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MEAN FIELD EFFECTS FOR COUNTERPROPAGATING TRAVELING WAVE SOLUTIONS OF REACTION-DIFFUSION SYSTEMS*

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עד מאה ועשרים

This paper is dedicated to Joseph Keller on the occasion of his 70th birthday.

Abstract. In many problems, e.g., in combustion or solidification, one observes traveling waves that propagate with constant velocity and shape in the x direction, say, are independent of y and z and describe transitions between two equilibrium states, e.g., the burned and the unburned reactants. As parameters of the system are varied, these traveling waves can become unstable and give rise to waves having additional structure, such as traveling waves in the y and z directions, which can themselves be subject to instabilities as parameters are further varied. To investigate this scenario we consider a system of reaction-diffusion equations with a traveling wave solution as a basic state. We determine solutions bifurcating from the basic state that describe counterpropagating traveling waves in directions orthogonal to the direction of propagation of the basic state and determine their stability. Specifically, we derive long wave modulation equations for the amplitudes of the counterpropagating traveling waves that are coupled to an equation for a mean field, generated by the translation of the basic state in the direction of its propagation. The modulation equations are then employed to determine stability boundaries to long wave perturbations for both unidirectional and counterpropagating traveling waves. The stability analysis is delicate because the results depend on the order in which transverse and longitudinal perturbation wavenumbers are taken to zero. For the unidirectional wave we demonstrate that it is sufficient to consider the cases of (i) purely transverse perturbations, (ii) purely longitudinal perturbations, and (iii) longitudinal perturbations with a small transverse component. These yield Eckhaus type, zigzag type, and skew type instabilities, respectively. The latter arise as a specific result of interaction with the mean field. We also consider the degenerate case of very small group velocity, as well as other degenerate cases, which yield several additional instability boundaries. The stability analysis is then extended to the case of counterpropagating traveling waves.

Key words. reaction-diffusion equations, traveling waves, mean field effects, interface problems

AMS subject classifications. 35K57, 80A22, 80A25

1. Introduction.

Mean field effects play an important role in the dynamics and pattern formation of physical systems. For example, in Rayleigh–Benard convection, mean flow modes (corresponding to a zero eigenvalue) with nonzero vertical vorticity may interact with convection roll solutions that are spatially periodic in one horizontal direction. This interaction leads to a significant reduction of the interval of wavenumbers for which the rolls are stable [1]–[3]. The resulting instability, referred to as the skew varicose instability, leads to transitions to two dimensionally modulated rolls, due to perturbations whose wavenumbers are oblique to that of the rolls. Similar effects have been studied for longitudinal seismic waves in a viscoelastic medium [4],

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in which a zero mode interacts with a single traveling wave mode. The range of stable wave numbers of the traveling wave is again dramatically reduced due to the interaction between the zero mode and the traveling wave.

In this paper we consider mean field effects in a system of general reaction-diffusion equations in three spatial dimensions in which the basic state is a traveling wave propagating along a given axis. Our study is motivated by the fact that in many problems, e.g., in combustion or solidification, one observes such waves propagating with constant velocity and shape in the x direction, say, independent of y and z, which describe transitions between two equilibrium states, e.g., the burned and the unburned reactants. As parameters of the system are varied, these waves can become unstable and give rise to waves having additional structure, such as traveling waves in the y and z directions, which can themselves be subject to instabilities as parameters are further varied. This system, in two spatial dimensions, with only weak mean field effects considered, was analyzed in [5]. Solutions bifurcating from the basic state include counterpropagating traveling waves in directions orthogonal to the direction of propagation of the basic state, and an underlying zero mode (mean field), corresponding to translation of the basic state along its axis of propagation. In [5] traveling waves with an $\mathcal{O}(1)$ group velocity were considered. Since the waves propagate on a fast time scale, a wave propagating in one direction "sees" on a slower time scale the average of the wave propagating in the opposite direction. Employing these averages, coupled nonlocal complex Ginzburg–Landau amplitude equations were derived that decouple from the equation for the zero mode. Similar analyses for problems with $\mathcal{O}(1)$ group velocities in gaseous combustion, in gasless solid fuel combustion, and in water waves were carried out in [6]–[8], respectively. Quantitative changes in stability results, due to the nonlocal nature of the equations, were derived in [9] for standing waves, and for quasiperiodic waves, which include standing waves as a special case, in [5]–[7].

We consider both $\mathcal{O}(1)$ and small group velocities. We derive evolution equations for the modulated amplitudes of two bifurcating counterpropagating traveling waves, which are coupled to an evolution equation for the zero mode corresponding to translation of the basic state along its axis of propagation. The modulation equations are then employed to determine how translation of the basic state affects the dynamics and pattern formation in the system.

In studying mean field effects on the stability of plane wave solutions, we consider perturbations along the wave vector $\mathbf{k}$ of the plane waves (longitudinal), orthogonal to $\mathbf{k}$ (transverse) and combinations thereof (oblique). We derive equations to describe modulations of the plane waves and then employ the equations to derive stability boundaries for both traveling wave and quasiperiodic plane wave solutions. Specifically, we consider stability in the limit of long wave perturbations. The stability analysis is delicate because the results depend on the order in which transverse and longitudinal perturbation wavenumbers are taken to zero. For the unidirectional wave we demonstrate that it is sufficient to consider the cases of (i) purely transverse perturbations, (ii) purely longitudinal perturbations, and (iii) longitudinal perturbations with a small transverse component. These yield Eckhaus type, zigzag type, and skew type instabilities, respectively. The latter arise as a specific result of interaction with the mean field. We also consider the degenerate case of very small group velocity, as well as other degenerate cases, which yield several additional instability boundaries. The stability analysis is then extended to the case of counterpropagating traveling waves.

In §2 we present the mathematical formulation of the problem, describe a traveling wave solution (which serves as a basic state), and describe conditions for its linear instability. In § 3 we consider a nonlinear analysis of the problem in a neighborhood of the minimum of the neutral stability curve. We derive a set of three coupled equations for the evolution of long wave modulations of the amplitudes of counterpropagating traveling waves in directions orthogonal to the direction of propagation of the basic state and a mean field generated by translation of the basic state along its direction of propagation. Solutions of these equations and their stability are discussed in § 4. Specifically, we determine Eckhaus type instabilities by considering longitudinal perturbations, zigzag type instabilities by considering transverse perturbations, and skew type instabilities by considering oblique perturbations. The latter arise as a specific result of the interaction with the mean field and are not present without this interaction. Finally, § 5 contains a discussion of results obtained.

2. Formulation. We consider the general reaction-diffusion system

$$\frac{\partial \tilde{u}}{\partial t}=D\left(\frac{\partial^2 \tilde{u}}{\partial x^2}+\frac{\partial^2 \tilde{u}}{\partial y^2}+\frac{\partial^2 \tilde{u}}{\partial z^2}\right)+f(\tilde{u};\lambda), \qquad t>0, \quad -\infty<x,y,z<\infty, \tag{2.1}$$

$$\tilde{u} \text{ bounded as } x,y,z \to \pm\infty.$$

where t and x, y, z are the temporal and spatial variables, respectively, and $\tilde{u}(x,y,t;\lambda)$ and $f(\tilde{u};\lambda)$ are vector functions

$$\tilde{u}=(\tilde{u}_1,\tilde{u}_2,\ldots,\tilde{u}_m), \qquad f(\tilde{u};\lambda)=(f_1,f_2,\ldots,f_m). \tag{2.2}$$

The matrix D is diagonal with nonnegative elements on the diagonal. The parameter λ is real.

We assume that there is a traveling wave solution $w(\xi)$ where $\xi=x+ct$, which propagates along the x-axis with constant velocity c. We note that in general c depends on both w and λ. We refer to the traveling wave solution $w(\xi)$ as the basic solution, which satisfies

$$Dw''-cw'+f(w;\lambda)=0, \tag{2.3}$$

where $'$ indicates differentiation with respect to the moving coordinate ξ.

We assume that at a critical value $\lambda=\lambda_0$ the basic solution loses stability. To determine conditions for the instability of the basic solution and the types of solutions that appear as a result of the instability, we introduce the perturbation

$$u=\tilde{u}-w \tag{2.4}$$

and linearize (2.1), written in terms of the moving coordinate system attached to the traveling wave, about $u=0$, to obtain

$$\frac{\partial u}{\partial t}=D\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-c\frac{\partial u}{\partial \xi}+\alpha_\lambda u, \tag{2.5}$$

where $\alpha_\lambda(w)$ is defined as the Jacobi matrix

$$\alpha_\lambda(w)\equiv\left(\frac{\partial f_i(w;\lambda)}{\partial w_j}\right) \qquad (i,j=1,\ldots,m). \tag{2.6}$$

We substitute

$$u=e^{\mu t+ik_y y+ik_z z}v(\xi), \qquad (k_y)^2+(k_z)^2\equiv k^2 \tag{2.7}$$

into (2.5) to obtain

$$Dv'' - cv' - k^2Dv + \alpha_\lambda v = \mu v. \tag{2.8}$$

Here k and $\mu = \kappa + i\omega$ are the wave number and growth rate, respectively, of the perturbation u. Below we consider modulations of solutions that are spatially periodic in the y direction only; that is, we consider the specific case $k_y = k$, $k_z = 0$. That is, we will consider modulations of traveling waves in the y direction.

The parameter μ is an eigenvalue of the operator in (2.8). We assume that there exists a critical value of the parameter λ, say $\lambda_0 = 0$, such that the following conditions hold.

(1) For $\lambda < 0$ and all $k \neq 0$, all eigenvalues μ have negative real parts.

(2) For $\lambda = 0$, $k = k_0 \neq 0$, there exist a pair of purely imaginary eigenvalues $\mu_\pm = \pm i\omega_0 \neq 0$, and all other eigenvalues μ have negative real parts.

(3) For each $\lambda > 0$ there is a range of wave numbers $k_- < k < k_+$ such that for each k in this range there exist eigenvalues $\mu_\pm(k, \lambda) = \kappa \pm i\omega$ with $\kappa > 0$, and $\kappa < 0$ for all other k. In addition, the real parts of all other eigenvalues are negative for $k \neq 0$.

(4) For $k = 0$ and all values of λ, there exists an eigenvalue $\mu = 0$, which corresponds to the eigenfunction w_0', while all other eigenvalues have negative real parts.

These assumptions indicate that the basic solution, which is stable for $\lambda < 0$, loses its stability when λ passes through the critical value $\lambda_0 = 0$ via a Hopf bifurcation. There exists a continuous band of wave numbers of perturbations for the instability region $\lambda > 0$. The curves $k_\pm$ form the neutral stability curve that has a minimum at $k = k_0$. For $\lambda = 0$ the system (2.5) has the solutions

$$e^{i(\omega_0 t + k_0 y)} v_0(\xi) \equiv e_1 v_0(\xi), \qquad e^{i(\omega_0 t - k_0 y)} v_0(\xi) \equiv e_2 v_0(\xi), \quad w_0'(\xi), \tag{2.9}$$

as well as $\overline{e_1 v_0(\xi)}$ and $\overline{e_2 v_0(\xi)}$, where $\bar{\ }$ indicates complex conjugate, v_0 is a solution of (2.8) for $k = k_0$, $\lambda = 0$, $\mu = i\omega_0$, and w_0' is a solution of (2.8) for $\lambda = k = \mu = 0$, which corresponds to translation.

3. Nonlinear analysis. We perform a local analysis about the point $(\lambda = 0, k = k_0)$, which corresponds to the minimum of the neutral stability curve, and seek modulated solutions that bifurcate from the basic solution. We define the small parameter ϵ as

$$\lambda \equiv \epsilon^2 \nu, \tag{3.1}$$

where $\nu = 1(-1)$ when $\lambda > (<)\ 0$, employ the scaled slow variables

$$T \equiv \epsilon t, \quad \tau \equiv \epsilon^2 t, \quad \eta \equiv \epsilon y, \quad \zeta \equiv \epsilon^{1/2} z, \tag{3.2}$$

and expand as

(3.3)
$$u \sim \epsilon U_1(t, T, \tau, \xi, y, \eta, \zeta) + \epsilon^2 U_2(t, T, \tau, \xi, y, \eta, \zeta) + \epsilon^3 U_3(t, T, \tau, \xi, y, \eta, \zeta) + \cdots.$$

We expand the nonlinear function $f(\tilde{u}; \lambda)$ about the basic solution w as

(3.4)
$$f(\tilde{u}; \lambda) = f(w + u; \lambda) \sim f(w; \lambda) + \alpha_\lambda u + \beta_\lambda(u, u) + \gamma_\lambda(u, u, u) + \delta_\lambda(u, u, u, u) + \cdots,$$

where α_λ is defined in (2.6) and

$$\beta_\lambda(u,u) \equiv \frac{1}{2!}\left\{\frac{\partial^2 f_i(w;\lambda)}{\partial w_j \partial w_k} u_j u_k\right\} \tag{3.5}$$

$$\gamma_\lambda(u,u,u) \equiv \frac{1}{3!}\left\{\frac{\partial^3 f_i(w;\lambda)}{\partial w_j \partial w_k \partial w_l} u_j u_k u_l\right\} \tag{3.6}$$

$$\delta_\lambda(u,u,u,u) \equiv \frac{1}{4!}\left\{\frac{\partial^4 f_i(w;\lambda)}{\partial w_j \partial w_k \partial w_l \partial w_n} u_j u_k u_l u_n\right\} \tag{3.7}$$
$$i,j,k,l,n = 1,\dots,m,$$

where repeated subscripts indicate summation. Since α_λ, β_λ, γ_λ, and δ_λ, as well as c, are in general functions of λ, we expand as

$$\alpha_\lambda \sim \alpha_0 + \lambda\alpha_1 + \cdots, \tag{3.8}$$

$$\beta_\lambda \sim \beta_0 + \lambda\beta_1 + \cdots, \tag{3.9}$$

$$\gamma_\lambda \sim \gamma_0 + \lambda\gamma_1 + \cdots, \tag{3.10}$$

$$\delta_\lambda \sim \delta_0 + \lambda\delta_1 + \cdots, \tag{3.11}$$

$$c \equiv c_\lambda \sim c_0 + \lambda c_1 + \cdots, \tag{3.12}$$

where $\alpha_0 = \alpha_\lambda|_{\lambda=0}$, $\beta_0 = \beta_\lambda|_{\lambda=0}$, $\gamma_0 = \gamma_\lambda|_{\lambda=0}$, $\delta_0 = \delta_\lambda|_{\lambda=0}$, $c_0 = c_\lambda|_{\lambda=0}$, and

$$\begin{gathered} \alpha_1 = \left.\frac{\partial\alpha_\lambda}{\partial\lambda}\right|_{\lambda=0}, \quad \beta_1 = \left.\frac{\partial\beta_\lambda}{\partial\lambda}\right|_{\lambda=0}, \quad \gamma_1 = \left.\frac{\partial\gamma_\lambda}{\partial\lambda}\right|_{\lambda=0}, \\ \delta_1 = \left.\frac{\partial\delta_\lambda}{\partial\lambda}\right|_{\lambda=0}, \quad c_1 = \left.\frac{\partial c_\lambda}{\partial\lambda}\right|_{\lambda=0}. \end{gathered} \tag{3.13}$$

Substituting $\tilde{u} = w + u$ and the above expansions into (2.1), written in the moving coordinate system (ξ, y, t), we equate like powers of ϵ to obtain (2.3) at leading order, and the sequence of equations

$$\begin{gathered} \mathscr{L}U_j \equiv \frac{\partial U_j}{\partial t} - D\left(\frac{\partial^2 U_j}{\partial \xi^2} + \frac{\partial^2 U_j}{\partial y^2}\right) + c_0\frac{\partial U_j}{\partial \xi} - \alpha_0 U_j = r_j \\ U_j \text{ bounded as } \xi, y \to \pm\infty, \qquad j = 1,2,\dots, \end{gathered} \tag{3.14}$$

where $r_1 = 0$, and the nonzero inhomogeneous terms r_j ($j = 2$, 3, 4) are given in Appendix A.

We first consider the problem corresponding to $j = 1$, that is $\mathscr{L}U_1 = 0$. The general long time solution is given by

$$U_1 = (R_1 e_1 + S_1 e_2)v_0(\xi) + \text{c.c.} + \Psi_1 w_0'(\xi), \tag{3.15}$$

where c.c. denotes complex conjugate. At this order the complex coefficients $R_1(T,\tau,\eta,\zeta)$ and $S_1(T,\tau,\eta,\zeta)$ are undetermined, as is the real coefficient $\Psi_1(T,\tau,\eta,\zeta)$. Since a nontrivial solution exists for the homogeneous problem (3.14) with $j = 1$, certain solvability conditions must be satisfied for solutions of the inhomogeneous problems ($j \geq 2$) to exist. We introduce the inner product

$$\langle g, h \rangle \equiv \frac{\omega_0 k_0}{4\pi^2} \int_0^{2\pi/\omega_0} \int_0^{2\pi/k_0} \int_{-\infty}^{\infty} g\bar{h}\, d\xi\, dy\, dt, \tag{3.16}$$

where g and h are bounded functions of ξ on $(-\infty, \infty)$ whose product $g\bar{h} \to 0$ as $\xi \to \pm\infty$ in such a way that the above integral exists. As noted above, we wish to study modulations of solutions that are periodic in t and y, so that we confine consideration in (3.16) to functions g and h that are periodic in t and y, as is the case for (3.15).

To state the solvability condition, we first determine the long time solutions to the problem adjoint to (3.14) with $j = 1$

$$\begin{gathered} \frac{\partial U^*}{\partial t} - D\left(\frac{\partial^2 U^*}{\partial \xi^2} + \frac{\partial^2 U^*}{\partial y^2}\right) - c_0 \frac{\partial U^*}{\partial \xi} - \alpha_0^* U^* = 0, \\ U^* \to 0 \quad \text{as } \xi \to \pm\infty, \end{gathered} \tag{3.17}$$

which have the same periodicity in the original temporal and spatial variables t and y as the long time solutions (3.15) to $\mathscr{L}\mathscr{U} = 0$. Here α_0^* is the matrix adjoint to α_0. Thus the solutions of (3.17) are given by

$$\Phi_0 \equiv \phi_0(\xi), \quad \Phi_1 \equiv \phi_1(\xi) e_1, \quad \Phi_2 \equiv \phi_1(\xi) e_2, \tag{3.18}$$

and their complex conjugates. The functions ϕ_0 and ϕ_1 are solutions of

$$D\phi_0'' + c_0 \phi_0' + \alpha_0^* \phi_0 = 0, \tag{3.19}$$

$$-D\phi_1'' + Dk_0^2 \phi_1 - c_0 \phi_1' - \alpha_0^* \phi_1 + i\omega_0 \phi_1 = 0. \tag{3.20}$$

The solvability conditions are then given by

$$\langle r_j, \Phi_k \rangle = 0 \quad \text{for } k = 0, 1, 2 \quad \text{and} \quad j \geq 2. \tag{3.21}$$

In applying these conditions for $j \geq 2$ we assume that ϕ_k ($k = 0$, 1) have been normalized so that

$$\langle v_0, \phi_1 \rangle = \langle w_0', \phi_0 \rangle = 1. \tag{3.22}$$

For $j = 2$ we obtain

$$\mathscr{L}_1 R_1 \equiv \frac{\partial R_1}{\partial T} - \omega_0' \frac{\partial R_1}{\partial \eta} - \langle Dv_0, \phi_1 \rangle \frac{\partial^2 R_1}{\partial \zeta^2} = 0, \tag{3.23}$$

$$\mathscr{L}_2 S_1 \equiv \frac{\partial S_1}{\partial T} + \omega_0' \frac{\partial S_1}{\partial \eta} - \langle Dv_0, \phi_1 \rangle \frac{\partial^2 S_1}{\partial \zeta^2} = 0, \tag{3.24}$$

$$\frac{\partial \Psi_1}{\partial T} = H_0\left(|R_1|^2 + |S_1|^2\right) + b_1 \frac{\partial^2 \Psi_1}{\partial \zeta^2}, \tag{3.25}$$

where

$$b_1 \equiv \langle Dw_0', \phi_0 \rangle; \qquad H_0 \equiv \langle \beta_0(v_0, \overline{v_0}), \phi_0 \rangle, \tag{3.26}$$

and the (real) group velocity ω_0' is

$$\omega_0' \equiv \left.\frac{\partial \omega}{\partial k}\right|_{k = k_0} = 2ik_0 \langle Dv_0, \phi_1 \rangle. \tag{3.27}$$

Below we will consider the group velocity ω_0' to be $\mathcal{O}(1)$ as well as small and will derive stability results for each case. When (3.23)–(3.25) are satisfied, the solution to the problem for $j=2$ is given by

$$
\begin{aligned}
U_2 &= 2\big(|R_1|^2+|S_1|^2\big)p_1 \\
&\quad + \big[(R_1^2e_1^2+S_1^2e_2^2)p_2 + R_1S_1e_1e_2p_3 + R_1\overline{S}_1\overline{e_2}e_1p_4 \\
&\qquad\qquad + 2\Psi_1(R_1e_1+S_1e_2)p_5 + \text{c.c.}\big] \\
&\quad + \Psi_1^2p_6 + \left[\left(2ik_0\frac{\partial R_1}{\partial\eta}e_1 - 2ik_0\frac{\partial S_1}{\partial\eta}e_2\right)p_7\right. \\
&\qquad\qquad \left. + \left(\frac{\partial^2R_1}{\partial\zeta^2}e_1+\frac{\partial^2S_1}{\partial\zeta^2}e_2\right)p_8 + \frac{\partial^2\Psi_1}{\partial\zeta^2}p_9 + \text{c.c.}\right] \\
&\quad + [(R_2e_1+S_2e_2)v_0+\text{c.c.}] + \Psi_2w_0',
\end{aligned}
\tag{3.28}
$$

where $R_2e_1v_0$, $S_2e_2v_0$ and Ψ_2w_0' are solutions to the homogeneous equation (3.14), and the equations for $p_j(\xi)$ are given in Appendix A. Proceeding as in the case $j=2$, we find solvability conditions for $j=3$, which are

$$
\begin{aligned}
\mathcal{L}_1R_2 &= -\frac{\partial R_1}{\partial\tau} + a_1R_1 + a_2R_1|R_1|^2 + a_3R_1|S_1|^2 \\
&\quad + a_5\frac{\partial^2R_1}{\partial\eta^2} + ik_0a_4R_1\frac{\partial\Psi_1}{\partial\eta} + a_6\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial R_1}{\partial\zeta} \\
&\quad + a_7R_1\frac{\partial^2\Psi_1}{\partial\zeta^2} + ik_0a_8\frac{\partial^3R_1}{\partial\zeta^2\partial\eta} + a_9\frac{\partial^4R_1}{\partial\zeta^4},
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
\mathcal{L}_2S_2 &= -\frac{\partial S_1}{\partial\tau} + a_1S_1 + a_2S_1|S_1|^2 + a_3S_1|R_1|^2 + a_5\frac{\partial^2S_1}{\partial\eta^2} - ik_0a_4S_1\frac{\partial\Psi_1}{\partial\eta} \\
&\quad + a_6\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial S_1}{\partial\zeta} + a_7S_1\frac{\partial^2\Psi_1}{\partial\zeta^2} - ik_0a_8\frac{\partial^3S_1}{\partial\zeta^2\partial\eta} + a_9\frac{\partial^4S_1}{\partial\zeta^4},
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
\mathcal{L}_3(R_2,S_2,\Psi_2) &\equiv \frac{\partial\Psi_2}{\partial T} - H_0\big(R_2\overline{R}_1+S_2\overline{S}_1+\text{c.c.}\big) - b_1\frac{\partial^2\Psi_2}{\partial\zeta^2} \\
&= -\frac{\partial\Psi_1}{\partial\tau} + b_1\frac{\partial^2\Psi_1}{\partial\eta^2} + \left[ik_0b_2\left(\overline{R}_1\frac{\partial R_1}{\partial\eta} - \overline{S}_1\frac{\partial S_1}{\partial\eta}\right) + \text{c.c.}\right] \\
&\quad + \left[b_3\left(\overline{R}_1\frac{\partial^2R_1}{\partial\zeta^2} - \overline{S}_1\frac{\partial^2S_1}{\partial\zeta^2}\right) + \text{c.c.}\right] + b_5\left(\frac{\partial\Psi_1}{\partial\zeta}\right)^2 \\
&\quad + b_4\frac{\partial^4\Psi_1}{\partial\zeta^4} + b_6\left[\frac{\partial R_1}{\partial\zeta}\frac{\partial\overline{R_1}}{\partial\zeta} + \frac{\partial S_1}{\partial\zeta}\frac{\partial\overline{S_1}}{\partial\zeta}\right],
\end{aligned}
\tag{3.31}
$$

and for $j=4$, which are

$$
\begin{aligned}
\mathcal{L}_1R_3 &= -\frac{\partial R_2}{\partial\tau} + a_1R_2 + a_2\big(R_1^2\overline{R}_2 + 2R_2|R_1|^2\big) + a_3\big(R_2|S_1|^2 + R_1\big(S_2\overline{S}_1 + S_1\overline{S}_2\big)\big) \\
&\quad + a_5\frac{\partial^2R_2}{\partial\eta^2} + ik_0a_4R_2\frac{\partial\Psi_1}{\partial\eta} + ik_0a_4R_1\frac{\partial\Psi_2}{\partial\eta}
\end{aligned}
$$

$$
\begin{aligned}
&+a_6\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial R_2}{\partial\zeta}+a_6\frac{\partial\Psi_2}{\partial\zeta}\frac{\partial R_1}{\partial\zeta}+a_7R_1\frac{\partial^2\Psi_2}{\partial\zeta^2}+a_7R_2\frac{\partial^2\Psi_1}{\partial\zeta^2}+ik_0a_8\frac{\partial^3R_2}{\partial\zeta^2\partial\eta}\\
&+a_9\frac{\partial^4R_2}{\partial\zeta^4}+ik_0d_1|R_1|^2\frac{\partial R_1}{\partial\eta}+ik_0d_2|S_1|^2\frac{\partial R_1}{\partial\eta}+d_3R_1\frac{\partial^2\Psi_1}{\partial\eta^2}\\
&+ik_0d_4\frac{\partial^3R_1}{\partial\eta^3}+d_5\frac{\partial R_1}{\partial\eta}\frac{\partial\Psi_1}{\partial\eta}+ik_0d_6\overline{S_1}R_1\frac{\partial S_1}{\partial\eta}+ik_0d_7\frac{\partial\overline{S_1}}{\partial\eta}R_1S_1\\
&+ik_0d_8R_1^2\frac{\partial\overline{R_1}}{\partial\eta}+ik_0d_9R_1\frac{\partial^3\Psi_1}{\partial\zeta^2\partial\eta}-d_{10}\frac{\partial^4R_1}{\partial\zeta^2\partial\eta^2}+ik_0d_{11}\frac{\partial^5R_1}{\partial\zeta^4\partial\eta}-ik_0d_{12}\frac{\partial R_1}{\partial\eta}\\
(3.32)\quad&+d_{13}\frac{\partial^2R_1}{\partial\zeta^2}-d_{14}|R_1|^2\frac{\partial^2R_1}{\partial\zeta^2}-d_{15}R_1^2\frac{\partial^2\overline{R_1}}{\partial\zeta^2}+ik_0d_{16}\frac{\partial R_1}{\partial\eta}\frac{\partial^2\Psi_1}{\partial\zeta^2}+d_{17}R_1\frac{\partial^4\Psi_1}{\partial\zeta^4}\\
&+d_{18}\frac{\partial^6R_1}{\partial\zeta^6}+d_{19}R_1S_1\frac{\partial^2\overline{S_1}}{\partial\zeta^2}+d_{20}R_1\overline{S_1}\frac{\partial^2S_1}{\partial\zeta^2}+d_{21}|S_1|^2\frac{\partial^2R_1}{\partial\zeta^2}\\
&+ik_0d_{22}\frac{\partial R_1}{\partial\zeta}\frac{\partial^2\Psi_1}{\partial\zeta\partial\eta}+ik_0d_{23}\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial^2R_1}{\partial\zeta\partial\eta}+ik_0d_{24}\frac{\partial\Psi_1}{\partial\eta}\frac{\partial^2R_1}{\partial\zeta^2}+d_{25}\frac{\partial R_1}{\partial\zeta}\frac{\partial\overline{R_1}}{\partial\zeta}R_1\\
&+d_{26}\left(\frac{\partial R_1}{\partial\zeta}\right)^2\overline{R_1}+d_{27}\frac{\partial R_1}{\partial\zeta}\frac{\partial S_1}{\partial\zeta}\overline{S_1}+d_{28}\frac{\partial R_1}{\partial\zeta}\frac{\partial\overline{S_1}}{\partial\zeta}S_1+d_{29}\frac{\partial S_1}{\partial\zeta}\frac{\partial\overline{S_1}}{\partial\zeta}R_1\\
&+ik_0d_{30}\frac{\partial^3\Psi_1}{\partial\zeta^2\partial\eta}R_1+d_{31}\left(\frac{\partial\Psi_1}{\partial\zeta}\right)^2R_1+d_{32}\frac{\partial^3\Psi_1}{\partial\zeta^3}\frac{\partial R_1}{\partial\zeta}\\
&+d_{33}\frac{\partial^2\Psi_1}{\partial\zeta^2}\frac{\partial^2R_1}{\partial\zeta^2}+d_{34}\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial^3R_1}{\partial\zeta^3},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}_2S_3=&-\frac{\partial S_2}{\partial\tau}+a_1S_2+a_2\left(S_1^2\overline{S}_2+2S_2|S_1|^2\right)+a_3\left(S_2|R_1|^2+S_1\left(R_2\overline{R}_1+R_1\overline{R}_2\right)\right)\\
&+a_5\frac{\partial^2S_2}{\partial\eta^2}-ik_0a_4S_2\frac{\partial\Psi_1}{\partial\eta}-ik_0a_4S_1\frac{\partial\Psi_2}{\partial\eta}+a_6\frac{\partial\Psi_2}{\partial\zeta}\frac{\partial S_1}{\partial\zeta}\\
&+a_6\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial S_2}{\partial\zeta}+a_7S_1\frac{\partial^2\Psi_2}{\partial\zeta^2}+a_7S_2\frac{\partial^2\Psi_1}{\partial\zeta^2}-ik_0a_8\frac{\partial^3S_2}{\partial\zeta^2\partial\eta}+a_9\frac{\partial^4S_2}{\partial\zeta^4}\\
&-ik_0d_1|S_1|^2\frac{\partial S_1}{\partial\eta}-ik_0d_2|R_1|^2\frac{\partial S_1}{\partial\eta}+d_3S_1\frac{\partial^2\Psi_1}{\partial\eta^2}\\
&-ik_0d_4\frac{\partial^3S_1}{\partial\eta^3}+d_5\frac{\partial S_1}{\partial\eta}\frac{\partial\Psi_1}{\partial\eta}-ik_0d_6\overline{R_1}S_1\frac{\partial R_1}{\partial\eta}-ik_0d_7\frac{\partial\overline{R_1}}{\partial\eta}R_1S_1\\
&-ik_0d_8S_1^2\frac{\partial\overline{S_1}}{\partial\eta}-ik_0d_9S_1\frac{\partial^3\Psi_1}{\partial\zeta^2\partial\eta}-d_{10}\frac{\partial^4S_1}{\partial\zeta^2\partial\eta^2}-ik_0d_{11}\frac{\partial^5S_1}{\partial\zeta^4\partial\eta}+ik_0d_{12}\frac{\partial S_1}{\partial\eta}\\
(3.33)\quad&+d_{13}\frac{\partial^2S_1}{\partial\zeta^2}-d_{14}|S_1|^2\frac{\partial^2S_1}{\partial\zeta^2}-d_{15}S_1^2\frac{\partial^2\overline{S_1}}{\partial\zeta^2}-ik_0d_{16}\frac{\partial S_1}{\partial\eta}\frac{\partial^2\Psi_1}{\partial\zeta^2}+d_{17}S_1\frac{\partial^4\Psi_1}{\partial\zeta^4}
\end{aligned}
$$

$$
\begin{aligned}
&+ d_{18}\frac{\partial^6 S_1}{\partial\zeta^6} + d_{19}R_1S_1\frac{\partial^2\overline{R_1}}{\partial\zeta^2} + d_{20}S_1\overline{R_1}\frac{\partial^2 R_1}{\partial\zeta^2} + d_{21}|R_1|^2\frac{\partial^2 S_1}{\partial\zeta^2}\\
&- ik_0 d_{22}\frac{\partial S_1}{\partial\zeta}\frac{\partial^2\Psi_1}{\partial\zeta\partial\eta} - ik_0 d_{23}\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial^2 S_1}{\partial\zeta\partial\eta} - ik_0 d_{24}\frac{\partial\Psi_1}{\partial\eta}\frac{\partial^2 S_1}{\partial\zeta^2} + d_{25}\frac{\partial S_1}{\partial\zeta}\frac{\partial\overline{S_1}}{\partial\zeta}S_1\\
&+ d_{26}\left(\frac{\partial S_1}{\partial\zeta}\right)^2\overline{S_1} + d_{27}\frac{\partial S_1}{\partial\zeta}\frac{\partial R_1}{\partial\zeta}\overline{R_1} + d_{28}\frac{\partial S_1}{\partial\zeta}\frac{\partial\overline{R_1}}{\partial\zeta}R_1 + d_{29}\frac{\partial R_1}{\partial\zeta}\frac{\partial\overline{R_1}}{\partial\zeta}S_1\\
&- ik_0 d_{30}\frac{\partial^3\Psi_1}{\partial\zeta^2\partial\eta}S_1 + d_{31}\left(\frac{\partial\Psi_1}{\partial\zeta}\right)^2 S_1 + d_{32}\frac{\partial^3\Psi_1}{\partial\zeta^3}\frac{\partial S_1}{\partial\zeta}\\
&+ d_{33}\frac{\partial^2\Psi_1}{\partial\zeta^2}\frac{\partial^2 S_1}{\partial\zeta^2} + d_{34}\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial^3 S_1}{\partial\zeta^3},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}_3(R_3,S_3,\Psi_3) = &-\frac{\partial\Psi_2}{\partial\tau} + H_0\big(|R_2|^2+|S_2|^2\big) + b_1\frac{\partial^2\Psi_2}{\partial\eta^2}\\
&+\left[ik_0b_2\left(\overline{R_2}\frac{\partial R_1}{\partial\eta} - \overline{S_2}\frac{\partial S_1}{\partial\eta}\right) + \text{c.c.}\right]\\
&+\left[b_3\left(\overline{R_1}\frac{\partial^2 R_2}{\partial\zeta^2} + \overline{S_1}\frac{\partial^2 S_2}{\partial\zeta^2}\right) + \text{c.c.}\right]\\
&+\left[ik_0b_2\left(\overline{R_1}\frac{\partial R_2}{\partial\eta} - \overline{S_1}\frac{\partial S_2}{\partial\eta}\right) + \text{c.c.}\right]\\
&+\left[b_3\left(\overline{R_2}\frac{\partial^2 R_1}{\partial\zeta^2} + \overline{S_2}\frac{\partial^2 S_1}{\partial\zeta^2}\right) + \text{c.c.}\right]\\
&+2b_5\frac{\partial\Psi_1}{\partial\zeta}\frac{\partial\Psi_2}{\partial\zeta} + b_6\left[\frac{\partial R_1}{\partial\zeta}\frac{\partial\overline{R_2}}{\partial\zeta} + \frac{\partial S_2}{\partial\zeta}\frac{\partial\overline{S_1}}{\partial\zeta} + \text{c.c.}\right] + b_4\frac{\partial^4\Psi_2}{\partial\zeta^4}\\
&+f_1\left[\frac{\partial\Psi_1}{\partial\eta}\right]^2 + k_0f_2\frac{\partial\Psi_1}{\partial\eta}\big[|R_1|^2-|S_1|^2\big]\\
&+\left(f_3\left[\frac{\partial^2 R_1}{\partial\eta^2}\overline{R_1} + \frac{\partial^2 S_1}{\partial\eta^2}\overline{S_1}\right] + \text{c.c.}\right)\\
&+f_4\big[|R_1|^4+|S_1|^4\big] + f_5|R_1|^2|S_1|^2 + f_6\big[|R_1|^2+|S_1|^2\big]\\
&+f_7\left[\frac{\partial S_1}{\partial\eta}\frac{\partial\overline{S_1}}{\partial\eta} + \frac{\partial R_1}{\partial\eta}\frac{\partial\overline{R_1}}{\partial\eta}\right]\\
&-\left(ik_0f_8\left[\frac{\partial^2 R_1}{\partial\zeta^2}\frac{\partial\overline{R_1}}{\partial\eta} - \frac{\partial^2 S_1}{\partial\zeta^2}\frac{\partial\overline{S_1}}{\partial\eta}\right] + \text{c.c.}\right)\\
&-f_9\frac{\partial^2\Psi_1}{\partial\zeta^2}\big[|R_1|^2+|S_1|^2\big]
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&-\left(ik_0 f_{10}\left[\overline{R_1}\frac{\partial^3 R_1}{\partial\zeta^2\partial\eta}-\overline{S_1}\frac{\partial^3 S_1}{\partial\zeta^2\partial\eta}\right]+\text{c.c.}\right)\\
&-\left(f_{11}\left[\overline{R_1}\frac{\partial^4 R_1}{\partial\zeta^4}+\overline{S_1}\frac{\partial^2 S_1}{\partial\zeta^4}\right]+\text{c.c.}\right)\\
&+f_{12}\frac{\partial^4\Psi_1}{\partial\zeta^2\partial\eta^2}+f_{13}\frac{\partial^2\Psi_1}{\partial\zeta^2}-f_{14}\frac{\partial^6\Psi_1}{\partial\zeta^6}\\
&+f_{15}\left(\frac{\partial^2\Psi_1}{\partial\zeta^2}\right)^2+\left[f_{16}\left(\frac{\partial^2 R_1}{\partial\zeta^2}\frac{\partial^2\overline{R_1}}{\partial\zeta^2}+\frac{\partial^2 S_1}{\partial\zeta^2}\frac{\partial^2\overline{S_1}}{\partial\zeta^2}\right)+\text{c.c.}\right]\\
&+\left[f_{17}\frac{\partial\Psi_1}{\partial\zeta}\left(\frac{\partial R_1}{\partial\zeta}\overline{R_1}+\frac{\partial S_1}{\partial\zeta}\overline{S_1}\right)+\text{c.c.}\right]\\
&+\left[ik_0 f_{18}\left(\frac{\partial^2 R_1}{\partial\zeta\partial\eta}\frac{\partial\overline{R_1}}{\partial\zeta}-\frac{\partial^2 S_1}{\partial\zeta\partial\eta}\frac{\partial\overline{S_1}}{\partial\zeta}\right)+\text{c.c.}\right]\\
&+\left[f_{19}\left(\frac{\partial^3 R_1}{\partial\zeta^3}\frac{\partial\overline{R_1}}{\partial\zeta}+\frac{\partial^3 S_1}{\partial\zeta^3}\frac{\partial\overline{S_1}}{\partial\zeta}\right)+\text{c.c.}\right]+f_{20}\frac{\partial^3\Psi_1}{\partial\zeta^3}\frac{\partial\Psi_1}{\partial\zeta},
\end{aligned}
$$

where a_j, b_j, d_j, and f_j are given in Appendix A. We note that the solvability conditions for $j=4$ depend on U_3 (see Appendix A), which exists only if the solvability conditions for $j=3$ are satisfied.

We now define $R \equiv R_1+\epsilon R_2+\epsilon^2 R_3$, $S \equiv S_1+\epsilon S_2+\epsilon^2 S_3$, and $\Psi \equiv \Psi_1+\epsilon\Psi_2+\epsilon^2\Psi_3$, and combine the solvability conditions for $j=2$, 3, 4 to obtain evolution equations for R, S, and Ψ:

$$
\begin{aligned}
0 = &-\frac{\partial R}{\partial\tau}+a_1 R+a_2 R|R|^2+a_3 R|S|^2+a_5\frac{\partial^2 R}{\partial\eta^2}+ik_0 a_4 R\frac{\partial\Psi}{\partial\eta}\\
&+a_6\frac{\partial\Psi}{\partial\zeta}\frac{\partial R}{\partial\zeta}+a_7 R\frac{\partial^2\Psi}{\partial\zeta^2}+ik_0 a_8\frac{\partial^3 R}{\partial\zeta^2\partial\eta}+a_9\frac{\partial^4 R}{\partial\zeta^4}-\frac{1}{\epsilon}\mathscr{L}_1 R\\
&+\epsilon\Bigg(ik_0 d_1|R|^2\frac{\partial R}{\partial\eta}+ik_0 d_2|S|^2\frac{\partial R}{\partial\eta}+d_3 R\frac{\partial^2\Psi}{\partial\eta^2}\\
&+ik_0 d_4\frac{\partial^3 R}{\partial\eta^3}+d_5\frac{\partial R}{\partial\eta}\frac{\partial\Psi}{\partial\eta}+ik_0 d_6\overline{S}R\frac{\partial S}{\partial\eta}+ik_0 d_7\frac{\partial\overline{S}}{\partial\eta}RS\\
&+ik_0 d_8 R^2\frac{\partial\overline{R}}{\partial\eta}+ik_0 d_9 R\frac{\partial^3\Psi}{\partial\zeta^2\partial\eta}-d_{10}\frac{\partial^4 R}{\partial\zeta^2\partial\eta^2}\\
&+ik_0 d_{11}\frac{\partial^5 R}{\partial\zeta^4\partial\eta}-ik_0 d_{12}\frac{\partial R}{\partial\eta}\\
&+d_{13}\frac{\partial^2 R}{\partial\zeta^2}-d_{14}|R|^2\frac{\partial^2 R}{\partial\zeta^2}-d_{15}R^2\frac{\partial^2\overline{R}}{\partial\zeta^2}+ik_0 d_{16}\frac{\partial R}{\partial\eta}\frac{\partial^2\Psi}{\partial\zeta^2}\\
&+d_{17}R\frac{\partial^4\Psi}{\partial\zeta^4}+d_{18}\frac{\partial^6 R}{\partial\zeta^6}+d_{19}RS\frac{\partial^2\overline{S}}{\partial\zeta^2}+d_{20}R\overline{S}\frac{\partial^2 S}{\partial\zeta^2}+d_{21}|S|^2\frac{\partial^2 R}{\partial\zeta^2}
\end{aligned}
\tag{3.35}
$$

$$+ ik_0 d_{22} \frac{\partial R}{\partial \zeta} \frac{\partial^2 \Psi}{\partial \zeta \partial \eta} + ik_0 d_{23} \frac{\partial \Psi}{\partial \zeta} \frac{\partial^2 R}{\partial \zeta \partial \eta} + ik_0 d_{24} \frac{\partial \Psi}{\partial \eta} \frac{\partial^2 R}{\partial \zeta^2}$$

$$+ d_{25} \frac{\partial R}{\partial \zeta} \frac{\partial \bar{R}}{\partial \zeta} R + d_{26} \left(\frac{\partial R}{\partial \zeta} \right)^2 \bar{R} + d_{27} \frac{\partial R}{\partial \zeta} \frac{\partial S}{\partial \zeta} \bar{S} + d_{28} \frac{\partial R}{\partial \zeta} \frac{\partial \bar{S}}{\partial \zeta} S$$

$$+ d_{29} \frac{\partial S}{\partial \zeta} \frac{\partial \bar{S}}{\partial \zeta} R + ik_0 d_{30} \frac{\partial^3 \Psi}{\partial \zeta^2 \partial \eta} R$$

$$+ d_{31} \left(\frac{\partial \Psi}{\partial \zeta} \right)^2 R + d_{32} \frac{\partial^3 \Psi}{\partial \zeta^3} \frac{\partial R}{\partial \zeta} + d_{33} \frac{\partial^2 \Psi}{\partial \zeta^2} \frac{\partial^2 R}{\partial \zeta^2} + d_{34} \frac{\partial \Psi}{\partial \zeta} \frac{\partial^3 R}{\partial \zeta^3} \Bigg),$$

$$
\begin{aligned}
0 = &-\frac{\partial S}{\partial \tau} + a_1 S + a_2 S|S|^2 + a_3 S|R|^2 + a_5 \frac{\partial^2 S}{\partial \eta^2} - ik_0 a_4 S \frac{\partial \Psi}{\partial \eta} \\
&+ a_6 \frac{\partial \Psi}{\partial \zeta} \frac{\partial S}{\partial \zeta} + a_7 S \frac{\partial^2 \Psi}{\partial \zeta^2} - ik_0 a_8 \frac{\partial^3 S}{\partial \zeta^2 \partial \eta} + a_9 \frac{\partial^4 S}{\partial \zeta^4} - \frac{1}{\epsilon} \mathscr{L}_2 S \\
&+ \epsilon \Bigg(-ik_0 d_1 |S|^2 \frac{\partial S}{\partial \eta} - ik_0 d_2 |R|^2 \frac{\partial S}{\partial \eta} + d_3 S \frac{\partial^2 \Psi}{\partial \eta^2} \\
&\quad - ik_0 d_4 \frac{\partial^3 S}{\partial \eta^3} + d_5 \frac{\partial S}{\partial \eta} \frac{\partial \Psi}{\partial \eta} - ik_0 d_6 \bar{R} S \frac{\partial R}{\partial \eta} - ik_0 d_7 \frac{\partial \bar{R}}{\partial \eta} RS \\
&\quad - ik_0 d_8 S^2 \frac{\partial \bar{S}}{\partial \eta} - ik_0 d_9 S \frac{\partial^3 \Psi}{\partial \zeta^2 \partial \eta} - d_{10} \frac{\partial^4 S}{\partial \zeta^2 \partial \eta^2} - ik_0 d_{11} \frac{\partial^5 S}{\partial \zeta^4 \partial \eta} \\
&\quad + ik_0 d_{12} \frac{\partial S}{\partial \eta} + d_{13} \frac{\partial^2 S}{\partial \zeta^2} - d_{14} |S|^2 \frac{\partial^2 S}{\partial \zeta^2} - d_{15} S^2 \frac{\partial^2 \bar{S}}{\partial \zeta^2} \\
&\quad - ik_0 d_{16} \frac{\partial S}{\partial \eta} \frac{\partial^2 \Psi}{\partial \zeta^2} + d_{17} S \frac{\partial^4 \Psi}{\partial \zeta^4} + d_{18} \frac{\partial^6 S}{\partial \zeta^6} \\
&\quad + d_{19} RS \frac{\partial^2 \bar{R}}{\partial \zeta^2} + d_{20} S \bar{R} \frac{\partial^2 R}{\partial \zeta^2} + d_{21} |R|^2 \frac{\partial^2 S}{\partial \zeta^2} \\
&\quad - ik_0 d_{22} \frac{\partial S}{\partial \zeta} \frac{\partial^2 \Psi}{\partial \zeta \partial \eta} - ik_0 d_{23} \frac{\partial \Psi}{\partial \zeta} \frac{\partial^2 S}{\partial \zeta \partial \eta} \\
&\quad - ik_0 d_{24} \frac{\partial \Psi}{\partial \eta} \frac{\partial^2 S}{\partial \zeta^2} + d_{25} \frac{\partial S}{\partial \zeta} \frac{\partial \bar{S}}{\partial \zeta} S \\
&\quad + d_{26} \left(\frac{\partial S}{\partial \zeta} \right)^2 \bar{S} + d_{27} \frac{\partial S}{\partial \zeta} \frac{\partial R}{\partial \zeta} \bar{R} + d_{28} \frac{\partial S}{\partial \zeta} \frac{\partial \bar{R}}{\partial \zeta} R + d_{29} \frac{\partial R}{\partial \zeta} \frac{\partial \bar{R}}{\partial \zeta} S \\
&\quad - ik_0 d_{30} \frac{\partial^3 \Psi}{\partial \zeta^2 \partial \eta} S + d_{31} \left(\frac{\partial \Psi}{\partial \zeta} \right)^2 S + d_{32} \frac{\partial^3 \Psi}{\partial \zeta^3} \frac{\partial S}{\partial \zeta} \\
&\qquad + d_{33} \frac{\partial^2 \Psi}{\partial \zeta^2} \frac{\partial^2 S}{\partial \zeta^2} + d_{34} \frac{\partial \Psi}{\partial \zeta} \frac{\partial^3 S}{\partial \zeta^3} \Bigg),
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
0 = &-\frac{\partial \Psi}{\partial \tau} + b_1 \frac{\partial^2 \Psi}{\partial \eta^2} + \left[i k_0 b_2 \left(\bar{R} \frac{\partial R}{\partial \eta} - \bar{S} \frac{\partial S}{\partial \eta} \right) + \text{c.c.} \right] \\
&+ \left[b_3 \left(\bar{R} \frac{\partial^2 R}{\partial \zeta^2} + \bar{S} \frac{\partial^2 S}{\partial \zeta^2} \right) + \text{c.c.} \right] \\
&+ b_5 \left(\frac{\partial \Psi}{\partial \zeta} \right)^2 + b_6 \left[\left(\frac{\partial R}{\partial \zeta} \right) \left(\frac{\partial \bar{R}}{\partial \zeta} \right) + \left(\frac{\partial S}{\partial \zeta} \right) \left(\frac{\partial \bar{S}}{\partial \zeta} \right) \right] \\
&+ b_4 \frac{\partial^4 \Psi}{\partial \zeta^4} + \frac{1}{\epsilon} \left[-\frac{\partial \Psi}{\partial T} + H_0 (|R|^2 + |S|^2) + b_1 \frac{\partial^2 \Psi}{\partial \zeta^2} \right] \\
&+ \epsilon \Bigg(f_1 \left[\frac{\partial \Psi}{\partial \eta} \right]^2 + k_0 f_2 \frac{\partial \Psi}{\partial \eta} [|R|^2 - |S|^2] + \left(f_3 \left[\frac{\partial^2 R}{\partial \eta^2} \bar{R} + \frac{\partial^2 S}{\partial \eta^2} \bar{S} \right] + \text{c.c.} \right) \\
&\quad + f_4 [|R|^4 + |S|^4] + f_5 |R|^2 |S|^2 + f_6 [|R|^2 + |S|^2] + f_7 \left[\frac{\partial S}{\partial \eta} \frac{\partial \bar{S}}{\partial \eta} + \frac{\partial R}{\partial \eta} \frac{\partial \bar{R}}{\partial \eta} \right] \\
&\quad - \left(i k_0 f_8 \left[\frac{\partial^2 R}{\partial \zeta^2} \frac{\partial \bar{R}}{\partial \eta} - \frac{\partial^2 S}{\partial \zeta^2} \frac{\partial \bar{S}}{\partial \eta} \right] + \text{c.c.} \right) \\
&\quad - f_9 \frac{\partial^2 \Psi}{\partial \zeta^2} [|R|^2 + |S|^2] - \left(i k_0 f_{10} \left[\bar{R} \frac{\partial^3 R}{\partial \zeta^2 \partial \eta} - \bar{S} \frac{\partial^3 S}{\partial \zeta^2 \partial \eta} \right] + \text{c.c.} \right) \\
&\quad - \left(f_{11} \left[\bar{R} \frac{\partial^4 R}{\partial \zeta^4} + \bar{S} \frac{\partial^4 S}{\partial \zeta^4} \right] + \text{c.c.} \right) + f_{12} \frac{\partial^4 \Psi}{\partial \zeta^2 \partial \eta^2} + f_{13} \frac{\partial^2 \Psi}{\partial \zeta^2} - f_{14} \frac{\partial^6 \Psi}{\partial \zeta^6} \\
&\quad + f_{15} \left(\frac{\partial^2 \Psi}{\partial \zeta^2} \right)^2 + \left[f_{16} \left(\frac{\partial^2 R}{\partial \zeta^2} \frac{\partial^2 \bar{R}}{\partial \zeta^2} + \frac{\partial^2 S}{\partial \zeta^2} \frac{\partial^2 \bar{S}}{\partial \zeta^2} \right) + \text{c.c.} \right] \\
&\quad + \left[f_{17} \frac{\partial \Psi}{\partial \zeta} \left(\frac{\partial R}{\partial \zeta} \bar{R} + \frac{\partial S}{\partial \zeta} \bar{S} \right) + \text{c.c.} \right] \\
&\quad + \left[i k_0 f_{18} \left(\frac{\partial^2 R}{\partial \zeta \partial \eta} \frac{\partial \bar{R}}{\partial \zeta} - \frac{\partial^2 S}{\partial \zeta \partial \eta} \frac{\partial \bar{S}}{\partial \zeta} \right) + \text{c.c.} \right] \\
&\quad + \left[f_{19} \left(\frac{\partial^3 R}{\partial \zeta^3} \frac{\partial \bar{R}}{\partial \zeta} + \frac{\partial^3 S}{\partial \zeta^3} \frac{\partial \bar{S}}{\partial \zeta} \right) + \text{c.c.} \right] + f_{20} \frac{\partial^3 \Psi}{\partial \zeta^3} \frac{\partial \Psi}{\partial \zeta} \Bigg).
\end{aligned}
\tag{3.37}
$$

Equations (3.35)–(3.37) are the modulation equations that we will employ to investigate the stability of plane wave solutions bifurcating from the basic state. We observe that had we scaled $\zeta = \epsilon z$, rather than $\zeta = \epsilon^{1/2} z$ as in (3.2), many terms in the evolution equations (3.35)–(3.37) would drop out since they correspond to higher-order terms. The stability of the plane wave solutions would be unaffected by the rescaling; however, the nonlinear dynamics of (3.35)–(3.37) may well differ from their rescaled version.

A useful observation about (3.35)–(3.37) is that the reflection symmetries in y and z yield symmetries for the system. Specifically, reflection in z implies invariance under

$$\zeta \to -\zeta, \tag{3.38}$$

whereas reflection in y implies invariance under

$$R \to S, \quad S \to R, \quad \eta \to -\eta. \tag{3.39}$$

4. Plane wave solutions and their stability. Plane wave solutions to (3.35)–(3.37) can be found by considering solutions of the form of a temporal modulation of a traveling wave. By specifying the spatial periodicity, we can reduce the system from a set of partial differential equations to a set of autonomous ordinary differential equations. Moreover, the amplitudes decouple from the phase and mean field yielding a two component system that is easily analyzed for the existence and stability of steady states.

In the calculation below we assume without loss of generality that ω_0', $k_0 > 0$. In addition we restrict ourselves to the physically relevant case that the initial bifurcation to a traveling wave is supercritical (a necessary condition for stability), which implies that a_1^r, $a_5^r > 0$ and $a_2^r < 0$.

Consider a solution of the form

$$R \equiv r(\tau)e^{i\phi_R(\tau)+ik_1(\eta+\omega_0'T)}, \quad S \equiv s(\tau)e^{i\phi_S(\tau)+ik_2(\eta-\omega_0'T)}, \quad \Psi_{PW} \equiv \psi_0(T)+\psi_1(\tau), \tag{4.1}$$

where r, s are real amplitudes, ϕ_R, ϕ_S are real phases, and ψ_0, ψ_1 are spatially uniform displacements of the basic state.

We introduce the notation

$$a_j = \operatorname{Re} a_j + i \operatorname{Im} a_j \equiv a_j^r + i a_j^i, \tag{4.2}$$

and use similar notation for the coefficients b_j, f_j, and d_j. Substituting (4.1) into (3.35)–(3.37) we find that the amplitudes are governed by

$$\begin{aligned} r_\tau = r\big[& a_1^r + a_2^r r^2 + a_3^r s^2 - k_1^2 a_5^r \\ &+ \epsilon(-k_0k_1d_1^r r^2 + k_0k_1^3 d_4^r + k_0k_1d_8^r r^2 + k_0k_1d_{12}^r \\ &\qquad -k_0d_6^r k_2 s^2 + k_0k_2d_7^r s^2 - k_0k_1d_2^r s^2)\big], \end{aligned} \tag{4.3}$$

$$\begin{aligned} s_\tau = s\big[& a_1^r + a_2^r s^2 + a_3^r r^2 - k_2^2 a_5^r \\ &+ \epsilon(k_0k_2d_1^r s^2 - k_0k_2^3 d_4^r - k_0k_2d_8^r s^2 - k_0k_2d_{12}^r \\ &\qquad +k_0d_6^r k_1 r^2 - k_0k_1d_7^r r^2 + k_0k_2d_2^r r^2)\big], \end{aligned} \tag{4.4}$$

which we note have decoupled from the equations for the phases and the mean field,

$$\begin{aligned} (\phi_R)_\tau = \big[& a_1^i + a_2^i r^2 + a_3^i s^2 - k_1^2 a_5^i \\ &+ \epsilon(-k_0k_1d_1^i r^2 + k_0k_1^3 d_4^i + k_0k_1d_8^i r^2 + k_0k_1d_{12}^i \\ &\qquad -k_0d_6^i k_2 s^2 + k_0k_2d_7^i s^2 - k_0k_1d_2^i s^2)\big], \end{aligned} \tag{4.5}$$

$$\begin{aligned} (\phi_S)_\tau = s\big[& a_1^i + a_2^i s^2 + a_3^i r^2 - k_2^2 a_5^i \\ &+ \epsilon(k_0k_2d_1^i s^2 - k_0k_2^3 d_4^i - k_0k_2d_8^i s^2 - k_0k_2d_{12}^i \\ &\qquad +k_0d_6^i k_1 r^2 - k_0k_1d_7^i r^2 + k_0k_2d_2^i r^2)\big], \end{aligned} \tag{4.6}$$

$$(\psi_0)_T = (s^2 + r^2)H_0, \tag{4.7}$$

$$\begin{aligned}(\psi_1)_t = &-2b_2^r\big(k_0(k_1 r^2 - k_2 s^2)\big)\\ &+\epsilon\big(-2f_3^r(r^2k_1^2 + s^2k_2^2) + f_4(r^4 + s^4)\\ &+f_5 r^2 s^2 + f_6(r^2 + s^2) + f_7(k_1^2 r^2 + k_2^2 s^2)\big).\end{aligned} \tag{4.8}$$

Equations (4.3), (4.4) are two coupled cubic Landau equations whose equilibria and stability are well understood (cf. [12]). There are three types of equilibria: (i) the zero solution ($R = S = 0$); (ii) unidirectional traveling wave solutions ($R \neq 0$, $S = 0$ or $R = 0$, $S \neq 0$); and (iii) quasiperiodic wave solutions ($R \neq 0$, $S \neq 0$) describing counter-propagating traveling waves. The zero solution is unstable to wavenumbers k_1 and k_2 satisfying

$$k_1^2, k_2^2 < \frac{a_1^r}{a_5^r} \tag{4.9}$$

as we are considering the situation where there has been a supercritical bifurcation.

We consider a unidirectional traveling wave of the form $R \neq 0$, $S = 0$ and note that the stability of the state $R = 0$, $S \neq 0$ can be deduced from the symmetry (3.39). The solution is given by

$$r(t) = \rho = \left(\frac{a_1^r - k_1^2 a_5^r}{-a_2^r}\right)^{1/2} + \mathscr{O}(\epsilon) \qquad s(t) = \sigma = 0. \tag{4.10}$$

The stability of this state can be computed by linearizing the system (4.3), (4.4) about the equilibrium and computing the two eigenvalues Σ.

$$\Sigma = 2a_2^r\rho^2, \qquad a_1^r - k_2^2 a_5^r - (a_1^r - k_1^2 a_5^r)\frac{a_3^r}{a_2^r}. \tag{4.11}$$

The first eigenvalue is always negative as $a_2^r < 0$, whereas the second is maximized for $k_2 = 0$. For this eigenvalue to be negative, corresponding to stability, it is necessary that

$$a_3^r < a_2^r, \qquad k_1^2 < \frac{a_1^r a_2^r}{a_5^r a_3^r}\left(1 - \frac{a_2^r}{a_3^r}\right). \tag{4.12}$$

Similarly, a quasiperiodic wave solution can be found

$$r(t) = \rho = \left(\frac{a_3^r(a_1^r - k_2^2 a_5^r) - a_2^r(a_1^r - k_1^2 a_5^r)}{(a_2^r)^2 - (a_3^r)^2}\right)^{1/2} + \mathscr{O}(\epsilon), \tag{4.13}$$

$$s(t) = \sigma = \left(\frac{a_3^r(a_1^r - k_1^2 a_5^r) - a_2^r(a_1^r - k_2^2 a_5^r)}{(a_2^r)^2 - (a_3^r)^2}\right)^{1/2} + \mathscr{O}(\epsilon), \tag{4.14}$$

where the radicands in (4.13), (4.14) must be positive for the traveling wave to exist.

Linearizing about this equilibrium yields two eigenvalues

$$\Sigma = a_2^r(\rho^2 + \sigma^2) \pm \sqrt{\big(a_2^r(\rho^2 + \sigma^2)\big)^2 + 4\sigma^2\rho^2\big((a_3^r)^2 - (a_2^r)^2\big)}, \tag{4.15}$$

which implies that the quasiperiodic solution is stable if and only if

$$|a_3^r| < |a_2^r|. \tag{4.16}$$

In summary, for the restricted system (4.3)–(4.8), if $a_3^r < a_2^r < 0$ the only stable equilibrium is a band of unidirectional traveling waves specified by (4.12) or their images under the symmetry (3.39). If $|a_3^r| < |a_2^r|$ the only stable equilibria are a band of quasiperiodic traveling waves specified by (4.13), (4.14), where the wavenumbers must be chosen to make the radicands positive. If $a_3^r > -a_2^r > 0$ no stable equilibria exist although unstable unidirectional and quasiperiodic solutions can be found.

Note that solutions to the phase and the mean field equations can be found corresponding to each of the equilibria above. Substituting

$$\phi_R = \omega_1 \tau, \quad \phi_S = \omega_2 \tau, \quad \psi_0 = h_0 T, \quad \psi_1 = h_1 \tau \tag{4.17}$$

into (4.5)–(4.8) yields a solution directly in terms of σ and ρ at the equilibrium.

In the next two subsections we consider the stability of unidirectional traveling waves ($R \neq 0$, $S = 0$) and of quasiperiodic waves ($R \neq 0$, $S \neq 0$) to modulational perturbations.

4.1. Stability of traveling waves. We first study the modulational stability of the single mode traveling wave solution (4.10). Note that we can take the amplitude of $S \equiv 0$ in this case as the stability to perturbations in S decouples and yields the criteria (4.12).

We write the amplitude R as

$$R = r(\eta, \tau, \zeta) e^{ik_1 \omega_0' T + i\omega_1 \tau + i\Omega(\eta, \tau, \zeta)} \tag{4.18}$$

and substitute into (3.35) and (3.37) above, denoting differentiation by subscripts, to obtain

$$\begin{aligned}
r_\tau = {} & a_1^r r + a_2^r r^3 - k_0 a_4^i r \Psi_\eta + a_5^r\left(-(\Omega_\eta)^2 r + r_{\eta\eta}\right) - a_5^i(2\Omega_\eta r_\eta + \Omega_{\eta\eta}) \\
& + \frac{\omega_0'}{\epsilon} r_\eta + a_7^r r \Psi_{\zeta\zeta} - k_0 a_8^r r_{\zeta\zeta}\Omega_\eta + a_8^i k_0 r \Omega_{\zeta\zeta\eta} + \frac{\omega_0'}{2k_0\epsilon}\Omega_{\zeta\zeta} r \\
& + \epsilon\Big(-k_0 d_1^i r^2 r_\eta - \Omega_\eta r^3 k_0 d_1^r + d_3^r \Psi_{\eta\eta} r + d_5^r r_\eta \Psi_\eta - d_5^i \Omega_\eta r \Psi_\eta \\
& \quad - k_0 d_4^i\left(r_{\eta\eta\eta} - 3\Omega_\eta^2 r_\eta - 3\Omega_{\eta\eta}\Omega_\eta r\right) \\
& \quad - k_0 d_4^r\left(3\Omega_\eta r_{\eta\eta} + 3\Omega_{\eta\eta} r_\eta + \Omega_{\eta\eta\eta} r - \Omega_\eta^3 r\right) \\
& \quad + k_0 d_{12}^i r_\eta + k_0 d_{12}^r \Omega_\eta r - k_0 d_8^i r^2 r_\eta + k_0 d_8^r r^3 \Omega_\eta \\
& \quad + d_{10}^r \Omega_\eta^2 r_{\zeta\zeta} - d_{10}^i \Omega_\eta^2 \Omega_{\zeta\zeta} r + d_{13}^r r_{\zeta\zeta} - d_{13}^i \Omega_{\zeta\zeta} r - (d_{14}^r + d_{15}^r) r_{\zeta\zeta} r^2 \\
& \qquad + (d_{14}^i - d_{15}^i)\Omega_{\zeta\zeta} r^3 - d_{16}^r k_0 \Psi_{\zeta\zeta}\Omega_\eta r\Big),
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\frac{k_1\omega_0'}{\epsilon} + \omega_1 + \Omega_\tau = {} & a_1^i + a_2^i r^2 + k_0 a_4^r \Psi_\eta + a_5^i\left(-(\Omega_\eta)^2 + \frac{r_{\eta\eta}}{r}\right) \\
& + a_5^r\left(\frac{2\Omega_\eta r_\eta}{r} + \Omega_{\eta\eta}\right) + \frac{1}{\epsilon}(\omega_0'\Omega_\eta) + a_7^i \Psi_{\zeta\zeta} \\
& - k_0 a_8^i r_{\zeta\zeta}\frac{\Omega_\eta}{r} - k_0 a_8^r \Omega_\eta \Omega_{\zeta\zeta} - \frac{\omega_0'}{2\epsilon k_0 r} r_{\zeta\zeta} \\
& + \epsilon\Big(k_0 d_1^r r r_\eta - \Omega_\eta r^2 k_0 d_1^i + d_3^i \Psi_{\eta\eta} + d_5^i \frac{r_n}{r}\Psi_\eta + d_5^r \Omega_\eta \Psi_\eta
\end{aligned}$$

$$
\begin{aligned}
(4.20)\qquad &+k_0 d_4^r\left(\frac{r_{\eta\eta\eta}}{r}-3\Omega_\eta^2\frac{r_\eta}{r}-3\Omega_{\eta\eta}\Omega_\eta\right)\\
&-k_0 d_4^i\left(3\Omega_\eta\frac{r_{\eta\eta}}{r}+3\Omega_{\eta\eta}\frac{r_\eta}{r}+\Omega_{\eta\eta\eta}-\Omega_\eta^3\right)\\
&-k_0 d_{12}^r\frac{r_\eta}{r}+k_0 d_{12}^i\Omega_\eta+k_0 d_8^r r r_\eta+k_0 d_8^i r^2\Omega_\eta\\
&+\Bigg(d_{10}^i\Omega_\eta^2\frac{r_{\zeta\zeta}}{r}+d_{10}^r\Omega_\eta^2\Omega_{\zeta\zeta}+d_{13}^i\frac{r_{\zeta\zeta}}{r}+d_{13}^r\Omega_{\zeta\zeta}-(d_{14}^r+d_{15}^i)r_{\zeta\zeta}r\\
&\qquad-(d_{14}^r-d_{15}^r)\Omega_{\zeta\zeta}r^2-d_{16}^i k_0\Omega_\eta\Psi_{\zeta\zeta}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\epsilon}\Psi_T+\Psi_\tau={}&b_1\Psi_{\eta\eta}-2k_0 b_2^i r_\eta r-2k_0 b_2^i\Omega_\eta r^2\\
&+\frac{H_0}{\epsilon}r^2+2b_3^r r_{\zeta\zeta}r-2b_3^i\Omega_{\zeta\zeta}r^2+\frac{b_1}{\epsilon}\Psi_{\zeta\zeta}\\
(4.21)\qquad &+\epsilon\Big(k_0 f_2\Psi_\eta r^2+2f_3^r\left(r_{\eta\eta}r-\Omega_\eta^2 r^2\right)-2f_3^i\left(2\Omega_\eta r_\eta r+\Omega_{\eta\eta}r^2\right)\\
&\quad+f_4 r^4+f_6 r^2+f_7\left(r_\eta^2+\Omega_\eta^2 r^2\right)-2k_0 f_8^i\Omega_{\zeta\zeta}\Omega_\eta r^2\\
&\quad-2k_0 f_8^r r_{\zeta\zeta}\Omega_\eta r-f_9 r^2\Psi_{\zeta\zeta}+2k_0 f_{10}^r\Omega_\eta r_{\zeta\zeta}r\\
&\quad-2k_0 f_{10}^i\Omega_\eta r^2\Omega_{\zeta\zeta}+f_{13}\Psi_{\zeta\zeta}\Big).
\end{aligned}
$$

We note that in the equations above we have retained only those terms that enter the stability analysis. For example, terms such as

$$\frac{\partial\Psi}{\partial\eta}\frac{\partial^2 R}{\partial\zeta^2}$$

involve only products of perturbations to the plane wave solution, which are dropped in the linearization. We study the stability of the plane wave solution by writing

$$r=\rho+\hat{r}e^{\Sigma\tau+i\Gamma_1\eta+i\Gamma_2\zeta}, \tag{4.22}$$

$$\Omega=k_1\eta+\hat{\Omega}e^{\Sigma\tau+i\Gamma_1\eta+i\Gamma_2\zeta}, \tag{4.23}$$

$$\Psi=h_0T+h_1\tau+\hat{\Psi}e^{\Sigma\tau+i\Gamma_1\eta+i\Gamma_2\zeta} \tag{4.24}$$

and linearizing (4.19)–(4.21) about $\hat{\Psi}=\hat{r}=\hat{\Omega}=0$. We find that the stability of the single mode, plane wave solution is determined by the eigenvalues Σ of the matrix $\mathscr{M}$ given by

$$\mathscr{M}\equiv\begin{bmatrix}A & B & C\\ F & G & H\\ J & K & L\end{bmatrix}, \tag{4.25}$$

where

$$
\begin{aligned}
A\equiv{}&a_1^r+3a_2^r\rho^2-k_1^2a_5^r+\epsilon(-3k_1\rho^2k_0d_1^r+k_1^3k_0d_4^r+k_0d_{12}^rk_1+3k_1d_8^r\rho^2k_0)\\
(4.26)\qquad &+i\Gamma_1\left(-2a_5^ik_1+\frac{\omega_0'}{\epsilon}+\epsilon(-k_0d_1^i\rho^2+3k_0k_1^3d_4^i+k_0d_{12}^i+k_0d_8^i\rho^2)\right)\\
&+\Gamma_1^2(-a_5^r+\epsilon(3k_0d_4^rk_1))\\
&+\Gamma_2^2\left(k_0a_8^rk_1+\epsilon(-d_{10}^rk_1^2-d_{13}^r+(d_{14}^r+d_{15}^r)\rho^2)\right),
\end{aligned}
$$

$$
\begin{aligned}
B \equiv\ & i\Gamma_1\big(-2a_5^r k_1\rho + \epsilon(-\rho^3 k_0 d_1^r + 3k_1^2\rho k_0 d_4^r + k_0 d_{12}^r\rho + k_0 d_8^r\rho^3)\big) \\
& + \Gamma_1^2\big(a_5^i\rho + \epsilon(-3k_0 d_4^i k_1\rho)\big) \\
& + \Gamma_2^2\left(-k_0 a_8^i k_1\rho - \frac{\omega_0'\rho}{2k_0\epsilon} + \epsilon\big(d_{10}^i k_1^2\rho + d_{13}^i\rho - (d_{14}^i - d_{15}^i)\rho^3\big)\right),
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
C \equiv\ & i\Gamma_1\big(-k_0 a_4^i\rho + \epsilon(-d_5^i k_1\rho)\big) + \Gamma_1^2(\epsilon d_3^r\rho) \\
& + \Gamma_2^2(-a_7^r\rho + \epsilon d_{16}^r k_0 k_1\rho),
\end{aligned} \tag{4.28}
$$

$$
\begin{aligned}
F \equiv\ & 2a_2^i\rho + \epsilon(-2k_1\rho k_0 d_1^i + 2k_1 d_8^i\rho k_0) \\
& + i\Gamma_1\left(\frac{2a_5^r k_1}{\rho} + \epsilon\left(k_0 d_1^r\rho - \frac{3k_0 k_1^2 d_4^r}{\rho} - \frac{k_0 d_{12}^r}{\rho} + k_0 d_8^r\rho\right)\right) \\
& + \Gamma_1^2\left(-\frac{a_5^i}{\rho} + \epsilon\left(\frac{3k_0 d_4^i k_1}{\rho}\right)\right) \\
& + \Gamma_2^2\left(\frac{k_0 a_8^i k_1}{\rho} + \frac{\omega_0'}{2k_0\epsilon\rho} + \epsilon\left(\frac{-d_{10}^i k_1^2}{\rho} - \frac{d_{13}^i}{\rho} + (d_{14}^i + d_{15}^i)\rho\right)\right),
\end{aligned} \tag{4.29}
$$

$$
\begin{aligned}
G \equiv\ & i\Gamma_1\left(-2a_5^i k_1 + \frac{\omega_0'}{\epsilon} + \epsilon(-\rho^2 k_0 d_1^i + 3k_1^2 k_0 d_4^i + k_0 d_{12}^i + k_0 d_8^i\rho^2)\right) \\
& + \Gamma_1^2(-a_5^r + \epsilon(3k_0 d_4^r k_1)) \\
& + \Gamma_2^2\big(k_0 a_8^r k_1 + \epsilon(-d_{10}^r k_1^2 - d_{13}^r + (d_{14}^i - d_{15}^i)\rho^2)\big),
\end{aligned} \tag{4.30}
$$

$$
\begin{aligned}
H \equiv\ & i\Gamma_1(k_0 a_4^r + \epsilon(d_5^r k_1)) + \Gamma_1^2(-\epsilon d_3^i) \\
& + \Gamma_2^2(-a_7^i + \epsilon d_{16}^i k_0 k_1),
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
J \equiv\ & -4k_0 b_2^r k_1\rho + \frac{2H_0\rho}{\epsilon} + \epsilon(-4k_1 f_3^r\rho + 4f_4\rho^3 + 2f_6\rho + 2f_7 k_1^2\rho) \\
& + i\Gamma_1\big(-2k_0 b_2^i\rho + \epsilon(-4f_3^i k_1\rho)\big) + \Gamma_1^2\epsilon(-2f_3^r\rho) \\
& + \Gamma_2^2\big(-2b_3^r\rho + \epsilon(-2k_0 f_8^i\rho + 2k_0 f_8^r k_1\rho - 2k_0 f_{10}^r k_1\rho)\big),
\end{aligned} \tag{4.32}
$$

$$
\begin{aligned}
K \equiv\ & i\Gamma_1\big(-2k_0 b_2^r\rho^2 + \epsilon(-4f_3^r k_1\rho^2 + 2f_7 k_1\rho^2)\big) + \Gamma_1^2\epsilon(2f_3^i\rho) \\
& + \Gamma_2^2\big(2b_3^i\rho^2 + \epsilon(-2k_0 f_8^r\rho^2 + 2k_0 f_{10}^i k_1\rho^2 - 2k_0 f_8^r k_1\rho^2)\big),
\end{aligned} \tag{4.33}
$$

$$
L \equiv i\Gamma_1\epsilon(k_0 f_2\rho^2) - \Gamma_1^2 b_1 + \Gamma_2^2\left(-\frac{b_1}{\epsilon} + \epsilon(f_9\rho^2 - f_{13})\right). \tag{4.34}
$$

The eigenvalues Σ are roots of the characteristic equation for the matrix $\mathscr{M}$. Stability is considered in the limit of long wave perturbations, $\Gamma_1, \Gamma_2 \to 0$. The particular distinguished limit taken is very important; by dividing the problem into two overlapping cases a complete set of long wave stability criteria can be established. First, transverse and oblique perturbations are considered subject to the restriction $\Gamma_1^2/\epsilon \ll \Gamma_2^2 \ll \epsilon$. Second, nearly longitudinal perturbations are considered with $\Gamma_2^2 \ll |\Gamma_1| \ll \epsilon$, which overlaps the general oblique case. Finally, we consider near degenerate cases that occur in certain parameter limits such as $\omega_0' \to 0$.

First note that for $\Gamma_1 = \Gamma_2 = 0$ the eigenvalues of $\mathscr{M}$ are given by $\Sigma = 2a_2^r\rho^2 + \mathscr{O}(\epsilon)$, 0, 0, where the first eigenvalue follows from the analysis in the previous section. The two zero eigenvalues correspond to translation along and orthogonal to the direction of propagation of the basic state. It is the perturbations of these eigenvalues from zero into the right half plane that may lead to instability. The third eigenvalue is negative and plays no role in the stability analysis.

The symmetries in the problem place some restrictions on the form of the characteristic equation for the eigenvalues. First note that the reflection symmetry in the transverse direction (3.38) implies invariance under the change of variable,

$$\Gamma_2 \to -\Gamma_2. \tag{4.35}$$

The perturbations (4.22)–(4.24) are complex perturbations to a real system, so the resulting characteristic equation is invariant under complex conjugation. This, with (4.35) implies invariance under the change of variables

$$\Sigma \to \bar{\Sigma}, \qquad \Gamma_1 \to -\Gamma_1. \tag{4.36}$$

We now proceed to study the two long wave limits described above.

4.1.1. Transverse and skew stability. In this case we consider both pure transverse ($\Gamma_1 = 0$, $\Gamma_2 \neq 0$) and oblique perturbations ($\Gamma_1 \neq 0$, $\Gamma_2 \neq 0$) subject to the restriction that we are not near the purely longitudinal case ($\Gamma_1 \neq 0$, $\Gamma_2 = 0$). Specifically, we require that

$$\Gamma_1^2/\epsilon \ll \Gamma_2^2 \ll \epsilon. \tag{4.37}$$

Expanding the characteristic equation under these conditions yields an equation of the form

$$\Sigma^2 + \epsilon^{-1}(e_1 i\Gamma_1 + e_2\Gamma_2^2)\Sigma + \epsilon^{-2}(e_3\Gamma_1^2 + e_4 i\Gamma_1\Gamma_2^2 + e_5\Gamma_2^4) = \mathscr{E}, \tag{4.38}$$

where $\mathscr{E}$ is an error term that is cubic in Σ, $i\Gamma_1/\epsilon$, $i\Gamma_2^2/\epsilon$,

$$e_1 = -\omega_0' - \frac{H_0 k_0 a_4^i}{a_2^r}, \quad e_2 = \frac{-1}{2a_2^r}\left(2H_0 a_7^r + \frac{a_2^i \omega_0'}{k_0} - 2b_1 a_2^r\right), \quad e_3 = -\frac{\omega_0' H_0 k_0 a_4^i}{a_2^r}$$

$$e_4 = \frac{H_0\omega_0'}{2a_2^r}(2a_7^r - a_4^r) - \omega_0' b_1, \qquad e_5 = \frac{\omega_0'}{2k_0 a_2^r}(H_0 a_7^i - b_1 a_2^i),$$

and the e_i's have $\mathscr{O}(\epsilon)$ corrections.

Note that this equation respects the symmetries (4.35), (4.36). The conditions (4.37) together with the assumption that certain combinations of the $e_i's$ don't vanish (discussed in §4.1.3) is sufficient to assure that the error term will only contribute higher-order corrections to the roots.

One might expect that the full solution to (4.38) needs to be computed. However, by completing the square, we have shown that necessary and sufficient conditions for the roots Σ to have negative real parts, corresponding to stability, are

$$e_2 > 0, \qquad e_5 > 0, \tag{4.39}$$

$$e_1^2 + 4e_3 > 0, \tag{4.40}$$

$$e_4^2 > e_1 e_2 e_4 + e_3 e_2^2. \tag{4.41}$$

Note that if any of these conditions vanish to leading order, it may be necessary to consider higher-order corrections.

A more intuitive interpretation of these conditions can be given by showing that they correspond to particular types of perturbations. This will prove useful by providing a framework for considering the quasiperiodic case in the next section where necessary and sufficient conditions are difficult to obtain analytically.

The conditions (4.39) correspond to purely transverse perturbations ($\Gamma_1 = 0$). The eigenvalues can be expanded as

$$\Sigma \sim \Sigma_2 \frac{\Gamma_2^2}{\epsilon} + \cdots . \tag{4.42}$$

For Σ_2 to have a negative real part, corresponding to stability, requires that the two conditions (4.39) which can be written, respectively, as

$$2H_0 a_7^r + \frac{a_2^i \omega_0'}{k_0} - 2b_1 a_2^r > 0 \quad \text{and} \quad \frac{\omega_0'}{k_0}(H_0 a_7^i - b_1 a_2^i) < 0, \tag{4.43}$$

be satisfied. Note that these criteria are independent of the wavenumber k_1.

The conditions (4.40) and (4.41) correspond to general (skew) type instabilities in the limit $\Gamma_2^2 \ll \Gamma_1 \ll \epsilon$ (subject also to (4.37)). These can be thought of as transverse perturbations of a purely longitudinal mode. We expand Σ in the form

$$\Sigma \sim \Sigma_1 \frac{i\Gamma_1}{\epsilon} + \Sigma_{22} \frac{\Gamma_2^2}{\epsilon} + \cdots . \tag{4.44}$$

The symmetries (4.35), (4.36) imply that Σ_1 has either two real values or a complex conjugate pair. In the former case, the next term Σ_{22} is real and its sign determines stability. The latter case always leads to instability.

Substituting (4.44) into (4.38) and collecting terms $O((\Gamma_1/\epsilon)^2)$ yields the quadratic for Σ_1

$$\Sigma_1^2 + e_1 \Sigma_1 - e_3 = 0. \tag{4.45}$$

This equation will have real roots when the discriminant is positive. This is condition (4.40), which is satisfied by the discriminant of (4.45). The two roots can be found explicitly. They are Σ_1^T, which is associated with the longitudinal translation of the traveling wave and Σ_1^M, which is associated with the mean flow, where

$$\Sigma_1^T = \omega_0', \qquad \Sigma_1^M = \frac{H_0 k_0 a_4^i}{a_2^r}. \tag{4.46}$$

The discriminant (4.40) is the square of the difference of these two roots,

$$e_1^2 + 4e_3 = (\Sigma_1^M - \Sigma_1^T)^2, \tag{4.47}$$

which will be positive unless the two roots are nearly equal, in which case higher-order corrections to e_1 and e_3 may become important. This case is discussed in § 4.1.3.

Substituting (4.44) into (4.38) and collecting terms $O(\Gamma_1 \Gamma_2^2/\epsilon^2)$ yields an expression for Σ_{22},

$$\Sigma_{22} = -\left(\frac{e_4 + e_2 \Sigma_1}{e_1 + 2\Sigma_1} \right). \tag{4.48}$$

A necessary and sufficient condition for stability in this case is that Σ_{22} is negative for both values of Σ_1. Note that these conditions are equivalent to condition (4.41). If we assume that $e_2 > 0$, which is necessary for transverse stability (cf. (4.43)), and consider only the case $\Sigma_1^T > \Sigma_1^M$, which will be shown to be a necessary condition for stability a posteriori, then this expression can be simplified. For $\Sigma_1 = \Sigma_1^T$, a necessary condition for stability is that $\Sigma_1^T > -e_4/e_2$ or

$$\omega_0' a_2^i + H_0 k_0 a_4^r > 0. \tag{4.49}$$

For $\Sigma_1 = \Sigma_1^M$, the condition is $\Sigma_1^M < -e_4/e_2$ or

$$\frac{H_0 k_0 a_4^i}{a_2^r} < \frac{\omega_0' k_0 [H_0(2a_7^r - a_4^r) - 2b_1 a_2^r]}{\omega_0' a_2^i + 2k_0(H_0 a_7^r - b_1 a_2^r)}. \tag{4.50}$$

To recapitulate, when (4.37) holds, we find two sets of stability criteria. The first set (4.43) corresponds to stability to purely transverse perturbations. The second set (4.49), (4.50) corresponds to stability to oblique perturbations of the form of a transverse perturbation of a longitudinal disturbance.

For completeness it is necessary to consider cases when the error terms in (4.38) may contribute at leading order. Thus, we consider two such cases. The first is the case of nearly longitudinal instability (§ 4.1.2) when (4.37) is violated, and the error terms in (4.38) must be considered. The second (§ 4.1.3) is the degenerate case when one or more of the inequalities (4.39), (4.40), (4.41) is sufficiently close to equality that higher-order terms can affect the stability boundaries.

4.1.2. Nearly longitudinal stability. Here we consider nearly longitudinal perturbations for which

$$\Gamma_2^2 \ll |\Gamma_1| \ll \epsilon. \tag{4.51}$$

This allows Σ to be Taylor expanded in the form

$$\Sigma \sim \frac{i\Gamma_1}{\epsilon}\Sigma_1 + \frac{\Gamma_1^2}{\epsilon^2}\Sigma_{21} + \frac{\Gamma_2^2}{\epsilon}\Sigma_{22} + \cdots. \tag{4.52}$$

Note that the region of validity of this expansion overlaps the region where the expansion (4.44) is valid. Consequently, solving for Σ_1 and Σ_{22} at order unity yields exactly the same values (4.46) and (4.48). However, the value of Σ_{21} must be determined by considering the higher-order terms in (4.38).

If the value of Σ_1 is taken to be the mean flow eigenvalue Σ_1^M, Σ_{21} is negative, corresponding to stability, if

$$0 < \Sigma_1^M < \Sigma_1^T. \tag{4.53}$$

This condition can be written explicitly as

$$0 < \frac{H_0 k_0 a_4^i}{a_2^r} < \omega_0'. \tag{4.54}$$

Note these stability criteria are independent of the wavenumber k_1.

The second eigenvalue Σ_1^T corresponds to the translational mode of the traveling wave. Physically, it represents a perturbation to the wave's phase being carried along with the group velocity. The value of Σ_{21} enters at $\mathscr{O}(\epsilon^2)$. For it to be negative, corresponding to stability, the appropriate condition (assuming that (4.54) is satisfied) is

$$\mathscr{A} + \mathscr{B} k_1^2 > 0, \tag{4.55}$$

where

$$\mathscr{A} = 2\frac{a_1^r}{a_2^r}\left(\omega_0'(a_2^r a_5^r + a_2^i a_5^i) + H_0 k_0 (a_4^i a_5^r - a_4^r a_5^i)\right) \tag{4.56}$$

and

$$\mathscr{B} = -\frac{a_5^r}{a_1^r}\mathscr{A} - 4\omega_0'(a_5^r)^2\left[1 + \left(\frac{\omega_0' a_2^i + H_0 k_0 a_4^r}{\omega_0' a_2^r - H_0 k_0 a_4^i}\right)^2\right]. \tag{4.57}$$

If $\mathscr{A}$ is positive, then $\mathscr{B}$ is negative, and a band of stable wavenumbers exists, $|k_1| < \sqrt{-\mathscr{A}/\mathscr{B}}$. If $\mathscr{A}$, $\mathscr{B} < 0$ there are no stable traveling waves. If $\mathscr{A} < 0$, $\mathscr{B} > 0$ there is a possibility of the existence of an inverted band of stable waves $|k_1| > \sqrt{-\mathscr{A}/\mathscr{B}}$. We have also verified that in the limit when the coupling to the mean field is suppressed, this stability band reduces to results well known for the Eckhaus type sideband instability for the Ginzburg–Landau equation [10], [11].

In summary, the unidirectional traveling wave can exhibit longitudinal instability due to coupling with the mean field (4.54) or due to standard sideband type instabilities that yields a band of stable wavenumbers (4.55) that reduce to the Ginzburg–Landau result in the absence of a mean field.

4.1.3. Degenerate cases. Here we consider two special cases, which occur (i) when the group velocity vanishes or is very small, $\omega_0' \to 0$ and (ii) when the two leading-order longitudinal eigenvalues are nearly equal, $\Sigma_1^T \approx \Sigma_1^M$. In these cases the transverse, skew and longitudinal criteria may vanish at leading order so that higher-order corrections must be considered.

For transverse modulations we need only consider the case $\omega_0' \to 0$. We note that the first condition in (4.43) does not vanish in this limit and therefore need not be considered to higher order. The second condition is degenerate since e_5 vanishes as $\omega_0' \to 0$. Rewriting the condition $e_5 > 0$ including the correction to $\mathbf{O}(\epsilon^2)$ yields the condition for stability

(4.58)
$$e_5 = \frac{\omega_0'}{2k_0 a_2^r}(H_0 a_7^i - b_1 a_2^i) + \frac{\epsilon k_0\left(H_0(a_7^i a_8^i + a_7^r a_8^r) - a_2^r a_8^r b_1 k_0\right)k_1 + \epsilon^2\mathscr{C}}{a_2^r} > 0$$

where $\mathscr{C}$ is given by

$$\begin{aligned}\mathscr{C} \equiv\; & -(a_7^r H_0 - a_2^r b_1)(d_{13}^r + (d_{14}^r - d_{15}^r)(a_1^r/a_2^r)) \\ & + (a_7^i H_0 + a_2^i b_1)\left(d_{13}^i + (d_{14}^i - d_{15}^i)(a_1^r/a_2^r)\right) \\ & + 2b_3^i(a_1^r/a_2^r)(a_7^i a_2^r - a_7^r a_2^i) + a_8^i(d_1^i - d_8^i) + O(k_1).\end{aligned} \tag{4.59}$$

In the limit $\omega_0' \to 0$, (4.58) defines a critical value for $k_1 = \mathscr{O}(\epsilon)$ at which waves become unstable to transverse perturbations.

The second degenerate case that we consider occurs when the leading-order expression for the discriminant of the quadratic (4.45) vanishes at leading order. If higher-order corrections lead to a negative discriminant, the two values of Σ_1 will be a complex conjugate pair that will always lead to instability. Expanding the discriminant to $\mathscr{O}(\epsilon)$ yields the condition for stability

$$e_1^2 + 4e_3 = (\Sigma_1^M - \Sigma_1^T)^2 + \epsilon k_1 \mathscr{D} + \epsilon^2 \mathscr{F} > 0, \tag{4.60}$$

where

$$
(4.61)\quad
\begin{aligned}
\mathscr{D} &\equiv \frac{1}{a_2^r}\big[\Sigma_1^T(-4a_2^r a_5^i - 2H_0 d_5^i \omega_0' + 4k_0^2 b_2^r a_4^i + 4a_2^i a_5^i)\\
&\qquad + \Sigma_1^M(4a_2^r a_5^i + 2H_0 d_5^i \omega_0' - 4k_0^2 b_2^r a_4^i + 4a_2^i a_5^i) + 4H_0 k_0 a_4^r a_5^r\big],\\
\mathscr{F} &\equiv 4a_4^r d_{12}^r H_0 k_0^2 + 4a_4^r d_8^r H_0 k_0^2 + 4a_4^r d_1^r H_0 k_0^2 (a_1^r/a_2^r)\\
&\quad - 8(a_2^r a_4^r + a_2^i a_4^i) b_2^r k_0^2 (a_1^r/a_2^r) + 4a_4^i H_0 k_0^2\big[d_{12}^i + (d_1^i - d_8^i)(a_1^r/a_2^r)\big]\\
&\quad + \omega_0'\big[4a_4^i k_0\{f_6 - 2f_4(a_1^r/a_2^r)\} - 2f_2 a_2^r k_0 (a_1^r/a_2^r)\big] + \mathscr{O}(k_1),
\end{aligned}
$$

so that if $\Sigma_1^M - \Sigma_1^T = \mathscr{O}(\sqrt{\epsilon})$, an instability will occur if

$$
(4.62)\qquad -\mathscr{D}k_1 > \frac{(\Sigma_1^M - \Sigma_1^T)^2}{\epsilon} + \epsilon\mathscr{F}.
$$

As $(\Sigma_1^M - \Sigma_1^T) \to 0$ this condition yields a stability boundary for $k_1 = \mathscr{O}(\epsilon)$.

If (4.62) is violated, then Σ_1 is real. However, the possibility of skew and longitudinal instabilities still exist for $(\Sigma_1^M - \Sigma_1^T) \sim \mathscr{O}(\epsilon)$. For this case we expand Σ_1 as

$$
(4.63)\qquad \Sigma_1 = \Sigma_{11} \pm \Sigma_{12},
$$

where

$$
(4.64)\qquad \Sigma_{11} = -\tfrac{1}{2}e_1 = (\Sigma_1^T + \Sigma_1^M)/2 + \mathscr{O}(\epsilon),
$$

$$
(4.65)\qquad \Sigma_{12} = \tfrac{1}{2}\sqrt{e_1^2 + 4e_3} = \tfrac{1}{2}\sqrt{(\Sigma_1^T - \Sigma_1^M)^2 - \epsilon\mathscr{D}k_1 + \epsilon^2\mathscr{F}} = O(\sqrt{\epsilon}).
$$

We now consider skew type instabilities and nearly longitudinal instabilities as in the nondegenerate cases.

To consider skew instabilities we note from (4.48) that

$$
(4.66)\qquad \Sigma_{22} = -\left(\frac{e_4 + e_2\Sigma_1}{e_1 + 2\Sigma_1}\right) = \mp\frac{e_4 + e_2\Sigma_{11}}{2\Sigma_{12}} - \frac{e_2}{2}.
$$

For stability it is necessary that $e_2 > 0$, as must be the case for transverse stability (4.43), and

$$
(4.67)\qquad \Sigma_{12}^2 > \left(\frac{e_4}{e_2} + \Sigma_{11}\right)^2.
$$

For $\omega_0' = \mathscr{O}(1)$ this condition is generally violated as Σ_{12}^2 is $\mathscr{O}(\epsilon)$, but the right-hand side of (4.67) is $\mathscr{O}(1)$ and positive unless parameters are chosen so that it vanishes identically. However, for $\omega_0' \to 0$ and $(\Sigma_1^M - \Sigma_1^T) \to 0$, both values of Σ_{22} are negative if

$$
(4.68)\qquad -\mathscr{D}k_1 > \frac{(\Sigma_1^M - \Sigma_1^T)^2}{\epsilon} + \epsilon\mathscr{F} - \frac{1}{\epsilon}\left[(\Sigma_1^M - \Sigma_1^T) - \frac{\omega_0' H_0 a_4^r}{2H_0 a_7^r - 2a_2^r b_1}\right]^2,
$$

which yields a critical value of $k_1 = \mathscr{O}(\epsilon)$. Note that this condition is more restrictive than (4.62). In the limit of small ϵ and fixed $(\Sigma_1^M - \Sigma_1^T)$, the skew stability criteria (4.49), (4.50) can be recovered from (4.68).

For longitudinal stability, it is necessary to compute higher-order correction terms from the characteristic equation,

$$\pm \Sigma_{12}\Sigma_{21} = -\frac{\Sigma_1}{2a_2^r \rho^2}(\omega_0' - \Sigma_1)^2 - e_8, \tag{4.69}$$

where e_8 is the coefficient of $i\Gamma_1^3/\epsilon$ in the characteristic equation. For all values of ω_0' we find that $\Sigma_{21} > 0$, so that an instability is always present in the near degenerate case. Unfortunately, we do not see how to recover the longitudinal criteria (4.54), (4.55) from (4.69).

To summarize, we have shown that for long wave stability of a traveling wave it is sufficient to consider transverse, oblique, and longitudinal perturbations of traveling wave solutions. For the nondegenerate cases, we find that the skew and transverse instabilities are independent of the wave number k_1. For purely longitudinal perturbations, there are two criteria: the first is associated with the mean flow coupling and is independent of k_1, and the second is associated with the translational mode of the wave and yields an order unity band of stable wavenumbers bounded by an Eckhaus type stability boundary. For the degenerate cases, in which the leading-order eigenvalues are nearly equal, or if the group velocity ω_0' is small, we find skew and transverse instabilities for $k_1 = \mathcal{O}(\epsilon)$, while for nearly longitudinal perturbations the traveling waves apparently are always unstable in this limit.

4.2. Stability of quasiperiodic waves. We now consider the stability of quasiperiodic waves ($R \neq 0$, $S \neq 0$). We proceed as for the unidirectional traveling waves, but now take into account perturbations to both waves. As above, stability is determined by the growth rate Σ of the perturbation to the plane wave solution (4.1). In this case the characteristic equation for Σ is a fifth-order polynomial equation corresponding to the five by five matrix given in Appendix B.

When $\Gamma_1 = \Gamma_2 = 0$, we find that there are three zero eigenvalues, two of which correspond to longitudinal translation of each of the two superposed traveling waves and a third corresponding to a translation in the direction of propagation of the basic state. The remaining two eigenvalues are given by the analysis in the beginning of § 4,

$$\Sigma = a_2^r(\rho^2 + \sigma^2) \pm \sqrt{\left(a_2^r(\rho^2 + \sigma^2)\right)^2 + 4\sigma^2\rho^2(a_3^{r2} - a_2^{r2})}, \tag{4.70}$$

where ρ and σ are given by (4.13) and (4.14). We limit our study to the case $|a_3^r| < |a_2^r|$ when the traveling waves are stable inside the reduced system (4.3), (4.4), and to the values of k_1, k_2 for which the quasiperiodic state (4.13), (4.14) exists.

A cubic equation can be derived for the near-zero eigenvalues; however, it is much easier to examine stability by substituting Taylor series expansions of the eigenvalues in the particular cases examined in the previous sections into the fifth-order determinant. While this does not guarantee the examination of all cases, it reduces the calculation to manageable size.

The symmetry (3.39) implies that the characteristic equation, and consequently the stability criteria are invariant under the change of variables

$$\rho \to \sigma, \quad \sigma \to \rho, \quad k_1 \to k_2, \quad k_2 \to k_1, \quad \Gamma_1 \to -\Gamma_1. \tag{4.71}$$

The symmetries (4.35), (4.36) must also be obeyed.

4.2.1. Transverse stability. We seek a Taylor series expansion for the eigenvalues of the same form as (4.42)

$$\Sigma \sim \Sigma_2 \frac{\Gamma_2^2}{\epsilon} + \cdots. \tag{4.72}$$

Substituting this expansion into the determinant in Appendix B yields a cubic equation for Σ_2

$$(2(a_2^r + a_3^r)\Sigma_2^2 + j_1\Sigma_2 + j_2)\left(2(a_2^r - a_3^r)\Sigma_2 - \frac{\omega_0'}{k_0}(a_2^i - a_3^i)\right) = 0, \tag{4.73}$$

where

$$j_1 = 2b_1(a_2^r + a_3^r) - \frac{\omega_0'}{k_0}(a_2^i + a_3^i) - 4H_0 a_7^r, \tag{4.74}$$

$$j_2 = \frac{\omega_0'}{k_0}\left[2H_0 a_7^i - b_1(a_2^i + a_3^i)\right]. \tag{4.75}$$

The factorization of the cubic can be explained by noting that it is independent of k_1, k_2 and consequently unchanged by the symmetry (4.71). This allows us to decompose the matrix into a similar matrix with two blocks; one invariant under the symmetry and one that is mapped into its negative. The quadratic factor is the determinant of the first block and the linear factor the determinant of the second.

The coefficients j_1 and j_2 must be negative for stability, whereas for the second factor a necessary condition for stability is

$$a_2^i > a_3^i. \tag{4.76}$$

Note that all these criteria are independent of the wave numbers k_1, k_2.

4.2.2. Nearly longitudinal stability. For nearly longitudinal perturbations, we again use the expansion (4.52) for the eigenvalues

$$\Sigma \sim \frac{i\Gamma_1}{\epsilon}\Sigma_1 + \frac{\Gamma_1^2}{\epsilon^2}\Sigma_{21} + \frac{\Gamma_2^2}{\epsilon}\Sigma_{22}\cdots. \tag{4.77}$$

The cubic equation for Σ_1 is such that two values of Σ_1 are $\mathscr{O}(1)$

$$\Sigma_1 = \omega_0', -\omega_0'. \tag{4.78}$$

These roots are associated with the translation invariance of the two traveling waves and are mapped into each other by the symmetry (4.71). The third eigenvalue is zero to leading order and is associated with the mean flow. It vanishes at leading order due to the cancellation of the contributions of the counterpropagating waves and is discussed below.

For $\Sigma_1 = \omega_0'$, the calculation of Σ_{21} and Σ_{22} is straightforward, with Σ_{21} entering at $\mathscr{O}(\epsilon)$ as in the unidirectional case

$$\Sigma_{21} = \epsilon k_1 \mathscr{D}_1 + \epsilon^2 \mathscr{G}_1, \qquad \Sigma_{22} = -\frac{\mathscr{H}}{(a_2^r)^2 - (a_3^r)^2}, \tag{4.79}$$

where

$$\mathscr{D}_1 = \frac{2a_5^r}{\left((a_3^r)^2 - (a_2^r)^2\right)\sigma^2}\left[\frac{\mathscr{H}k_0(H_0 k_0 a_4^i - \omega_0' a_2^r)}{\omega_0'(a_3^r)^2 - (a_2^r)^2} + H_0 k_0 a_4^r + \omega_0' a_2^i\right], \tag{4.80}$$

$$\mathscr{H} = H_0\left[a_4^r(a_3^r - a_2^r) + a_4^i(a_3^i - a_2^i)\right] - (\omega_0'/k_0)(a_2^r a_2^i - a_3^r a_3^i). \tag{4.81}$$

Since the expression for Σ_{21} in (4.79) yields an instability with $k_1 = \mathscr{O}(\epsilon)$, we write $\mathscr{G}_1$ only for $k_1 = \mathscr{O}(\epsilon)$.

$$\begin{aligned} &\left((a_2^r)^2 - (a_3^r)^2\right)\sigma^2\rho^2(\omega_0')^2\mathscr{G}_1 \\ &\quad = -8\rho^2\sigma^2\omega_0'^2\left[\left((a_2^r)^2 - (a_3^r)^2\right)a_5^r + a_5^i(a_2^i a_2^r - a_3^i a_3^r)\right] \qquad (4.82)\\ &\quad + 8k_0(\omega_0')^2\rho^2\left[(d_{12}^i - d_1^i\rho^2 + d_8^i\rho^2)(a_2^r\omega_0' - a_4^i H_0 k_0)\right. \\ &\qquad\qquad \left. + (d_2^r\sigma^2 - d_8^r\rho^2 + d_1^r\rho^2 - d_{12}^r)\right](a_2^i\omega_0' + a_4^r H_0 k_0) \\ &\quad - 8H_0k_0\omega_0'\left[a_4^r a_5^i(a_2^r - a_3^r)\rho^2\sigma^2 + a_4^i a_5^i(a_2^i - a_3^i)\rho^2\sigma^2\right]. \qquad (4.83) \end{aligned}$$

The corresponding values for $\Sigma_1 = -\omega_0'$, that is $\Sigma_{21} = k_2\mathscr{D}_2 + \epsilon\mathscr{G}_2$ can be found by using the symmetry (4.71).

Consequently, the conditions for stability are

$$k_1\mathscr{D}_1 < -\epsilon\mathscr{G}_1, \quad k_2\mathscr{D}_2 < -\epsilon\mathscr{G}_2, \quad \mathscr{H} > 0. \qquad (4.84)$$

The remaining eivenvalue must be examined more closely. The third eigenvalue is given by

$$\Sigma_1 = \epsilon\frac{\Sigma_{11}}{\omega_0'\left((a_2^r)^2 - (a_3^r)^2\right)}, \qquad (4.85)$$

where

$$\begin{aligned} \Sigma_{11} = &-2H_0a_5^r k_0(k_1 - k_2)\left(a_4^i(a_3^i - a_2^i) - a_4^r(a_2^r - a_3^r)\right) \\ &+ 2\omega_0'^2\Bigg[-H_0(k_1 - k_2)(2d_5^i(a_2^r - a_3^r) + a_4^i k_0^2(-3d_1^r + 3d_8^r)) \\ &\quad + (k_1^3 - k_2^3)k_0H_0d_4^r + (\sigma^2k_1 - \rho^2k_2)H_0d_{12}^r \\ &\quad + 4k_0b_2^r(k_1 - k_2)a_4^i(a_3^r + a_2^r) \qquad (4.86)\\ &\quad - 2H_0a_4^i k_0(d_7^r - d_6^r + d_2^r)(k_1 - k_2) + k_0^2 a_4^i\Bigg(d_2^r\left(-k_2\frac{\rho^2}{\sigma^2} + k_1\frac{\sigma^2}{\rho^2}\right) \\ &\qquad + \left(k_2\frac{\sigma^2}{\rho^2} - k_1\frac{\rho^2}{\sigma^2}\right)(d_6^r - d_7^r)\Bigg)\Bigg]; \end{aligned}$$

from this it follows that Σ_{22} and Σ_{21} are given by

$$\begin{aligned} \Sigma_{22} &= -b_1 + \frac{2H_0\left[(a_3^i - a_2^i)a_4^i + (a_4^r - 2a_7^r)(a_3^r - a_2^r)\right]}{(a_2^r)^2 - (a_3^r)^2}, \\ \Sigma_{21} &= \epsilon\frac{\omega_0' k_0 a_4^i a_4^r(\sigma^2 + \rho^2)}{2\rho^2\sigma^2\left((a_2^r)^2 - (a_3^r)^2\right)}. \end{aligned} \qquad (4.87)$$

For stability, it is necessary that Σ_{21} and Σ_{22} are negative; the second of these conditions reduces to $a_4^i a_4^r < 0$.

In summary, we have examined the stability of quasiperiodic waves to perturbations of the same form as was appropriate for the unidirectional case. Stability to transverse perturbations yielded three conditions independent of wave numbers.

Nearly longitudinal perturbations lead to three possible pure imaginary leading-order eigenvalues ($i\Sigma_1\Gamma_1$). Two of these are associated with translation of the phase of the two wave components, the third is associated with coupling to the mean flow. For the translational mode to be stable to a small transverse perturbation we obtain criteria independent of wave number. Purely longitudinal perturbations of each of the translational modes yield a stability boundary for the wavenumber (k_1 or k_2), that is, $\mathcal{O}(\epsilon)$. For the mean flow mode, nearly longitudinal perturbations yield two criteria independent of wave number.

5. Discussion. We have studied mean field effects in a general system of reaction-diffusion equations by deriving coupled evolution equations for the amplitudes of counterpropagating traveling waves in directions orthogonal to the direction of propagation of the basic state, and a zero mode describing the mean field, which corresponds to translation of the basic state in its direction of propagation. We employed these equations to determine the stability of plane wave solutions, including both traveling waves and quasiperiodic waves. The latter include standing waves as a special case. The stability results are given in terms of the perturbation wave number and the parameters of the model.

For unidirectional waves, most of the criteria are independent of wave number except for an Eckhaus type boundary corresponding to purely longitudinal perturbations. In certain degenerate limits (such as very small group velocity) stability boundaries with perturbation wave numbers $k_1 = \mathcal{O}(\epsilon)$ were found. For quasiperiodic waves, both wave number independent and $k_1 = \mathcal{O}(\epsilon)$ wave number boundaries were found.

Siggia and Zippelius [1] and Busse and Bolton [2] first studied mean flow effects in Rayleigh–Benard convection between stress-free upper and lower boundaries. Their conflicting results for the stability boundaries of stationary convection rolls were resolved in [3], where the skew varicose instability boundaries of stationary cellular solutions were given by $k > k_0 + \epsilon^2\alpha_1$ and $k < k_0 + \epsilon^2\alpha_2$. Here k_0 is the critical wave number at the onset of convection and α_1 and α_2 are $\mathcal{O}(1)$ quantities. Higher-order terms, corresponding to $\mathcal{O}(\epsilon^4)$ terms in this paper, were included in [3], where it was shown that if higher-order terms are not included, as in [1], the resulting expressions for α_1 and α_2 are not correct.

In [3] the evolution equation for the amplitude of the stationary cellular solutions is coupled to an evolution equation for the mean flow field. The equation for the amplitude of the stationary cellular solution is similar in form to the equations for R and S in this paper. Differences result from the facts that in [3] stationary cellular solutions are considered, while here traveling waves are analyzed, and that the nonlinearities are different. In addition, the evolution equation for the mean flow field in [3] is different from that for the drift Ψ in this model, since in [3] the mean field effects are governed by the fluid dynamical equations, while here Ψ satisfies a diffusion equation.

In [4] short-wave instabilities in systems with a zero mode were studied in the context of longitudinal seismic waves in a viscoelastic medium. Evolution equations were obtained for the amplitudes of both stationary cellular solutions and unidirectional traveling wave solutions that bifurcate from a basic state of no flow. In each case the amplitude equations couple to an evolution equation for a zero mode solution. The resulting evolution equations are similar to those given in §3 up to $\mathcal{O}(\epsilon^3)$.

The nonlocal amplitude equations in [5]–[9], couple to the mean field equation only if transverse modulations are considered. In these studies the dependence of the mean field on the evolution time scale (τ in this paper) was not included, and higher-order terms were not considered in the stability analyses. Thus they did not describe all the instabilities described here. As noted in [5]–[9], standing wave solutions are more stable for the nonlocal (averaged) equations than for the same equations with the averages removed. Similarly they are more stable in the context of the nonlocal equations than in the context of the evolution equations derived in this paper. That is, the considerations in this paper lead to descriptions of instabilities not described in [5]–[9].

To obtain the evolution equations in this paper, we combined the solvability conditions for $\mathscr{O}(\epsilon^j)(j=2, 3, 4)$. One might have expected that only the solvability conditions up to $\mathscr{O}(\epsilon^3)$ need be considered. However, the higher-order terms contribute to many of the $k_1=\mathscr{O}(\epsilon)$ boundaries for both the degenerate case of the unidirectional waves and the nearly longitudinal cases of the quasiperiodic waves.

In conclusion, we derived the governing modulation equations for counterpropagating traveling waves in directions orthogonal to the direction of propagation of the basic state, coupled to the mean field induced by local displacement of the basic state. These equations support both unidirectional and counterpropagating wave solutions. We determined instabilities of these solutions to long wave perturbations by considering the delicate balance of longitudinal and transverse perturbations to the waves. Specifically, we determined Eckhaus type instabilities by considering longitudinal perturbations, zigzag type instabilities by considering transverse perturbations, and skew type instabilities by considering oblique perturbations. The latter arise as a specific result of the interaction with the mean field and are not present without this interaction.

Appendix A. The right-hand sides of (3.14) for $j=2, 3, 4$ are given by

(A.1)
$$r_2=\beta_0(U_1,U_1)+2D\frac{\partial^2 U_1}{\partial y\partial\eta}-\frac{\partial U_1}{\partial T}+D\frac{\partial^2 U_1}{\partial\zeta^2},$$

$$\begin{aligned}r_3={}&2D\frac{\partial^2 U_2}{\partial y\partial\eta}-\frac{\partial U_2}{\partial T}+D\frac{\partial^2 U_1}{\partial\eta^2}-\frac{\partial U_1}{\partial\tau}+D\frac{\partial^2 U_2}{\partial\zeta^2}\\&-c_1\nu\frac{\partial U_1}{\partial\xi}+\nu\alpha_1 U_1+2\beta_0(U_1,U_2)+\gamma_0(U_1,U_1,U_1),\end{aligned}$$

(A.2)
$$\begin{aligned}r_4={}&2D\frac{\partial^2 U_3}{\partial y\partial\eta}-\frac{\partial U_3}{\partial T}+D\frac{\partial^2 U_2}{\partial\eta^2}-\frac{\partial U_2}{\partial\tau}+D\frac{\partial^2 U_3}{\partial\zeta^2}\\&-c_1\nu\frac{\partial U_2}{\partial\xi}+\nu\alpha_1 U_2+2\beta_0(U_1,U_3)+3\gamma_0(U_1,U_1,U_2)\\&+\nu\beta_1(U_1,U_1)+\delta_0(U_1,U_1,U_1,U_1)+\beta_0(U_2,U_2).\end{aligned}$$

When the solvability conditions for $j=3$ are satisfied, the solution to the problem for $j=3$ is

$$\begin{aligned}U_3={}&U_{31}+U_{32}\\&+\Big[(R_1^3e_1^3+S_1^3e_2^3)s_1+(R_1^2S_1e_1^2e_2+S_1^2R_1e_2^2e_1)s_2\\&+\left(R_1^2\overline{S}_1e_1^2\overline{e}_2+S_1^2\overline{R}_1e_2^2\overline{e}_1\right)s_3\end{aligned}$$

$$
\begin{aligned}
&+8ik_0\left(R_1\frac{\partial R_1}{\partial \eta}e_1^2 - S_1\frac{\partial S_1}{\partial \eta}e_2^2\right)s_4 + (R_1^2e_1^2 + S_1^2e_2^2)\Psi_1 s_5\\
&+(R_2R_1e_1^2 + S_2S_1e_2^2)s_6 + 8R_1S_1\Psi_1 s_7 e_1e_2\\
&+4ik_0\left(S_1\frac{\partial R_1}{\partial \eta} - R_1\frac{\partial S_1}{\partial \eta}\right)e_2e_1s_8\\
&+2(R_2S_1 + R_1S_2)s_9e_1e_2 + 4ik_0\bar{S}_1\frac{\partial R_1}{\partial \eta}s_{10}\bar{e}_2e_1\\
&+4R_1\Psi_1\bar{S}_1s_{11}e_1\bar{e}_2 + 2\left(R_2\bar{S}_1 + R_1\bar{S}_2\right)s_{12}e_1\bar{e}_2 + 4ik_0R_1\frac{\partial \bar{S}_1}{\partial \eta}s_{13}e_1\bar{e}_2\\
&+2\left(R_1\frac{\partial^2 R_1}{\partial \zeta^2}e_1^2 + S_1\frac{\partial^2 S_1}{\partial \zeta^2}e_2^2\right)h_1 + \left(2\left(\frac{\partial R_1}{\partial \zeta}\right)^2 e_1^2 + 2\left(\frac{\partial S_1}{\partial \zeta}\right)^2 e_2^2\right)h_2\\
&+\left(2S_1\frac{\partial^2 R_1}{\partial \zeta^2} + 2R_1\frac{\partial^2 S_1}{\partial \zeta^2}\right)h_3e_1e_2\\
&+\left(2\bar{S}_1\frac{\partial^2 R_1}{\partial \zeta^2}h_4 + 2R_1\frac{\partial^2 \bar{S}_1}{\partial \zeta^2}h_7\right)e_1\overline{e_2}\\
&+4\frac{\partial R_1}{\partial \zeta}\frac{\partial S_1}{\partial \zeta}h_5e_1e_2 + 4\frac{\partial R_1}{\partial \zeta}\frac{\partial \bar{S}_1}{\partial \zeta}h_6e_1\overline{e_2}\\
&\qquad\qquad\qquad\qquad\left. + (R_3e_1 + S_3e_2)v_0 + \text{c.c.}\right] + \Psi_3 w_0',
\end{aligned}
\tag{A.3}
$$

where

$$
\begin{aligned}
U_{31} = {}&(R_1e_1 + S_1e_2)q_1 + \left(R_1|R_1|^2e_1 + S_1|S_1|^2e_2\right)q_2 + \left(S_1|R_1|^2e_2 + R_1|S_1|^2e_1\right)q_3\\
&+(R_1e_1 + S_1e_2)\Psi_2q_4 + 4ik_0\Psi_1\left(\frac{\partial R_1}{\partial \eta}e_1 - \frac{\partial S_1}{\partial \eta}e_2\right)q_5\\
&-4\left(\frac{\partial^2 R_1}{\partial \eta^2}e_1 + \frac{\partial^2 S_1}{\partial \eta^2}e_2\right)q_6 + 4ik_0\left(\frac{\partial \Psi_1}{\partial \eta}R_1e_1 - \frac{\partial \Psi_1}{\partial \eta}S_1e_2\right)q_7\\
&+\Psi_1(R_2e_1 + S_2e_2)q_8 + \Psi_1^2(R_1e_1 + S_1e_2)q_9 + 2ik_0\left(\frac{\partial R_2}{\partial \eta}e_1 - \frac{\partial S_2}{\partial \eta}e_2\right)g_1\\
&+4\frac{\partial \Psi_1}{\partial \zeta}\left(\frac{\partial R_1}{\partial \zeta}e_1 + \frac{\partial S_1}{\partial \zeta}e_2\right)g_2\\
&+2\frac{\partial^2 \Psi_1}{\partial \zeta^2}(R_1e_1 + S_1e_2)g_3 + 2ik_0\left(\frac{\partial^3 R_1}{\partial \zeta^2\partial \eta}e_1 - \frac{\partial^2 S_1}{\partial \zeta^2\partial \eta}e_2\right)g_4\\
&+\left(\frac{\partial^4 R_1}{\partial \zeta^4}e_1 + \frac{\partial^4 S_1}{\partial \zeta^4}e_2\right)g_5 + \left(\frac{\partial^2 R_2}{\partial \zeta^2}e_1 + \frac{\partial^2 S_2}{\partial \zeta^2}e_2\right)g_6
\end{aligned}
\tag{A.4}
$$

$$+2\Psi_1\left(\frac{\partial^2 R_1}{\partial \zeta^2}e_1+\frac{\partial^2 S_1}{\partial \zeta^2}e_2\right)g_7-2\left(\frac{\partial R_1}{\partial T}e_1+\frac{\partial S_1}{\partial T}e_2\right)\Psi_1 g_{13}$$

$$-2ik_0\left(\frac{\partial^2 R_1}{\partial T\partial\eta}e_1-\frac{\partial^2 S_1}{\partial T\partial\eta}e_2\right)g_{14}+\text{c.c.},$$

$$\begin{aligned}U_{32}&=\Psi_1 q_{10}+\Psi_1\left(|R_1|^2+|S_1|^2\right)q_{11}+\Psi_1^3 q_{12}+\left[2\left(R_2\overline{R}_1+S_2\overline{S}_1\right)q_{13}+\text{c.c.}\right]\\&\quad+\frac{\partial^2\Psi_1}{\partial\eta^2}q_{14}+\left[4ik_0\left(\frac{\partial R_1}{\partial\eta}\overline{R}_1-\frac{\partial S_1}{\partial\eta}\overline{S}_1\right)+\text{c.c.}\right]q_{15}+2\Psi_1\Psi_2 q_{16}\\&\quad+\left(\left(\overline{R}\frac{\partial^2 R_1}{\partial\zeta^2}+\overline{S}\frac{\partial^2 S_1}{\partial\zeta^2}\right)g_8+\text{c.c.}\right)\\&\quad+\frac{\partial^4\Psi_1}{\partial\zeta^4}g_9+\frac{\partial^2\Psi_2}{\partial\zeta^2}g_{10}+\Psi_1\frac{\partial^2\Psi_1}{\partial\zeta^2}g_{11}+2\left(\frac{\partial\Psi_1}{\partial\zeta}\right)^2 g_{12},\end{aligned}\tag{A.5}$$

where the equations for $p_j(\xi)$ that appear in the definition of U_2 are

$$\begin{aligned}&K_w p_1\equiv -Dp_1''+c_0p_1'-\alpha_0p_1=\beta_0(v_0,\overline{v_0})-\frac{H_0}{2}w_0',\\&K_v p_5\equiv i\omega_0p_5-Dp_5''+k_0^2p_5+c_0p_5'-\alpha_0p_5=\beta_0(v_0,w_0'),\\&K_w p_6=\beta_0(w_0',w_0'),\\&K_v p_7=Dv_0-\langle Dv_0,\phi_1\rangle v_0,\\&\quad p_8=p_7,\\&K_w p_9=Dw_0'-\langle Dw_0',\phi_0\rangle w_0',\end{aligned}\tag{A.6}$$

and $p_j(\xi)$ for $j=2$, 3, 4 satisfy

$$\begin{aligned}2i\omega_0p_2-Dp_2''+4k_0^2Dp_2+c_0p_2'-\alpha_0p_2&=\beta_0(v_0,v_0),\\2i\omega_0p_3-Dp_3''+c_0p_3'-\alpha_0p_3&=\beta_0(v_0,v_0),\\-Dp_4''+4k_0^2Dp_4+c_0p_4'-\alpha_0p_4&=\beta_0(v_0,\overline{v_0}).\end{aligned}\tag{A.7}$$

The coefficients $q_j(\xi)$ and $g_j(\xi)$ appearing in the definition of U_3 satisfy

$$\begin{aligned}&K_vq_1=\nu(-c_1v_0'+\alpha_1v_0)-\nu\langle -c_1v_0'+\alpha_1v_0,\phi_1\rangle v_0;\\&K_vq_2=3\gamma_0(v_0,v_0,\overline{v_0})+4\beta_0(v_0,p_1)+2\beta_0(p_2,\overline{v_0})-2H_0p_5\\&\qquad-\langle 3\gamma_0(v_0,v_0,\overline{v_0})+4\beta_0(v_0,p_1)\\&\qquad+2\beta_0(p_2,\overline{v_0})-2H_0p_5,\phi_1\rangle v_0;\\&K_vq_3=6\gamma_0(v_0,v_0,\overline{v_0})+4\beta_0(v_0,p_1)\\&\qquad+4\beta_0(p_3,\overline{v_0})+4\beta_0(p_4,v_0)-2H_0p_5\\&\qquad-\langle 6\gamma_0(v_0,v_0,\overline{v_0})+4\beta_0(v_0,p_1)+4\beta_0(p_3,\overline{v_0})\\&\qquad+4\beta_0(p_4,v_0)-2H_0p_5,\phi_1\rangle v_0;\\&K_vq_6=-4k_0^2Dp_7+Dv_0+\langle 4k_0^2Dv_0,\phi_1\rangle p_7-a_5v_0;\\&K_vq_7=Dp_5-\langle Dp_5,\phi_1\rangle v_0;\end{aligned}\tag{A.8}$$

$$K_w q_{10} = \nu(-c_1 w_0'' + \alpha_1 w_0');$$
$$K_w q_{14} = Dw_0' - \langle Dw_0', \phi_0 \rangle w_0';$$
$$K_w q_{15} = \beta_0(p_7, \overline{v_0}) - p_1 - \frac{b_2}{4} w_0';$$
$$q_{16} = p_6, \quad q_4 = v_0', \quad q_5 = \frac{p_7'}{2}, \quad q_8 = v_0', \quad q_9 = \frac{v_0''}{2};$$
$$q_{11} = 2p_1', \quad q_{12} = \frac{w_0'''}{6}, \quad q_{13} = p_1;$$
$$g_1 = p_7, \quad g_4 = 2g_5, \quad g_{11} = p_9', \quad g_3 = g_7 = p_7'/2, \quad g_6 = p_7;$$
$$K_v g_2 = Dp_5 - \langle Dp_5, \phi_1 \rangle v_0;$$
$$K_v g_5 = 2Dp_8 - \langle 2Dv_0, \phi_1 \rangle p_8 - a_9 v_0;$$
$$K_w g_8 = Dp_1 - \frac{H_0}{2} p_9 + 2\beta_0(p_8, \overline{v}_0) - \langle Dv_0, \phi_1 \rangle p_1 - \frac{b_3}{2} w_0';$$
$$\text{(A.9)} \qquad K_w g_9 = Dp_9 - \langle Dw_0', \phi_0 \rangle p_9 - b_4 w_0';$$
$$K_w g_{10} = Dw_0' - b_1 w_0';$$
$$K_w g_{12} = Dp_6 - \langle Dp_6, \phi_0 \rangle w_0';$$
$$K_v g_{13} = p_5 - \langle p_5, \phi_1 \rangle v_0;$$
$$K_v g_{14} = p_7 - \langle p_7, \phi_1 \rangle v_0.$$

The coefficients $h_j(\xi)$ which appear in the definition of U_3 satisfy

$$2i\omega_0 h_1 - Dh_1'' + 4k_0^2 Dh_1 + c_0 h_1' - \alpha_0 h_1 = \beta_0(v_0, p_8) + Dp_2 - \langle Dv_0, \phi_1 \rangle p_2;$$
$$2i\omega_0 h_2 - Dh_2'' + 4k_0^2 Dh_2 + c_0 h_2' - \alpha_0 h_2 = Dp_2;$$
$$2i\omega_0 h_3 - Dh_3'' + c_0 h_3' - \alpha_0 h_3 = \beta_0(v_0, p_8) + Dp_3 - \langle Dv_0, \phi_1 \rangle p_3;$$
$$\text{(A.10)} \qquad -Dh_4'' + 4k_0^2 Dh_4 + c_0 h_4' - \alpha_0 h_4 = \beta_0(p_8, \overline{v_0}) + Dp_4 - \langle Dv_0, \phi_1 \rangle p_4;$$
$$2i\omega_0 h_5 - Dh_5'' + c_0 h_5' - \alpha_0 h_5 = Dp_3$$
$$-Dh_7'' + 4k_0^2 Dh_7 + c_0 h_7' - \alpha_0 h_7 = \beta_0(\overline{p}_8, v_0) + Dp_4 - \langle Dv_0, \phi_1 \rangle p_4$$
$$-Dh_6'' + 4k_0^2 Dh_6 + c_0 h_6' - \alpha_0 h_6 = Dp_4.$$

For $j = 4, \ldots, 13$ the coefficients $s_j(\xi)$ are given by

$$\text{(A.11)} \qquad \begin{gathered} s_5 = p_2', \quad s_6 = p_2, \quad s_7 = \frac{p_3'}{4}, \quad s_9 = p_3, \quad s_{11} = \frac{p_4'}{2}, \quad s_{12} = p_4, \\ s_4 = \tfrac{1}{2}(h_1 + h_2), \quad s_8 = h_3 + h_5, \quad s_{10} = h_4 + h_6, \quad s_{13} = h_6 + h_7, \end{gathered}$$

and the equations for the remaining $s_j(\xi)$ are

$$3i\omega_0 s_1 - Ds_1'' + 9k_0^2 Ds_1 + c_0 s_1' - \alpha_0 s_1$$
$$= 2\beta_0(v_0, p_2) + \gamma_0(v_0, v_0, v_0);$$
$$\text{(A.12)} \qquad 3i\omega_0 s_2 - Ds_2'' + k_0^2 Ds_2 + c_0 s_2' - \alpha_0 s_2$$
$$= 2\beta_0(v_0, p_2) + \gamma_0(v_0, v_0, v_0) + 4\beta_0(v_0, p_3);$$
$$i\omega_0 s_3 - Ds_3'' + 9k_0^2 Ds_3 + c_0 s_3' - \alpha_0 s_3$$
$$= 2\beta_0(\overline{v_0}, p_2) + \gamma_0(v_0, v_0, \overline{v_0}) + 4\beta_0(v_0, p_4).$$

The coefficients in the evolution equations are given by

$$
\begin{aligned}
a_1 &= \langle -\nu c_1 v_0' + \nu\alpha_1 v_0, \phi_1\rangle,\\
a_2 &= \langle 3\gamma_0(v_0, v_0, \overline{v_0}) + 4\beta_0(v_0, p_1) + 2\beta_0(p_2, \overline{v_0}) - 2H_0 p_5, \phi_1\rangle,\\
a_3 &= \langle 6\gamma_0(v_0, v_0, \overline{v_0}) + 4\beta_0(v_0, p_1)\\
&\qquad + 4\beta_0(p_3, \overline{v_0}) + \beta_0(p_4, v_0) - 2H_0 p_5, \phi_1\rangle,\\
a_4 &= 4\langle Dp_5, \phi_1\rangle,\\
a_5 &= \langle Dv_0 - 4k_0^2 Dp_7 + 4k_0^2 p_7\langle Dv_0, \phi_1\rangle, \phi_1\rangle,\\
a_6 &= 4\langle Dp_5, \phi_1\rangle,\\
a_7 &= 2\langle Dp_5 - \langle Dw_0', \phi_0\rangle p_5, \phi_1\rangle \;,\\
a_8 &= 2\langle (D - \langle Dv_0, \phi_1\rangle)(p_7 + p_8), \phi_1\rangle,\\
a_9 &= 2\langle (D - \langle Dv_0, \phi_1\rangle) p_8, \phi_1\rangle;
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
b_1 &= \langle Dw_0', \phi_0\rangle,\\
b_2 &= 4\langle \beta_0(p_7, \overline{v_0}) - p_1, \phi_0\rangle,\\
b_3 &= 2\left\langle Dp_1 - \frac{H_0}{2} p_9 + 2\beta_0(p_8, \overline{v_0}) - p_1\left\langle Dv_0, \phi_1\right\rangle, \phi_0\right\rangle,\\
b_4 &= \langle Dp_9 - \langle Dw_0', \phi_0\rangle p_9, \phi_0\rangle,\\
b_5 &= \langle Dp_6, \phi_0\rangle,\\
b_6 &= 4\langle Dp_1, \phi_0\rangle;
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
d_1 &= 4\langle H_0(-q_5 + q_7) + Dq_2 + 3\gamma_0(p_7, \bar{v}_0, v_0)\\
&\qquad + 2\beta_0(q_{15}, v_0) + \beta_0(p_1, p_7), \phi_1\rangle\\
&\qquad - 4a_2\langle Dv_0 + p_7, \phi_1\rangle - 2b_2\langle p_5, \phi_1\rangle - 4\langle Dv_0, \phi_1\rangle\langle q_2, \phi_1\rangle;\\
d_2 &= 4\left\langle -H_0 q_5 + \frac{Dq_3}{2} + 3\gamma_0(p_7, \bar{v}_0, v_0) + 2\beta_0(s_8, \bar{v}_0) + 2\beta_0(s_{10}, v_0)\right.\\
&\qquad\qquad \left. + \beta_0(p_1, p_7), \phi_1\right\rangle - 2a_3\langle p_7, \phi_1\rangle - 2\langle Dv_0, \phi_1\rangle\langle q_3, \phi_1\rangle;\\
d_3 &= 2\langle -4k_0^2 Dq_7 + Dp_5 + \beta_0(q_{14}, v_0), \phi_1\rangle - 4a_4 k_0^2\langle p_7, \phi_1\rangle;\\
d_4 &= 2\langle -4Dq_6 + Dp_7, \phi_1\rangle - 2a_5\langle p_7, \phi_1\rangle + 8\langle Dv_0, \phi_1\rangle\langle q_6 - Dg_{14}, \phi_1\rangle;\\
d_5 &= -8k_0^2\langle Dq_5 + Dq_7, \phi_1\rangle + 2k_0^2 a_4\langle p_7, \phi_1\rangle\\
&\qquad - 8k_0^2\langle Dv_0, \phi_1\rangle\langle q_7 + Dg_{13}, \phi_1\rangle;\\
d_6 &= \langle 4H_0 q_7 + 2Dq_3 - 12\gamma_0(p_7, \bar{v}_0, v_0) - 8\beta_0(p_7, p_4) - 8\beta_0(q_{15}, v_0)\\
&\qquad - 8\beta_0(s_8, \bar{v}_0), \phi_1\rangle + 8b_2\langle p_5, \phi_1\rangle\\
&\qquad - 8a_3\langle (Dv_0 + p_7), \phi_1\rangle - 2\langle Dv_0, \phi_1\rangle\langle q_3, \phi_1\rangle;\\
d_7 &= 4\left\langle H_0 q_7 + \frac{Dq_3}{2} + 3\gamma_0(v_0, \bar{p}_7, v_0) + 2\beta_0(\bar{p}_7, p_3) + 2\beta_0(q_{16}, v_0)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad +2\beta_0(s_{13},v_0),\phi_1\Big\rangle - 2a_3\langle p_7,\phi_1\rangle \\
&\qquad - 2\langle Dv_0,\phi_1\rangle\langle q_3,\phi_1\rangle - 2\bar{b}_2\langle p_5,\phi_1\rangle; \\
d_8 &= \langle 4H_0q_7 + 2Dq_2 - 6\gamma_0(\bar{p}_7,v_0,v_0) - 2\beta_0(\bar{p}_8,p_2) \\
&\qquad - 8\beta_0(q_{16},v_0),\phi_1\rangle - 4a_2\langle p_7,\phi_1\rangle \\
&\qquad - 2\langle Dv_0,\phi_1\rangle\langle q_2,\phi_1\rangle + 2\bar{b}_2\langle p_5,\phi_1\rangle; \\
d_9 &= 4\langle Dg_3,\phi_1\rangle + (2a_7+a_4)\langle Dv_0,\phi_1\rangle - 2a_7\langle p_7,\phi_1\rangle; \\
d_{10} &= -4\langle k_0^2Dg_4 + Dq_6 - \frac{Dp_8}{4},\phi_1\rangle \\
&\qquad + 4\langle Dv_0,\phi_1\rangle\langle (q_6 - Dg_{14}),\phi_1\rangle - 2a_8\langle p_7,\phi_1\rangle; \\
d_{11} &= 2\langle Dg_5 + Dg_4,\phi_1\rangle - 2a_9\langle p_7,\phi_1\rangle - 2\langle Dv_0,\phi_1\rangle\langle g_4+g_5,\phi_1\rangle; \\
d_{12} &= -2\langle Dq_1 - c_1\nu p_7' + \nu\alpha_1p_7,\phi_1\rangle \\
&\qquad - 2a_1\langle Dv_0 + p_7,\phi_1\rangle - \langle q_1,\phi_1\rangle\langle Dv_0,\phi_1\rangle; \\
d_{13} &= \langle Dq_1 - c_1\nu p_8' + \nu\alpha_1p_8,\phi_1\rangle - \langle q_1,\phi_1\rangle\langle Dv_0,\phi_1\rangle; \\
d_{14} &= 2\langle Dq_2 + 3\gamma_0(p_8,v_0,\bar{v}_0) + 2\beta_0(\bar{v}_0,h_1) + 2\beta_0(g_8,v_0) \\
&\qquad +2\beta_0(p_8,p_1),\phi_1\rangle - 2\langle Dv_0,\phi_1\rangle\langle q_2,\phi_1\rangle - 2a_2\langle p_8,\phi_1\rangle \\
&\qquad + 2H_0\langle Dv_0,\phi_1\rangle\langle g_{13},\phi_1\rangle - 2H_0\langle g_7,\phi_1\rangle; \\
d_{15} &= \langle Dq_2 + 3\gamma_0(\bar{p}_8,v_0,v_0) + 2\beta_0(\bar{p}_8,p_2) + 4\beta_0(\bar{g}_8,v_0),\phi_1\rangle \\
&\qquad - \langle Dv_0,\phi_1\rangle\langle q_2,\phi_1\rangle - a_2\langle p_8,\phi_1\rangle; \\
d_{16} &= 4\langle Dq_5 + \beta_0(p_7,p_9),\phi_1\rangle - 4\langle Dv_0,\phi_1\rangle\langle g_3,\phi_1\rangle - 2a_7\langle p_8,\phi_1\rangle; \\
d_{17} &= 2\langle Dg_3 + 2\beta_0(g_9,v_0),\phi_1\rangle - 2b_1\langle g_3,\phi_1\rangle; \\
d_{18} &= \langle Dg_5,\phi_1\rangle - a_9\langle p_8,\phi_1\rangle; \\
d_{19} &= -a_3\langle p_8,\phi_1\rangle - \langle Dv_0,\phi_1\rangle\langle q_3,\phi_1\rangle \\
&\qquad + \langle 4\beta_0(p_3,\bar{p}_8) + Dq_3 + 6\gamma_0(v_0,v_0,\bar{p}_8) \\
&\qquad\qquad +4\beta_0(\bar{g}_8,v_0) + 4\beta_0(h_7,v_0),\phi_1\rangle; \\
d_{20} &= -a_3\langle p_8,\phi_1\rangle - \langle Dv_0,\phi_1\rangle\langle q_3,\phi_1\rangle \\
&\qquad + \langle 4\beta_0(p_4,p_8) + 6\gamma_0(v_0,\bar{v}_0,p_8) + \frac{Dq_3}{4} \\
&\qquad + 4\beta_0(g_8,v_0) + 4\beta_0(h_3,\bar{v}_0),\phi_1\rangle; \\
d_{21} &= -a_3\langle p_8,\phi_1\rangle - \langle Dv_0,\phi_1\rangle\langle q_3,\phi_1\rangle \\
&\qquad + 2H_0\langle Dv_0,\phi_1\rangle\langle g_{13},\phi_1\rangle - 2H_0\langle g_7,\phi_1\rangle \\
&\qquad + \langle 4\beta_0(v_0,h_4) + 6\gamma_0(v_0,\bar{v}_0,p_8) \\
&\qquad + \frac{Dq_3}{4} + 4\beta_0(\bar{v}_0,h_3) + 4\beta_0(p_8,p_1),\phi_1\rangle; \\
d_{22} &= 8\langle Dg_2 + Dq_7,v_0\rangle; \\
d_{23} &= 8\langle Dg_2 + Dg_5,\phi_1\rangle - 4\langle Dv_0,\phi_1\rangle\langle g_2,\phi_1\rangle;
\end{aligned} \tag{A.15}
$$

$$d_{24} = 4\langle Dg_7 + Dq_7, \phi_1\rangle - \langle Dv_0, \phi_1\rangle\langle q_7 + Dg_{13}, \phi_1\rangle - a_4\langle p_8, \phi_1\rangle;$$
$$d_{25} = 4\langle Dq_2, \phi_1\rangle - 2a_2\langle p_8, \phi_1\rangle - 4H_0\langle g_2, \phi_1\rangle - 4H_0\langle g_3, \phi_1\rangle;$$
$$d_{26} = 2\langle Dq_2 + 2\beta_0(h_2, \bar{v}_0), \phi_1\rangle - 2a_2\langle p_8, \phi_1\rangle - 4H_0\langle g_3, \phi_1\rangle - 4H_0\langle g_2, \phi_1\rangle;$$
$$d_{27} = 2\langle Dq_2 + 4\beta_0(h_5, \bar{v}_0), \phi_1\rangle - 2a_3\langle p_8, \phi_1\rangle - 4H_0\langle g_2, \phi_1\rangle;$$
$$d_{28} = 2\langle Dq_3 + 4\beta_0(h_6, v_0), \phi_1\rangle - 2a_3\langle p_8, \phi_1\rangle - 4H_0\langle g_2, \phi_1\rangle;$$
$$d_{29} = 2\langle Dq_3, \phi_1\rangle - 2a_3\langle p_8, \phi_1\rangle - 4H_0\langle g_3, \phi_1\rangle;$$
$$d_{31} = 2\langle Dq_9 + 2\beta_0(g_{12}, v_0), \phi_1\rangle;$$
$$d_{32} = 4\langle Dg_2 + Dg_3, \phi_1\rangle - (a_6 + 2a_7)\langle p_8, \phi_1\rangle - 4b_1\langle g_2, \phi_1\rangle;$$
$$d_{33} = 2\langle 4Dg_2 + Dg_3 + Dg_7 + \beta_0(p_9, p_8), \phi_1\rangle - (2a_6 + a_7)\langle p_8, \phi_1\rangle - 2\langle Dv_0, \phi_1\rangle\langle g_3, \phi_1\rangle - 2b_1\langle g_7, \phi_1\rangle + 2\langle Dv_0, \phi_1\rangle b_1\langle g_{13}, \phi_1\rangle;$$
$$d_{34} = 4\langle Dg_2 + Dg_7, \phi_1\rangle - a_6\langle p_8, \phi_1\rangle;$$
$$f_1 = 2\langle Dp_6, \phi_0\rangle;$$
$$f_2 = 16\mathrm{Im}\langle \beta_0(\bar{q}_7, v_0), \phi_0\rangle + 4\mathrm{Im}(a_4\langle p_1, \phi_0\rangle);$$
$$f_3 = -H_0\langle q_{14}, \phi_0\rangle - 8\langle \beta_0(q_6, \bar{v}_0), \phi_0\rangle + \langle Dp_1, \phi_0\rangle - 2a_5\langle p_1, \phi_0\rangle + 8k_0^2\langle Dv_0, \phi_0\rangle\langle q_{15}, \phi_0\rangle;$$
$$f_4 = -8\mathrm{Re}a_2\langle p_1, \phi_0\rangle + 4\langle \beta_0(p_1, p_1) + 6\delta(v_0, v_0, \bar{v}_0, \bar{v}_0), \phi_0\rangle - H_0\langle q_{11}, \phi_0\rangle + 2\langle \beta_0(p_2, \bar{p}_2) + 6\gamma_0(v_0, \bar{v}_0, p_1) + \mathrm{Re}(3\gamma_0(p_2, \bar{v}_0, \bar{v}_0) + 2\beta_0(q_2, \bar{v}_0)), \phi_0\rangle;$$
$$f_5 = -2H_0\langle q_{11}, \phi_0\rangle + 8\langle \beta_0(p_3, \bar{p}_3) + \beta_0(p_4, \bar{p}_4) + 3\gamma_0(p_1, \bar{v}_0, v_0) + \mathrm{Re}(-8a_3\langle p_1, \phi_0\rangle + 3\gamma_0(p_3, \bar{v}_0, \bar{v}_0) + 3\gamma_0(p_4, \bar{v}_0, v_0) + \beta_0(v_0, \bar{q}_3)), \phi_0\rangle + 8\langle \beta_0(p_1, p_1) + 9\delta_0(v_0, v_0, \bar{v}_0, \bar{v}_0), \phi_0\rangle;$$
$$f_6 = 2\langle -c_1\nu p_1' + \nu\alpha_1 p_1, \phi_0\rangle - 4a_1\langle p_1, \phi_0\rangle + \langle \beta_0(q_1, \bar{v}_0) + \beta_0(v_0, \bar{q}_1) + \nu\beta_1(v_0, \bar{v}_0), \phi_0\rangle - H_0\langle q_{10}, \phi_0\rangle - 4\bar{a}_1\langle p_1, \phi_0\rangle;$$
$$f_7 = 8k_0^2\langle \beta_0(p_7, \bar{p}_7), \phi_0\rangle + 4\langle Dp_9, \phi_0\rangle;$$
$$f_8 = 4\langle 2\beta_0(p_8, \bar{p}_7) + Dq_{16}, \phi_0\rangle + \bar{b}_2\langle p_9, \phi_0\rangle;$$
$$f_9 = -\langle 4\beta_0(p_1, p_9) + Dq_{11} + 6\gamma_0(v_0, p_9, \bar{v}_0) + 16\mathrm{Re}\beta_0(v_0, \bar{g}_3), \phi_0\rangle - H_0\langle g_{11}, \phi_0\rangle;$$
$$f_{10} = -4\langle Dq_{15} + \beta_0(g_4, v_0), \phi_0\rangle + b_2\langle p_9, \phi_0\rangle;$$
$$f_{11} = -2\langle Dg_8 + \beta_0(g_5, \bar{v}_0), \phi_0\rangle + b_3\langle p_9, \phi_0\rangle;$$
$$f_{12} = \langle Dp_9 + Dq_{14}, \phi_0\rangle - b_1\langle p_9, \phi_0\rangle;$$
$$f_{13} = \langle -c_1\nu p_9' + \nu\alpha_1 p_9 + Dq_{10}, \phi_0\rangle;$$

(A.16)
$$\begin{aligned}
f_{14} &= \langle Dg_9, \phi_0 \rangle - b_4 \langle p_9, \phi_0 \rangle - b_1 \langle g_9, \phi_0 \rangle; \\
f_{15} &= \langle \beta_0(p_9, p_9) + Dg_{11} + 2Dg_{12}, \phi_0 \rangle - b_1 \langle g_{11}, \phi_0 \rangle; \\
f_{16} &= 2\langle Dg_8, \phi_0 \rangle + (b_3 + 2b_6) \langle p_9, \phi_0 \rangle - 2H_0 \langle g_9, \phi_0 \rangle; \\
f_{17} &= 2\langle Dq_{11}, \phi_0 \rangle + 4\langle \beta_0(\bar{v}_0, g_2), \phi_0 \rangle - 4H_0 \langle Dg_{12}, \phi_0 \rangle; \\
f_{18} &= 8\langle Dq_{15}, \phi_0 \rangle - 2b_2 \langle p_9, \phi_0 \rangle; \\
f_{19} &= 4\langle Dg_8, \phi_0 \rangle - (2b_3 + b_6) \langle p_9, \phi_0 \rangle; \\
f_{20} &= 2\langle Dg_{11} + 2Dg_{12}, \phi_0 \rangle - 2b_5 \langle p_9, \phi_0 \rangle - 4b_1 \langle g_{12}, \phi_1 \rangle.
\end{aligned}$$

Appendix B. For quasiperiodic waves, the characteristic equation is a fifth-order polynomial equation corresponding to the matrix

(B.1)
$$\mathscr{M} \equiv \begin{bmatrix} A^* & B^* & Q^* & R^* & C^* \\ F^* & G^* & T^* & V^* & H^* \\ \hat{Q} & \hat{R} & \hat{A} & \hat{B} & \hat{C} \\ \hat{T} & \hat{V} & \hat{F} & \hat{G} & \hat{H} \\ J^* & K^* & \hat{J} & \hat{K} & L^* \end{bmatrix}$$

where

(B.2)
$$\begin{aligned}
A^* &= A + a_3^r \sigma^2 - \epsilon(k_0 d_2^r k_1 \sigma^2 + (d_6^r - d_7^r) k_0 k_2 \sigma^2 \\
&\qquad + i\Gamma_1 d_2^i k_0 \sigma^2 + d_{21}^r \sigma^2 \Gamma_2^2), \\
B^* &= B - \epsilon(i\Gamma_1 k_0 d_2^r \sigma^2 \rho - \Gamma_2^2 d_{21}^i \sigma^2 \rho), \\
C^* &= C, \quad H^* = H, \\
F^* &= F + \epsilon\left(i\Gamma_1 d_2^r \frac{\sigma^2}{\rho} - \Gamma_2^2 d_{21}^i \frac{\sigma^2}{\rho} \right), \\
G^* &= G + \epsilon(-i\Gamma_1 k_0 d_2^i \sigma^2 - \Gamma_2^2 d_{21}^r \sigma^2), \\
J^* &= J + 2\epsilon f_5 \sigma^2 \rho, \\
K^* &= K - 2\epsilon \Gamma_2^2 k_0 f_8^r k_1 \rho^2, \\
L^* &= L + \epsilon(-i\Gamma_1 k_0 f_2 \sigma^2 + \Gamma_2^2 f_9 \sigma^2), \\
Q^* &= 2\rho\sigma\big(a_3^r + \epsilon\big(-2k_0(d_6^r - d_7^r)k_2 - 2k_0 k_1 d_2^r \\
&\qquad - i\Gamma_1(d_6^i + d_7^i)k_0 - (d_{19}^r + d_{20}^r)\Gamma_2^2\big)\big), \\
R^* &= \epsilon\big(i\Gamma_1 k_0(d_7^r - d_6^r) - (d_{19}^i - d_{20}^i)\Gamma_2^2\big)\rho\sigma^2, \\
T^* &= 2\sigma\big(a_3^i + \epsilon\big(-2k_0(d_6^i - d_7^i)k_2 - 2k_0 k_1 d_2^i \\
&\qquad + i\Gamma_1(d_6^r + d_7^r)k_0 - (d_{19}^i + d_{20}^i)\Gamma_2^2\big)\big), \\
V^* &= \epsilon\big(i\Gamma_1 k_0(d_7^i - d_6^i) + (d_{19}^r - d_{20}^r)\Gamma_2^2\big)\sigma^2,
\end{aligned}$$

where A, B, C, F, G, H, J, K, and L are defined as in (4.26)–(4.34), and the quantities with $\hat{}$ are obtained by interchanging ρ and σ, interchanging k_1 and k_2, and replacing Γ_1 with $-\Gamma_1$ in the quantities with *.

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