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# The LSB Theorem Implies the KKM Lemma

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# The LSB Theorem Implies the KKM Lemma

**Gwen Spencer and Francis Edward Su**

Let $S^d$ be the unit $d$-sphere, the set of all points of unit Euclidean distance from the origin in $\mathbb{R}^{d+1}$. Any pair of points in $S^d$ of the form $x, -x$ is a pair of *antipodes* in $S^d$. Let $\Delta^d$ be the $d$-simplex formed by the convex hull of the standard unit vectors in $\mathbb{R}^{d+1}$. Equivalently, $\Delta^d = \{(x_1, \ldots, x_{d+1}) : \sum_i x_i = 1, x_i \geq 0\}$. The following are two classical results about closed covers of these topological spaces (for the first see [**6**] or [**3**], for the second see [**5**]):

**The LSB Theorem (Lusternik-Schnirelmann-Borsuk).** *If $S^d$ is covered by $d+1$ closed sets $A_1, \ldots, A_{d+1}$, then some $A_i$ contains a pair of antipodes.*

**The KKM Lemma (Knaster-Kuratowski-Mazurkiewicz).** *If $\Delta^d$ is covered by $d+1$ closed sets $C_1, C_2, \ldots, C_{d+1}$ such that each $x$ in $\Delta^d$ belongs to $\cup\{C_i : x_i > 0\}$, then the sets $C_i$ have a common intersection point (i.e., $\cap_{i=1}^{d+1} C_i$ is nonempty).*

A cover satisfying the condition in the KKM lemma is sometimes called a *KKM cover*. It can be described in an alternate way: associate labels $1, 2, \ldots, d+1$ to the vertices of $\Delta^d$ and demand that each face of $\Delta^d$ be covered by the sets that correspond to the vertices spanning that face. (Thus vertex $i$ is covered by set $C_i$, the edge between $i$ and $j$ are covered by $C_i \cup C_j$, etc.)

Both of the foregoing set-covering results are perhaps best known in connection with their equivalent formulations in topology: the LSB theorem is equivalent to the Borsuk-Ulam theorem [**3**], and the KKM lemma is equivalent to the Brouwer fixed point theorem [**5**]. Also, the LSB theorem has found spectacular application in proofs of the Kneser conjecture in combinatorics [**1**], [**4**]. The KKM lemma has numerous applications in economics (see, for example, [**2**]).

Since the Brouwer fixed point theorem can be obtained as a consequence of the Borsuk-Ulam theorem [**7**], it is natural to ask whether there is a direct proof of the KKM lemma using the LSB theorem. The purpose of this note is to provide such a proof.

**Theorem.** *The LSB theorem implies the KKM lemma.*

Consider $\Sigma^d$, a $d$-sphere under the $L^1$ norm:

$$\Sigma^d := \{(x_1, \ldots, x_{d+1}) : \sum_i |x_i| = 1\}.$$

Observe that the LSB theorem holds for $\Sigma^d$, since $\Sigma^d$ and $S^d$ are related by an antipode-preserving homeomorphism. Note that $\Sigma^2$ is just the boundary of a regular octahedron, while for general $d$, $\Sigma^d$ is the boundary of the $(d+1)$-*crosspolytope*. It is the union of $2^{d+1}$ facets that are simplices, one for each orthant of $\mathbb{R}^{d+1}$ (see Figure 1).

It will be convenient, then, to use the LSB theorem for $\Sigma^d$ to prove the KKM lemma, because $\Delta^d$ is naturally embedded in $\Sigma^d$; namely, $\Delta^d$ is the facet of $\Sigma^d$ for which $\sum_i x_i = 1$. Call this facet $F_{\text{top}}$, the "top" facet, and call the antipodal facet the "bottom" facet $F_{\text{bot}}$. Let $F_{\text{mid}}$ signify the complement of $F_{\text{top}} \cup F_{\text{bot}}$ in $\Sigma^d$, the "middle" band of

  

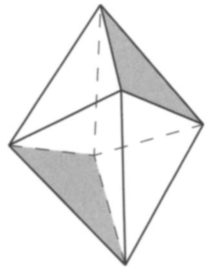

**Figure 1.** The 2-sphere $\Sigma^2$ in the $L^1$-norm, which is the boundary of an octahedron. The "top" and "bottom" facets are shaded.

the $d$-sphere. The strategy of our proof is to assume for the sake of contradiction that a KKM cover of $\Delta^d$ has no common intersection point. Then we extend these sets to construct a closed cover of $\Sigma^d$ whose sets contain no pair of antipodes, thereby contradicting the LSB theorem.

*Part 1 of proof: construction.* We first consider the case where a given KKM cover $C_1, \dots, C_{d+1}$ of $\Delta^d$ is *nondegenerate* (i.e., for each $x$ in $\Delta^d$ and set $C_i$, $x$ is in $C_i$ only if $x_i > 0$). In the alternate characterization of the KKM cover, this means that each face is covered *only* by the sets that correspond to the vertices spanning that face. For example, the figure at left in Figure 2 is degenerate because the white set covers a point on the bottom edge of the triangle.

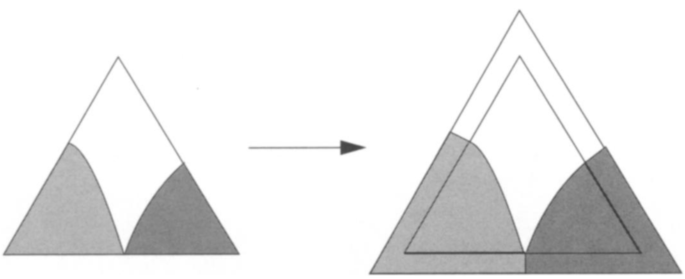

**Figure 2.** In these diagrams, the sets are closed and thus contain their boundaries. At left the simplex $\Delta^d$ has a degenerate KKM cover because the white (nonshaded set) covers a point on the bottom edge. At right, the same KKM cover of $\Delta^d$ has been "thickened" to form a nondegenerate KKM cover of a larger simplex $\Delta'^d$.

For the sake of contradiction, assume that there is no point common to all the sets $C_1, \dots, C_{d+1}$. For each $i$ let $-C_i$ be the set in $F_{\text{bot}}$ antipodal to $C_i$. Let $B_i$ be the complement of $-C_i$ in $F_{\text{bot}}$. By assumption every point of $F_{\text{top}}$ is excluded from at least one $C_i$. Hence the complementary sets $B_i$ form an open cover of $F_{\text{bot}}$ (in the relative topology). Moreover, the sets $B_i$ satisfy a certain kind of nondegeneracy that follows from the nondegeneracy of the $C_i$: for $x$ in $F_{\text{bot}}$, $x_i = 0$ implies that $x$ is covered by $B_i$. By normality, the sets $B_i$ can be shrunk to obtain closed subsets $E_i$ of $B_i$ that still cover $F_{\text{bot}}$ and satisfy the same nondegeneracy.

Now that $F_{\text{bot}}$ has been covered, we construct a cover of $F_{\text{top}} \cup F_{\text{mid}}$. For $x = (x_i)$ in $\Sigma^d$ let $\text{pos}(x) = \sum_{x_i>0} x_i$. Note that $\text{pos}(x) = 0$ on $F_{\text{bot}}$ but $\text{pos}(x) > 0$ on $F_{\text{top}}$ and $F_{\text{mid}}$. Define a function $f = (f_i)$ on $F_{\text{top}} \cup F_{\text{mid}}$ as follows:

$$f_i(x) = \frac{x_i + |x_i|}{2\,\text{pos}(x)}.$$

Then $f$ is a continuous function taking $F_{\text{top}} \cup F_{\text{mid}}$ to $F_{\text{top}}$, and it fixes $F_{\text{top}}$ pointwise.

The set $D_i := f^{-1}(C_i)$ is a closed subset of $F_{\text{top}} \cup F_{\text{mid}}$ in the relative topology. We may think of $D_i$ as extending the set $C_i$ on $F_{\text{top}}$ to cover $F_{\text{mid}}$. In fact, $D_i$ extends the boundary of $C_i$ in a linear fashion across $F_{\text{mid}}$ (see Figure 3). We record some observations about the sets $D_i$:

**Observation 1.** *Since the $C_i$ cover $F_{\text{top}}$, the $D_i$ cover $F_{\text{top}} \cup F_{\text{mid}}$.*

**Observation 2.** *Since $f$ fixes $F_{\text{top}}$, each $D_i$ restricted to $F_{\text{top}}$ is just $C_i$.*

**Observation 3.** *If $x$ is in $D_i$, then $x_i > 0$.*

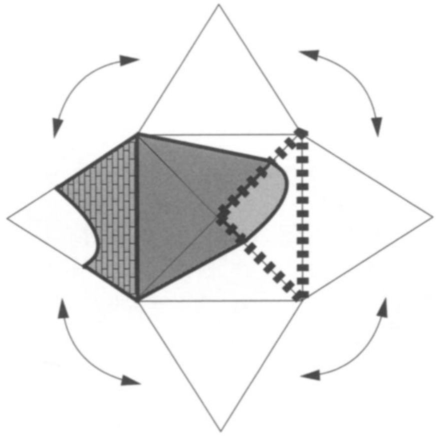

**Figure 3.** The octahedral 2-sphere $\Sigma^2$ unfolded, with shaded set $A_i$ derived from a set $C_i$ in the KKM cover. (The set $A_i$ consists of three regions: light-shaded, dark-shaded, and bricked. The light-shaded region is $C_i$; it sits in $F_{\text{top}}$, the triangle with dashed outline. The set $D_i$ extends $C_i$ and includes both the light-shaded and dark-shaded regions of $A_i$. The bricked region is $E_i$; it sits in the facet antipodal to $F_{\text{top}}$. Note its relation with $C_i$.)

The first two observations are apparent from the definition of $f$, and the last follows by noting that if $x$ is in $D_i$, then $f(x)$ is in $C_i$, so the nondegeneracy of $C_i$ implies that $f_i(x) > 0$. But this can only occur if $x_i > 0$.

Now let $A_i = D_i \cup E_i$. We verify that the $A_i$ cover $\Sigma^d$ and are closed sets, but that no $A_i$ contains a pair of antipodes. This verification will contradict the LSB theorem, forcing us to reject our initial assumption that the KKM cover had no common intersection point. ■

*Part 2 of proof: verification.* Clearly the $A_i$ cover $\Sigma^d$. This is a consequence of Observation 1 and the fact that the $E_i$ cover $F_{\text{bot}}$.

To show that $A_i$ is closed, note that $E_i$ is a closed subset of $\Sigma^d$ and $D_i$ is closed in $F_{\text{top}} \cup F_{\text{mid}}$ (but not necessarily $\Sigma^d$). Thus it suffices to show that any limit points of $D_i$ in $F_{\text{bot}}$ must lie in $E_i$. Observation 3 implies that a limit point $x$ of $D_i$ must satisfy $x_i \geq 0$, but since points in $F_{\text{bot}}$ have no positive coordinates, a limit point of $D_i$ in $F_{\text{bot}}$ must satisfy $x_i = 0$. By the nondegeneracy of $E_i$, $x$ must be in $E_i$.

To show that $A_i$ contains no pair of antipodes, we observe that $E_i$ cannot contain a pair of antipodes. In light of Observation 3, neither can $D_i$, because $x_i$ and $-x_i$ cannot both be positive when $x$ lies in $D_i$. All that remains for us to check is that there is no $x$ in $D_i$ such that $-x$ is in $E_i$. But this can only occur if $x$ is in $F_{\text{top}}$. By construction $C_i$ cannot have antipodes in $E_i$, so Observation 2 shows that $D_i$ has no antipodes in

$E_i$. Hence the $A_i$ form a cover of $\Sigma^d$ by $d+1$ closed sets, yet no $A_i$ contains a pair of antipodes. This contradicts the LSB theorem. ∎

*Part 3 of proof: degenerate KKM covers.* Finally, we consider the case where the KKM cover is degenerate. We claim that a degenerate cover of $\Delta^d$ can be made nondegenerate by "thickening" up the boundary and extending the cover in a way that introduces no new common intersection point.

Fix some $\epsilon > 0$, and let $\Delta'^d$ be a regular $d$-simplex that shares the same barycenter as $\Delta^d$ but is enlarged by factor $1+\epsilon$. Note that $\Delta'^d$ is a "thickened" version of $\Delta^d$. Given a KKM cover of $\Delta^d$ by $\{C_1, C_2, \dots, C_{d+1}\}$, we construct a KKM cover $\{C'_1, C'_2, \dots, C'_{d+1}\}$ of $\Delta'^d$ that is nondegenerate. Consider any $x$ in $\Delta'^d \setminus \Delta^d$. The line from $x$ to the barycenter intersects the boundary of $\Delta^d$ at a unique point, call it $r(x)$. (In fact, $r(x)$ retracts $\Delta'^d \setminus \Delta^d$ onto the boundary of $\Delta^d$.) Let

$$C'_i = C_i \cup \{x : r(x) \in C_i \text{ and } r_i(x) > 0\},$$

where $r_i(x)$ is the $i$th coordinate of $r(x)$. One can check that the $C'_i$ are closed, and by construction there are no points of $\cap C'_i$ in $\Delta'^d \setminus \Delta^d$ (see Figure 2). This "thickened," nondegenerate cover can then be used as in the first part of this proof. ∎

We remark that, although our proof of the KKM lemma is nonconstructive, the asserted KKM intersection is hiding in our construction in the following way. When we assume (falsely) that the asserted KKM intersection does not exist, we are (wrongly) led to conclude that the $B_i$ cover the bottom facet of $\Sigma^d$. In actuality, these open sets do not cover the bottom facet; the set of points that are exposed are precisely the points whose antipodes comprise the asserted KKM intersection in the top facet.

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