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A CONSTRUCTIVE PROOF OF KY FAN'S GENERALIZATION OF TUCKER'S LEMMA

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ABSTRACT. We present a constructive proof of Ky Fan's combinatorial lemma concerning labellings of triangulated spheres. Our construction works for triangulations of S^n that contain a flag of hemispheres. As a consequence, we produce a constructive proof of Tucker's lemma that holds for a larger class of triangulations than previous constructive proofs.

1. INTRODUCTION

Tucker's lemma is a combinatorial analogue of the Borsuk-Ulam theorem with many useful applications. For instance, it can provide elementary routes to proving the Borsuk-Ulam theorem [1] and the Lusternik-Schnirelman-Borsuk set covering theorem [6], Kneser-type coloring theorems [11], and "fair division" theorems in game theory [8]. Moreover, any *constructive* proof of Tucker's lemma provides algorithmic interpretations of these results.

Although Tucker's lemma was originally stated for triangulations of an n-ball (for $n = 2$ in [9] and general n in [7]), in this paper we shall consider an equivalent version on triangulations of a sphere:

Tucker's Lemma ([9, 7])**.** *Let K be a barycentric subdivision of the octahedral subdivision of the n-sphere S^n. Suppose that each vertex of K is assigned a label from $\{\pm 1, \pm 2, \ldots \pm n\}$ in such a way that labels at antipodal vertices sum to zero. Then some pair of adjacent vertices of K have labels that sum to zero.*

The original version on the n-ball can be obtained from this by restricting the above statement to a hemisphere of K. This gives a triangulation of the n-ball in which the antipodal condition holds for vertices on the boundary of the ball. It is relatively easy to show that Tucker's Lemma is equivalent to the Borsuk-Ulam theorem, which says that any continuous function $f : S^n \to \mathbb{R}^n$ must map some pair of opposite points to the same point in the range [1]. In fact, this equivalence shows that the triangulation need not be a refinement of the octahedral subdivision; it need only be symmetric.

However, all known constructive proofs of Tucker's lemma seem to require some condition on the triangulation. For instance, the first constructive proof, due to Freund and Todd [6], requires the triangulation to be a refinement of the octahedral subdivision, and the constructive proof of Yang [10] depends on the AS-triangulation that is closely related to the octahedral subdivision.

In this paper, we give a constructive proof of Tucker's lemma for triangulations with a weaker condition: that it only contain a *flag of hemispheres*. Our proof (see Theorem 2)

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arises as a consequence of a constructive proof that we develop for the following theorem of Fan:

Ky Fan's combinatorial lemma ([4]). *Let K be a barycentric subdivision of the octahedral subdivision of the n-sphere S^n. Suppose that each vertex of K is assigned a label from $\{\pm 1, \pm 2, ... \pm m\}$ in such a way that (i) labels at antipodal vertices sum to zero and (ii) labels at adjacent vertices do not sum to zero. Then there are an odd number of n-simplices whose labels are of the form $\{k_0, -k_1, k_2, \ldots, (-1)^n k_n\}$, where $1 \leq k_0 < k_1 < \cdots < k_n \leq m$. In particular, $m \geq n+1$.*

Our constructive version of Fan's lemma (see Theorem 1) only requires that the triangulation contain a flag of hemispheres. We use the contrapositive (with $m = n$) to obtain a constructive proof of Tucker's lemma. This yields an algorithm for Tucker's lemma that is quite different in nature than that of Freund and Todd [6].

Our approach may provide new techniques for developing constructive proofs of certain generalized Tucker lemmas (such as the Z_p-Tucker lemma of Ziegler [11] or the generalized Tucker's lemma conjectured by Simmons-Su [8]) as well as provide new interpretations of algorithms that depend on Tucker's lemma (see [8] for applications to cake-cutting, Alon's necklace-splitting problem, team-splitting, and other fair division problems).

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2. Terminology

Let S^n denote the n-sphere, which we identify with the unit n-sphere $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. If A is a set in S^n, let $-A$ denote the *antipodal* set.

A *flag of hemispheres* in S^n is a sequence $H_0 \subset \cdots \subset H_n$ where each H_d is homeomorphic to a d-ball, and for $1 \leq d \leq n$, $\partial H_d = \partial(-H_d) = H_d \cap -H_d = H_{d-1} \cup -H_{d-1} \cong S^{d-1}$, $H_n \cup -H_n = S^n$, and $\{H_0, -H_0\}$ are antipodal points. One can think of a flag of hemispheres in the following way: decompose S^n into two balls that intersect along an equatorial S^{n-1}. Each ball can be thought of as a hemisphere. By successively decomposing equators in this fashion (since they are spheres) and choosing one such ball in each dimension, we obtain a flag of hemispheres.

A triangulation K of S^n is *(centrally) symmetric* if when a simplex σ is in K, then $-\sigma$ is in K. A symmetric triangulation of S^n is said to be *aligned with hemispheres* if we can find a flag of hemispheres such that H_d is contained in the d-skeleton of the triangulation. The *carrier hemisphere* of a simplex σ in K is the minimal H_d or $-H_d$ that contains σ.

A *labeling* of the triangulation assigns an integer to each vertex of the triangulation. We will say that a symmetric triangulation has an *anti-symmetric* labeling if each pair of antipodal vertices have labels that sum to zero. We say an edge is a *complementary edge* if the labels at its endpoints sum to zero.

We call a simplex in a labelled triangulation *alternating* if its vertex labels are distinct in magnitude and alternate in sign when arranged in order of increasing absolute value, i.e., the labels have the form

$$\{k_0, -k_1, k_2, \ldots, (-1)^n k_n\} \quad \text{or} \quad \{-k_0, k_1, -k_2, \ldots, (-1)^{n+1} k_n\}$$

where $1 \leq k_0 < k_1 < \cdots < k_n \leq m$. The *sign* of an alternating simplex is the sign of k_0, that is, the sign of the smallest label in absolute value. For instance, a simplex with labels $\{3, -5, -2, 9\}$ is a negative alternating simplex, since the labels can be reordered

$\{-2, 3, -5, 9\}$. A simplex with labels $\{-2, 2, -5\}$ is not alternating because the vertex labels are not distinct in magnitude.

We also define a simplex to be *almost-alternating* if it is not alternating, but by deleting one of the vertices, the resulting simplex (a facet) is alternating. The *sign* of an almost-alternating simplex is defined to be the sign any of its alternating facets (it is easy to check that this is well-defined). For example, a simplex with labels $\{-2, 3, 4, -5\}$ is not alternating, but it is almost-alternating because deleting 3 or 4 would make the resulting simplex alternating. Also, a simplex with labels $\{-2, 3, 3, -5\}$ is almost-alternating because deleting either 3 would make the resulting simplex alternating. Finally, a simplex with labels $\{-2, 2, 3, -5\}$ is almost-alternating because deleting 2 would make the resulting simplex alternating. However, this type of simplex will not be allowed by the conditions of Fan's lemma (since complementary edges are not allowed). See Figure 1.

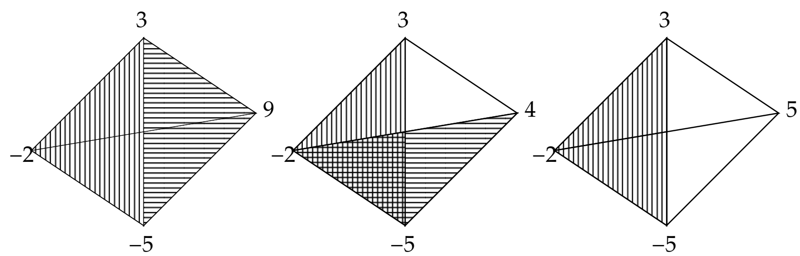

FIGURE 1. The first simplex is alternating and the other two are almost-alternating simplices. Their shaded facets are the facets that are also alternating simplices. The last simplex has a complementary edge.

Note that in an almost-alternating simplex with no complementary edge, there are exactly two vertices each of whose removal makes the resulting simplex alternating, and their labels must be adjacent to each other when the labels are ordered by increasing absolute value (e.g., see the second simplex in Figure 1). Thus any such almost-alternating simplex must have exactly two facets which are alternating.

3. Fan's Combinatorial Lemma

We now present a constructive proof of Fan's lemma, stated here for more general triangulations than Fan's original version.

Theorem 1. *Let K be a symmetric triangulation of S^n aligned with hemispheres. Suppose K has (i) an anti-symmetric labelling by labels $\{\pm 1, \pm 2, \ldots \pm m\}$ and (ii) no complementary edge (an edge whose labels sum to zero).*

Then there are an odd number of positive alternating n-simplices and an equal number of negative alternating n-simplices. In particular, $m \geq n+1$. Moreover, there is a constructive procedure to locate an alternating simplex of each sign.

Fan's proof in [4] used a non-constructive parity argument and induction on the dimension n. Freund and Todd's constructive proof of Tucker's lemma [6] does not appear to generalize to a proof of Fan's lemma, since their construction uses $m = n$ in an inherent way. Cohen [2] implicitly proves a version of Fan's lemma for $n = 2$ and $n = 3$ in order to prove Tucker's lemma; his approach differs from our proof in that the paths of his search procedure can pair up alternating simplices with non-alternating simplices (for instance, $\{1, -2, 3\}$ can be paired up with $\{1, -2, -3\}$). Cohen hints, but does not explicitly say, how his method would extend to higher dimensions; moreover, such an approach would only be semi-constructive, since as he points out, finding one asserted edge in dimension n would require knowing the location of "all relevant simplices" in dimension $n-1$.

Our strategy for proving Theorem 1 constructively is to identify paths of simplices whose endpoints are alternating n-simplices or alternating 0-simplices (namely, H_0 or $-H_0$). Then one can follow such a path from H_0 to locate an alternating n-simplex.

Proof. Suppose that the given triangulation K of S^n is aligned with the flag of hemispheres $H_0 \subset \cdots \subset H_n$. Call an alternating or almost-alternating simplex *agreeable* if the sign of that simplex matches the sign of its carrier hemisphere. For instance, the simplex with labels $\{-2, 3, -5, 9\}$ in Figure 1 is agreeable if its carrier hemisphere is $-H_d$ for some d.

We now define a graph G. A simplex σ carried by H_d is a node of G if it is one of the following:

(1) an agreeable alternating $(d-1)$-simplex,
(2) an agreeable almost-alternating d-simplex, or
(3) an alternating d-simplex.

Two nodes σ and τ are adjacent in G if all the following hold:

(a) one is a facet of the other,
(b) $\sigma \cap \tau$ is alternating, and
(c) the sign of the carrier hemisphere of $\sigma \cup \tau$ matches the sign of $\sigma \cap \tau$.

We claim that G is a graph in which every vertex has degree 1 or 2. Furthermore, a vertex has degree 1 if and only if its simplex is carried by $\pm H_0$ or is an n-dimensional alternating simplex. To see why, we consider the three kinds of nodes in G:

(1) An agreeable alternating $(d-1)$-simplex σ with carrier $\pm H_d$ is the facet of exactly two d-simplices, each of which must be an agreeable alternating or an agreeable almost-alternating simplex in the same carrier. These satisfy the adjacency conditions (a)-(c) with σ, hence σ has degree 2 in G.
(2) An agreeable almost-alternating d-simplex σ with carrier $\pm H_d$ is adjacent in G to its two facets that are agreeable alternating $(d-1)$-simplices. (Adjacency condition (c) is satisfied because σ is agreeable and an almost-alternating d-simplex must have the same sign as its alternating facets.)
(3) An alternating d-simplex σ carried by $\pm H_d$ has one alternating facet τ whose sign agrees with the sign of the carrier hemisphere of σ. That facet is obtained by deleting either the highest or lowest label (by magnitude) of σ so that the remaining simplex satisfies condition (c). (For instance, the first simplex in Figure 1 has two alternating facets, but only one of them can have a sign that agrees with the carrier hemisphere.) Thus σ is adjacent to τ in G.

    Also, σ is the facet of exactly two simplices, one in H_{d+1} and one in $-H_{d+1}$, but it is adjacent in G to exactly one of them; which one is determined by the sign of σ, since the adjacency condition (c) must be satisfied.

    Thus σ has degree 2 in G, unless $d = 0$ or $d = n$: if $d = 0$, then σ is the point $\pm H_0$ and it has no facets, so σ has degree 1; and if $d = n$, then σ is not the facet of any other simplex, and is therefore of degree 1.

Every node in the graph therefore has degree two with the exception of the points at $\pm H_0$ and all alternating n-simplices. Thus G consists of a collection of disjoint paths with endpoints at $\pm H_0$ or in the top dimension.

Note that the antipode of any path in G is also a path in G. No path can have antipodal endpoints (else the center edge or node of the path would be antipodal to itself); thus a path is never identical to its antipodal path. So all the paths in G must come in pairs, implying that the number of endpoints of paths in G must be a multiple of four. Since exactly two such endpoints are the nodes at H_0 and $-H_0$, there are twice an odd number of alternating n-simplices. And, because every positive alternating n-simplex has a negative alternating n-simplex as its antipode, exactly half of the alternating n-simplices are positive. Thus there are an odd number of positive alternating n-simplices (and an equal number of negative alternating n-simplices).

To locate an alternating simplex, follow the path that begins at H_0; it cannot terminate at $-H_0$ (since a path is never its own antipodal path), so it must terminate in a (negative or positive) alternating simplex. The antipode of this simplex will be an alternating simplex of the opposite sign. $\square$

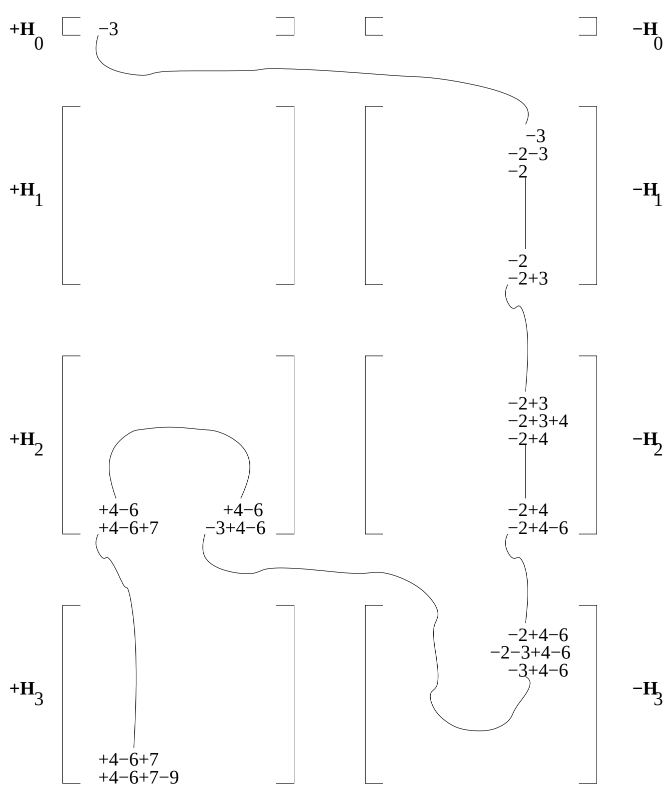

FIGURE 2. An example of what sets of labels of simplices along a path in G could look like. Repeated labels are not shown.

Figure 2 shows an example of how a path may wind through the various hemispheres of a triangulated 3-sphere. Note how the sign of each simplex agrees with the sign of its carrier hemisphere ("agreeability"), unless the path connects a d-hemisphere with a $d+1$-hemisphere, in which case the sign of the d-simplex specifies which $(d+1)$-hemisphere the path should connect to. These facts follow from adjacency condition (c).

Our approach is related to that of another paper of Fan [5], which studied labelled triangulations of an d-manifold M and derived a set of paths that pair up alternating simplices in the interior of M with positive alternating simplices on the boundary of M. When $M = H_d$, the paths of Fan coincide with the restriction of our paths in G to H_d. By itself, this is only semi-constructive, since finding one alternating d-simplex necessitates locating all positive alternating $(d-1)$-simplices on the boundary of H_d. To make Fan's approach fully constructive for S^n, one might attempt to use Fan's approach in each d-hemisphere of S^n and then glue all the hemispheres in each dimension together, thereby gluing all the paths. But this results in paths that branch (where positive alternating simplices in H_d are glued to paths in *both* H_{d+1} and $-H_{d+1}$) or paths that terminate prematurely (where a path ends in a negative alternating d-simplex where $d < n$).

By contrast, the path we follow in G from the point H_0 to an alternating n-simplex is well-defined, has no branching, and need not pass through all the alternating $(d-1)$-simplices on the boundary of H_d for each d. In our proof, the use of the flag of hemispheres controls the branching that would occur in paths of G if one ignored the property of "agreeability" and adjacency condition (c). In that sense, it serves a similar function in controlling branching as the use of the flag of polytope faces in the constructive proof of the polytopal Sperner lemma of DeLeora-Peterson-Su [3].

Note that the contrapositive of Theorem 1 implies Tucker's Lemma, since if $m = n$ and condition (i) holds, then condition (ii) must fail. In fact, if we remove condition (ii) in the statement of Theorem 1, the graph G can have additional nodes of degree 1, namely, agreeable almost-alternating simplices with a complementary edge.

This gives a constructive proof for Tucker's lemma by starting at H_0 and following the associated path in G. Because $m = n$, there are not enough labels for the existence of any alternating n-simplices, so there must be an odd number of agreeable positive almost-alternating simplices with a complementary edge. (Note that this says nothing about the parity of the number of complementary edges, since several such simplices could share one edge.)

It is of some interest that our constructive proof allows for a larger class of triangulations than previous constructive proofs of Tucker's lemma, so for completeness we state it carefully here:

Theorem 2. *Let K be a symmetric triangulation of S^n aligned with hemispheres. Suppose K has an anti-symmetric labelling by labels $\{\pm 1, \pm 2, \dots, \pm n\}$. Then there are an odd number of positive (negative) almost-alternating simplices which contain a complementary edge. Moreover, there is a constructive procedure to locate one such edge.*

The hypothesis that K can be aligned with a flag of hemispheres is weaker than, for instance, requiring K to refine the octahedral subdivision (e.g., Freund-Todd's proof of Tucker's lemma). If a triangulation refines the octahedral subdivision, then the octahedral orthant hyperplanes contain a natural flag of hemispheres. But the converse is not true: there are triangulations aligned with hemispheres that are not refinements of the octahedral subdivision. For instance, consider the triangulated 2-sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ whose 1-skeleton is cut out by intersections with the plane $z = 0$ and half-planes $\{x = 0, z \geq 0\}$ and $\{y = 0, z \leq 0\}$. This triangulation has 4 vertices at $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$, it is symmetric, and it contains a flag of hemispheres. But it is combinatorially equivalent to the boundary of a 3-simplex, and hence does not refine the octahedral subdivision of S^2.

We remark that the AS-triangulation, used by Yang [10] to prove Tucker's lemma, is closely related to an octahedral subdivision and contains a natural flag of hemispheres.

It is an interesting open question as to whether any symmetric triangulation of S^n can be aligned with a flag of hemispheres, and if so, how to find such a flag. Together with our arguments this would yield a constructive proof of Tucker's lemma for any symmetric triangulation.

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