sum_{\rho, \rho^1, \mu^1} c_{\rho, \rho^1}^{\mu^1} (-1)^{|\mu^1|} e^{-|\mu^1|t_1} \mathcal{W}_{(\mu^1)t} (q) | s_{\rho^1} \rangle \langle s_{\rho} | \right) \\
 & \left( \sum_{\rho', \rho^3, \mu^3} c_{\rho', \rho^3}^{\mu^3} (-1)^{|\mu^3|} e^{-|\mu^3|t_3} \mathcal{W}_{(\mu^3)t} (q) | s_{\rho'} \rangle \langle s_{\rho^3} | \right) W | s_{\mu^2} \rangle.
 \end{aligned}$$

Here we have used the explicit expression for the topological vertex (8) and the identity (10). We now write

$$\begin{aligned}
 & \sum_{\rho, \rho^1, \mu} c_{\rho, \rho^1}^{\mu} (-1)^{|\mu|} e^{-|\mu|t_1} \mathcal{W}_{\mu t} (q) | s_{\rho^1} \rangle \langle s_{\rho} | \\
 & = \sum_{\rho, \mu} (-1)^{|\mu|} e^{-|\mu|t_1} \mathcal{W}_{\mu t} (q) | s_{\mu/\rho} \rangle \langle s_{\rho} | \\
 & = \sum_{\rho, \mu} s_{\mu} (u^{\xi}) | s_{\mu/\rho} \rangle \langle s_{\rho} | \\
 (53) \quad & = |u^{\xi}\rangle \langle u^{\xi}|,
 \end{aligned}$$

where

$$(54) \quad u_n = e^{-nt_1} \beta_n.$$

and in the last step of (53) use has been made of (47) and of the fact that, under  $t_n \rightarrow at_n$  one has

$$s_{\mu}(at) = a^{|\mu|} s_{\mu}(t).$$

Following the same steps for the second bracket in (52) one finds,

$$Z_1(\lambda; \mathbf{t}) = \sum_{\mu} (-1)^{|\mu|} e^{-|\mu|t_2} \langle s_{\mu t} | W | u^{\xi} \rangle \langle u^{\xi} | v^{\xi} \rangle \langle v^{\xi} | W | s_{\mu} \rangle.$$

with

$$(55) \quad v_n = e^{-nt_3} \beta_n.$$

Notice that

$$|u^\xi\rangle = \sum_{\mu} (-1)^{|\mu|} e^{-|\mu|t_1} \mathcal{W}_{\mu^t}(q) |s_{\mu}\rangle,$$

and using again that  $\mathcal{W}_{\mu^t}(q) = q^{-\kappa_{\mu}/2} \mathcal{W}_{\mu}(q)$  one can write

$$|u^\xi\rangle = q^{-\kappa/2} |u^\xi\rangle.$$

We have similar equations for  $|v^\xi\rangle$ . Since  $\langle u^\xi | v^\xi \rangle$  is a product of coherent states, we only have to evaluate

$$\sum_{\mu} (-1)^{|\mu|} e^{-|\mu|t_2} \langle v^\xi | q^{-\kappa/2} \mathcal{W} q^{-\kappa/2} |s_{\mu}\rangle \langle s_{\mu^t} | q^{-\kappa/2} \mathcal{W} q^{-\kappa/2} |u^\xi\rangle,$$

where we used that  $\kappa_{\mu} = -\kappa_{\mu^t}$ . The last step involves writing

$$(56) \quad \sum_{\mu} (-1)^{|\mu|} e^{-|\mu|t_2} |s_{\mu}\rangle \otimes |s_{\mu^t}\rangle = \exp\left(-\sum_{n=1}^{\infty} \frac{e^{-nt_2}}{n} \alpha_{-n}^{(1)} \otimes \alpha_{-n}^{(2)}\right) |0\rangle_1 \otimes |0\rangle_2$$

which is an element of  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , and we have introduced explicit indices 1, 2 to label the factors in the tensor product. We first take the scalar product of this state with  $\langle v^\xi | q^{-\kappa/2} \mathcal{W} q^{-\kappa/2} \in \mathcal{H}_1^*$  to obtain a state in  $\mathcal{H}_2$ . In order to do that, we can regard (56) as a coherent state with  $t_n = -e^{-nt_2} \alpha_{-n}^{(2)}$ , therefore we can use the formula (50) to obtain the element in  $\mathcal{H}_2$

$$\begin{aligned} & \exp\left(\sum_{n=1}^{\infty} -\frac{v_n}{n[n]} + \frac{(-1)^n e^{-nt_2}}{n[n]} \alpha_{-n}^{(2)} + \frac{(-1)^{n+1} e^{-nt_2} v_n}{n} \alpha_{-n}^{(2)}\right) |0\rangle \\ &= \exp\left(-\sum_{n=1}^{\infty} \frac{e^{-nt_3}}{n[n]^2}\right) |w\rangle \end{aligned}$$

where  $|w\rangle$  is a coherent state in  $\mathcal{H}_2$  given by

$$w_n = (-1)^{n+1} \beta_n e^{-nt_2} (e^{-nt_3} - 1).$$

The remaining step is to compute  $\langle w | q^{-\kappa/2} \mathcal{W} q^{-\kappa/2} |u^\xi\rangle$ , which can be done again with the help of (50). Collecting all terms, one finds

$$(57) \quad Z_1(\lambda; \mathbf{t}) = \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n[n]^2} (e^{-nt_1} + e^{-nt_2} + e^{-nt_3} - e^{-n(t_1+t_2)} - e^{-n(t_2+t_3)} - e^{-n(t_3+t_1)} + e^{-n(t_1+t_2+t_3)})\right\}.$$

This completes the proof. □

### 6.3 A chain of rational curves

Let  $F_N^2(\lambda; \mathbf{t}_1)$  be defined as in Section 5.1, where  $N \geq 2$ . It can be viewed as a generating function of formal Gromov–Witten invariants of a chain of  $(N - 1)$  rational curves with normal bundles  $\mathcal{O} \oplus \mathcal{O}(-2)$ . In this section, we will compute

$$Z_N^2(q; t_{1,2}, \dots, t_{1,N}) = \exp(F_N^2(\lambda; \mathbf{t}_1))$$

using vertex techniques.

We have

$$(58) \quad Z_N^2(q; t_{1,2}, \dots, t_{1,N}) = \sum_{\mu^2, \dots, \mu^N} \prod_{i=2}^{N+1} \mathcal{W}_{(\mu^{i-1})^t, \emptyset, \mu^i}(q) q^{-\kappa_{\mu^i}/2} e^{-|\mu^i|t_{1,i}} = \sum_{\mu^2, \dots, \mu^N} \prod_{i=2}^{N+1} q^{-\kappa_{\mu^{i-1}/2}} \mathcal{W}_{\mu^{i-1}, \mu^i}(q) q^{-\kappa_{\mu^i}/2} e^{-|\mu^i|t_{1,i}},$$

where  $\mu^1 = \mu^{N+1} = \emptyset$ . Using the above techniques, in particular (49) and (46), we can write the above expression as

$$Z_N^2(q; t_{1,2}, \dots, t_{1,N}) = \langle u^2 | \left( \prod_{i=3}^N q^{-\kappa/2} W q^{-\kappa/2} \mathcal{O}_i \right) | u^{N+1} \rangle,$$

where  $|u^2\rangle, |u^{N+1}\rangle$  are coherent states defined by

$$u_n^2 = \frac{(-1)^{n+1} e^{-nt_{1,2}}}{[n]}, \quad u_n^{N+1} = \frac{(-1)^{n+1}}{[n]}$$

and

$$\mathcal{O}_i = \sum_{\mu^i} |s_{\mu^i}\rangle e^{-|\mu^i|t_{1,i}} \langle s_{\mu^i}|, \quad i = 3, \dots, N.$$

We first compute

$$\langle u^2 | q^{-\kappa/2} W q^{-\kappa/2} \mathcal{O}_2,$$

which is an element of  $\mathcal{H}^*$ . We proceed as in the computation following (56) above, to obtain

$$\langle u^2 | q^{-\kappa/2} W q^{-\kappa/2} \mathcal{O}_3 = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n[n]} u_n^2\right) \langle u^3 |,$$



where  $\langle u^3 |$  is a coherent state defined by

$$u_n^3 = e^{-nt_{1,3}} \left( u_n^2 + \frac{(-1)^{n+1}}{[n]} \right).$$

We can now compute  $Z_N^2(q; t_{1,2}, \dots, t_{1,N})$  recursively, defining the coherent state  $|u^i\rangle$  as

$$\langle u^{i-1} | q^{-\kappa/2} W q^{-\kappa/2} \mathcal{O}_i = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n[n]} u_n^{i-1} \right) \langle u^i |,$$

where

$$(59) \quad u_n^i = e^{-nt_{1,i}} \left( u_n^{i-1} + \frac{(-1)^{n+1}}{[n]} \right), \quad i = 3, \dots, N.$$

One then finds, by using (50) repeatedly, that

$$\begin{aligned} Z_N^2(q; t_{1,2}, \dots, t_{1,N}) &= \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n[n]} \sum_{i=2}^{N-1} u_n^i + \sum_{n=1}^{\infty} \frac{1}{n} u_n^N u_n^{N+1} \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n[n]} \sum_{i=2}^N u_n^i \right). \end{aligned}$$

The recursion relation defining  $u_n^i$  is easily solved:

$$u_n^i = \frac{(-1)^{n+1}}{[n]} \left( \sum_{j=2}^i e^{-n(t_{1,j} + \dots + t_{1,i})} \right), \quad i = 2, \dots, N,$$

and putting everything together we finally obtain

$$Z_N^2(q; t_{1,2}, \dots, t_{1,N}) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \left( \sum_{2 \leq i \leq j \leq N} e^{-n(t_{1,i} + \dots + t_{1,j})} \right) \right\},$$

or equivalently,

$$F_N^2(\lambda; \mathbf{t}_1) = \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \left( \sum_{2 \leq i \leq j \leq N} e^{-n(t_{1,i} + \dots + t_{1,j})} \right)$$

which agrees with (22).

Notice that the non-trivial Gopakumar–Vafa invariants for this geometry occur for Kähler classes which are in one-to-one correspondence with the positive roots of the Lie algebra  $A_{N-1}$ .

### 6.4 Minimal trivalent configuration

Let us finally consider the minimal trivalent configuration. We will allow the three chains of  $\mathbb{P}^1$ 's to have different lengths  $N_1, N_2, N_3$ :

$$Y^{N_1, N_2, N_3} = \bigcup_{1 \leq i \leq N_1} A_i \cup \bigcup_{1 \leq j \leq N_2} B_j \cup \bigcup_{1 \leq k \leq N_3} C_k.$$

So we have

$$\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3), \quad \mathbf{d}_i = (d_{i,1}, \dots, d_{i,N_i}),$$

$$\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3), \quad \mathbf{t}_i = (t_{i,1}, \dots, t_{i,N_i}),$$

and we define

$$u_n^i(q; \mathbf{t}) = \frac{1}{[n]} \left( 1 + \sum_{k=2}^{N_i} e^{-n(t_{i,2} + \dots + t_{i,k})} \right).$$

The rules of the topological vertex give the following expression,

$$Z_{N_1, N_2, N_3}(\lambda; \mathbf{t})$$

$$= \sum_{\mu^{i,j}} (-1)^{\sum_{i=1}^3 |\mu^{i,1}|} e^{-\sum_{i=1}^3 |\mu^{i,1}| |t_{i,1}|} q^{\sum_{i=1}^3 \kappa_{\mu^{i,1}/2}} \mathcal{W}_{(\mu^{3,1})^t, (\mu^{2,1})^t, (\mu^{1,1})^t}(q)$$

$$\cdot \prod_{i=1}^3 \prod_{j=2}^{N_i+1} q^{-\kappa_{\mu^{i,j-1}/2}} \mathcal{W}_{\mu^{i,j-1}, \mu^{i,j}}(q) q^{-\kappa_{\mu^{i,j}/2}}(q) e^{-|\mu^{i,j}| |t_{i,j}|}$$

where  $\mu^{i, N_i+1} = \emptyset$ . We will show that this expression can be simplified as follows:

**Proposition 20**

$$(60) \quad Z_{N_1, N_2, N_3}(\lambda; \mathbf{t}) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \sum_{i=1}^3 \sum_{2 \leq k_1 \leq k_2 \leq N_i} e^{-n(t_{i,k_1} + \dots + t_{i,k_2})} \right)$$

$$\cdot \sum_{\bar{\mu}} \mathcal{W}_{\mu^1, \mu^2, \mu^3}(q) \prod_{i=1}^3 (-1)^{|\mu^i|} e^{-|\mu^i| |t_{i,1}|} s_{(\mu^i)^t}(u^i(q, \mathbf{t}_i))$$

Equation (5) in Section 1 corresponds to the case  $N_1 = N_2 = N_3 = N$ .

**Proof** The sum over the partitions  $\mu^{i,j}$ ,  $2 \leq j \leq N_i$ , can be performed by following the same steps that we made before, and making use of (51). After writing  $\mu^{i,1} \rightarrow (\mu^i)^t$ ,

the resulting expression takes the following form:

$$(61) \quad Z_{N_1, N_2, N_3}(\lambda; \mathbf{t}) = \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n[n]} \sum_{i=1}^3 \sum_{j=2}^{N_i} v_n^{i,j} \right\} \\ \cdot \sum_{\vec{\mu}} (-1)^{\sum_{i=1}^3 |\mu^i|} e^{-\sum_{i=1}^3 |\mu^i| t_{i,1}} \mathcal{W}_{\mu^3, \mu^2, \mu^1}(q) \prod_{i=1}^3 q^{-\kappa_{\mu^i/2}} s_{\mu^i}(u^i).$$

In (61), the variables  $v_n^{i,j}$  are defined recursively by

$$v_n^{i, N_i} = \frac{(-1)^{n+1} e^{-n t_{i, N_i}}}{[n]}, \\ v_n^{i, j-1} = e^{-n t_{i, j-1}} \left( v_n^{i, j} + \frac{(-1)^{n+1}}{[n]} \right), \quad j = 3, \dots, N_i, \\ u_n^i = \frac{1}{[n]} + (-1)^{n+1} v_n^{i, 2}$$

The recursion defining  $v_n^{i,j}$  can be easily solved:

$$v_n^{i, j} = \frac{(-1)^{n+1}}{[n]} \left( \sum_{k=j}^{N_i} e^{-n(t_{i, j} + \dots + t_{i, k})} \right), \quad j = 2, \dots, N_i, \\ u_n^i = \frac{1}{[n]} \left( 1 + \sum_{k=2}^{N_i} e^{-n(t_{i, 2} + \dots + t_{i, k})} \right).$$

Therefore

$$(62) \quad Z_{N_1, N_2, N_3}(\lambda, \mathbf{t}) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n[n]^2} \sum_{i=1}^3 \sum_{2 \leq j \leq k \leq N_i} e^{-n(t_{i, j} + \dots + t_{i, k})} \right\} \\ \cdot \sum_{\vec{\mu}} (-1)^{\sum_{i=1}^3 |\mu^i|} e^{-\sum_{i=1}^3 |\mu^i| t_{i,1}} \mathcal{W}_{\mu^3, \mu^2, \mu^1}(q) \prod_{i=1}^3 q^{-\kappa_{\mu^i/2}} s_{\mu^i}(u^i).$$

Recall that

$$\mathcal{W}_{(\mu^1)^t, (\mu^2)^t, (\mu^3)^t}(q) = q^{-\sum_{i=1}^3 \kappa_{\mu^i/2}} \mathcal{W}_{\mu^3, \mu^2, \mu^1}(q),$$

so (62) is equivalent to (60).  $\square$

The closed topological vertex (Section 6.2) and chain of rational curves (Section 6.3) can be obtained taking limits of  $Z_{N_1, N_2, N_3}(\lambda; \mathbf{t})$ :

(1) Let  $t_{i,j} \rightarrow \infty$  for  $i = 1, 2, 3, j \geq 2$ . One has

$$v_n^{i,j} = 0, \quad i = 1, 2, 3, \quad j \geq 2.$$

and  $u_n^i = \beta_n$  for  $i = 1, 2, 3$ , so  $s_{\mu^i}(u^i) = \mathcal{W}_{\mu^i}(q) = q^{\kappa_{\mu^i}/2} \mathcal{W}_{(\mu^i)'}(q)$  and  $Z_{N_1, N_2, N_3}(\lambda; \mathbf{t})$  becomes the closed topological vertex.

(2) Let  $t_{i,j} \rightarrow \infty$  for  $i = 2, 3$ , and  $t_{1,1} \rightarrow \infty$ . We recover a chain of spheres with Kähler parameters  $t_{1,2}, \dots, t_{1, N_1}$ .

Unfortunately, the sum over partitions in (62) can not be evaluated in close form as we did before. In fact, explicit computations show that  $Z_{N_1, N_2, N_3}(\lambda; \mathbf{t})$  involves Gopakumar–Vafa invariants at higher genera, and seem to indicate that there are infinitely many degrees  $d_{i,j}$  for which the Gopakumar–Vafa invariants are non-vanishing. If we write  $Z_{N_1, N_2, N_3}(\lambda; \mathbf{t})$  in the Gopakumar–Vafa form

$$Z_{N_1, N_2, N_3}(\lambda; \mathbf{t}) = \exp\left(\sum_{\ell=1}^{\infty} \sum_{g=0}^{\infty} \sum_{\mathbf{d}} n_{\mathbf{d}}^g \frac{1}{\ell} [\ell]^{2g-2} e^{-\ell \mathbf{d} \cdot \mathbf{t}}\right)$$

where  $\mathbf{d} \cdot \mathbf{t} = \sum_{i,j} d_{i,j} t_{i,j}$ , then, for  $N = 2$  we find for example

$$\begin{aligned} n_{(1,1),(1,1),(1,1)}^0 &= -1, \\ n_{(2,1),(1,1),(1,1)}^0 &= 1, \\ n_{(1,0),(2,1),(2,1)}^0 &= -2, \\ n_{(1,1),(2,1),(2,1)}^0 &= -2, \\ n_{(2,1),(2,1),(2,1)}^0 &= 4, \end{aligned}$$

and they vanish for  $g > 0$ . Due to the cyclic symmetry of the configuration, the same values are obtained for cyclic permutations of the three sets of degrees.

If, say,  $t_{3,j} \rightarrow \infty$  for  $j \geq 1$ , so we have two lines of spheres joined by a two-vertex, then one can perform the sum over one of the two remaining partitions. This is because  $\mathcal{W}_{\emptyset, \mu^2, \mu^1}(q) = \mathcal{W}_{\mu^2, (\mu^1)'}(q) q^{\kappa_{\mu^1}/2}$ , and (61) reads

$$\begin{aligned} Z_{N_1, N_2}(\lambda; \mathbf{t}_1, \mathbf{t}_2) &= \exp\left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n[n]} \sum_{i=1}^2 \sum_{j=2}^{N_i} v_n^{i,j} \right\} \\ &\quad \sum_{\mu^1, \mu^2} (-1)^{\sum_{i=1}^2 |\mu^i|} e^{-\sum_{i=1}^2 |\mu^i| t_{i,1}} s_{\mu^2}(u^2) q^{-\kappa_{\mu^2}/2} \mathcal{W}_{\mu^2, (\mu^1)'}(q) s_{\mu^1}(u^1). \end{aligned}$$

Using again (51), and relabelling  $\mu^1 \rightarrow \mu$ , we finally obtain

$$Z_{N_1, N_2}(\lambda; \mathbf{t}_1, \mathbf{t}_2) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n[n]} \left(v_n^{2,1} + \sum_{i=1}^2 \sum_{j=2}^{N_i} v_n^{i,j}\right) + \frac{1}{n} v_n^{2,1} v_n^{2,2}\right) \sum_{\mu} q^{-\kappa_{\mu}/2} (-1)^{|\mu|} e^{-t_{1,1}|\mu|} s_{\mu}(\hat{u}^2) s_{\mu}(u^1),$$

where

$$v_n^{2,1} = \frac{(-1)^n e^{-nt_{2,1}}}{[n]}, \quad \hat{u}_n^2 = \frac{1}{[n]} - e^{-nt_{2,1}} u_n^2 = \frac{1}{[n]} \left(1 - \sum_{k=1}^{N_1} e^{-n(t_{2,1} + \dots + t_{2,k})}\right).$$

We conclude that

**Proposition 21**

$$Z_{N_1, N_2}(\lambda; \mathbf{t}_1, \mathbf{t}_2) = \exp\left(F_{N_2}^1(\lambda; \mathbf{t}_2) + F_{N_1}^2(\lambda; \mathbf{t}_1) + F_{N_2}^2(\lambda; \mathbf{t}_2)\right) \sum_{\mu} q^{-\kappa_{\mu}/2} (-1)^{|\mu|} e^{-t_{1,1}|\mu|} s_{\mu}(\hat{u}^2) s_{\mu}(u^1),$$

where

$$F_{N_i}^1(\lambda; \mathbf{t}_i) = \sum_{n>0} \frac{-1}{n[n]^2} \sum_{k=1}^{N_i} e^{-n(t_{i,1} + \dots + t_{i,k})}$$

$$F_{N_i}^2(\lambda; \mathbf{t}_i) = \sum_{n>0} \frac{1}{n[n]^2} \sum_{2 \leq k_1 \leq k_2 \leq N_i} e^{-n(t_{i,k_1} + \dots + t_{i,k_2})}$$

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*Department of Mathematics, University of California at Berkeley  
California 94720-3840, USA*

*Department of Mathematics, Northwestern University  
Evanston, Illinois 60208-2370, USA*

*Department of Physics, CERN  
Geneva 23, CH-1211, Switzerland*

[dkarp@math.berkeley.edu](mailto:dkarp@math.berkeley.edu), [ccliu@math.northwestern.edu](mailto:ccliu@math.northwestern.edu),  
[marcos@mail.cern.ch](mailto:marcos@mail.cern.ch)

Proposed: Jim Bryan  
Seconded: Paul Goerss, Eleny Ionel

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