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Existence of Solutions for a Wave Equation with Non-monotone Nonlinearity and a Small Parameter

José F. Caicedo, Alfonso Castro and Rodrigo Duque*

Abstract. We provide sufficient conditions for the existence of solutions to a semilinear wave equation with non-monotone nonlinearity involving a small parameter. Our results are based on the analysis of a an operator that characterizes the projection onto the kernel of the wave operator subject to *periodic-Dirichlet* boundary conditions. Such a kernel is infinite dimensional which makes standard compactness arguments inapplicable.

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Keywords. Semilinear wave equation, characteristic line, infinite dimensional kernel.

1. Introduction

Based on the results of [1] and the methods introduced in [6], we study the equation

$$\begin{cases} \Box u = \epsilon(u^{2k} + h(t,x) + R(t,x,u)) \\ u(t,0) = u(t,\pi) = 0 \\ u(t,x) = u(t+2\pi,x) \end{cases} \tag{1}$$

where $\Box = \partial_{tt} - \partial_{xx}$ denotes the D'Alembert operator, k is a positive integer, $t \in \mathbb{R},\ x \in [0,\pi]$ and $R \in C^0(\mathbb{R} \times [0,\pi] \times \mathbb{R})$ is 2π-periodic in its first variable. We assume that R is differentiable in its third variable, and that

$$R(t,x,0) = 0 \quad \text{and} \quad \lim_{u\to 0} \frac{R_u(t,x,u)}{u^{2k-1}} = 0, \tag{2}$$

uniformly for $(t,x) \in \mathbb{R} \times [0,\pi]$.

The key feature of equation (1) is that, regardless of the size of ϵ, the derivative of the nonlinearity includes the eigenvalue 0 which has infinite multiplicity (see (3) below) making compactness arguments not applicable. All the results of this paper

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extend to the case where, in (1), u^{2k} is replaced by $\beta(x)u^{2k}$ with β a positive continuous function such $\beta(x) = \beta(\pi - x)$. For the sake of simplicity in the presentation we restrict ourselves to the case $\beta(x) \equiv 1$.

Throughout this paper all functions are 2π-periodic in the variable t.

The kernel of $\Box$ subject to the boundary conditions in (1) is

$$\begin{aligned} N &= \{v(t,x) = p(t+x) - p(t-x);\ p \in L^2([0,2\pi]),\ \int_0^{2\pi} p(s)ds = 0\} \\ &= \{v(t,x) = p(t+x) - p(t-x);\ p \in L^2([0,2\pi])\}. \end{aligned} \tag{3}$$

Let $\Omega = [0,2\pi] \times [0,\pi]$,

$$\begin{aligned} N^{\perp} &= \{w : \mathbb{R} \times [0,\pi] \to \mathbb{R}; w \in L^2(\Omega),\ \int_\Omega wv = 0 \quad \forall v \in N\}, \\ \mathbf{H^1} &= \{w : \mathbb{R} \times [0,\pi] \to \mathbb{R}; w, w_t, w_x \in L^2(\Omega),\ w(t,0) = w(t,\pi) = 0\}, \end{aligned} \tag{4}$$

For $1 \le p < \infty$, the norm in $L^p(\Omega)$ will be denoted by $\|\cdot\|_p$; the norms in L^∞ and C^0 will simply be denoted by $\|\cdot\|$. The norm in the space $\mathbf{H^1}$ will be denoted by $\|\cdot\|_{1,2}$ and is defined as

$$\|w\|_{1,2} = \left(\int_\Omega (w_t^2(t,x) + w_x^2(t,x))dtdx\right)^{1/2}. \tag{5}$$

An elementary argument based on Fourier expansions shows that for each $f \in N^{\perp}$ there exists a unique $w \in \mathbf{H^1} \cap N^{\perp}$ such that $\Box(w) = f$ in the sense of distributions. Moreover, the transformation $f \to w \equiv \Box^{-1}(f)$ is continuous as an operator from $L^2(\Omega) \cap N^{\perp}$ into $\mathbf{H^1} \cap N^{\perp}$, from $L^2(\Omega) \cap N^{\perp}$ into $L^\infty \cap N^{\perp}$, and from $L^\infty(\Omega) \cap N^{\perp}$ into $C^{0,1} \cap N^{\perp}$. Thus there exists a constant c_0 such that

$$\|\Box^{-1} f\|_{1,2} \le c_0\|f\|_2, \|\Box^{-1} f\|_\infty \le c_0\|f\|_2, \text{ and } \|\Box^{-1} f\|_{C^{0,1}} \le c_0\|f\|_\infty, \tag{6}$$

see [1], (2.3).

For the rest of this paper $h \in N^{\perp}$. Letting $H = \Box^{-1}(h) + v$, with $v \in N$, subtracting $\Box(\epsilon H) = \epsilon h$ from both sides of the first equation in (1) and replacing $u - \epsilon H$ by u that equation becomes

$$\Box u = \epsilon(u + \epsilon H)^{2k} + \epsilon R(t,x,u + \epsilon H), \tag{7}$$

subject to the the boundary conditions in (1).

We establish the solvability of (7) in terms of the operator $L_J : C^0((0,2\pi)) \to C^0((0,2\pi))$ defined by

$$\begin{aligned} (L_J(p))(r) = p(r) &\int_0^\pi \{J(r+x,x) + J(r-x,x)\}dx \\ &- \int_0^\pi \{p(r+2x)J(r+x,x) + p(r-2x)J(r-x,x)\}dx \end{aligned} \tag{8}$$

where $p(t+x) - p(t-x) = v(t,x) \in N, \quad r \in [0,2\pi]$.

In fact, our main result is:

Theorem 1. *Suppose* (2) *is satisfied and let* Π_N *denote the* $L^2(\Omega)$*-orthogonal projection onto* N*. If*

a) $k = 1$ *and* L_H *is invertible, or*
b) $k > 1$*, and there exists* $\hat{v} \in N$ *such that* $\|\hat{v}\| \leq O(\epsilon)$, $\Pi_N(\hat{v} + \epsilon H)^{2k} = 0$ *and* L_J *is invertible for* $J = (\hat{v} + \epsilon H)^{2k-1} - \hat{v}^{2k-1}$ *with* $\epsilon^{1-2k}\|L_J\|$ *bounded away from zero,*

then there exist $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0)$ *the equation* (7) *has a solution* $u \in C^0(\Omega)$.

Our next result shows that Theorem 1 includes the existence results in [1]. In fact, we have:

Theorem 2. *If, for some* $v \in N$*,* H *is continuous and* $H(t, x) > 0$ *for all* $(t, x) \in \Omega$*, then* L_H *is invertible. If, in addition,* $k > 1$ *then there exists* $\hat{v} \in N$ *satisfying part b) in Theorem* 1*. Hence there exist* $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0)$ *the equation* (7) *has a solution* $u \in C^0(\Omega)$.

The positivity of H is not a necessary condition for the invertivility of L_H. For example if $h(t, x) = 9\sin(3x) + h_1(t, x)$ with h_1 small enough then $H = \Box^{-1}h + v$ changes sign for any $v \in N$ and yet L_H is invertible. This is a case where Theorem 1 applies but not the results of [1]. See Theorem 3 in Section 5 below.

The operator L_H was introduced in [5] to prove the nonexistence of continuous solutions to

$$\Box u = g(u) + \lambda u + h(t, x), \quad u(t, x) = u(t + 2\pi, x) = u(t, x + 2\pi) \tag{9}$$

when λ is not an eigenvalue of $\Box$ subject to the periodicity condition in (9), g has compact support, $\lambda u + g(u)$ is not monotone, and h is a large multiple of $\sin(t + x)$. The operator L_H was also used in [6] to prove the existence of solutions to (9) when h does not vanish on sets of positive measure in any characteristic line. For results on (9) the reader is referred to [2, 3, 12], for the other studies on the non-monotone case see [4, 9, 13].

Following the results in [8], if the functions h and R satisfy the symmetry $h(t, x) = h(t + \pi, \pi - x), R(t, x, u) = R(t + \pi, \pi - x, u)$ then one may restrict the study of equation (1) to spaces of functions u with this symmetry. Since no non-zero element in N satisfies this symmetry, $\Pi_N((u + \epsilon H)^{2k} + R(t, x, u + \epsilon H)) = 0$ for any u satisfying the symmetry (see (10) below). This reduces the solvability of (1) to the solvability of the range equation (11) which is easily solved for ϵ small under no additional hypothesis on h, see Section 2.

The solutions given by Theorem 1 satisfy $\|u\| \leq O(\epsilon)$. This cannot happen if $h \notin N^\perp$, see [1], Remark 1.1. Thus the assumption $h \in N^\perp$ is necessary. If $H > 0$ and smooth the solutions to (1) are smooth (see [1], Theorem 2.) The regularity of the solutions here obtained for H changing sign is yet to be studied.

2. Solvability in $N^{\perp}$

Let $V = N \cap C^0(\Omega)$, $W = N^{\perp} \cap C^0(\Omega)$ and Π_N, $\Pi_{N^{\perp}}$ the orthogonal projections from $L^2(\Omega)$ onto N and $N^{\perp}$, respectively.

Setting $u = v + w$ with $v \in V$ and $w \in W$, the problem (7) is equivalent to solving the kernel and the range equations

$$\Pi_N((v + w + \epsilon H)^{2k} + R(t, x, v + w + \epsilon H)) = 0, \tag{10}$$

$$\epsilon \Box^{-1} \Pi_{N^{\perp}}((v + w + \epsilon H)^{2k} + R(t, x, v + w + \epsilon H)) = w. \tag{11}$$

In order to solve the kernel equation, (10), we will follow the methods introduced in [5, 6]. The following proposition summarizes the solvability of the range equation, (11). We omit its proof as, up to minor details, it is given by the proof of Proposition 3.2 of [1].

Proposition 1. *There exist $\hat{\epsilon} > 0$ and $\delta_0 > 0$ such that if $v \in V$ with $\|v\|_{C^0} \leq \delta_0$ and $|\epsilon| < \hat{\epsilon}$ then (11) has a unique solution $w(v, \epsilon) \in N^{\perp}$. Moreover there exists $\alpha > 0$ such that*

$$\|w(v, \epsilon)\| \leq \alpha|\epsilon|(\|v\|^{2k} + |\epsilon|^{2k}) \quad \text{and} \quad \|w(v_1, \epsilon) - w(v_2, \epsilon)\| \leq |\epsilon|\alpha\|v_1 - v_2\|, \tag{12}$$

for all v, v_1, v_2 with $\|v\|, \|v_1\|, \|v_2\| \leq \delta_0$, and $|\epsilon| \leq \hat{\epsilon}$.

3. Proof of Theorem 1

We prove in detail the case $k > 1$; the case $k = 1$ follows the same pattern with $\hat{v} = 0$ making the calculations a lot simpler.

Let $\hat{v}$ be as in part b) of Theorem 1, and $v = \hat{v} + \zeta$. Since the product of an even number of elements in N is in $N^{\perp}$, $\Pi_N(\hat{v}^{2k-1}\zeta) = 0$ (see [1, Lemma 2.4]). Therefore (10) is equivalent to

$$\begin{aligned} 0 &= \Pi_N\Big(-2k\hat{v}^{2k-1}\zeta + (\hat{v} + \epsilon H)^{2k} + 2k(\hat{v} + \epsilon H)^{2k-1}\zeta + 2k(\hat{v} + \epsilon H)^{2k-1}w \\ &\quad + \sum_{j=2}^{2k} C_j(\hat{v} + \epsilon H)^{2k-j}(\zeta + w)^j + R(t, x, \hat{v} + \zeta + w + \epsilon H)\Big) \\ &= \Pi_N\Big(2kJ\zeta + 2k(\hat{v} + \epsilon H)^{2k-1}w + \sum_{j=2}^{2k} C_j(\hat{v} + \epsilon H)^{2k-j}(\zeta + w)^j \\ &\quad + R(t, x, \hat{v} + \zeta + w + \epsilon H)\Big) \\ &\equiv \Pi_N\Big(2kJ\zeta + Q(\zeta, \epsilon, H)\Big), \end{aligned} \tag{13}$$

where C_j is the binomial coefficient $2k$ choose j, and $J = (\hat{v} + \epsilon H)^{2k-1} - \hat{v}^{2k-1}$.

Let $z : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $\zeta(t, x) = z(t + x) - z(t - x)$ with $\int_0^{2\pi} z(s)ds = 0$.

As in [6], for $0 \le r \le s \le 2\pi$ let $\chi_{[r,s]}$ be the 2π-periodic function such that

$$\chi_{[r,s]}(t) = \begin{cases} 1, & t \in [r,s] \\ 0, & t \in [0,2\pi] - [r,s] \end{cases} \tag{14}$$

and let ϕ be the function defined by

$$\phi(t,x) = \chi_{[r,s]}(t+x) - \chi_{[r,s]}(t-x) \in N \tag{15}$$

Also we let

$$\begin{aligned} A = \{(t,x) \in \Omega;\ & x \in [0,\pi],\ t \in \bigcup_j [r + 2j\pi - x, s + 2j\pi - x], \\ & j = 0, 1\} \\ B = \{(t,x) \in \Omega;\ & x \in [0,\pi],\ t \in \bigcup_j [r + 2j\pi + x, s + 2j\pi + x], \\ & j = -1, 0\}. \end{aligned} \tag{16}$$

From now on, for the sake of simplicity in the notations, we write $dtdx = d\sigma$.

Multiplying ζJ by ϕ and integrating on Ω we obtain

$$\begin{aligned} \int_\Omega \zeta J \phi d\sigma &= \int_\Omega z(t+x)J(t,x)\phi(t,x)d\sigma - \int_\Omega z(t-x)J(t,x)\phi(t,x)d\sigma \\ &= \int_B z(t+x)J(t,x)d\sigma - \int_A z(t+x)J(t,x)d\sigma \\ &\quad - \int_B z(t-x)J(t,x)d\sigma + \int_A z(t-x)J(t,x)d\sigma \\ &= \int_0^\pi \int_r^s \Big(z(\eta)J(\eta - x, x) - z(\eta + 2x)J(\eta + x, x) \Big) d\eta dx \\ &\quad - \int_0^\pi \int_r^s \Big(z(\eta - 2x)J(\eta - x, x) - z(\eta)J(\eta + x, x) \Big) d\eta dx. \end{aligned} \tag{17}$$

By the Lebesgue differentiation theorem (see (49) in [2]), for almost every $r \in [0, 2\pi]$,

$$\begin{aligned} \lim_{s \to r} \frac{1}{s - r} \int_\Omega \zeta J \phi d\sigma &= \int_0^\pi (z(r) - z(r - 2x))J(r - x, x)dx \\ &\quad + \int_0^\pi (z(r) - z(r + 2x))J(r + x, x)dx \\ &= [L_J(z)](r). \end{aligned} \tag{18}$$

Similarly, multiplying $Q(\zeta, \epsilon, H)$ by ϕ, integrating on Ω, dividing by $s - r$, and taking limit as s tend to r we have

$$\begin{aligned} \lim_{s \to r} \frac{1}{s - r} \int_\Omega Q(\zeta, \epsilon, H)\phi d\sigma &= \int_0^\pi Q(\zeta, \epsilon, H)(r - x, x)dx \\ &\quad + \int_0^\pi Q(\zeta, \epsilon, H)(r + x, x))dx \\ &\equiv [\Gamma(z)](r). \end{aligned} \tag{19}$$

Hence if $v = \hat{v}+\zeta$ and $w = w(v,\epsilon)$ satisfies (10) then $2kL_J(z) = \Gamma(z)$ (see Proposition 1). Conversely, if z satisfies $z = (1/(2k))L_J^{-1}\Gamma(z) \equiv \Gamma_1(z)$ then $v + w(v,\epsilon)$ satisfies (10) (see [6, Lemma 1.3].) Therefore $v + w(v,\epsilon)$ solves (1).

Let $M > 0$ be such that $\|\hat{v}\| + \epsilon\|H\| \le M\epsilon$ for all $\epsilon \in (0,\hat{\epsilon})$. Due to assumption b), by further restricting $\hat{\epsilon}$ if needed, there exists $m > 0$ such that $\epsilon^{2k-1}\|L^{-1}\| \le m$ for all $\epsilon \in (0,\hat{\epsilon})$. Since $k > 1$, there exist $\epsilon_3 \in (0,\hat{\epsilon})$ and $\tau > 0$ be such that

$$\begin{aligned} 2kM^{2k-1}\alpha\mu\epsilon_3^{2k} + \sum_{j=2}^{2k} D_j(2\tau + \alpha\mu\epsilon_3^{2k})^j &\le \frac{\tau}{4\pi}, \\ m\left(2k\alpha M^{2k-1}\epsilon_3 + \sum_{j=2}^{2k} jD_j(2\tau + \alpha\epsilon_3^{2k}\mu)^{j-1}\right) &\le \frac{1}{8\pi}, \end{aligned} \tag{20}$$

where $D_j = C_jM^{2k-j}$ and $\mu = (M + 2\tau)^{2k} + 1$. Next we choose $\gamma > 0$ be such that

$$4\pi\gamma < \min\left\{\frac{\tau}{(M + 2\tau + \alpha\epsilon_3^{2k}\mu)^{2k}}, \frac{1}{2m(1 + \alpha\epsilon_3)(2\tau + M + \alpha\mu\epsilon_3^{2k})^{2k-1}}\right\}. \tag{21}$$

By (2), there exists $\delta > 0$ such that if $|s| < \delta$ then $|R(t,x,s)| < \gamma s^{2k}$ and $|R_u(t,x,s)| \le \gamma|s|^{2k-1}$. Finally we take $\epsilon_4 \in (0,\epsilon_3)$ such that

$$\epsilon_4(2M + 2\tau + \alpha\epsilon_4^{2k}) < \delta. \tag{22}$$

Now for $\|z\| \le \tau\epsilon$ we have $\|\zeta\| \le 2\tau\epsilon$, and $\|w(\hat{v} + \zeta,\epsilon)\| \le \alpha(M + 2\tau)^{2k}\epsilon^{2k+1}$. Therefore

$$\begin{aligned} \|Q(\zeta,\epsilon,H)\| &\le 2k\|(\hat{v} + \epsilon H)^{2k-1}w\| + \sum_{j=2}^{2k} C_j(M\epsilon)^{2k-j}\|\zeta + w\|^j \\ &\quad + \|R(t,x,\hat{v} + \zeta + w + \epsilon H)\| \\ &\le 2kM^{2k-1}\alpha(M + 2\tau)^{2k}\epsilon^{4k} \\ &\quad + \sum_{j=2}^{2k} D_j\epsilon^{2k-j}(2\tau\epsilon + \alpha(M + 2\tau)^{2k}\epsilon^{2k+1})^j \\ &\quad + \gamma\|\hat{v} + \zeta + w + \epsilon H)\|^{2k} \\ &\le \epsilon^{2k}\big(2kM^{2k-1}\alpha(M + 2\tau)^{2k}\epsilon^{2k} \\ &\quad \sum_{j=2}^{2k} D_j(2\tau + \alpha(M + 2\tau)^{2k}\epsilon^{2k})^j \\ &\quad + \gamma(M + 2\tau + \alpha\mu\epsilon^{2k})\big) \\ &\le \tau\epsilon^{2k}. \end{aligned} \tag{23}$$

Let $\zeta_i(t,x) = z_i(t+x) - z_i(t-x)$ for $i = 1,2$, and $w_i = w(\hat{v} + \zeta_i, \epsilon)$ with $\|z_i\| \leq \tau\epsilon$. Thus, from the definition of Q, Proposition 1, (20), (21) and (22), we have

$$\begin{aligned} \|Q(\zeta_1,\epsilon,H) - Q(\zeta_2,\epsilon,H)\| &\leq 2k\|(\hat{v}+\epsilon H)^{2k-1}\|\|w_1 - w_2\| \\ &\quad + \sum_{j=2}^{2k} C_j (M\epsilon)^{2k-j}\|(\zeta_1+w_1)^j - (\zeta_2+w_2)^j\| \\ &\quad + \|R(t,x,\hat{v}+\zeta_1+w_1+\epsilon H) - R(t,x,\hat{v}+\zeta_2+w_2+\epsilon H)\| \\ &\leq 2k(M\epsilon)^{2k-1}\alpha\epsilon\|\zeta_1-\zeta_2\| \\ &\quad + \Big(\sum_{j=2}^{2k} D_j \epsilon^{2k-j}(1+\epsilon\alpha)\sum_{i=0}^{j-1}\|(\zeta_1+w_1)^{j-1-i}(\zeta_2+w_2)^i\| \\ &\quad + \gamma(2M+1+\alpha)^{2k-1}\epsilon^{2k-1}\Big)\|\zeta_1-\zeta_2\| \\ &\leq \epsilon^{2k-1}\Big(\sum_{j=2}^{2k} D_j(1+\epsilon\alpha)j(2\tau+\alpha\mu\epsilon^{2k})^{j-1} \\ &\quad + 2kM^{2k-1}\alpha\epsilon + \gamma(1+\alpha\epsilon)(2\tau+M+\alpha\mu\epsilon^{2k})^{2k-1}\Big)\|\zeta_1-\zeta_2\| \\ &\leq \frac{1}{2\|L_J^{-1}\|}\|\zeta_1-\zeta_2\|. \end{aligned} \tag{24}$$

From (23) we see that $(\epsilon^{1-2k}/(2k))L_J^{-1}\Gamma$ transforms the metric space $\{z; \|z\| \leq \tau\epsilon\}$ into itself. Also (24) proves that $(\epsilon^{1-2k}/(2k))L_J^{-1}\Gamma$ is a contraction. Hence it has a unique fixed point which proves Theorem 1.

4. Proof of Theorem 2

Let

$$X_2 = \Big\{p \in C^0(\mathbb{R}); p(x) = p(x+2\pi), \int_0^{2\pi} p(s)ds = 0, \|p\|_{C^0} = 1\Big\}$$

For each $p \in X_2$, let $r_p \in [0, 2\pi]$ be such that $|p(r_p)| = 1$. We claim that

$$\inf_{p\in X_2} |L_H(p)(r_p)| > 0. \tag{25}$$

Let us assume that there exists a sequence $\{p_n\}$ in X_2 such that$|L_H(p_n)(r_{p_n})| < 1/n$. Without loss of generality we may assume that $p_n(r_{p_n}) = 1$. Hence

$$\begin{aligned} \frac{1}{n} &\geq L_H(p_n)(r_{p_n}) \\ &= \int_0^{\pi}(1-p_n(r_{p_n}-2x))H(r_{p_n}-x,x)dx \\ &\quad + \int_0^{\pi}(1-p_n(r_{p_n}+2x))H(r_{p_n}+x,x)dx \\ &\geq \int_0^{\pi}(1-p_n(r_{p_n}-2x))H(r_{p_n}-x,x)dx. \end{aligned} \tag{26}$$

Therefore

$$\{\sqrt{(1-p_n(r_{p_n}+2x))H(r_{p_n}+x,x)}\} \to 0, \text{ in } L^2. \tag{27}$$

Hence there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that

$$\{(1-p_{n_k}(r_{p_{n_k}}+2x))H(r_{p_{n_k}}+x,x)\} \to 0 \tag{28}$$

almost everywhere on $[0,\pi]$. Since $H(t,x) > 0$, $\forall (t,x) \in \Omega$, then the sequence $\{p_{n_k}\} \to 1$ a.e. $[0,2\pi]$. But this is a contradiction, because $\int_0^{2\pi} p_{n_k}(s)ds = 0$ for all k. Hence (25) is proven, which proves that L_H is invertible. Thus the first statement in Theorem 2 is proven.

In order to prove the second statement in Theorem 2 we define $\hat{v} = \epsilon V$, with V as given by Lemma 3. Since $(\hat{v}+\epsilon H)^{2k-1} - \hat{v}^{2k-1} \geq 2((\epsilon H)/2)^{2k-1}$ the second stament in Theorem 2 is proven. The third statement in Theorem 2 follows from Theorem 1, which completes the proof of Theorem 2.

5. Examples of invertivility of L when H changes sign

In this section we make use of Fourier expansions to provide examples in which $L_H \equiv L$ (see (8), Theorem 1) is invertible in the space of continuous functions and yet H changes sign. In fact we show that this is the case for $H(t,x) = \sin(3x)$, and explicitely calculate L^{-1}. This and Theorem 2 prove that our results properly include those of [1].

Let

$$\begin{aligned} H(t,x) &= \sum_{j=1,l=0}^{\infty,\infty} (a_{jl}\sin(jx)\sin(lt) + b_{jl}\sin(jx)\cos(lt)), \\ p(t) &= \sum_{k=1}^{\infty} (c_k\sin(kt) + d_k\cos(kt)). \end{aligned} \tag{29}$$

Elementary calculations show that

$$\begin{aligned} L(\sin(kr)) = 8 \sum_{j+l \ odd} \Bigg(& \frac{ja_{jl}(k^2-kl)}{(j^2-l^2)((2k-l)^2-j^2)}\cos(kr-lr) \\ & - \frac{ja_{jl}(k^2+kl)}{(j^2-l^2)((2k+l)^2-j^2)}\cos(kr+lr) \\ & + \frac{jb_{jl}(k^2-kl)}{(j^2-l^2)((2k-l)^2-j^2)}\sin(kr-lr) \\ & + \frac{jb_{jl}(k^2+kl)}{(j^2-l^2)((2k+l)^2-j^2)}\sin(kr+lr)\Bigg). \end{aligned} \tag{30}$$

Similarly

$$\begin{aligned}L(\cos(kr)) = 8\sum_{j+l\ odd}\Bigg(&\frac{jb_{jl}(k^2-kl)}{(j^2-l^2)((2k-l)^2-j^2)}\cos(kr-lr)\\&+\frac{jb_{jl}(k^2+kl)}{(j^2-l^2)((2k+l)^2-j^2)}\cos(kr+lr)\\&-\frac{ja_{jl}(k^2-kl)}{(j^2-l^2)((2k-l)^2-j^2)}\sin(kr-lr)\\&+\frac{ja_{jl}(k^2+kl)}{(j^2-l^2)((2k+l)^2-j^2)}\sin(kr+lr)\Bigg).\end{aligned}\tag{31}$$

In particular, if $H(t,x)=\sin(3x)$

$$L(\sin(kr)) = \frac{16k^2}{3(4k^2-9)}\sin(kr)\tag{32}$$

and

$$L(\cos(kr)) = \frac{16k^2}{3(4k^2-9)}\cos(kr).\tag{33}$$

Hence, for $H(t,x)=\sin(3x)$

$$\begin{aligned}L(p(t)) &= L\left(\sum_{k=1}^{\infty}(c_k\sin(kt)+d_k\cos(kt))\right)\\&=\sum_{k=1}^{\infty}\frac{16k^2}{3(4k^2-9)}(c_k\sin(kt)+d_k\cos(kt))\\&=\frac{16}{3}\sum_{k=1}^{\infty}\left(1+\frac{9/4}{k^2-9/4}\right)(c_k\sin(kt)+d_k\cos(kt)).\end{aligned}\tag{34}$$

Clearly we have that if $p\in C[0,2\pi]$, then $p\in L^2[0,2\pi]$ and, by (34), $L(p(t))\in L^2[0,2\pi]$. Now see us that $L(p(t))$ is continuous in $[0,2\pi]$. Fom (34) we have

$$\begin{aligned}L(p(t)) &= \frac{16\pi}{3}p(t)+\frac{16}{3}\sum_{k=1}^{\infty}\frac{9/4}{k^2-9/4}c_k\sin(kt)\\&\quad+\frac{16}{3}\sum_{k=1}^{\infty}\frac{9/4}{k^2-9/4}d_k\cos(kt)\\&\equiv\frac{16}{3}(p(t)+S_1(t)+S_2(t)).\end{aligned}\tag{35}$$

Let now $t_n \to t$. Thus

$$\begin{aligned} |S_1(t_n) - S_1(t)| &\leq \sum_{k=1}^{\infty} \frac{9/4}{|k^2 - 9/4|} |c_k| |\sin(kt_n) - \sin(kt)| \\ &\leq \frac{9}{4} \sum_{k=1}^{\infty} \frac{k}{|k^2 - 9/4|} |c_k| |\cos(\zeta)| |t_n - t| \\ &\leq 9 \sum_{k=1}^{\infty} \frac{1}{k} |c_k| |t_n - t| \\ &\leq 9 \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{1/2} \left(\sum_{k=1}^{\infty} (c_k)^2 \right)^{1/2} |t_n - t| \\ &\leq c|t_n - t|, \end{aligned} \tag{36}$$

where the constant c is independent of t and t_n. Hence S_1 is a continuous function. Similarly, S_2 is also a continuous function. Hence, by (35) and (36), $L(p) \in C[0, 2\pi]$ if $p \in C[0, 2\pi]$.

Furthermore for all k positive integer, $1 + \frac{9/4}{k^2 - 9/4} \neq 0$ and

$$\begin{aligned} p(t) &= \sum_{k=1}^{\infty} (c_k \sin(kt) + d_k \cos(kt)) \\ &= \sum_{k=1}^{\infty} \left(1 - \frac{9}{4k^2}\right) \left(1 + \frac{9/4}{k^2 - 9/4}\right) (c_k \sin(kt) + d_k \cos(kt)). \end{aligned} \tag{37}$$

Therefore, following the arguments in (34), (35) and (36) we have that the operator $L^{-1} : C[0, 2\pi] \to C[0, 2\pi]$ defined by

$$L^{-1}(q(t)) = \frac{3}{16\pi} \sum_{k=1}^{\infty} \left(1 - \frac{9}{4k^2}\right) (f_k \sin(kt) + g_k \cos(kt)) \tag{38}$$

is the inverse of L. Here $\sum_{k=1}^{\infty} (f_k \sin(kt) + g_k \cos(kt))$ is the Fourier series of $q(t)$. So, by Theorem 1, there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$ the equation (7) has solution $u \in C^0(\Omega)$.

Lemma 1. *If $h(t, x) = 9 \sin(3x)$ and $H = \Box^{-1}(h) + v$, with $v \in N$, then H changes sign.*

Proof. By the definition of H, $H(t, x) = \sin 3x + v(t, x)$ with $v(t, x) = p(t + x) - p(t - x)$ (see (3)). Assuming that $H(t, x) > 0$ for all $x \in (0, \pi)$, $t \in [0, 2\pi]$, we have that $H(t, \pi/2) = -1 + p(t + \pi/2) - p(t - \pi/2) > 0$. Hence

$$\begin{aligned} \int_0^{2\pi} p(\pi/2 + t) dt &> \int_0^{2\pi} (1 + p(t - \pi/2)) dt \\ &= 2\pi + \int_0^{2\pi} p(t - \pi/2) dt. \end{aligned} \tag{39}$$

Since p is 2π-periodic and $\int_0^{2\pi} p(t)dt = 0$, (39) is a contradiction. On the other hand, if we assume that $H(t,x) < 0$ for all $x \in (0,\pi)$, $t \in [0,2\pi]$, taking $x = \pi/6$ we also reach a contradiction. Hence H changes sign. □

Taking $h(t,x) = 9\sin(3x)$ and $H(t,x) = \sin(3x)$, by (38) and Theorem 1, the equation (1) has a solution for ϵ small. On the other hand, by Lemma 1, Theorem 1 of [1] does not apply because neither h nor H are of one sign. These arguments easily extend to any function $h(t,x)$ of the form $\sin(kx)$ with k odd and positive. This provides a large class of examples for which Theorem 1 applies but not Theorem 1 of [1].

Since the set of invertible of operators in a Banach space is open in the algebra of such operators, if $H_1(t,x)$ is small then $L + L_{H_1}$ is also invertible. Thus we have:

Theorem 3. *There exists $\delta > 0$ such that if $\|h_1\| \leq \delta$ then there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0,\epsilon_0)$, $h(t,x) = 9\sin(3x) + h_1(t,x) \in N^{\perp}$, and $k = 1$ the equation* (1) *has a solution. Moreover, every solution to $\Box H = h$ satisfying the boundary condition in* (1) *changes sign.*

6. Appendix

The purpose of this appendix is to establish the existence of $\hat{v} = \epsilon V$ as used in the proof of the case $k > 1$ in Theorem 1 when H is positive (see Lemma 3 below.)

Lemma 2. *There exists Δ such that for any $v \in N \cap L^{2k}(\Omega)$*

$$\int_\Omega v^{2k}(t,x)d\sigma \leq \Delta \int_{\Omega_1} v^{2k}(t,x)d\sigma, \tag{40}$$

where $\Omega_1 = \{(t,x); |x - \pi/2| \leq \pi/4, t \in [0,2\pi]\}$.

Proof. For $v \in N \cap L^{2k}(\Omega)$, let

$$A = \{x \in [\pi/4, 3\pi/4]; \int_0^{2\pi} v^{2k}(t,x)dt \geq \frac{20}{\pi} \int_{\Omega_1} v^{2k}(t,x)d\sigma\}. \tag{41}$$

By Fubini's Theorem $m(A) \leq \pi/20$. Let $B = [\pi/4, 3\pi/4] - A$. Hence

$$\begin{aligned} m(B) &\geq 9\pi/20, \\ m([\pi/4,\pi/2] \cap B) &\geq m(B) - m([\pi/2, 3\pi/4]) \\ &\geq \pi/5, \\ m([\pi/2, 3\pi/4] \cap B) &\geq \pi/5. \end{aligned} \tag{42}$$

Assuming that there is $x \in [0,\pi/4)$ such that, for all $z \in B \cap [\pi/2, 3\pi/4]$, $z - x \notin B$ we see that $B_1 = \{z - x; z \in B \cap [\pi/2, 3\pi]/4\} \subset A$. Hence $m(A) \geq m(B_1) \geq \pi/5$ which contradicts that $m(A) \leq \pi/20$. Thus for all $x \in [0,\pi/4)$, there exists $z \in B \cap [\pi/2, 3\pi/4]$ such that $z - x \in B$. Now for $t \in [0,2\pi]$, $x \in [0,\pi/4)$ we have

$$\begin{aligned} |v^{2k}(t,x)| &= |v(t-z,z-x)+v(t-x,0)-v(t+x-z,z)|^{2k} \\ &\leq |v(t-z,z-x)-v(t-x-z,z)|^{2k} \\ &\leq 2^{2k-1}(v^{2k}(t-z,z-x)+v^{2k}(t-x-z,z)). \end{aligned} \tag{43}$$

Thus

$$\begin{aligned} \int_0^{2\pi} v^{2k}(t,x)dt &\leq 2^{2k-1}\left(\int_0^{2\pi} v^{2k}(t,z-x)dt + \int_0^{2\pi} v^{2k}(t,z)dt\right) \\ &\leq 2^{2k}\frac{20}{\pi}\int_{\Omega_1} v^{2k}(t,x)d\sigma. \end{aligned} \tag{44}$$

Similarly, for all $x \in (3\pi/4, \pi]$,

$$\int_0^{2\pi} v^{2k}(t,x)dt \leq 2^{2k}\frac{20}{\pi}\int_{\Omega_1} v^{2k}(t,x)d\sigma. \tag{45}$$

Consequently, by Fubini's theorem, (44), and (45)

$$\begin{aligned} \int_\Omega v^{2k}(t,x)d\sigma &= \int_0^{\pi/4}\int_0^{2\pi} v^{2k}(t,x)dtdx + \int_{\Omega_1} v^{2k}(t,x)d\sigma \\ &\quad + \int_{3\pi/4}^{\pi}\int_0^{2\pi} v^{2k}(t,x)dtdx \\ &\leq \left(1+5\cdot 2^{2k+1}\right))\int_{\Omega_1} v^{2k}(t,x)d\sigma \\ &\equiv \Delta\int_{\Omega_1} v^{2k}(t,x)d\sigma, \end{aligned} \tag{46}$$

which proves Lemma (2). □

Lemma 3. *If H is continuous and positive in $\mathbb{R}\times(0,\pi)$then there exists $V \in N \cap L_\infty$ such that $\Pi_N(V+H)^{2k} = 0$.*

Proof. Let

$$\begin{aligned} g(s,t,x) &= (s+H(t,x))^{2k+1} - s^{2k+1} \\ &= \sum_{j=1}^{2k+1} \frac{(2k+1)!}{(2k+1-j)!j!} s^{2k+1-j} H^j(t,x). \end{aligned} \tag{47}$$

As long as $H(t,x) \geq 0$, g is a convex function of its first variable. Hence $f(v) = \int_\Omega g(v(t,x),t,x)d\sigma$ defines a convex functional on $N \cap L^{2k}(\Omega)$. By the continuity of of H, there exists a positive constant C such that $H(t,x) \geq C$ for all $(t,x) \in \Omega_1$. This and Lemma 2 imply that $\lim_{\|v\|_{2k}\to\infty} f(v) = +\infty$. Therefore there exists $V \in N \cap L^{2k}(\Omega)$ such that $f(V) = \min\{f(v); v \in N \cap L^{2k}(\Omega)\}$ (see [10], Theorem 7.3.4).

Let us see that V is in $L^\infty(\Omega)$. Let $p : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that $V(t,x) = p(t+x) - p(t-x)$. Let φ be as in (15). Since $\varphi \in N \cap L^{2k}(\Omega)$ and $V^{2k} \in N^\perp$, $0 = \int_\Omega \varphi(V+H)^{2k}d\sigma = \int_\Omega \varphi((V+H)^{2k} - V^{2k})d\sigma$ (see Lemma 2.4 in [1]).

Arguing as in (17)-(18) we have

$$\begin{aligned}0 &= \int_0^\pi (V+H)^{2k}(r+x,x)dx - \int_0^\pi (V+H)^{2k}(r-x,x)dx \\ &= \int_0^\pi \sum_{j=1}^{2k} \frac{(2k)!}{(2k+1-j)!j!}(p(r+2x)-p(r))^{2k-j}H^j(r+x,x)dx \\ &\quad - \int_0^\pi \sum_{j=1}^{2k} \frac{(2k)!}{(2k+1-j)!j!}(p(r)-p(r-2x))^{2k-j}H^j(r-x,x)dx \qquad (48) \\ &= -2kp^{2k-1}(r)\left(\int_0^\pi (H(r+x,x)+H(r-x,x))dx\right) \\ &\quad + \sum_{j=2}^{2k} p^{2k-j}(r)q_j(r),\end{aligned}$$

where the q_j's are bounded periodic functions. Since also we are assuming H to be continuous and positive, there exists a positive constant c such that $\int_0^\pi (H(r+x,x)+H(r-x,x))dx \geq c$ for all $r \in [0,2\pi]$. This and (48) imply that $p \in L^\infty(\mathbb{R})$. Hence $V \in L^\infty(\mathbb{R})$, which proves Lemma 3. □

References

[1] Berti, M. and Biasco, L. *Forced vibrations of wave equations with non-monotone nonlinearities*, Ann. Inst. H. Poincaré Anal. Non Linaire, 23 (2006), no. 4, 439–474.

[2] Brézis, H. and Nirenberg, L., *Forced vibrations for a nonlinear wave equation*, Comm. Pure Appl. Math., 31 (1978), 1-30.

[3] Brézis, H. and Nirenberg, L., *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Sc. Norm. Sup. di Pisa, (1978), 225-326.

[4] Caicedo, J. F. and Castro, A. *A semilinear wave equation with derivate of nonlinearity containing multiple eigenvalues of infinite multiplicity*, Contemporary Math., 208 (1997),111-132.

[5] Caicedo, J. F. and Castro, A., *A semilinear wave equation with smooth data and no resonance having no continuous solution*, Disc. and Cont. Dynamical Systems, 24 (2009), 653-658.

[6] Castro, A. and Preskill B., *Existence of solutions for a semilinear wave equation with non-monotone nonlinearity*, Continuous and Discrete Dynamical Systems, Series A, Vol. 28, No. 2, (2010), 649-658.

[7] Castro, A. and Unsurangsie S., *A semilinear wave equation with nonmonotone nonlinearity*, Pacific Jour. of Math., 132 (1988), 215-225.

[8] Coron, J. M. *Periodic solutions of a nonlinear wave equation without assumption of monotonicity*, Math. Ann. 262 (1983), no. 2, 273285.

[9] Hofer, H., *On the range of a wave operator with nonmonotone nonlinearity*, Math. Nachr., 106 (1980), 327-340.

[10] Kurdila, A. and Zabarankin, M., *Convex functional analysis, Volume 1*, Birkhauser.

[11] Lovicarová, H., *Periodic solutions of a weakly wave equation in one dimension*, Czech. Math. Jour., 19 (1969), 324-342.

[12] Rabinowitz, P.H., *Large amplitude time periodic solutions of a semilinear wave equation,* Comm. Pure Appl. Math. 37 (1984), no. 2, 189–206.

[13] Willem, M., *Density of the range of potential operators,* Proc. Amer. Math. Soc. **83** (1981), no. 2, 341–344.

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