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An Inverse Function Theorem via Continuous Newton's Method

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Abstract

We prove an inverse function theorem of the Nash-Moser type. The main difference between our method and that of [4] is that we use continuous steepest descent while [4] uses a combination of Newton type iterations and approximate inverses. We bypass the *loss of derivatives problem* by working on finite dimensional subspaces of infinitely differentiable functions.

Key words: Inverse Function Theorem Continuous Newton's Method

1 Introduction

Inverse function theorems have long had a prominent role in analysis, particularly in the study of differential equations. In [4] there is an inverse function theorem whose proof uses a form of conventional Newton's method. In the present note we use a version of continuous Newton's method to give a new inverse function theorem. Our hypothesis is suggested from [4] but a direct comparison has not yet been established. We believe, however, that the present result covers a substantial portion of the cases covered by the hypothesis in [4].

Suppose each of H and K is a Banach space, $r > 0$ and F is a C^1 function from the open ball $B_r(0)$ in H, centered at 0, so that $F(0) = 0$. We intend to give a condition on a member g of K so that there exists $\gamma > 0$ such that $tg \in R(F)$ if $0 \leq t \leq \gamma$. In intended applications, H may be the Sobolev space $H^{1,2}(\Omega)$ for some region Ω in a Euclidean space and K may be $L_2(\Omega)$. The function F then may be a nonlinear differential operator.

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For a motivating example we essentially follow [4] by choosing F defined by

$$F(u) = u_1 \text{ for } u \in H^{1,2}(\Omega) \tag{1}$$

where here $\Omega = [0,1]^2$ and the subscript on u indicates partial differentiation in the first argument of u. Some reflection yields that the range of F can not be all of $L_2(\Omega)$ since many members g of that space lack sufficient smoothness to be in that range. For some members g of $L_2(\Omega)$, namely those members of $L_2(\Omega)$ which are also in $H^{1,2}(\Omega)$, there is a solution u to

$$F(u) = g$$

which is in $H^{1,2}(\Omega)$. To be more specific, suppose $r > 0$ and H' is a subset of $H^{1,2}(\Omega)$, uniformly bounded in the norm of that space. Observe that there is $\gamma > 0$ so that if $0 \leq t \leq \gamma$ and $g \in H'$, then there is

$$u \in H^{1,2}(\Omega) \text{ with } \|u\|_{H^{1,2}(\Omega)} \leq r$$

so that

$$F(u) = tg.$$

2 An Inverse Function Theorem

Return to the general setting of the introduction, that is suppose $r > 0$,

$$F: \; B_r(0) \subset H \to K,$$

F is so that $F(0) = 0$ and F is C^1.

Theorem 1 *Suppose $g \in K$ and there is a function h with domain $B_r(0) \subset H$ which is Lipschitz continuous so that*

$$F'(x)h(x) = g \text{ for } \|x\|_H \leq r.$$

Then there is $\gamma > 0$ so that if $0 \leq t \leq \gamma$, there is $u \in B_r(0)$ so that

$$F(u) = tg.$$

Proof: Under the hypothesis of the theorem, denote by γ a positive number so that there is a unique solution z on $[0, \gamma]$ to

$$z(0) = 0 \text{ and } z'(t) = h(z(t)) \text{ with } \|z(t)\| \leq r \text{ for } t \in [0, \gamma] \tag{2}$$

(that there is such a number γ follows from the basic existence and uniqueness theorem for ordinary differential equations). Then

$$F'(z(t))z'(t) \;=\; F'(z(t))(h(z(t)) \;=\; g,$$

i.e.,

$$(F(z))'(t) \;=\; g \text{ for } t \in [0,\gamma].$$

Hence,

$$F(z(t)) \;=\; tg \text{ for } \in [0,\gamma],$$

and the argument is complete.

Note that we have not required uniqueness of solution $k \in H$ to

$$F'(x)k \;=\; g \tag{3}$$

for any $x \in B_r(0)$, but rather that solutions to 3 for various $x \in B_r(0)$ can be fit together in a smooth enough way in order to provide a function h satisfying the hypothesis of the theorem.

3 Discussion

Here we attempt to justify calling the above process a version of continuous Newton's method. Suppose here that F is a function from H to K so that $F(0) = 0$ and $(F'(x))^{-1}$ exists and is in $L(K,H)$ for each $x \in H$. Suppose furthermore that $(F'(\cdot))^{-1}$ is locally Lipschitz. Given $g \in K$, conventional continuous Newton's method for finding $u \in H$ such that $F(u) = g$ might consist of first finding $z : [0,\infty) \to H$ so that

$$z(0) = 0 \text{ and } z'(t) \;=\; -(F'(z(t)))^{-1}(F(z(t)) - g) \text{ for } t \geq 0 \tag{4}$$

and then seeking

$$u \;=\; \lim_{t\to\infty} z(t)$$

so that $F(u) = g$.

Assuming we have z satisfying (4), we then have

$$F'(z(t))z'(t) \;=\; -(F(z(t)) - g) \text{ for } t \geq 0,$$

and consequently

$$(F(z) - g)'(t) \;=\; -(F(z(t)) - g) \text{ for } t \geq 0.$$

Hence we have

$$F(z(t)) - g = e^{-t}(F(z(0)) - g) \text{ for } t \geq 0. \tag{5}$$

Using 5 and the fact that $F(z(0)) = 0$ we substitute in 4 to obtain an alternate expression

$$z(0) = 0,\ z'(t) = e^{-t}(F'(z(t))^{-1}g \text{ for } t \geq 0.$$

Deleting the exponential factor in the above just changes the time scale from $[0, \infty)$ to $[0, 1)$, which leaves us with

$$z(0) = 0,\ z'(t) = (F'(z(t)))^{-1}g \text{ for } t \in [0, 1).$$

This provided motivation for the process 2 in which it is not assumed that $(F'(x))^{-1}$ for $x \in B_r(0)$, exist, but rather that for a fixed $g \in K$ and any $x \in B_r(0)$, there is $k \in H$ such that

$$F'(x)k = g$$

(and these solutions k for $x \in B_r(0)$ can be fit together to make a function h as in the hypothesis of the theorem).

4 Application: range of the sum of two maximal monotone operators

Finally we consider the semilinear boundary value problem

$$-\Delta u + f(u) = g \text{ in } \Omega,\quad u(x) = 0 \text{ for } x \in \partial\Omega, \tag{6}$$

where Ω is a smooth bounded region in R^n with $n > 2$, and $f : R \to R$ is a monotonically increasing function with supercritical growth (i.e.,

$$\liminf_{|t|\to\infty} f(t)/t^{\rho} > 0$$

for some $\rho > (n+2)/(n-2)$) and has a locally Lipschitzian derivative. For the sake of simplicity in the calculations we assume that $f(0) = 0$. Due to the growth of f, the sum of the maximal monotone operators defined by $-\Delta$ and f is not maximal monotone (see [1]) and general theory does not provide adequate information on its range. However letting $p > \min\{1, n/2\}$ and taking H to be the Sobolev space $H_0^{2,p}(\Omega)$ of functions having second order derivatives in $L^p(\Omega)$ and vanishing on the boundary of Ω, and K the space $L^p(\Omega)$ we see that for each $u \in H, g \in K$ the equation

$$-\Delta v + f'(u)v = g \text{ in } \Omega,\quad u(x) = 0 \text{ for } x \in \partial\Omega, \tag{7}$$

has a unique solution $v = h(u)$ (see Thoerem 9.15 of [3]). In order to apply Theorem 1 we prove the following result.

Lemma 1 *The function $h : H \to H$ given by $h(u) = v$, where u, v are as in (7), is bounded on bounded sets and locally Lipzchitzian.*

Proof: Let B be a bounded subset of H. By the Sobolev imbedding theorem (see Corollary 7.11 of [3]), without loss of generality we may assume that $\|u\|_\infty \leq 1$ for $u \in B$. Let $t > ((n-2)p - n)/n$ be an odd positive integer. Multiplying (7) by v^t, and using that $f' \geq 0$ and Holder's inequality we have

$$\begin{aligned} \int_\Omega \nabla v^{(t+1)/2} \cdot \nabla v^{(t+1)/2}\, dx &\leq \frac{(t+1)^2}{4\,t} \int_\Omega g v^t\, dx \\ &\leq \frac{(t+1)^2}{4\,t} \|g\|_s \|v\|^t_{(t+1)n/(n-2)}, \end{aligned} \tag{8}$$

where $s = (t+1)n/(n+2t) \leq n/2$. Let $C > 0$ (see again Corollary 7.11 of [3]) be a constant such that

$$\Big(\int_\Omega |w|^{2n/(n-2)} dx\Big)^{(n-2)/(2n)} \leq C\Big(\int_\Omega \nabla w \cdot \nabla w dx\Big)^{1/2}, \tag{9}$$

for all $w \in H_0^{1,2}$. Taking $w = v^{(t+1)/2}$ and using (8) we see that

$$\|v\|_{n(t+1)/(n-2)} \leq \frac{C^2(t+1)^2}{4\,t} \|g\|_s. \tag{10}$$

Since $s \leq n/2 < p$, we see that there exists a constant M such that

$$\|v\|_p \leq M \|g\|_{n/2}. \tag{11}$$

Thus $\|g - f'(u)v\|_p \leq \|g\|_p + M_1 \|v\|_p$, where $M_1 =$ $\leq \|u\|_\infty +$ $1\}$. Therefore by *a priori* estimes for elliptic bounaary vaiue problems (see Theorem 9.15 of [3]) we infer $\|v\|_H \leq M_2$, with M_2 depending on M_1, Ω, and p. Thus h is bounded in bounded sets.

Let $u_1, u_2, v_1, v_2 \in H$ be such that

$$-\Delta v_i + f'(u_i) v_i = g \tag{12}$$

for $i = 1, 2$, with $\|u_1 - u_2\|_H \leq 1$. An elementary algebraic manipulation shows that $-\Delta(v_1 - v_2) + f'(u_1)(v_1 - v_2) = (f'(u_2) - f'(u_1))v_2$. From Lemma 9.17

of [3], there exists a positive constant C_q for each $q \in (1, \infty)$ such that if

$$-\Delta w + f'(u_1)w = y \tag{13}$$

with $y \in L^q(\Omega)$ then $w \in H_0^{2,q}(\Omega)$ and

$$\|w\|_{H_0^{2,q}(\Omega)} \leq C_q \|y\|_q. \tag{14}$$

Also by the Sobolev imbedding theorem there exists a constant $\bar{C}$ such that

$$\|z\|_\infty \leq \bar{C}\|z\|_H, \tag{15}$$

for all $z \in H$. Therefore

$$\begin{aligned} \|v_1 - v_2\|_H &\leq C_p\|(f'(u_1) - f'(u_2))v_2\|_K \leq AC_p\|u_1 - u_2\|_\infty \|v_2\|_K \\ &\leq AC_pM_2\|u_1 - u_2\|_\infty \leq AC_pM_2\bar{C}\|u_1 - u_2\|_H, \end{aligned} \tag{16}$$

where A is a Lipschitz constant for f' on $[-\bar{C}(\|u\|_\infty + 1), \bar{C}(\|u\|_\infty + 1)]$. This proves that h is a locally Lipchitzian function, which proves the lemma.

Now combining Lemma 1 and Theorem 1 we prove that for any $g \in K$ the equation (6) has a solution.

Theorem 2 *Under the above assumptions, for each $g \in K$ the equation (6) has a solution.*

Proof: Let $F : H \to K$ be the operator defined by $F(u) = -\Delta u + f \circ u$. Let $z(t)$ be as in (2). Multiplying the equation $F(z(t)) = tg$ by $|f(z(t))|^{p-2} f(z(t))$ we see that

$$\int_\Omega |f(z(t))|^p dx \leq t \int_\Omega |g||f(z(t))|^{p-1} dx.$$

Hence, by Holder's inequality, we see that $\|f(z(t))\|_K$ is bounded on bounded intervals of $[0, \infty)$. Thus, by Theorem 9.15 of [3], $\|z(t)\|_H$ is bounded on bounded intervals. Since h is defined on all of H it follows that z is defined on $[0, \infty)$. In particular $F(z(1)) = g$, which proves the theorem.

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