1-1-1999

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Arthur T. Benjamin
Harvey Mudd College

Matthew T. Fluet '99
Harvey Mudd College

The

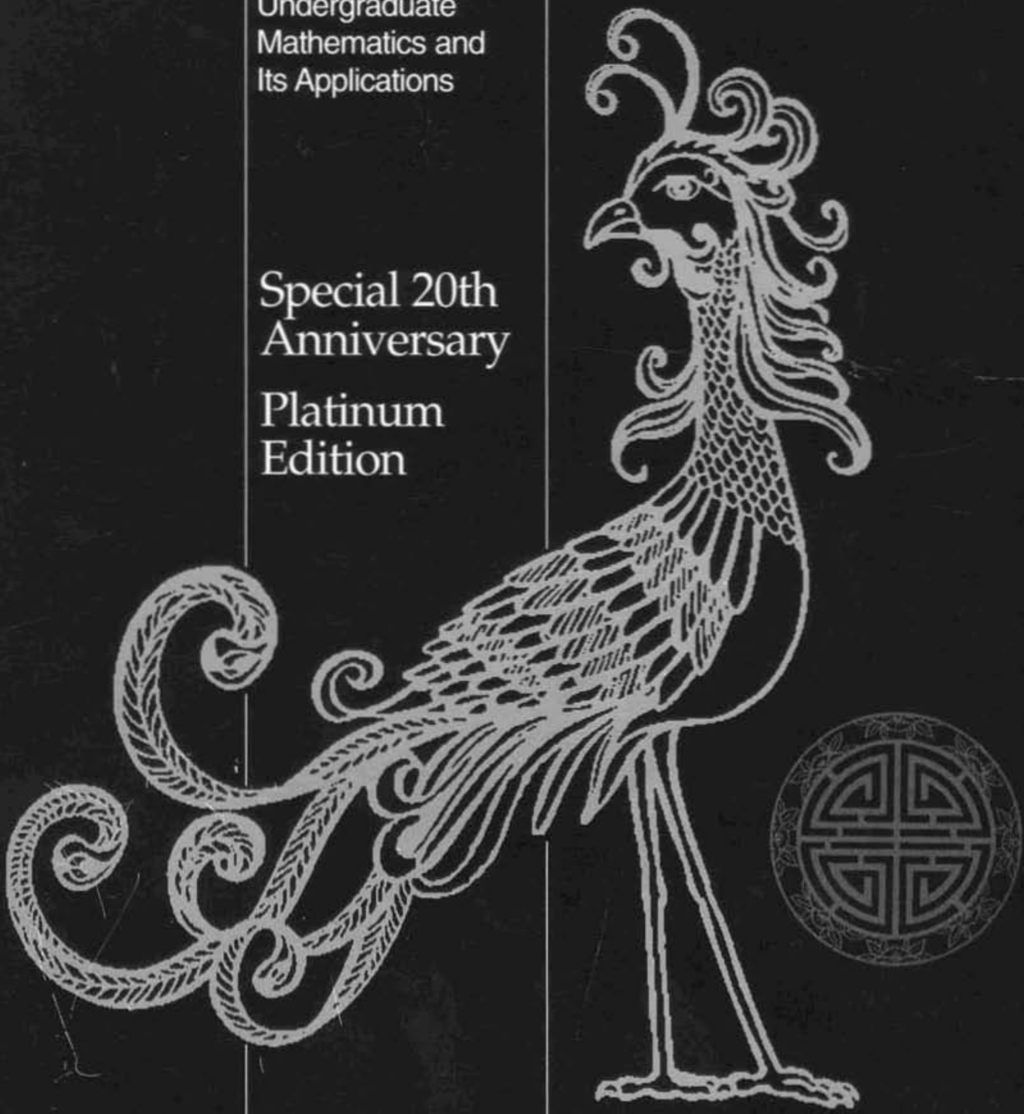
UMAP

Journal

The Journal of
Undergraduate
Mathematics and
Its Applications

Special 20th
Anniversary

Platinum
Edition



COMAP

Spring 1999

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Bounds on a Bug

Arthur T. Benjamin
Matthew T. Fluet
Dept. of Mathematics
Harvey Mudd College
Claremont, CA 91711
{benjamin, matthew_fluet} @hmc.edu

Introduction

In the game of Cootie[1], players race to construct a "cootie bug" by rolling a die to collect component parts. Each cootie bug is composed of a body, a head, two eyes, one nose, two antennae, and six legs. Players must first acquire the body of the bug by rolling a 1. Next, they must roll a 2 to add the head to the body. Once the body and head are both in place, the remaining body parts can be obtained in any order by rolling two 3s for the eyes, one 4 for the nose, two 5s for the antennae, and six 6s for the legs. This game raises the question:

If the game lasts for T turns, what is $E[T]$, the theoretical expected value of the number of rolls required to make a cootie?

Two previous articles (Benjamin and Fluet [1999], Deng and Whalen [1988]) have addressed this question by determining the expected value exactly and showing that $E[T] = 48.95242+$. However, both methods required extensive computer calculations to derive this result, methods that provide no intuitive explanation for this number. In this article, we employ a different strategy to obtain close approximations of $E[T]$ by simple "back of the envelope" calculations. Through these calculations, the exact value of $E[T]$ becomes much less mysterious.

Envelope 1: A Quick Approximation

The rules of Cootie naturally break an analysis of the playing time T into three parts:

$$T = B + H + T',$$

The UMAP Journal 20 (1) (1999) 5–9.

[1]Cootie is a trademark of Milton Bradley Co., Springfield, MA.

where B and H denote the number of rolls to obtain the body and head, respectively, and T' is the number of rolls to subsequently obtain two 3s, one 4, two 5s, and six 6s. Since $E[B] = E[H] = 6$, we have, by the linearity of expectation,

$$E[T] = 12 + E[T']. \tag{1}$$

At this point, we exploit the fact that in the vast majority of Cootie games, we will complete the eyes, nose, and antennae before completing the six legs. Thus, we express

$$T' = L + R,$$

where L denotes the time to roll six 6s (once the body and head are in place) and R denotes the number of rolls required to clean up any 3s, 4s, and 5s that remain after rolling six 6s. Since $E[L] = 36$, we have

$$E[T] = 48 + E[R].$$

Since $R \geq 0$, we immediately have $E[T] \geq 48$.

Envelope 2: A Better Approximation

To improve this approximation, we compute $P[R > 0]$. This is the probability that we rolled fewer than two 3s or fewer than one 4 or fewer than two 5s in the process of rolling six 6s. If we let X_3 be the event of having some 3s left to roll after rolling six 6s, let X_{34} be the event of having both some 3s and some 4s left to roll, and let X_4, X_5, X_{35}, X_{45}, and X_{345} be defined similarly, then by the Principle of Inclusion–Exclusion,

$$P[R > 0] = P[X_3] + P[X_4] + P[X_5] - P[X_{34}] - P[X_{35}] - P[X_{45}] + P[X_{345}]. \tag{2}$$

Notice that X_4 is the event that no 4 was rolled before the six 6s. To calculate $P[X_4]$, we observe that on any sequence of rolls, the probability is $\frac{1}{2}$ that we roll a 6 before we roll a 4. Thus, to roll six or more 6s before the first 4 has probability $\frac{1}{2^6} = \frac{1}{64}$, so $P[X_4] = \frac{1}{64}$. Likewise, for X_3, we must roll zero or one 3 before our six 6s. The latter event can happen 6 ways, each occurrence having probability $\frac{1}{2^7} = \frac{1}{128}$. Thus,

$$P[X_3] = \frac{1}{64} + \frac{6}{128} = \frac{1}{16}$$

We can exploit the symmetrical roles that 3 and 5 play in the game to state that $P[X_5] = P[X_3] = \frac{1}{16}$.

We are now in a position to make a rough approximation of $E[R]$. Note that if we have any 3s, 4s, or 5s left after rolling six 6s, then the expected number of rolls needed to achieve them is at least 6. On the other hand, if we assume that all intersections of these events have negligible probability, then $P[R > 0] \approx P[X_3] + P[X_4] + P[X_5]$, and

$$E[R] \approx 6(P[X_3] + P[X_4] + P[X_5]) = 6 \cdot \frac{9}{64} = \frac{27}{32} = 0.84375.$$

Thus, $E[T] \approx 48.84375$.

Envelope 3: A Quick Lower Bound

For a more rigorous approximation and lower bound, we will need to finish evaluating $P[R > 0]$. The event X_{34} requires that we roll at most one 3 and no 4 before our six 6s. As before, on any sequence of rolls, the probability is $\frac{1}{3}$ that we roll a 6 before a 3 or a 4. Consequently, the probability of no 3 and no 4 before six 6s is $\frac{1}{3^6} = \frac{1}{729}$. The probability of exactly one 3 and no 4 is $\frac{6}{3^7}$: Of the 3^7 equally likely length-7 sequences of 3s, 4s and 5s, exactly 6 of them consist of one 3 and six 6s where the 3 is not last. Thus, $P[X_{34}] = \frac{1}{3^6} + \frac{6}{3^7} = \frac{1}{243}$. By symmetry, $P[X_{45}] = \frac{1}{243}$. Similarly, $P[X_{35}]$ is computed like $P[X_{34}]$ with an additional $\frac{6}{3^7}$ term, but we have one more possibility, namely, the occurrence of exactly one 3 and one 5 before the sixth 6. Among the 3^8 equally likely outcomes, there are $2 \cdot \binom{7}{2} = 42$ ways this can occur. Thus, $P[X_{35}] = \frac{1}{243} + \frac{6}{3^7} + \frac{42}{6561} = \frac{29}{2187}$. Using similar reasoning, we get

$$P[X_{345}] = \frac{1}{4^6} + 2 \cdot 6 \cdot \frac{1}{4^6} + 7 \cdot 6 \cdot \frac{1}{4^8} = \frac{53}{32768}.$$

Consequently, by **(2)**,

$$\begin{aligned} P[R > 0] &= \frac{1}{16} + \frac{1}{64} + \frac{1}{16} - \frac{1}{243} - \frac{29}{2187} - \frac{1}{243} + \frac{53}{32768} \\ &= \frac{8653511}{2^{15}3^7} \\ &= \frac{8653511}{71663616} \\ &= 0.12075+. \end{aligned}$$

To place a lower bound on $E[R]$, we again note that if we have any 3s, 4s, or 5s left after rolling six 6s, then the expected number of rolls needed to achieve them is greater than 6. Thus,

$$E[R] > 6 \cdot P[R > 0] = \frac{8653511}{11943936} = 0.72451+,$$

and $E[T] > 48.72451$.

Envelope 4: A Better Lower Bound

With just a little more work, we are able to get an even tighter lower bound on $E[R]$. Consider a variation of the game in which we play as in Cootie until we obtain a sixth 6. If the 3s, 4s, and 5s are all obtained at this point, we are done. If there are any rolls left, then we finish just one of the numbers remaining. We must choose a number that has at least as many needed rolls as any other number. If there is a tie, we must complete the lower number. (For instance, if we are left with one 3 and two 5s, we have to get the two 5s. If we are left with one 3 and one 4, we need to get a 3.)

We note that this game breaks down into two cases: endgames with two 3s or two 5s remaining, which have an expected completion time of 12 rolls, and endgames with one 3 or one 4 or one 5 remaining, which have an expected completion time of 6 rolls. If we let R' be the time to complete this game after obtaining six 6s, we note that $P[R > 0] = P[R' > 0]$ and $E[R] > E[R']$. Further, if we let Z be the event of having two 3s or two 5s remaining, then

$$P[Z] = \frac{1}{2^6}P[\text{two 3s remain}] + \frac{1}{2^6}P[\text{two 5s remain}] - \frac{1}{3^6}P[\text{two 3s and two 5s remain}].$$

Thus,

$$\begin{aligned} E[R] > E[R'] &= 12 \cdot P[Z] + 6 \cdot \left(P[R' > 0] - P[Z]\right) \\ &= \frac{10794695}{11943936} \\ &= 0.90378+, \end{aligned}$$

and $E[T] > 48.90378$.

Envelope 5: A Quick Upper Bound

For a tight upper bound, we explore another variation of the game, in which we again play as in Cootie until we obtain a sixth 6. If the 3s, 4s, and 5s are all obtained at this point, we are done. If there are any rolls left, then we must first obtain the remaining 3s, then obtain the remaining 4s, and finally obtain the remaining 5s. If we let R^* be the time to complete this game after obtaining six 6s, then $R \leq R^*$and we must have $E[R] < E[R^*]$. Further, letting R_3^*, R_4^*, and R_5^* be the time to obtain the 3s, 4s, and 5s after the six 6s, then $R^* = R_3^* + R_4^* + R_5^*$ and $E[R^*] = E[R_3^*] + E[R_4^*] + E[R_5^*]$. The probability that we have two 3s remaining is $\frac{1}{2^6}$, with an expected completion time of 12 rolls; and the probability that we have one 3 remaining is $6 \cdot \frac{1}{2^7}$, with an expected completion time of 6 rolls. Thus, $E[R_3^*] = 12 \cdot \frac{1}{64} + 6 \cdot 6 \cdot \frac{1}{2^7} = \frac{15}{32}$.

By symmetry, $E[R_5^*] = \frac{15}{32}$. The probability that we have one 4 remaining is $\frac{1}{2^6}$ with an expected completion time of 6 rolls, so $E[R_4^*] = 6 \cdot \frac{1}{2^6} = \frac{3}{32}$. Thus,

$$E[R] < E[R^*] = \frac{15}{32} + \frac{3}{32} + \frac{15}{32} = \frac{33}{32} = 1.03125.$$

Hence, $E[T] < 49.03125$.

Conclusion

Thus, by elementary means, we have

$$48.90378 < E[T] < 49.03125.$$

Acknowledgment

We thank Professor Janet Myhre and the Institute for Decision Sciences for supporting this research.

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About the Authors

Arthur Benjamin, famous as a "mathemagician" and human calculator, is the all-time point leader of the American backgammon Tour. He is an associate professor of mathematics at Harvey Mudd College, where he has taught since earning his Ph.D. in Mathematical Sciences from Johns Hopkins in 1989. He has co-authored with students several previous articles in *The UMAP Journal* about games.

Matthew Fluet is a senior mathematics major at Harvey Mudd College, pursuing the Mathematics with Computer Science option. In addition to mathematics, he is completing a humanities concentration in Medieval Studies. After graduation, he plans to begin graduate studies in Computer Science.