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# A Polytopal Generalization of Sperner's Lemma

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We prove the following conjecture of Atanassov (*Studia Sci. Math. Hungar.* **32** (1996), 71–74). Let $T$ be a triangulation of a $d$-dimensional polytope $P$ with $n$ vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Label the vertices of $T$ by $1, 2, \ldots, n$ in such a way that a vertex of $T$ belonging to the interior of a face $F$ of $P$ can only be labelled by $j$ if $\mathbf{v}_j$ is on $F$. Then there are at least $n - d$ full dimensional simplices of $T$, each labelled with $d + 1$ different labels. We provide two proofs of this result: a non-constructive proof introducing the notion of a *pebble set* of a polytope, and a constructive proof using a path-following argument. Our non-constructive proof has interesting relations to minimal simplicial covers of convex polyhedra and their chamber complexes, as in Alekseyevskaya (*Discrete Math.* **157** (1996), 15–37) and Billera *et al.* (*J. Combin. Theory Ser. B* **57** (1993), 258–268). © 2002 Elsevier Science (USA)

*Key Words:* Sperner's lemma; polytopes; path-following; simplicial algorithms.

## 1. INTRODUCTION

Sperner's lemma is a combinatorial statement about labellings of triangulated simplices whose claim to fame is its equivalence with the

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topological fixed-point theorem of Brouwer [12, 20]. In this paper, we prove a generalization of Sperner's lemma that settles a conjecture proposed by Atanassov [2].

Consider a convex polytope $P$ in $\mathbb{R}^d$ defined by $n$ vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$. For brevity, we will call such polytope an $(n, d)$-*polytope*. Throughout the paper we will follow the terminology of the book [24]. By a *triangulation* $T$ of the polytope $P$ we mean a finite collection of distinct simplices such that: (i) the union of the simplices of $T$ is $P$, (ii) every face of a simplex in $T$ is in $T$, and (iii) any two simplices in $T$ intersect in a face common to both. The points $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are called *vertices of* $P$ to distinguish them from vertices of $T$, the triangulation. Similarly, a simplex spanned by vertices of $P$ will be called a *simplex of* $P$ to distinguish it from simplices involving other vertices of $T$. If $S$ is a subset of $P$, then the *carrier* of $S$, denoted $\text{carr}(S)$, is the smallest face $F$ of $P$ that contains $S$. In that case we say $S$ is *carried* by $F$. A *cover* $C$ of a convex polytope $P$ is a collection of full dimensional simplices in $P$ such that $\bigcup_{\sigma \in C} \sigma = P$. The *size of a cover* is the number of simplices in the cover.

Let $T$ be a triangulation of $P$, and suppose that the vertices of $T$ have a labelling satisfying these conditions: each vertex of $P$ is assigned a unique label from the set $\{1, 2, \ldots, n\}$, and each other vertex $v$ of $T$ is assigned a label of one of the vertices of $P$ in $\text{carr}(\{v\})$. Such a labelling is called a *Sperner labelling* of $T$. We say that a $d$-simplex in the triangulation is a *fully labelled simplex* or simply a *full cell* if all its labels are distinct. The following result was proved by Sperner [20] in 1928:

SPERNER'S LEMMA. *Any Sperner labelling of a triangulation of a* $d$*-simplex must contain an odd number of full cells*; *in particular*, *there is at least one*.

Constructive proofs of Sperner's lemma [8, 13, 17] emerged in the 1960s, and these were used to develop constructive methods for locating fixed points [22, 23]. Sperner's lemma and its variants continue to be useful in applications. For example, they have recently been used to solve *fair division* problems in game theory [18, 21].

The main purpose of this paper is to present a solution of the following conjecture: any Sperner labelling of a triangulation of an $(n, d)$-polytope must contain at least $n - d$ full cells. In 1996, Atanassov [2] stated this conjecture and gave a proof for the case where $d = 2$. Note that Sperner's lemma is exactly the case $n = d + 1$. In this paper we prove this conjecture for all $(n, d)$-polytopes:

THEOREM 1. *Any Sperner labelling of a triangulation* $T$ *of an* $(n, d)$*-polytope* $P$ *must contain at least* $n - d$ *full cells.*

We provide a non-constructive and a constructive proof of Theorem 1. The non-constructive proof that we give in Section 2 relies on a known result about the surjectivity of the piecewise linear map induced by a labelled triangulation of $P$ (Proposition 3, cf. [5, 14]) and the notion of a *pebble set* that we develop. Pebble sets are also used in Section 2 to prove the following result of independent interest in discrete geometry.

THEOREM 2. *Let $c(P)$ denote the covering number of an $(n, d)$-polytope $P$, which is the size of the smallest cover of $P$. Then, $c(P) \geqslant n - d$. This result is best possible as the equality is attained for stacked polytopes.*

This result, which bounds the size of a polytope cover, is somewhat reminiscent of Barnette's lower bound theorem [3] for bounding the number of facets of simplicial polytopes. Other results bounding the size of polytope covers may be found in [6, 7, 16] and references within.

We emphasize that Theorem 2 holds for *all* covers, not just for those covers needed in the proof of Theorem 1. To be specific, Theorem 1 uses the fact that the collection of full cells in $T$ corresponds to a cover of $P$ under the piecewise linear map that sends each vertex of $T$ to the vertex of $P$ that shares the same label. In such a cover, any pair of simplices is connected by a sequence of simplices that meet face-to-face, and it is easier to prove a result like Theorem 2 for such covers.

However, not all covers are necessarily of this type. For example, in the left-hand side of Fig. 1 we specify a cover of an 8-vertex, three-dimensional convex polytope that cannot be obtained from a piecewise linear map of $P$ to $P$. Nevertheless, Theorem 2 still applies.

In principle, failing to satisfy this "face-to-face" property can lead to very small covers; in Fig. 1 we display a star-shaped 12-gon that can be covered with just two triangles. Thus, the assertion of Theorem 2 is not true for covers of non-convex polygons (even though it is true for triangulations of non-convex polygons). The significance of Theorem 2 is that it holds for *all covers of convex polytopes*, not just triangulations or "face-to-face" covers.

Section 3 develops background on path-following arguments in polytopes that is closely related to classical path-following arguments for Sperner's lemma [8, 13, 22]. This is applied to give a constructive proof of Theorem 1 for simplicial polytopes. In Section 4, we extend the construction to prove the conjecture for arbitrary polytopes. Section 5 of the paper is devoted to applications, remarks, and open questions.

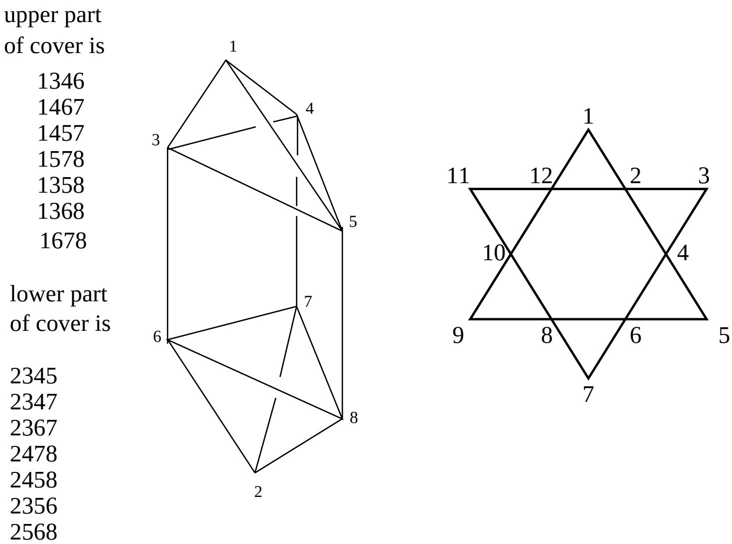


**FIG. 1.** Pathological covers.

## 2. A NON-CONSTRUCTIVE PROOF USING PEBBLE SETS

Our non-constructive proof of Theorem 1 relies on the notion of a *pebble set* that we develop in this section. Constructing such a set will yield the desired lower bound for the number of full cells.

In what follows, we will use the notion of chamber complex of a polytope $P$ (see [1] and [4]): let $\Sigma$ be the set of all $d$-simplices of $P$. Denote by $\mathrm{bdry}(\sigma)$ the boundary of simplex $\sigma$. Consider the set of open polyhedra $P - \bigcup_{\sigma \in \Sigma} \mathrm{bdry}(\sigma)$. A *chamber* is the closure of one of these components. The chamber complex of $P$ is the polyhedral complex given by all chambers and their faces.

Let $P$ be an $(n, d)$-polytope with Sperner-labelled triangulation $T$. Consider the piecewise linear (PL) map $f : P \to P$ that maps each vertex of $T$ to the vertex of $P$ that shares the same label, and is linear on each $d$-simplex of $T$. The next proposition will be very useful:

PROPOSITION 3. *The map $f : P \to P$ defined as above is surjective, and thus the collection of full cells in $T$ forms a cover of $P$ under $f$.*

The proof of the surjectivity of $f$ that we show here is taken directly from the forthcoming book [5]. The surjectivity of $f$ can also be proved as a consequence of the KKM-type result of [14, Theorem 10]. Similar surjectivity results for maps arising from labellings by facets of $P$ (rather than vertices of $P$) can be found in [10, Theorem 4; 23, Theorem 14.5.3].

*Proof.* First, note that because of the Sperner labelling of the triangulation $T$ of $P$, the map $f$ satisfies $f(F) \subseteq F$ for any face $F$ of $P$.

Since this condition is hereditary for faces, to show that $f$ is surjective, it suffices to show that each point $y$ in the interior of $P$ has a pre-image. For contradiction, suppose that some $y \in \text{int}\, P$ is not in the image of $f$. For $x \in P$, consider the ray emanating from $f(x)$ and passing through $y$, and let $g(x)$ be the unique intersection of that ray with the boundary of $P$. This $g$ is a well defined and continuous map $P \to P$, and by Brouwer's fixed point theorem, there is an $x_0 \in P$ with $g(x_0) = x_0$. The point $x_0$ lies on the boundary of $P$, in some proper face $F$. But $f(x_0)$ cannot lie in $F$ (because the segment from $x_0 = g(x_0)$ to $f(x_0)$ passes through the point $y$ outside $F$) which contradicts the fact that $f(F) \subseteq F$.

Thus $f$ is surjective. Moreover, the collection of images of full cells under $f$ suffices to cover $P$ because interiors of chambers of $P$ are only covered by images of full cells, and boundaries of chambers are covered by any simplex that covers an adjacent chamber. ∎

As a consequence of Proposition 3, if we can find a set of points in $P$ such that any $d$-simplex spanned by $(d+1)$ vertices of $P$ contains at most one such point in its interior, then the pre-image of each such point will correspond to a full cell in $P$ (in fact, an odd number, because the number of pre-images, counted with sign, is 1). Thus finding full cells in Theorem 1 corresponds precisely to looking for the following kind of finite point set:

DEFINITION. A *pebble set* of a $(n, d)$-polytope $P$ is a finite set of points (*pebbles*) such that each $d$-simplex of $P$ contains at most one pebble interior to chambers.

It is worth noting two facts about pebble sets. Firstly, the larger the pebble set, the more full cells we can identify, i.e., the number of full cells is at least the cardinality of the largest size pebble set in $P$. Secondly, by the definition of a chamber, only one pebble can exist within a chamber and when choosing a pebble $p$ we have the freedom to replace it by any point $p'$ in the interior of the same chamber because $p$ and $p'$ are contained by the same set of $d$-simplices. We now show that a pebble set of size $n-d$ exists for any $(n, d)$-polytope $P$ by a "facet-pivoting" construction.

In the simplest situation, if one of the facets of $P$ is a simplex, call this simplex the *base facet*. Choose any point $\mathbf{q}_0$ (the *basepoint*) in the interior of this base facet. Now for each vertex $\mathbf{v}_i$ not in the base facet, choose a point $\mathbf{p}_i$ along a line between $\mathbf{q}_0$ and $\mathbf{v}_i$ but very close to $\mathbf{v}_i$. Exactly how close will be specified in the proof. The collection of all such points $\{\mathbf{p}_i\}$ forms a pebble set; it is size $n - d$ because the simplicial base facet has $d$ vertices. See, for example, Fig. 2 for the case of a pentagon; it is a $(5, 2)$-polytope with pebble set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

If none of the facets are simplicial, then one must choose a non-simplicial facet as base. In this case, choose a pebble set $\{\mathbf{q}_i\}$ for the base facet (an inductive hypothesis is used here) and then use any one of them for a basepoint $\mathbf{q}_0$ to construct $\mathbf{p}_i$ as above. The remaining pebbles are obtained from the other $\mathbf{q}_i$ by perturbing them so they are interior to $P$. See Fig. 3.

THEOREM 4. *Any $(n, d)$-polytope contains a pebble set of size $n - d$.*

*Proof.* We induct on the dimension $d$. For dimension $d = 1$, a polytope is just a line segment spanned by two vertices. Hence $n - d = 1$ and clearly any point in the interior of the line segment forms a pebble set.

For any other dimension $d$, let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \in \mathbb{R}^d$ denote the vertices of the given $(n, d)$-polytope $P$. Choose any facet $F$ of $P$ as a "base facet", and suppose without loss of generality that it is the convex hull of the last $k$ vertices $\mathbf{v}_{n-k+1}, \ldots, \mathbf{v}_n \in \mathbb{R}^d$, $k \geqslant d$. Then $F$ is a $(d - 1)$-dimensional polytope with $k$ vertices, and by the inductive hypothesis, $F$ has a pebble set $Q_F$ with $(k - d + 1)$ pebbles $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_{k-d}$. (If $F$ is a simplex, then $k = d$ and $Q_F$ consists of one point $\mathbf{q}_0$, which can be taken to be any point on the interior of $F$.)

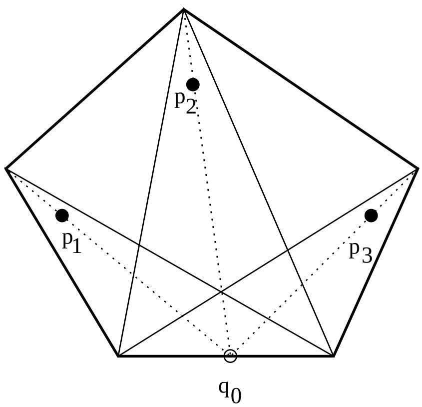


**FIG. 2.** A pebble set with pebbles $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$.

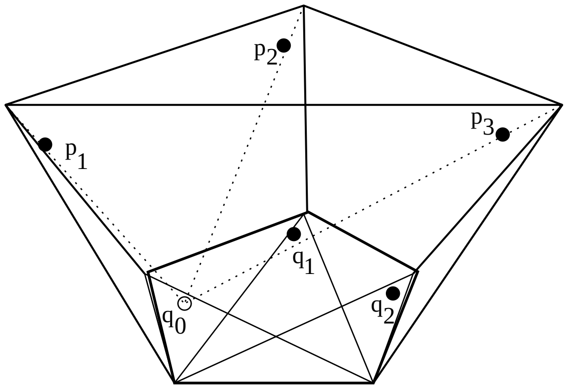


**FIG. 3.** A pebble set with pebbles $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{q}_1{}^\circ, \mathbf{q}_2{}^\circ$. The pebbles $\mathbf{q}_1{}^\circ, \mathbf{q}_2{}^\circ$ (not shown) lie just above $\mathbf{q}_1, \mathbf{q}_2$ on the base of the polytope. Note how $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2$ arise from the pebble set construction in Fig. 2.

Let $\mathrm{diam}(P)$ denote the diameter of the polytope $P$, i.e., the maximum pairwise distance between any two points in $P$. Let $H$ be the minimum distance between any vertex $\mathbf{v} \in V$ and the convex hull of the vertices in $V \backslash \{\mathbf{v}\}$. Since there are finitely many such distances and the vertices are in convex position, $H$ exists and is positive. Set

$$\varepsilon = \frac{H}{2\,\mathrm{diam}(P)}. \tag{1}$$

Using $\mathbf{q}_0$, let $Q = \{\mathbf{p}_1, \ldots, \mathbf{p}_{n-k}\}$ denote the collection of $n-k$ points defined by

$$\mathbf{p}_i = \varepsilon \mathbf{q}_0 + (1-\varepsilon)\mathbf{v}_i, \tag{2}$$

for $1 \leqslant i \leqslant n-k$, where $\varepsilon$ is a small positive constant given by (1). Therefore points in $Q$ lie along straight lines extending from $\mathbf{q}_0$ and very close to the vertices of $P$ not in $F$.

Because $\mathbf{q}_i$ is in $F$, it lies on the boundary of $P$ and borders exactly one chamber of $P$ (since by induction it is interior to a single chamber in the facet $F$). Ignoring $\mathbf{q}_0$ momentarily, for $1 \leqslant i \leqslant k-d$, let $\mathbf{q}_i{}^\circ$ denote a point obtained by "pushing" $\mathbf{q}_i$ into the interior of the unique chamber that it borders. Let $Q_F{}^\circ = \{\mathbf{q}^\circ{}_1, \ldots, \mathbf{q}^\circ{}_{k-d}\}$. We shall show that

$$Q \cup Q_F{}^\circ = \{\mathbf{p}_1, \ldots, \mathbf{p}_{n-k}, \mathbf{q}^\circ{}_1, \ldots, \mathbf{q}^\circ{}_{k-d}\}$$

is a pebble set for the polytope $P$. Note that if $P$ has a simplicial facet $F$, then with this facet as base, the set $Q$ suffices; for then $Q_F{}^\circ$ is empty and construction (2) yields the required number of pebbles by choosing any $\mathbf{q}_0$ in the interior of a simplicial facet $F$. First, we prove some important facts about the $\mathbf{p}_i$ and $\mathbf{q}_i{}^{\circ}$.

LEMMA 5. *Let $S$ be a $d$-simplex spanned by vertices of $P$. If $S$ contains $\mathbf{p}_i$, then $S$ must also contain $\mathbf{v}_i$ as one of its vertices.*

*Proof.* By construction, each $\mathbf{p}_i$ has the property that $\mathbf{p}_i$ is not in the convex hull of $V\backslash\{\mathbf{v}_i\}$. This follows because

$$\frac{||\mathbf{v}_i - \mathbf{p}_i||}{||\mathbf{v}_i - \mathbf{q}_0||} = \varepsilon = \frac{H}{2\,\mathrm{diam}(P)} \leqslant \frac{H}{2||\mathbf{v}_i - \mathbf{q}_0||},$$

hence $||\mathbf{v}_i - \mathbf{p}_i|| \leqslant H/2$, implying that the distance of $\mathbf{p}_i$ from the convex hull of $V\backslash\{\mathbf{v}_i\}$ is greater than or equal to $H/2$.

Since the convex hull of $V\backslash\{\mathbf{v}_i\}$ does not contain $\mathbf{p}_i$, if $S$ is to contain $\mathbf{p}_i$ it must contain $\mathbf{v}_i$ as one of its vertices. ∎

LEMMA 6. *Let $S$ be a non-degenerate $d$-simplex spanned by vertices of $P$. Then $\mathbf{q}_i{}^{\circ}$ is in $S$ if and only if $\mathbf{q}_i$ is in $S \cap F$.*

*Proof.* Since $\mathbf{q}_i{}^{\circ}$ is in the unique chamber of $P$ that $\mathbf{q}_i$ borders, any non-degenerate simplex containing $\mathbf{q}_i$ must contain $\mathbf{q}_i{}^{\circ}$. Conversely, any simplex $S$ containing $\mathbf{q}_i{}^{\circ}$ must contain its chamber and therefore contains $\mathbf{q}_i$. Since $\mathbf{q}_i$ is in $F$, then $\mathbf{q}_i$ is in $S \cap F$. ∎

The next three lemmas will show that $Q \cup Q_F{}^{\circ}$ is a pebble set for $P$.

LEMMA 7. *Any $d$-simplex $S$ spanned by vertices of $P$ contains no more than one pebble of $Q$.*

*Proof.* If $S$ is degenerate (i.e., the convex hull of those vertices is not full dimensional), then it clearly contains no pebbles of $Q$ because the $\mathbf{p}_i$ are by construction in the interior of a chamber. So we may assume that $S$ is non-degenerate.

Let $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1} \in V$ denote the vertices of $S$. Suppose by way of contradiction that $S$ contained more than one point of $Q$. Then $\mathbf{p}_{i'}$ and $\mathbf{p}_{j'}$ are contained in $S$ for distinct $i', j'$, where $1 \leqslant i', j' \leqslant n-k$. Lemma 5 implies that $\mathbf{v}_{i'}, \mathbf{v}_{j'}$, the vertices of $P$ associated to $\mathbf{p}_{i'}$ and $\mathbf{p}_{j'}$, must both be vertices of $S$. Without loss of generality, let $\mathbf{s}_1 = \mathbf{v}_{i'}$ and $\mathbf{s}_2 = \mathbf{v}_{j'}$. Let $A$ be a matrix whose columns consist of $\mathbf{q}_0$ and the vertices of $S$, adjoined with a row of 1's:

$$A = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_{d+1} & \mathbf{q}_0 \\ \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

This is a $(d+1) \times (d+2)$ matrix that has rank $(d+1)$ because the $\mathbf{s}_i$ are affinely independent (by the non-degeneracy of $S$). So the kernel of $A$, ker $(A)$ is one dimensional. Note that $\mathbf{p}_{i'} \in S$ implies that it is a convex combination of the first $(d+1)$ columns of $A$. On the other hand, by construction, it is also a convex combination of $\mathbf{s}_1$ and $\mathbf{q}_0$. Thus there exist constants $0 \leqslant x_1, x_2, \ldots, x_{d+1} \leqslant 1$ satisfying

$$\begin{bmatrix} \mathbf{p}_{i'} \\ \\ 1 \end{bmatrix} = A \begin{bmatrix} 1-\varepsilon \\ 0 \\ \vdots \\ 0 \\ \varepsilon \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d+1} \\ 0 \end{bmatrix}, \tag{3}$$

where the first equality follows from (2). Similarly, $\mathbf{p}_{j'} \in S$ implies

$$\begin{bmatrix} \mathbf{p}_{j'} \\ \\ 1 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1-\varepsilon \\ \vdots \\ 0 \\ \varepsilon \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{d+1} \\ 0 \end{bmatrix},$$

for some constants $0 \leqslant y_1, y_2, \ldots, y_{d+1} \leqslant 1$. The above equations show that $(x_1 + \varepsilon - 1, x_2, x_3 \ldots, x_{d+1}, -\varepsilon)^T$ and $(y_1, y_2 + \varepsilon - 1, y_3, \ldots, y_{d+1}, -\varepsilon)^T$ are both in $\ker(A)$. But since $\ker(A)$ is one dimensional, and the last coordinates of these vectors are equal, all entries of these vectors are identical. In particular, $x_1 + \varepsilon - 1 = y_1$.

We now claim that $x_1 + \varepsilon < 1$ and hence $y_1 < 0$, which would show that $\mathbf{p}_{j'}$ could not have been in $S$ after all, a contradiction.

To establish the claim, use Eqs. (2) and (3) to express $\mathbf{q}_0$ as an affine combination of the vertices of $S$:

$$\mathbf{q}_0 = \frac{1}{\varepsilon}((x_1 + \varepsilon - 1)\mathbf{s}_1 + x_2\mathbf{s}_2 + \cdots + x_{d+1}\mathbf{s}_{d+1}). \tag{4}$$

Since $x_2, \ldots, x_{d+1} \geqslant 0$ and $\mathbf{q}_0$ is not in the interior of $S$, then either (i) one of the coefficients of the $\mathbf{s}_i$ in (4) is equal to zero or (ii) $x_1 + \varepsilon < 1$.

If case (i) holds, then $\mathbf{q}_0$ is on some facet of $S$, hence $\mathbf{q}_0$ is spanned by $d$ vertices of $S$ which lie on the facet $F$ of $P$. Thus the vertices of $S$ include those $d$ vertices but by Lemma 5, $\mathbf{v}_{i'}$ and $\mathbf{v}_{j'}$ as well. Since $\mathbf{v}_{i'}, \mathbf{v}_{j'}$ were not on the facet $F$, we obtain a contradiction since $S$ cannot contain more than $d + 1$ vertices.

Hence case (ii) holds, namely that $x_1 + \varepsilon < 1$, which is the desired contradiction. ∎

LEMMA 8. *Any d-simplex S spanned by vertices of P contains no more than one pebble in* $Q_F{}^\circ$.

*Proof.* Since $S \cap F$ is a simplex in $F$ that contains at most one point of $Q_F$, then by Lemma 6, $S$ can contain at most one point of $Q_F{}^\circ$. ∎

LEMMA 9. *Any d-simplex S spanned by vertices of P cannot contain pebbles of Q and* $Q_F{}^\circ$ *simultaneously.*

*Proof.* Suppose $S$ contained a point $\mathbf{q}_i{}^\circ$ of $Q_F{}^\circ$. Then by Lemma 6, $S \cap F$ contains $\mathbf{q}_i$ of $Q_F$. Since $Q_F$ was a pebble set for the facet $F$, $S \cap F$ cannot also contain $\mathbf{q}_0$.

If $S$ also contained a pebble $\mathbf{p}_{i'}$ of $Q$, then by Lemma 5, $S$ contains $\mathbf{v}_{i'}$ as a vertex. Since $S \cap F$ contains $\mathbf{q}_i$ which is interior to a chamber of $F$, $S$ must also contain $d$ vertices of $F$. Since $\mathbf{q}_0$ is in $F$ (but not in $S \cap F$), $\mathbf{q}_0$ is expressible as a linear combination (but not convex combination) of those $d$ vertices. This linear combination, when substituted for $\mathbf{q}_0$ in (2), would show that the pebble $\mathbf{p}_{i'}$ is not a convex combination of $\mathbf{v}_{i'}$ and those $d$ vertices. This contradicts the fact that $\mathbf{p}_{i'}$ was in $S$ to begin with. ∎

Together, the three lemmas above show that $S$ cannot contain more than one point of $Q \cup Q_F{}^\circ$, which concludes the proof of Theorem 4. ∎

Theorem 1 now follows from Theorem 4, in light of the remarks following the proof of Proposition 3.

Theorem 2 also follows from our pebble set construction. Since each element of a cover can contain at most one pebble, Theorem 4 shows that any cover must have at least $n - d$ elements. This bound is best possible, because stacked polytopes have triangulations of size $n - d$ [16].

## 3. GRAPHS FOR PATH-FOLLOWING AND SIMPLICIAL POLYTOPES

Sperner's lemma has a number of constructive proofs which rely on "path-following" arguments (see, for example, the survey of Todd [22]). Path-following arguments work by using a labelling to determine a path through simplices in a triangulation, in which one endpoint is known and the other endpoint is a full cell. In this section, we adapt these ideas for Sperner-labelled polytopes, which are used in the next section to give a constructive "path-following" proof of Theorem 1.

Let $P$ be an $(n, d)$-polytope with triangulation $T$ and a Sperner labelling using the label set $L = \{1, 2, \ldots, n\}$. We define some further terminology and notation that we will use from now on. Let $L(\sigma)$, the *label set* of $\sigma$, denote the set of distinct labels of vertices of $\sigma$. Let $L(F)$ denote the label set of a face $F$ of $P$. As defined earlier, a $d$-simplex $\sigma$ in $T$ is a *full cell* if the vertex labels of $\sigma$ are all distinct. Similarly, a $(d-1)$-simplex $\tau$ in $T$ is a *full facet* if the vertex labels of $\tau$ are all distinct. Note that a full facet on the boundary of $P$ can be regarded as a full cell in that facet.

DEFINITION. Given a Sperner-labelled triangulation $T$ of a polytope $P$, we define three useful graphs:

1. The *nerve graph* $G$ is an undirected graph with nodes that are simplices of $T$ whose label set is of size at least $d$. Two nodes in $G$ are adjacent if (as simplices) one is a facet of the other.

2. If $K$ is a subset of the label set $L = \{1, 2, \ldots, n\}$ of size $(d-1)$, the *derived graph* $G_K$ is the subgraph of the nerve graph $G$ consisting of nodes in $G$ whose label sets contain $K$.

3. Let $G'$ denote the *full cell graph*, whose nodes are full cells in the nerve graph $G$. Two full cells $\sigma, \tau$ are adjacent in $G'$ if there exists a path from $\sigma$ to $\tau$ in $G$ that does not intersect any other full cell. If $\bar{G}$ is a connected component of $G$, construct the full cell graph $\bar{G}'$ similarly.

Thus the nodes of $G$ and $G_K$ are either full cells, full facets, or $d$-simplices with exactly one repeated label. The full cell graph $G'$ only has full cells as nodes.

EXAMPLE. The pentagon in Fig. 4 has dimension $d = 2$. Let $K = \{1\}$, a set of cardinality $(d-1)$. Then the derived graph $G_K$ consists of 1- and 2-simplices that appear darkly shaded, and it is a subgraph of the nerve graph $G$ consisting of the dark- and light-shaded 1- and 2-simplices in Fig. 4.

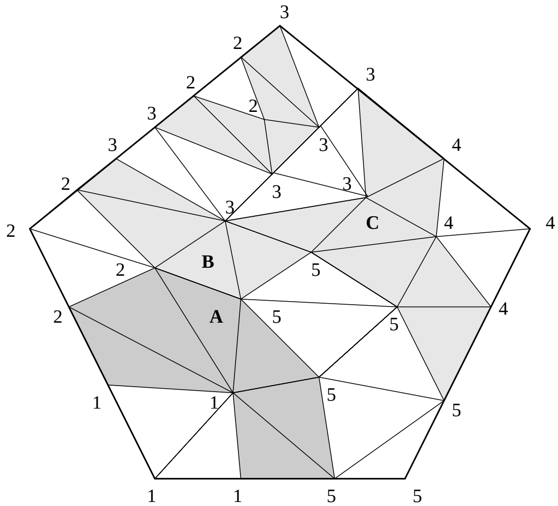


**FIG. 4.** A triangulated $(5, 2)$-polytope (a pentagon) with Sperner labelling. If $K = \{1\}$, nodes of $G_K$ consist of the dark-shaded simplices, nodes of $G$ consist of dark- and light-shaded simplices, and nodes of $G'$ consist of the three full cells marked by $A, B, C$.

In Fig. 4, $G'$ is a 3-node graph with nodes $A, B, C$, the full cells. In $G'$, $A$ is adjacent to $B$, and $B$ is adjacent to $C$, but $A$ is not adjacent to $C$.

As the example illustrates, the nerve graph $G$ branches in $(d+1)$ directions at full cells, while the derived graph $G_K$ is the subgraph consisting of paths or loops that "follow" the labels of $K$ along the boundary of the simplices in $G$. We prove these assertions.

LEMMA 10. *The nodes of the derived graph $G_K$ are either of degree* 1 *or* 2 *for any $K$ of size $d-1$. A node $\sigma$ is of degree* 1 *if and only if $\sigma$ is a full facet on the boundary of $P$. Hence $G_K$ is a graph whose connected components are either loops or paths that connect pairs of full facets on the boundary of $P$.*

*Proof.* Recall that each node $\sigma$ of $G_K$ has a label set containing $K$ and is either a full cell, full facet, or $d$-simplex with exactly one repeated label.

If $\sigma$ is a full cell, since $|K| = d-1$ we see that $L(\sigma)$ consists of labels in $K$ and two other labels $l_1, l_2$. There are exactly two facets of $\sigma$ whose label sets contain $K$; these are the full facets with label sets $K \cup l_1$ and $K \cup l_2$, respectively. Thus $\sigma$ has degree 2.

If $\sigma$ is a full facet with label set containing $K$, then it is the face of exactly two $d$-simplices, unless $\sigma$ is on the boundary of $P$, in which case it is the face of exactly one $d$-simplex. Thus $\sigma$ is degree 1 or 2 in $G_K$, and degree 1 when $\sigma$ is a full facet on the boundary of $P$.

If $\sigma$ is a $d$-simplex with exactly one repeated label, then it must possess exactly two full facets. Since $K \subset L(\sigma)$, these full facets must also have label sets that contain $K$. Hence these two full facets are the neighbors of $\sigma$ in $G_K$, so $\sigma$ has degree 2. ∎

LEMMA 11. *The nodes of the nerve graph G are of degree* 1, 2, *or* $d+1$. *A node* $\sigma$ *is of degree* 1 *if and only if* $\sigma$ *is a full facet on the boundary of P. A node* $\sigma$ *is of degree* $d+1$ *if and only if* $\sigma$ *is a full cell.*

*Proof.* As noted before, each node $\sigma$ of $G$ is either a full cell, full facet, or $d$-simplex with exactly one repeated label. The arguments for the latter two cases are identical to those in the proof of Lemma 10 by letting $K$ be the empty set.

If $\sigma$ is a full cell, then every facet of $\sigma$ is a full facet, hence the degree of $\sigma$ is $(d+1)$ in $G$. ∎

The nerve graph $G$ may have several components, as in Fig. 4. In Theorem 15, we will establish an interesting relation between the labels occurring in a component $\bar{G}$ and the number of full cells it contains. First, we show that all the labels in a component are carried by the full cells.

LEMMA 12. *If* $\sigma$ *is adjacent to* $\tau$ *in G, then* $L(\sigma) \subseteq L(\tau)$, *unless* $\sigma$ *is a full cell, in which case* $L(\tau) \subset L(\sigma)$. *Adjacent nodes in* $\bar{G}$ *carry exactly the same labels unless one of them is a full cell.*

*Proof.* If $\sigma$ is a $D$-simplex with exactly one repeated label, then it is adjacent to two full facets with exactly the same label set, so the conclusion holds. If $\sigma$ is a full cell, any simplex $\tau$ adjacent to $\sigma$ in $G$ is contained in $\sigma$ as a facet, so $L(\tau) \subset L(\sigma)$ in that case. Otherwise, if $\sigma$ is a full facet, then it is adjacent to two $d$-simplices that contain it as a facet, hence $L(\sigma) \subseteq L(\tau)$ for $\tau$ adjacent to $\sigma$.

These observations combine to show that adjacent nodes in $\bar{G}$ carry exactly the same labels unless one of them is a full cell. ∎

LEMMA 13. *Suppose* $\bar{G}$ *is connected component of G. If* $\bar{G}$ *contains at least one full cell as a node, then all the labels occurring in* $\bar{G}$ *are carried by its full cells.*

For example, in Fig. 4, $G$ has two components. One of them has no full cells. In the other component, all of its labels $\{1, 2, 3, 4, 5\}$ are carried by its full cells $A, B, C$.

*Proof.* Since $\bar{G}$ is connected and contains at least one full cell, each simplex $\sigma$ that is not a full cell is connected to a full cell $\tau$ via a path in $\bar{G}$ that does not intersect any other full cell in $\bar{G}$. Call this path $\{\sigma = \sigma_1, \sigma_2, \ldots, \sigma_p = \tau\}$. By Lemma 12, $L(\sigma_1) = L(\sigma_2) = \cdots = L(\sigma_{p-1}) \subset L(\tau)$. Therefore labels carried by the full cells contain all labels carried by any other node of the graph. ∎

Since the label information in a nerve graph is found in its full cells, it suffices to understand how the full cells connect to each other.

LEMMA 14. *Any two adjacent nodes in $G'$ are full cells in $T$ whose label sets contain at least $d$ labels in common.*

*Proof.* Let $\sigma_1$ and $\sigma_2$ be adjacent nodes in $G'$. By construction they must be simplices connected by a path in $G$; let $\tau$ be any such node along this path. Repeated application of Lemma 12 yields $L(\tau) \subset L(\sigma_1)$ and $L(\tau) \subset L(\sigma_2)$, so $L(\sigma_1) \cap L(\sigma_2)$ contains at least the $d$ labels in $L(\tau)$. ∎

We will say the full cell graph $G'$ is a *fully $d$-labelled graph* because it clearly satisfies four properties:

(a) all nodes in the graph are assigned $(d + 1)$ labels (simply assign to a node $\sigma$ of $G'$ the label set $L(\sigma)$),

(b) all edges are assigned $d$ labels (assign an edge $(\sigma_1, \sigma_2)$ of $G'$ the $d$ labels specified in Lemma 14),

(c) the label set of an edge $(\sigma, \tau)$ (denoted by $L(\sigma, \tau)$) is contained in $L(\sigma) \cap L(\tau)$, and

(d) if $\tau, \tau'$ are nodes each adjacent to $\sigma$, then $L(\tau, \sigma) \neq L(\tau', \sigma)$ (facets of a full cell $\sigma$ must have different label sets).

PROPOSITION 15. *Suppose $G'$ is a connected fully $d$-labelled graph. Let $L(G')$ denote the set of all labels carried by simplices in $G'$ and $|G'|$ the number of nodes in $G'$. Then*

$$|G'| \geqslant |L(G')| - d.$$

We shall use this theorem for graphs $G'$ arising as a full cell graph of one connected component of a nerve graph $G$. In Fig. 4, the full cell graph $G'$ has just one connected component, and $L(G') = 5$, $|G'| = 3$ and $d = 2$, and indeed $3 \geqslant 5 - 2$.

*Proof.* We induct on $|G'|$. If $|G'| = 1$, the one full cell in $G'$ has $d + 1$ labels. Hence $|L(G')| - d = (d + 1) - d = 1$, so the statement holds.

We now assume the statement holds for fully $d$-labelled graphs with less than $j$ nodes, and show it holds for fully $d$-labelled graphs $G'$ with $|G'| = j$. Assume $G'$ has $j$ full cells. We claim that it is possible to remove a vertex $v$ from $G'$ and leave $G'$ connected. This is true because $G'$ contains a maximal spanning tree, and the removal of any leaf from this tree will leave the rest of the nodes in $G'$ connected by a path in this tree.

Now $G'$ with $v$ and all its incident edges removed is a new graph (denoted by $G' - v$) with $j - 1$ nodes. Note that this new graph is still fully $d$-labelled, so by the inductive hypothesis, $|G' - v| \geqslant |L(G' - v)| - d$.

Clearly $|G' - v| = |G'| - 1$, and $|L(G' - v)| \geqslant L(G') - 1$ because $v$ has at least $d$ labels in common with some vertex in $G' - v$, by Lemma 14. Hence $|G'| - 1 \geqslant L(G') - 1 - d$. Adding 1 to both sides gives the desired conclusion. ∎

This will prove the following useful result.

Theorem 16. *Let $T$ be a Sperner-labelled triangulation of an $(n, d)$-polytope $P$. If the nerve graph $G$ has a component $\bar{G}$ that carries all the labels of $G$, then $T$ contains at least $n - d$ full cells.*

*Proof.* Suppose that there is a component $\bar{G}$ such that $L(\bar{G}) = n$. Use $\bar{G}$ to construct the full cell graph $\bar{G}'$ as above, which is a fully $d$-labelled graph. Note that if $\bar{G}$ is connected then $\bar{G}'$ is also connected. By Lemma 13, $L(\bar{G}) = L(\bar{G}')$. Using Proposition 15, we have

$$|\bar{G}'| \geqslant |L(\bar{G})| - d = n - d,$$

which shows there are at least $n - d$ full cells in $\bar{G}$, and hence in $G$ itself. ∎

Thus to prove Atanassov's conjecture for a given $(n, d)$-polytope it suffices to find some component $\bar{G}$ of the nerve graph $G$ for which $L(\bar{G}) = n$. This is the central idea of the proofs in the next sections.

We now use path-following ideas to outline a proof of Atanassov's conjecture in the special case where the polytope is simplicial. This will motivate the proof of Theorem 1 for arbitrary $(n, d)$-polytopes in the subsequent section.

Theorem 17. *If $P$ is a simplicial polytope, there is some component $\bar{G}$ of the nerve graph $G$ which meets every facet of $P$, and hence carries all labels of $G$.*

*Proof.* Let $F$ be a simplicial facet of the polytope $P$. Let $\chi(G, F)$ count the number of nodes of $G$ that are simplices in $F$. This may be thought of as the number of endpoints of paths in $G$ that terminate on the facet $F$.

Consider two "adjacent" facets $F_1, F_2$ of $P$, whose intersection is a *ridge* of the polytope $P$, i.e., a co-dimension two face of $P$ spanned by $(d-1)$ vertices of $P$. These vertices have distinct labels; let $K$ be their label set. By Lemma 10, the derived graph $G_K$ consists of loops or paths whose endpoints in $G_K$ must be full facets in $F_1$ or $F_2$, since the Sperner labelling guarantees that no other facet of $P$ has a label set containing $K$.

Since every facet of $P$ is simplicial, all the full facets in $F_1$ and $F_2$ contain $K$ in their label set. Thus all the nodes of $G$ that are full facets in $F_1$ and $F_2$ must also be nodes in the graph $G_K$. Since $G_K$ is a subgraph of $G$ and consists of paths that pair up full facets in $F_1$ and $F_2$, we see that $\chi(G, F_1) \equiv \chi(G, F_2) \bmod 2$. In fact, since paths in $G_K$ are contained within a single connected component of $G$, this argument shows that

$$\chi(\bar{G}, F_1) \equiv \chi(\bar{G}, F_2) \bmod 2$$

for any connected component $\bar{G}$ of $G$.

Since $F_1$ and $F_2$ were arbitrary, the same argument holds for any two adjacent facets. This yields the somewhat surprising conclusion that the parity of $\chi(\bar{G}, F)$ is independent of the facet $F$. We denote this parity by $\rho(\bar{G})$. Since $\chi(G, F)$ is also independent of facet, we can define $\rho(G)$ similarly.

Since $\chi(G, F)$ is the sum of $\chi(\bar{G}, F)$ over all connected components $\bar{G}$ of $G$, it follows that $\rho(G) \equiv \sum \rho(\bar{G}) \bmod 2$ over all connected components $\bar{G}$ of $G$. Moreover, $\rho(G) \equiv 1 \bmod 2$ because the usual Sperner's lemma applied to (any) simplicial facet $F$ shows that there are an odd number of full facets of $T$ in the facet $F$.

Hence there must be some $\bar{G}$ such that $\rho(\bar{G}) \equiv 1 \bmod 2$, i.e., this $\bar{G}$ meets every facet of $P$. Because the facets of $P$ are simplicial, $\bar{G}$ carries every label, i.e., $|L(\bar{G})| = n$. ∎

THEOREM 18. *Any Sperner-labelled triangulation of a simplicial $(n, d)$-polytope must contain at least $n - d$ full cells.*

*Proof.* This follows immediately from Theorems 17 and 16. ∎

To extend this proof for non-simplicial polytopes requires some new ideas but follows the basic pattern: (1) find a function $\chi$ that counts the number of times a component $\bar{G}$ of $G$ meets a certain facet in a certain way, and show that this function only depends on $\bar{G}$, and (2) appeal to the usual Sperner's lemma for simplices in a lower dimension to constructively show that the parity of $\chi$ summed over all components $\bar{G}$ must be odd. For the non-simplicial case, we cannot guarantee that any faces of $P$ except those in dimension 1 are simplicial. How to connect dimension 1 to dimension $d$ is tackled in the next section, and the *flag graph* introduced there gives a constructive procedure for finding certain full cells. Then we construct a counting function $\chi$ to show that there are at least $n-d$ full cells for an $(n, d)$-polytope.

## 4. THE FLAG GRAPH AND ARBITRARY POLYTOPES

Throughout this section, let the symbol $\equiv$ denote equivalence mod 2. Recall that $L(F)$ denotes the label set of a face $F$. We call an $i$-dimensional face of $P$ an *$i$-face* of $P$. Let $\mathscr{F}$ denote a *flag* of the polytope $P$, i.e., a choice of faces $F_1 \subset F_2 \subset \cdots \subset F_d$ where $F_i$ is an $i$-face of $P$. When the choice of $F_i$ is not understood by context, we refer to the $i$-face of a particular flag $\mathscr{F}$ by writing $F_i(\mathscr{F})$.

Given a flag $\mathscr{F}$, it will be extremely useful to construct "super-paths" containing simplices of $P$ of various dimensions whose endpoints are either on a one-dimensional edge or a $d$-dimensional full cell.

Definition. Let $P$ be an $(n, d)$-polytope with a Sperner-labelled triangulation $T$. Let $\mathscr{F} = \{F_1 \subset F_2 \subset \cdots \subset F_d\}$ be a flag of $P$. We define the *flag graph* $G_{\mathscr{F}}$ in the following way. For $1 \leqslant k \leqslant d$, a $k$-simplex $\sigma \in T$ is a node in the graph $G_{\mathscr{F}}$ if and only if $\sigma$ is one of four types:

(I) the $k$-simplex $\sigma$ is carried by the $k$-face $F_k$ and

$$|L(\sigma) \cap L(F_i)| = i + 1 \qquad \text{for all } 1 \leqslant i \leqslant k,$$

(II) the $k$-simplex $\sigma$ is carried by the $(k+1)$-face $F_{k+1}$ and

$$|L(\sigma) \cap L(F_i)| = i + 1 \qquad \text{for all } 1 \leqslant i \leqslant k,$$

(III) the $k$-simplex $\sigma$ is carried by the $k$-face $F_k$ and

$$|L(\sigma) \cap L(F_k)| = k$$

and

$$|L(\sigma) \cap L(F_i)| = i + 1 \qquad \text{for all } 1 \leqslant i \leqslant k-1,$$

(IV) the $k$-simplex $\sigma$ is carried by the $k$-face $F_k$ and there is an $I$ such that

$$|L(\sigma) \cap L(F_k)| = k + 1,$$

$$|L(\sigma) \cap L(F_i)| = i + 2 \qquad \text{for all } I \leqslant i \leqslant k-1$$

and

$$|L(\sigma) \cap L(F_i)| = i + 1 \qquad \text{for all } 1 \leqslant i < I.$$

Two nodes $\sigma$ and $\tau$ carried by $F_k$ are adjacent in $G_{\mathscr{F}}$ if (as simplices) $\sigma$ is a facet of $\tau$. Nodes $\sigma$ carried by $F_{k-1}$ and $\tau$ carried by $F_k$ are adjacent in $G_{\mathscr{F}}$ if $\sigma$ is a facet of $\tau$ *and* $\sigma$ is of type (I). There are no other adjacencies in $G_{\mathscr{F}}$.

Note that if $\sigma$ is a type (I) simplex in $G_{\mathscr{F}}$, then it is a "non-degenerate" full cell of the $k$-face that it is carried in, i.e., in every $i$-face of the flag, the vertices of $P$ corresponding to the labels in $L(\sigma)$ span an $i$-dimensional simplex (rather than something of lower dimension). A type (II) simplex is a non-degenerate full facet in the $(k+1)$-face that it is carried in. A type (III) simplex has just one repeated label and satisfies a certain kind of non-degeneracy (that ensures its two full facets are non-degenerate). A type (IV) simplex is one kind of degenerate full cell in the $k$-face that it is carried in (but such that it has exactly two facets which are non-degenerate). Adjacencies in the graph $G_{\mathscr{F}}$ can only occur between two nodes $\sigma, \tau$ carried by the same $k$-face or carried by faces differing in dimension by 1. In the former case, the adjacency conditions guarantee that the lower-dimensional simplex $\sigma$ is of type (II). In the latter case, the simplex $\sigma$ carried by the lower-dimensional face is required to be of type (I).

EXAMPLE. Let $P$ be a $(7,3)$-polytope $P$, i.e., a three-dimensional polytope with 7 vertices, and suppose $T$ is a Sperner-labelled triangulation of $P$. Let $F_1 \subset F_2 \subset F_3$ be a flag $\mathscr{F}$ of $P$ with label sets $\{1,2\} \subset \{1,2,3,4,5\} \subset \{1,2,\ldots,7\}$, respectively.

Consider the following collection of simplices shown in Fig. 5. Let $\sigma_1, \ldots, \sigma_7$ be simplices with label sets: $L(\sigma_1) = \{1,2\}$, $L(\sigma_2) = \{1,2\}$ (with repeated label 2), $L(\sigma_3) = \{1,2\}$, $L(\sigma_4) = \{1,2,3\}$, $L(\sigma_5) = \{1,2,3,4\}$, $L(\sigma_6) = \{1,2,4\}$. $L(\sigma_7) = \{1,2,4,6\}$ such that in each pair $\{\sigma_i, \sigma_{i+1}\}$, one is a facet of the other. The face $F_1$ carries $\sigma_1$; the face $F_2$ carries $\sigma_2, \sigma_3, \sigma_4$; and the face $F_3$ carries $\sigma_5, \sigma_6, \sigma_7$. Each of these simplices is a node in the graph $G_{\mathscr{F}}$: simplices $\sigma_1, \sigma_4, \sigma_7$ are type (I), $\sigma_3, \sigma_6$ are of type (II), $\sigma_2$ is of type (III), and $\sigma_5$ is of type (IV). Furthermore, each pair $\sigma_i$ and $\sigma_{i+1}$ are adjacent in $G_{\mathscr{F}}$. Except for $\sigma_1$ and $\sigma_7$, each of these simplices has exactly 2 neighbors in $G_{\mathscr{F}}$ so the above sequence traces out a path.

The following result shows that $G_{\mathscr{F}}$ does, in fact, consist of a collection of loops or paths whose endpoints are either one and $d$ dimensional.

LEMMA 19. *Every node $\sigma$ of $G_{\mathscr{F}}$ has degree 1 or 2, and has degree 1 only when $\sigma$ is a 1-simplex or a $d$-simplex in $G_{\mathscr{F}}$.*

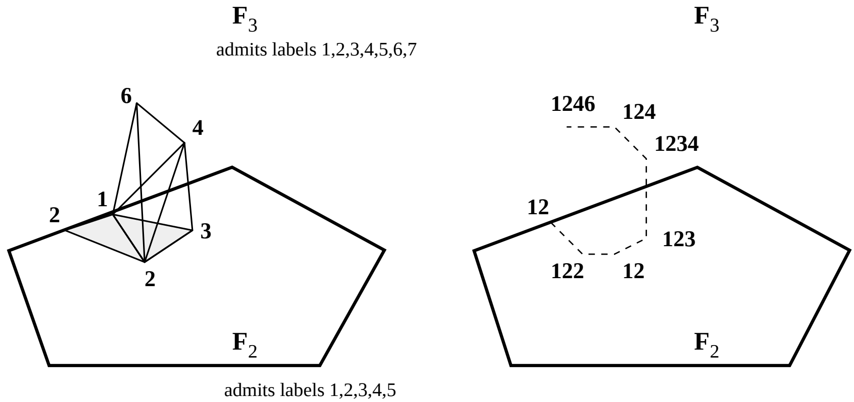


**FIG. 5.** A path in the flag-graph of a $(7, 3)$-polytope. The figure at left shows simplices along a path in the triangulation. Simplices carried by $F_2$ are shaded. The figure at right shows the label sets of the simplices along this path. Simplices $\sigma_1, \ldots, \sigma_7$ occur in counterclockwise order along this path.

*Proof.*

*Case* (I). Consider a $k$-simplex $\sigma$ of type (I). If $k \geqslant 2$, then $\sigma$ has a facet determined by the $k$ labels in $L(\sigma) \cap L(F_{k-1})$, and this facet is a $(k-1)$-simplex of type (I) or (II) so it is adjacent to $\sigma$. No other facets of $\sigma$ are types (I)–(IV). If $k \leqslant d-1$, then $\sigma$ is a facet of exactly one $(k+1)$-simplex $\tau$ (carried by $F_{k+1}$) that must be of type (I) or (III) or (IV). Thus a type (I) simplex has degree 2 unless $k = 1$ or $d$, in which case it has degree 1.

*Case* (II). A type (II) $k$-simplex $\sigma$ is the facet of exactly two $(k+1)$-simplices in $F_{k+1}$; these are either of type (I) or (III) or (IV) and are thus neighbors of $\sigma$ in $G_{\mathscr{F}}$. According to the adjacency rules for $G_{\mathscr{F}}$, if there were any other neighbor of $\sigma$, it would have to be a facet of $\sigma$ of type (I) in $F_k$ or type (II) in $F_{k+1}$. But because $\sigma$ is a $k$-simplex, a facet of $\sigma$ carried in $F_k$ cannot be of type (I), and a facet of $\sigma$ carried in $F_{k+1}$ cannot be of type (II). Thus type (II) vertices have degree 2.

*Case* (III). A type (III) $k$-simplex $\sigma$ has exactly two facets determined by the $k$ labels in $L(\sigma)$; each of these is a $(k-1)$-simplex adjacent to $\sigma$ in $G_F$ because it is either of type (I) in $F_{k-1}$ or of type (II) in $F_k$. No other facets of $\sigma$ are of type (I) or (II). Thus type (III) vertices have degree 2.

*Case* (IV). The labelling rules for a type (IV) simplex $\sigma$ show that the set $L(\sigma) \cap (L(F_l) \setminus L(F_{l-1}))$ is of size two. Call these labels $a$ and $b$. There is exactly one facet of $\sigma$ that omits the label $a$ and one facet which omits the label $b$; each of these is a $(k-1)$-simplex of type (I) in $F_{k-1}$ or of type (II) in $F_k$, so is adjacent to $\sigma$ in $G_{\mathscr{F}}$. No other facets of $\sigma$ are non-degenerate, hence the other facets cannot be of type (I) or (II). Thus type (IV) vertices have degree 2. ∎

Thus the components of $G_{\mathscr{F}}$ are paths that wind their way through nodes carried by faces of the flag. The adjacency rules for $G_{\mathscr{F}}$ require that a node $\sigma$ in a lower-dimensional face is adjacent to a node $\tau$ in the next higher-dimensional face only when $\sigma$ is of type (I). This condition is important because otherwise some nodes in $G_{\mathscr{F}}$ could have degree greater than 2. For instance, in a Sperner-labelled, triangulated $(9,4)$-polytope, suppose $F_1 \subset F_2 \subset F_3 \subset F_4$ is a flag $\mathscr{F}$ of $P$ with label sets $\{1,2\} \subset \{1,2,3,4,5\} \subset \{1,2,\dots,7\} \subset \{1,2,\dots,9\}$, respectively. If $\tau$ is a 4-simplex in $F_4$ with label set $\{1,2,3,4,6\}$ such that its face $\sigma$ with labels $\{1,2,3,4\}$ is carried in $F_3$, then both $\sigma$ and $\tau$ are of type (IV). Each already has two facets that are nodes in $G_{\mathscr{F}}$ of type (I) or (II), so we would not want to define $\sigma$ and $\tau$ to be adjacent to each other, even though one is a facet of the other.

THEOREM 20. *A Sperner-labelled triangulation of an $(n,d)$-polytope contains, for each edge $F_1$ of $P$, a non-degenerate full cell whose labels contain $L(F_1)$.*

*Proof.* For any flag $\mathscr{F}$ containing the edge $F_1$, Lemma 19 shows that $G_{\mathscr{F}}$ consist of loops or paths whose endpoints are non-degenerate full cells in $F_1$ or $F_d$; thus the total number of such endpoints (full cells in $F_1$ and in $F_d$) must be of the same parity. On the other hand, the one-dimensional Sperner's lemma shows that the number of full cells in $F_1$ is odd. So the number of non-degenerate full cells in $F_d$ in $G_{\mathscr{F}}$ must be odd. In particular there is at least one non-degenerate full cell in $F_d$ whose label set contains $L(F_1)$. ∎

Notice that the above proof is constructive; the graph $G_{\mathscr{F}}$ yields a method for locating a non-degenerate full cell for any choice of flag $\mathscr{F}$, by starting at one of the full cells on the edge $F_1$ (an odd number of them are available) and following its path component in $G_{\mathscr{F}}$. At most an even number of edges of $F_1$ are matched by paths in $G_{\mathscr{F}}$, so at least one of them is matched by a path to a non-degenerate full cell in $F_d$.

However, as we show now, more can be said about the location of full cells. Rather than locating all of them at the endpoints of paths in a flag-graph, we can show that there is some component $\bar{G}$ of the nerve graph $G$ that contains at least $n-d$ full cells. One can trace paths through this component to find them. We find a component $\bar{G}$ of $G$ that carries all labels. Theorem 15 will imply that the component must have $n-d$ full cells. As in the case for simplicial polytopes, the key rests on defining a function $\chi$ that counts the number of times that $\bar{G}$ meets a facet in a certain way, and then showing that the parity of $\chi$ exhibits a certain kind of invariance—it really only depends on $\bar{G}$. Any component with non-zero parity will be the desired component.

DEFINITION. Suppose $F$ is a facet of $P$ and $R$ is a ridge of $P$ that is a facet of $F$. Let $\chi(G, F, R)$ denote the number of nodes $\sigma$ of the nerve graph $G$ in the facet $F$ such that $|L(\sigma) \cap L(R)| = d - 1$.

Similarly if $K$ is any $(d-1)$-subset of $L(F)$, let $\chi(G, F, K)$ denote the number of nodes $\sigma$ of the graph $G$ in the facet $F$ such that $|L(\sigma) \cap K| = d - 1$.

If $\bar{G}$ is a connected component of $G$, define $\chi(\bar{G}, F, R)$ and $\chi(\bar{G}, F, K)$ similarly using $\bar{G}$ instead of $G$.

Thus $\chi(G, F, R)$ (resp. $\chi(\bar{G}, F, R)$) counts non-degenerate full cells of type (I) from $G_{\mathscr{F}}$ (resp. $\bar{G}_{\mathscr{F}}$) in the facet $F$, for all flags $\mathscr{F}$ of $P$ such that $F = F_{d-1}(\mathscr{F})$ and $R = F_{d-2}(\mathscr{F})$. It is easy to show that the parity of $\chi(G, F, R)$ is independent of both $F$ and $R$:

THEOREM 21. *Given any flag $\mathscr{F}$ of $P$, suppose $F = F_{d-1}(\mathscr{F})$ and $R = F_{d-2}(\mathscr{F})$. Then*

$$\chi(G, F, R) \equiv 1.$$

*Proof.* Consider the subgraph of $G_{\mathscr{F}}$ that contains simplices of dimension $(d-1)$ or lower. This subgraph must be a collection of loops or paths (since $G_{\mathscr{F}}$ is) whose endpoints (an even number of them) are non-degenerate full cells in either $F_1$ or $F_{d-1}$. But Sperner's lemma in one-dimension (or simple inspection) shows that the number of full cells in $F_1$ must be odd. Hence the number of non-degenerate full cells of $G_{\mathscr{F}}$ that meet $F_{d-1}$ must be odd as well. ∎

The next two theorems show that for *connected* components $\bar{G}$ of $G$, the parity of $\chi(\bar{G}, F, R)$ is also independent of $F$ and $R$. This fact does not follow directly from Theorem 21 since we do not know that endpoints of $G_{\mathscr{F}}$ for different flags are connected in $\bar{G}$. To establish this we need to trace connected paths in the nerve graph $G$ rather than the flag graph $G_{\mathscr{F}}$.

LEMMA 22. *Let $\bar{G}$ be a connected component of $G$. Suppose that $R, R'$ are ridges of $P$ and $F$ a facet of $P$ such that $R, R'$ are both facets of $F$. Then*

$$\chi(\bar{G}, F, R) \equiv \chi(\bar{G}, F, R').$$

*Proof.* First, assume that $R$ and $R'$ are "adjacent" ridges sharing a common facet $C$ (this has dimension $d-3$). We claim that

$$\sum_{A,x} \chi(\bar{G}, F, A \cup x) \equiv 0,$$

where $x$ runs over all labels in $L(F)$ that are not in $L(C)$ and $A$ runs over all $(d-2)$-subsets of $L(C)$. (Here we write $A \cup x$ instead of $A \cup \{x\}$ to reduce notation.) The above sum holds because it only counts fully labelled simplices $\sigma$ from $\bar{G}$ in $F$ that contain a $(d-2)$-subset $A$ of $L(C)$, and every such $\sigma$ appears exactly *twice* in this sum; if $L(\sigma) = A \cup a \cup b$, then $\sigma$ is counted once each in $\chi(\bar{G}, F, A \cup a)$ and in $\chi(\bar{G}, F, A \cup b)$.

On the other hand, if $K = A \cup x$ is not the label set of any ridge of $P$, then any fully labelled simplex on the boundary of $P$ that contains $K$ must be contained in the facet $F$. Since $K$ is of size $d-1$, by Lemma 10, we see that $\chi(\bar{G}, F, K) \equiv 0$ because there is an even number of endpoints of paths in $G_K$, and such paths are connected subgraphs of the connected graph $\bar{G}$.

Thus the only terms surviving the above sum correspond to label sets of the two ridges $R, R'$ that are facets of $F$ and share a common face $C$, i.e.,

$$\chi(\bar{G}, F, R) + \chi(\bar{G}, F, R') \equiv 0,$$

which yields the desired conclusion for neighboring ridges $R, R'$.

Since any two ridges of a facet $F$ are connected by a chain of adjacent ridges, the general conclusion holds. ∎

LEMMA 23. *Let $\bar{G}$ be a connected component of $G$. Let $F, F'$ be adjacent facets of the polytope $P$ bordering on a common ridge $R$. Then*

$$\chi(\bar{G}, F, R) \equiv \chi(\bar{G}, F', R).$$

*Proof.* Let $R$ denote the ridge common to both $F$ and $F'$. Let $K$ be any non-degenerate subset of $L(R)$ of size $(d-1)$, i.e., $K$ is not a subset of $L(F_i)$ for $i < (d-2)$. Consider the derived graph $G_K$. By Lemma 10, this graph consists of paths connecting full cells from $\bar{G}$ on the boundary of $P$ that contain the label set $K$. Since these paths are connected subgraphs of $\bar{G}$, there is an even number of endpoints of these paths in $\bar{G}$.

On the other hand, because of the Sperner labelling, all such endpoints must lie in facets of $P$ that contain $R$. There are exactly two such facets, $F$ and $F'$. Hence

$$\chi(\bar{G}, F, R) + \chi(\bar{G}, F', R) \equiv 0,$$

which produces the desired conclusion. ∎

Theorem 24. *Let $\bar{G}$ be a connected component of $G$. The parity of $\chi(\bar{G}, F, R)$ is independent of $F$ and $R$.*

*Proof.* Since all facet-ridge pairs $(F, R)$ are connected by a sequence of adjacent facets and ridges, the statement follows from Lemmas 22 and 23. ∎

Hence we may define the *parity* of $\bar{G}$ to be $\rho(\bar{G}) \equiv \chi(\bar{G}, F, R)$ for any facet-ridge pair $(F, R)$. Similarly, define the *parity* of $G$ to be $\rho(G) \equiv \chi(G, F, R)$ for any facet-ridge pair $(F, R)$, which is well defined and congruent to 1 in light of Theorem 21. Now we may prove

Theorem 25. *If $P$ is an $(n, d)$-polytope, there is some component $\bar{G}$ of $G$ which carries all labels of $P$.*

*Proof.* Fix some flag $\mathscr{F}$ of $P$, and let $F = F_{d-1}(\mathscr{F})$ and $R = F_{d-2}(\mathscr{F})$. Since $\chi(G, F, R)$ is the sum of $\chi(\bar{G}, F, R)$ over all connected components $\bar{G}$ of $G$, it follows that $\rho(G) \equiv \sum \rho(\bar{G})$ over all connected components $\bar{G}$ of $G$. Moreover, Theorem 21 shows that $\rho(G) \equiv 1$.

Hence there must be some $\bar{G}$ such that $\rho(\bar{G}) \equiv 1$, i.e., this $\bar{G}$ carries the labels in $L(R)$. Since the flag $\mathscr{F}$ was arbitrary, $\bar{G}$ must carry all labels of $P$. ∎

This concludes our alternate "path-following" proof of Theorem 1, because the $n - d$ count follows immediately from Theorems 25 and 16, while the covering property follows (as before) from Proposition 3.

## 5. CONCLUSION

As an application of Theorem 1, we establish a version of a KKM-type intersection result of [14, Theorem 10] which now includes more cardinality information.

Corollary 26. *Let $P$ be an $(n, d)$-polytope with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Let $\{C^j \mid j = 1, \ldots, n\}$ be a collection of closed sets covering the $(n, d)$-polytope $P$, such that each face $F$ is covered by $\cup\{C^h \mid \mathbf{v}^h \in F\}$.*

*Then, for each $p \in P$, there exists a subset $J_p \subset \{1, 2, \ldots, n\}$ such that (1) $p$ lies in the convex hull of the vertices $\mathbf{v}_j$ with $j \in J_p$, (2) $J_p$ has cardinality $d + 1$, (3) $\bigcap_{j \in J_p} C^j \neq \emptyset$, and (4) if $p$ and $q$ are interior points of the same chamber of $P$, then $J_p = J_q$. There are at least $c(P)$, the covering number, different such subsets, and the simplices of $P$ indicated by the labels in these subsets form a cover of $P$.*

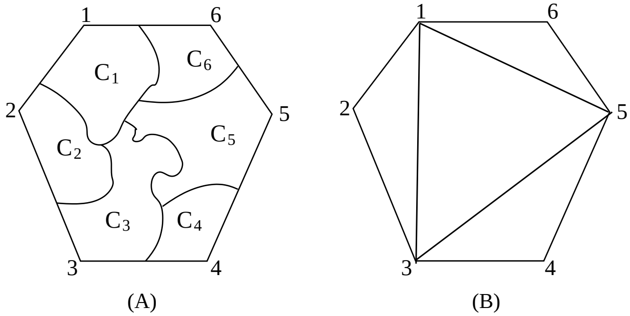


**FIG. 6.** Part (A) shows several closed sets covering a hexagon and their four intersection points. The points of intersection correspond to a cover of the hexagon, in this case a triangulation, illustrated in part (B).

Fig. 6 illustrates with an example the content of the above corollary.

*Proof of Corollary* 26. Let $C^j$, $j = 1, \ldots, n$ be the closed sets in the statement. Consider an infinite sequence of triangulations $T_k$ of the polytope $P$ with the property that the maximal diameter of their simplices tends to zero as $k$ goes to infinity. For each triangulation, we label a vertex $y$ of $T_k$ with $i = \min\{j \in \{1, 2, \ldots, n\} \mid y \in C^j\}$. This is clearly a Sperner labelling.

By Theorem 1, each triangulation $T_k$ specifies a collection of simplices of $P$ corresponding to full cells in $T_k$. There are only finitely many possible collections (since they are subsets of the set of all simplices of $P$), and because there are infinitely many $T_k$, some collection $C$ of simplices must be specified infinitely many times by a subsequence $T_{k_i}$ of $T_k$. By Proposition 3, this collection $C$ is a cover of $P$ and therefore has at least $c(P)$ elements.

For each simplex $\sigma$ in $C$, choose one full cell $\sigma_i$ in $T_{k_i}$ that shares the same label set. The $\sigma_i$ form a sequence of simplices decreasing in size. By the compactness of $P$, some subsequence of these triangles converges to a point, which (by the labelling rule) must be in the intersection of the closed sets $C^j$ with $j \in L(\sigma)$.

Thus given a point $p \in P$, choose any simplex $\sigma$ of $C$ that contains $P$ (since $C$ is a cover of $P$), and let $J_p = L(\sigma)$. Then the above remarks show that $J_p$ satisfies the conditions in the conclusion of the theorem. Moreover, there are at least $c(P)$ different such subsets, one for each $\sigma$ in $C$. ∎

We close with a couple of questions. For a specific polytope $P$, define the *pebble number* $p(P)$ to be the size of its largest pebble set. The $n - d$ lower bound of Theorem 1 is tight, achieved by stacked polytopes whose vertices are assigned different labels. But for a specific polytope $P$, the arguments of Section 2 show that the lower bound $n-d$ can be improved to $p(P)$. What can be said about the value of $p(P)$?

We can provide at least two upper bounds for this number. On the one hand $p(P) \leqslant c(P)$, because for a maximal pebble set, at most one pebble lies in each simplex of a minimal size cover. On the other hand, consider the simplex-chamber incidence $0/1$ matrix $M$ introduced in [1]. As the columns correspond to chambers a pebble selection is essentially a selection of a "row-echelon" submatrix; therefore the rank of $M$ is an upper bound on the size of pebble sets. Our pebble construction gives an algorithm for selecting an explicit independent set of columns of $M$ (although this may not always be a basis).

A related question is: for a specific polytope $P$, how can one determine the minimal cover size $c(P)$? Although Theorem 2 gives a general sharp lower bound for all polytopes, we know that sometimes minimal covers are much larger for specific polytopes, such as for cubes (as the volume arguments in [19] show). Also note that the minimal cover may be strictly smaller than the minimal triangulation (an example is contained in [6]).

Finding other explicit constructions of pebble sets (besides our "facet-pivoting" construction of Section 2) that work for specific polytopes may shed some light on these questions.

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## REFERENCES

1. T. V. Alekseyevskaya, Combinatorial bases in systems of simplices and chambers, *Discrete Math.* **157** (1996), 15–37.
2. K. T. Atanassov, On Sperner's lemma, *Studia Sci. Math. Hungar.* **32** (1996), 71–74.
3. D. Barnette, A proof of the lower bound conjecture for convex polytopes, *Pacific J. Math.* **46** (1973), 349–354.
4. L. J Billera, I. M. Gel'fand, and B. Sturmfels, Duality and minors of secondary polyhedra, *J. Combin. Theory Ser. B* **57** (1993), 258–268.
5. A. Björner, J. Matousek, and G. Ziegler, "Topological Combinatorics," book manuscript, in preparation.
6. A. Below, U. Brehm, J. A. De Loera, and J. Richter-Gebert, Minimal simplicial dissections and triangulations of convex 3-polytopes, *Discrete Comput. Geom.* **24** (2000), 35–48.
7. J. A. De Loera, F. Santos, and F. Takeuchi, Extremal properties for dissections of convex 3-polytopes, *SIAM J. Discrete Math.* **14** (2001), 143–161.

8. D. I. A. Cohen, On the Sperner lemma, *J. Combin. Theory* **2** (1967), 585–587.
9. Reference deleted in proof.
10. R. M. Freund, Combinatorial analogs of Brouwer's fixed-point theorem on a Bounded polyhedron, *J. Combin. Theory Ser. B* **47** (1989), 192–219.
11. Reference deleted in proof.
12. B. Knaster, C. Kuratowski, and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für $n$-dimensionale Simplexe, *Fund. Math.* **14** (1929), 132–137.
13. H. W. Kuhn, Simplicial approximation of fixed points, *Proc. Natl. Acad. Sci.* **61** (1968), 1238–1242.
14. G. van der Laan, D. Talman, and Z. Yang, Intersection theorems on polytopes, *Math. Programming* **84** (1999), 25–38.
15. E. Peterson, "Combinatorial Proofs of Generalizations of Sperner's Lemma," Senior thesis, Harvey Mudd College, 2000.
16. B. L. Rothschild and E. G. Straus, On triangulations of the convex hull of $n$ points, *Combinatorica* **5** (1985), 167–179.
17. H. Scarf, The approximation of fixed points of a continuous mapping, *SIAM J. Appl. Math.* **15** (1967), 1328–1343.
18. F. W. Simmons and F. E. Su, Consensus-halving via theorems of Borsuk-Ulam and Tucker, preprint.
19. W. D. Smith, A lower bound for the simplexity of the $n$-cube via hyperbolic volumes, special issue: Combinatorics of polytopes, *European J. Combin.* **21** (2000), 131–137.
20. E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg* **6** (1928), 265–272.
21. F. E. Su, Rental harmony: Sperner's lemma in fair division, *Amer. Math. Monthly* **106** (1999), 930–942.
22. M. J. Todd, "The Computation of Fixed Points and Applications," Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, New York, 1976.
23. Z. Yang, "Computing Equilibria and Fixed Points," Kluwer Academic Publishers, Boston, 1999.
24. G. M. Ziegler, "Lectures on Polytopes," Springer-Verlag, New York, 1995.