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Jose F. Caicedo
Universidad Nacional de Colombia

Alfonso Castro
Harvey Mudd College

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A SEMILINEAR WAVE EQUATION WITH SMOOTH DATA AND NO RESONANCE HAVING NO CONTINUOUS SOLUTION

José F. Caicedo

Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá, Colombia

Alfonso Castro

Department of Mathematics
Harvey Mudd College
Claremont, CA 91711, USA

Abstract. We prove that a boundary value problem for a semilinear wave equation with smooth nonlinearity, smooth forcing, and no resonance cannot have continuous solutions. Our proof shows that this is due to the nonmonotonicity of the nonlinearity.

1. **Introduction.**

Here we consider the hyperbolic boundary value problem

$$\begin{cases} \Box(u) + g(u) = p(x,t) = p(x,t+2\pi) = p(x+2\pi,t) & x,t \in \mathbf{R} \\ u(x,t) = u(x,t+2\pi) = u(x+2\pi,t) & x,t \in \mathbf{R}, \end{cases} \tag{1}$$

where $\Box$ denotes the D'Alembert operator $\partial_{tt} - \partial_{xx}$,

$$g(t) = \tau t + h(t) \quad \text{with} \quad \tau \in (0,\infty) - \{k^2 - j^2; k, j = 0, 1, \ldots\}, \tag{2}$$

and $h : \mathbf{R} \to \mathbf{R}$ is a differentiable function with support in $[0, D]$ and such that

$$h(D/2) < -\tau D/2. \tag{3}$$

Thus, for some $t \in (0, D)$, $g'(t) < 0$.

The wave operator $\Box$ subject to the boundary conditions in (1) has discrete spectrum. It is given by $\sigma(\Box) = \{k^2 - j^2; k, j = 0, 1, \ldots\}$. All the eigenvalues have finite multiplicity except for 0 whose eigenspace is spanned by

$$\{\alpha_{k,k}, \beta_{k,k}, \gamma_{k,k}, \delta_{k,k}, ; k = 0, 1, 2, \ldots\}, \tag{4}$$

where

$$\begin{aligned} &\alpha_{k,j}(x,t) = \sin(kx)\cos(jt), \beta_{k,j}(x,t) = \sin(kx)\sin(jt), \\ &\gamma_{k,j}(x,t) = \cos(kx)\cos(jt), \quad \text{and} \quad \delta_{k,j}(x,t) = \cos(kx)\sin(jt). \end{aligned} \tag{5}$$

In [2] it was shown that if g is monotone and $\lim_{|t|\to+\infty} g(t)/t = \tau$, the boundary value problem

$$\begin{cases} \Box(u) + g(u) = p(x,t) = p(x,t+2\pi) & (x,t) \in (0,\pi) \times \mathbf{R} \\ u(0,t) = u(\pi,t) = 0 \quad t \in \mathbf{R} \\ u(x,t) = u(x,t+2\pi), \quad (x,t) \in [0,\pi] \times \mathbf{R}, \end{cases} \tag{6}$$

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has a weak solution in $L^2([0,\pi]\times[0,2\pi])$. A related result for systems of equations is found in [1]. Also in [2] it is shown that if, in addition, there exists $\epsilon > 0$ such that $g'(z) \geq \epsilon > 0$ for all $z \in \mathbf{R}$ then such a solution is of class C^∞ when p is of class C^∞. Here we prove that such a result cannot be extended to (1) when g is nonmonotone. In fact we show that *the lack of monotonicity prevents even the existence of continuous solutions regardless of the smoothness of of p.*

Studies of (6) for non-monotone g may be found in [8] and [5] where is it proved that it has a solution for p in a dense set of $L^2([0,\pi]\times[0,2\pi])$. In [4], also for non-monotone g, sufficient conditions for the existence of a solution in the Sobolev space $H^1([0,\pi]\times[0,2\pi])$ are given in terms of the components of p in the kernel and range of the operator $\Box$. Here $H^1([0,\pi]\times[0,2\pi])$ denotes the Sobolev space of square integrable functions in $[0,\pi]\times[0,2\pi]$ having first order partial derivatives in $L^2([0,\pi]\times[0,2\pi])$ and satisfying the boundary condition in (1). Extensions of this result to cases where the period 2π is replaced by a number such that all the eigenvalues have infinite multiplicity were are found in [3]. For additional studies on solvability of equation (6) with multiple eigenvalues of infinite multiplicity the reader is referred to [7]. For a survey on boundary value problems for semilinear wave equations we refer the reader to [6].

2. **Preliminaries and statement of main result.**

Throughout this paper $\Omega = (0,2\pi)\times(0,2\pi)$, We denote the norm in $L^p(\Omega)$ by $\| \ \|_p$. We let N denote the closed subspace of $L^2(\Omega)$ spanned by $\{\alpha_{k,k},\beta_{k,k},\gamma_{k,k},\delta_{k,k}; k = 0,1,2,\ldots\}$, see (4). That is, N is the null space of the wave operator $\Box$ subject to the boundary conditions in (1). We let H denote the *Sobolev* space of functions u that are 2π-periodic in both x and t, and such that u as well as its first order partial derivatives belong to $L^2(\Omega)$. The norm in H is denoted by $\| \ \|_{1,2}$. We let Y denote the subspace of H of functions y such that

$$\int_\Omega y(x,t)v(x,t)dxdt = 0 \quad \text{for all} \quad v \in N. \tag{7}$$

We say that $u = y + v \in Y \oplus N$ is a weak solution of (1) if

$$\int_\Omega \{(y_t\hat{y}_t - y_x\hat{y}_x) - (g(u)-p)(\hat{y}+\hat{v})\}\, dxdt = 0, \tag{8}$$

for all $\hat{y}+\hat{v} \in Y \oplus N$. Our main result is:

Theorem 2.1. *There exists $c_0 \geq 0$ such that if $|c| > c_0$, and $p(x,t) = c\sin(x+t)$ then (1) has no continuous weak solution.*

Corollary 2.2. *There exists $c_0 \geq 0$ such that if $|c| > c_0$, and $p(x,t) = c\sin(x+t)$ then (1) has no weak solution in $H^1([0,2\pi]\times[0,2\pi])$.*

The corollary follows immediately from the theorem since every element u in $H^1([0,2\pi]\times[0,2\pi])$ may be written as $u = y+z$ with $y \in Y$ and $z(x,t) = z_1(x+t) + z_2(x-t)$ with $z_1, z_2 \in H^1([0,2\pi])$. Since the elements in $H^1([0,2\pi])$ are continuous function, z is continuous. Hence it cannot be a solution to (1).

3. **Regularity.**

Let $u = y + v$ be a weak solution to (1). We write $\alpha(x,t) = \sin(x+t)$, $v = a\alpha + w$, $a \in \mathbf{R}$, and $w = \bar{v} + z$ where

$$\int_\Omega \alpha w dxdt = 0, \quad \text{and} \quad 4\pi^2\bar{v} = \int_\Omega v dxdt = \int_\Omega w dxdt. \tag{9}$$

Since $z \in N$ we may write $z(x,t) = z_1(x+t) + z_2(x-t)$ with z_1, z_2 2π-periodic functions such that

$$\int_\Omega z_1(x+t)dxdt = \int_\Omega z_2(x+t)dxdt = 0. \tag{10}$$

Lemma 3.1. *Under the above assumptions,* $\|z_i\|_\infty \leq 3\|h\|_\infty/\tau$, *and* $|\bar{v}| \leq \|h\|_\infty/\tau$.

Proof. Taking $\hat{y} = 0$ and $\hat{v} = \alpha$ in (8) we have

$$\int_\Omega (\tau a\alpha + h(u))\alpha dxdt = \int_\Omega c\alpha^2 dxdt. \tag{11}$$

This and $\|\alpha\|_2 = \sqrt{2}\pi$ yield

$$|\tau a - c| \leq 2\|h\|_\infty \tag{12}$$

For b positive odd integer, it is easy to see that $\bar{z}_1(x,t) = z_1^b(x+t)$ and $\bar{z}_2(x,t) = z_2^b(x-t)$ are in N. Hence, taking $\hat{v} = \bar{z}_1$ in (8) we have

$$\begin{aligned} \tau\|z_1\|_{b+1}^{b+1} &= -\int_\Omega (h(u(x,t)) + \bar{v}\tau - (c-\tau a)\alpha(x,t))z_1^b(x,t)dxdt \\ &\leq 3\|h\|_\infty |\Omega|^{\frac{1}{b+1}} \left(\int_\Omega |z_1(x,t)|^{b+1}dxdt\right)^{\frac{b}{b+1}}, \end{aligned} \tag{13}$$

which yields

$$\tau\|z_1\|_{b+1} \leq 4\|h\|_\infty |\Omega|^{\frac{1}{b+1}}. \tag{14}$$

Since b may taken arbitrarily large and $\|z_1\|_\infty = \lim_{b\to\infty} \|z_1\|_{b+1}$ we have

$$\tau\|z_1\|_\infty \leq 4\|h\|_\infty. \tag{15}$$

Similarly $\tau\|z_2\|_\infty \leq 4\|h\|_\infty$. Since

$$4\pi^2\tau|\bar{v}| = \tau|\int_\Omega w(x,t)dxdt| = |\int_\Omega h(u(x,t))dxdt| \leq 4\pi^2\|h\|_\infty, \tag{16}$$

the lemma is proven. □

Lemma 3.2. *There exists* $K > 0$, *independent of* c *such that if* $u = y + v \in Y \oplus N$ *is a weak solution to* (1) *then* $|y(x,t)| \leq K\|h\|_\infty$ *for all* $(x,t) \in \Omega$, *and* $\|y\|_{1,2} \leq K$.

Proof. Let

$$\begin{aligned} y &= \sum_{k\neq j} a_{kj}\alpha_{k,j} + b_{kj}\beta_{k,j} + c_{kj}\gamma_{k,j} + d_{kj}\delta_{k,j} \quad \text{and} \\ P_Y(h(y+v)) &= \sum_{k\neq j} A_{kj}\alpha_{k,j} + B_{kj}\beta_{k,j} + C_{kj}\gamma_{k,j} + D_{kj}\delta_{k,j}. \end{aligned} \tag{17}$$

Since $\|P_Y(h(v+y))\|_2 \leq \|h(y+v)\|_2 \leq 2\pi\|h\|_\infty$, $a_{kj} = A_{kj}/(k^2 - j^2 + \tau)$, $b_{kj} = A_{kj}/(k^2 - j^2 + \tau)$, $c_{kj} = C_{kj}/(k^2 - j^2 + \tau)$, and $d_{kj} = D_{kj}/(k^2 - j^2 + \tau)$, by Parseval's identity we have

$$
\begin{aligned}
|y(x,t)| &= \Big|\sum_{k\neq j} a_{kj}\alpha_{k,j}(x,t) + b_{kj}\beta_{k,j}(x,t) + c_{kj}\gamma_{k,j}(x,t) + d_{kj}\delta_{k,j}(x,t)\Big| \\
&\leq \left(\sum_{k\neq j} A_{kj}^2 + B_{kj}^2 + C_{kj}^2 + D_{kj}^2\right)^{1/2} \left(\sum_{k\neq j} \frac{1}{(k^2-j^2+\tau)^2}\right)^{1/2} \\
&\leq 2\pi\|h\|_\infty \left(\sum_{k\neq j} \frac{1}{(k^2-j^2+\tau)^2}\right)^{1/2} \\
&\equiv K_1\|h\|_\infty,
\end{aligned} \tag{18}
$$

where we used that the last series in (18) converges. Similarly

$$
\begin{aligned}
\|y\|_{1,2}^2 &\leq 2\sum_{k\neq j} \frac{(k^2+j^2)(A_{kj}^2 + B_{kj}^2 + C_{kj}^2 + D_{kj}^2)}{(k^2-j^2+\tau)^2} \\
&\leq K_2\|h(u)\|_2^2 \\
&\leq 4\pi^2 K_2\|h\|_\infty^2
\end{aligned} \tag{19}
$$

Taking $K = \max\{K_1, 2\pi\sqrt{K_2}\}$ the lemma is proven. □

Let $D > 0$ be as in (3). Now (see (12))

$$
\begin{aligned}
|u(x,t)| &= |a\sin(x+t) + \bar{v} + z(x,t) + y(x,t)| \\
&\geq [(|c| - 2\|h\|_\infty)|\sin(x+t)| - (9 + K_1\tau)\|h\|_\infty]/\tau.
\end{aligned} \tag{20}
$$

Hence

$$
h(u(x,t)) = 0 \text{ if } \ |\sin(x+t)| \geq \frac{\tau D + (9+K_1\tau)\|h\|_\infty}{|c| - 2\|h\|_\infty}. \tag{21}
$$

Therefore there exists a positive constants c_0 and m such that if $|c| \geq c_0$ then

$$
m\{(x,t) \in \Omega; h(u(x,t)) \neq 0\} \leq \frac{m}{c}. \tag{22}
$$

Hence $\|h(u)\|_2 \leq m^{1/2}\|h\|_\infty c^{-1/2}$ for $|c| \geq c_0$. Replacing this in (18) we have

$$
|y(x,t)| \leq K\|h\|_\infty c^{-1/2}, \tag{23}
$$

for $|c| \geq c_0$. Also

$$
\begin{aligned}
\tau|\bar{v}| &= |\int_\Omega h(u(x,t))dxdt| \\
&\leq \|h\|_\infty m\{(x,t) \in \Omega; h(u(x,t)) \neq 0\} \\
&\leq \frac{m\|h\|_\infty}{c}.
\end{aligned} \tag{24}
$$

Similarly (see (12))

$$
|\tau a - c| \leq m\|h\|_\infty c^{-1}. \tag{25}
$$

For $0 \leq r \leq s \leq 2\pi$, let $\chi_{[r,s]}$ be the 2π-periodic function such that $\chi_{[r,s]}(t) = 1$ if $t \in [r,s]$, and $\chi_{[r,s]}(t) = 0$ if $t \in [0,2\pi] - [r,s]$. Let $\phi(x,t) = \chi_{[r,s]}(x-t)$,

$\bar{z}_1(x,t) = z_1(x+t)$, and $\bar{z}_2(x,t) = z_2(x-t)$. Using that $\phi \in N$ and the mean value theorem for integrals we have

$$\begin{aligned} 0 &= \int_\Omega \phi((a\tau - c)\alpha + \tau(\bar{z}_1 + \bar{z}_2) + \bar{v} + h(u))dxdt \\ &= 2\pi(s-r)\tau z_2(s_2) + \int_\Omega \phi h(u)dxdt + 2\pi\bar{v}(s-r), \end{aligned} \tag{26}$$

where $s_2 \in (r,s)$. Since $|\int_\Omega \phi h(u)dxdt| \le \|h\|_\infty (r-s)m/c$, we conclude

$$|z_2(r)| \le M\|h\|_\infty / c, \tag{27}$$

with M independent of c. Similarly, letting $\psi(x,t) = \chi_{[r,s]}(x+t)$ and multiplying (1) by ψ,

$$\begin{aligned} 0 &= \int_\Omega \psi((a\tau - c)\alpha + \tau(\bar{z}_1 + \bar{z}_2) + \bar{v} + h(u))dxdt \\ &= 2\pi(s-r)((a\tau - c)\alpha(0,s_3) + \tau z_1(s_1)) + \tau\bar{v}2\pi(s-r) \\ &\quad + \int_\Omega \psi(h(u) - h(a\alpha + \bar{z}_1))dxdt + \int_\Omega \psi h(a\alpha + \bar{z}_1)dxdt, \end{aligned} \tag{28}$$

with $s_1, s_3 \in (r,s)$. Letting $s \to r$,

$$\begin{aligned} 0 &= 2\pi((a\tau - c)\alpha(0,r) + \tau z_1(r) + h((a\alpha + \bar{z}_1)(0,r)) + \bar{v}) \\ &\quad + \int_0^{2\pi} (h(y + \bar{v} + \bar{z}_1 + a\alpha + \bar{z}_2) - h(a\alpha + \bar{z}_1))(x, r-x)dx \end{aligned} \tag{29}$$

Hence (see (23), (24), (27))

$$\tau z_1(r) + h(a\alpha(0,r) + z_1(r)) = O(c^{-1/2}) \tag{30}$$

4. **Proof of Theorem 2.1.**

Proof. Without loss of generality we may assume that $c > 0$. Since for c large $a\alpha(0,\pi/2) + z_1(\pi/2) > D$ and $a\alpha(0,3\pi/2) + z_1(3\pi/2) < 0$, there exists t_1, t_2 such that $\pi/2 < t_1 < t_2 < 3\pi/2$, $a\alpha(0,t_1) + z_1(t_1) = D/2$, and $a\alpha(0,t_2) + z_1(t_2) = 0$. From (30)

$$\tau z_1(t_1) = -h(D/2) + O(c^{-1/2}). \tag{31}$$

Thus $a\alpha(0,t_1) = D/2 - z_1(t_1) = D/2 + (h(D/2)/\tau) + O(c^{-1/2}) < 0$. On the other hand, by (30), $\tau z_1(t_2) = -h(0) + O(c^{-1/2})$ which implies that $a\alpha(0,t_2) = -z_1(t_2) = O(c^{-1/2}) > O(c^{-1/2}) + (D/2 + h(D/2)/\tau)/2 > a\alpha(0,t_1)$, which contradicts that $t \to \alpha(0,t)$ defines a decreasing function on $[\pi/2, 3\pi/2]$.

□

REFERENCES

[1] P. Bates and A. Castro, *Existence and uniqueness for a variational hyperbolic system without resonance,* Nonlinear Analysis TMA, **4** (1980), 1151–1156.

[2] H. Brezis and L. Nirenberg, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems,* Annali Scuola Norm. Sup. Pisa Cl Sci (4), **5** (1978), 225–236.

[3] J. Caicedo and A. Castro, *A semilinear wave equation with derivative of nonlinearity containing multiple eigenvalues of infinite multiplicity,* Harmonic analysis and nonlinear differential equations (Riverside, CA, 1995), 111–132, Contemp. Math., **208**, Amer. Math. Soc., Providence, RI, (1997).

[4] A. Castro and S. Unsurangsie, *A semilinear wave equation with nonmonotone nonlinearity,* Pacific J. Math., **132** (1988), 215–225.

[5] H. Hofer, *On the range of a wave operator with nonmonotone nonlinearity,* Math. Nachr., **106** (1982), 327–340.
[6] J. Mawhin, *Periodic solutions of some semilinear wave equations and systems: A survey,* Chaos, Solitons and Fractals, **5** (1995), 1651–1669.
[7] P. J. McKenna, *On solutions of a nonlinear wave equation when the ratio of the period to the length of the interval is irrational,* Proc. Amer. Math. Soc., **93** (1985), 59–64.
[8] M. Willem, *Density of the range of potential operators,* Proc. Amer. Math. Soc., **83** (1981), 341–344.

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E-mail address: `jfcaicedoc@matematicas.unal.edu.co`
E-mail address: `castro@math.hmc.edu`